Paradan's wall crossing formula for partition functions and Khovanski-Pukhlikov differential operator.

Arzu Boysal and Michèle Vergne

March 2008

Contents

1 Introduction

The function computing the number of ways one can decompose a vector as a linear combination with nonnegative integral coefficients of a fixed finite set of integral vectors is called a **partition function.** This problem can be expressed in terms of polytopes as follows. Let A be a r by N integral matrix with column vectors $\phi_1, \ldots, \phi_N$, and assume that the elements ϕ_k generate the lattice $\mathbb{Z}^r$. Let $a \in \mathbb{Z}^r$ be a r-dimensional integral column vector and let $P(\Phi, a) := \{y \in \mathbb{R}^N_{\geq 0} | Ay = a\}$ be the convex polytope associated to $\Phi = [\phi_1, \phi_2, \ldots, \phi_N]$ and a. The number of ways one can decompose a as a linear combination with nonnegative integral coefficients of the vectors ϕ_i is the number of integral points in $P(\Phi, a)$. The function $a \to |P(\Phi, a) \cap \mathbb{Z}^N|$ will be called the partition function $k(\Phi)(a)$. It is intuitively clear that $k(\Phi)(a)$ is related to the volume function $\mathrm{vol}(\Phi)(a) = \mathrm{volume}(P(\Phi, a))$. The latter varies polynomially as a function of a, provided the polytope $P(\Phi, a)$ does not change 'shape', that is, when a varies in a chamber $\mathfrak{c}$ for A. In short, there is a decomposition of $\mathbb{R}^r$ in closure of chambers $\mathfrak{c}_i$ and polynomial functions $v(\Phi, \mathfrak{c}_i)$ such that the function $\mathrm{vol}(\Phi)(a)$ coincide with the polynomial function $v(\Phi, \mathfrak{c}_i)(a)$ on each cone $\mathfrak{c}_i$. Similarly, there exists quasi-polynomial functions $k(\Phi, \mathfrak{c}_i)$ on $\mathbb{Z}^r$ such that the function $k(\Phi)(a)$ coincide with the quasi-polynomial function $k(\Phi, \mathfrak{c}_i)(a)$ on $\mathfrak{c}_i \cap \mathbb{Z}^r$.

When $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are adjacent chambers, P-E. Paradan [8] gave a remarkable formula for the quasi-polynomial function $k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2)$ as a convolution of distributions. There is an analogous formula for $v(\Phi, \mathfrak{c}_1) - v(\Phi, \mathfrak{c}_2)$. In this note, we give an elementary algebraic proof of Paradan's convolution formula for the jumps. We also express $k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2)$ and $v(\Phi, \mathfrak{c}_1) - v(\Phi, \mathfrak{c}_2)$ by one-dimensional residue formulae. Let us describe these residue formulae.

Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be two adjacent chambers lying on two sides of a hyperplane W (determined by a primitive vector E). Define $\Phi_0 = \Phi \cap W$. The intersection of $\overline{\mathfrak{c}_1}$ and $\overline{\mathfrak{c}_2}$ is contained in the closure of a chamber $\mathfrak{c}_{12}$ of Φ_0.

Theorem 1 • *Let $v_{12} = v(\Phi_0, \mathfrak{c}_{12})$ be the polynomial function on W associated to the chamber $\mathfrak{c}_{12}$ of Φ_0. Let V_{12} be any polynomial function* *on $\mathbb{R}^r$ extending v_{12}. Then, if $\langle E, \mathfrak{c}_1\rangle > 0$, we have for $a \in \mathbb{R}^r$*

$$v(\Phi,\mathfrak{c}_1)(a) - v(\Phi,\mathfrak{c}_2)(a) = \mathrm{Res}_{z=0}\left(V_{12}(\partial_x)\cdot \frac{e^{\langle a, x+zE\rangle}}{\prod_{\phi\in\Phi\setminus\Phi_0}\langle\phi, x+zE\rangle}\right)_{x=0}.$$

- *Suppose Φ is unimodular. Let $k_{12} = k(\Phi_0, \mathfrak{c}_{12})$ be the polynomial function on W associated to the chamber $\mathfrak{c}_{12}$ of Φ_0. Let K_{12} be any polynomial function on $\mathbb{R}^r$ extending k_{12}. Then, if $\langle E, \mathfrak{c}_1\rangle > 0$, we have for $a \in \mathbb{R}^r$*

$$k(\Phi,\mathfrak{c}_1)(a) - k(\Phi,\mathfrak{c}_2)(a) = \mathrm{Res}_{z=0}\left(K_{12}(\partial_x)\cdot \frac{e^{\langle a, x+zE\rangle}}{\prod_{\phi\in\Phi\setminus\Phi_0}(1-e^{-\langle\phi, x+zE\rangle})}\right)_{x=0}.$$

In fact we will give a general version of the second part of this theorem in Section 6, where Φ is not necessarily unimodular.

It is immediate to see that both formulae for the jumps $k(\Phi,\mathfrak{c}_1) - k(\Phi,\mathfrak{c}_2)$ and $v(\Phi,\mathfrak{c}_1) - v(\Phi,\mathfrak{c}_2)$ are related by the application of a generalized Khovanski-Pukhlikov differential operator [4], [7], [3].

We also demonstrate in various examples how to use these formulae to compute the functions $v(\Phi,\mathfrak{c})$ and $k(\Phi,\mathfrak{c})$.

List of Notations

U	r-dimensional real vector space; $x \in U$; dx Lebesgue measure on U.
V	dual of U; $a \in V$; da dual Lebesgue measure on V.
Γ	a lattice in V; $\gamma \in \Gamma$.
Γ^*	dual lattice in U; $\langle\Gamma, \Gamma^*\rangle \subset \mathbb{Z}$.
$T(\Gamma), T$	torus U/Γ^*.
Φ	a sequence of nonzero vectors in Γ all on one side of a hyperplane ;$\phi \in \Phi$.
$C(\Phi)$	cone generated by Φ.
$< \Phi >$	vector space generated by Φ.
$\mathbb{Z}\Phi$	lattice generated by Φ in $< \Phi >$.
$P(\Phi, a)$	convex polytope associated to Φ and a.
$\mathrm{vol}(\Phi, dx)(a)$	volume of $P(\Phi, a)$ for the quotient measure dt/da.
$k(\Phi)(a)$	number of integral points in $P(\Phi, a)$.

$\mathfrak{c}$	a chamber of $C(\Phi)$.
$v(\Phi, dx, \mathfrak{c})$	polynomial function coinciding with $\mathrm{vol}(\Phi, dx)$ on the cone $\mathfrak{c}$.
$k(\Phi, \mathfrak{c})$	quasi-polynomial function coinciding with $k(\Phi)$ on $\mathfrak{c} \cap \Gamma$.
K	quasi-polynomial function of a.
$\tau(\gamma)$	translation operator $(\tau(\gamma)K)(a) = K(a-\gamma)$.
$D(\phi)$	difference operator $(D(\phi)K)(a) = K(a) - K(a-\gamma)$.
W	hyperplane in V defined for a fixed $E \in U$ by $\{a \in V \| \langle a, E\rangle = 0\}$.
Ψ	a sequence of vectors in Γ not in W; $\psi \in \Psi$.
Δ^+	a sequence of vectors all on one side of W; $\alpha \in \Delta^+$.

2 Partition functions

2.1 Definitions and notations

Let U be a r-dimensional real vector space and V be its dual vector space. We assume that V is equipped with a lattice Γ. We will usually denote by x the variable in U and by a the variable in V. We will see an element P of $S(V)$ both as a polynomial function on U and a differential operator on V via the relation $P(\partial_a)e^{\langle a,x\rangle} = P(x)e^{\langle a,x\rangle}$.

Let $\Phi = [\phi_1, \phi_2, \ldots, \phi_N]$ be a sequence of non-zero, not necessarily distinct, linear forms on U lying in an open half space. Assume that all the $\phi_k \in \Phi$ belong to the lattice Γ. We denote by $< \Phi >$ the linear span of Φ. Then Φ generates a lattice in $< \Phi >$. We denote this lattice by $\mathbb{Z}\Phi \subset \Gamma$.

We consider $\mathbb{R}^N$ with basis $(\omega_1, \ldots, \omega_N)$ and let A be the linear map from $\mathbb{R}^N$ to the vector space $< \Phi >$ defined by $A(\omega_k) = \phi_k,\ 1 \leq k \leq N$. The vectors ϕ_k are the column vectors of the matrix A, and the map A is surjective onto $< \Phi >$. For $a \in < \Phi >$, we consider the convex polytope

$$P(\Phi, a) := \{t = (t_1, t_2, \ldots, t_N) \in \mathbb{R}^N_{\geq 0} |\ At = a\}.$$

In other words,

$$P(\Phi, a) = \{t = (t_1, t_2, \ldots, t_N) \in \mathbb{R}^N_{\geq 0} |\ \sum_i t_i\phi_i = a\}.$$

Any polytope can be realized as a polytope $P(\Phi, a)$.

Let $C(\Phi) \subset < \Phi >$ be the cone generated by $\{\phi_1, \ldots, \phi_N\}$. The cone $C(\Phi)$ is a pointed polyhedral cone. The dual cone $C(\Phi)^*$ of $C(\Phi)$ is defined by $C(\Phi)^* = \{x \in U\ |\langle \phi, x\rangle \geq 0 \text{ for all } \phi \in \Phi\}$ and its interior is non–empty.

The polytope $P(\Phi, a)$ is empty if a is not in $C(\Phi)$. If $a \in <\Phi>$ is in the relative interior of the cone $C(\Phi)$, then the polytope $P(\Phi, a)$ has dimension $d := N - \dim(<\Phi>)$.

We choose dx on $<\Phi>^*$ and denote by da the dual measure on $<\Phi>$. Let dt be the Lebesgue measure on $\mathbb{R}^N$. The vector space $\mathrm{Ker}(A) = A^{-1}(0)$ is of dimension $d = N - \dim(<\Phi>)$ and it is equipped with the quotient Lebesgue measure dt/da satisfying $(dt/da) \wedge da = dt$. For $a \in <\Phi>$, $A^{-1}(a)$ is an affine space parallel to $\mathrm{Ker}(A)$, thus also equipped with the Lebesgue measure dt/da. Volumes of subsets of $A^{-1}(a)$ are computed with this measure. In particular we can define for any $a \in <\Phi>$, the number $\mathrm{vol}(\Phi)(a, dx)$ as being the volume of the convex set $P(\Phi, a)$ in the affine space $A^{-1}(a)$ equipped with the measure dt/da. If dx is rescaled by $c > 0$, then $\mathrm{vol}(\Phi)(a, c dx) = c\, \mathrm{vol}(\Phi)(a, dx)$. By definition, if the dimension of $P(\Phi, a)$ is less than d, $\mathrm{vol}(\Phi)(a, dx)$ is equal to 0.

Definition 2 *Let $<\Phi>$ be the subspace of V generated by Φ.*

- *If $a \in <\Phi>$, define* $\mathrm{vol}(\Phi, dx)(a) = \mathrm{volume}(P(\Phi, a), dt/da)$.
- *If $a \in <\Phi>$, define* $k(\Phi)(a) = |P(\Phi, a) \cap \mathbb{Z}^N|$.

We extend the definition of the functions $\mathrm{vol}(\Phi, dx)(a)$ *and* $k(\Phi)(a)$ *as functions on V by defining* $\mathrm{vol}(\Phi, dx)(a) = 0$ *if $a \notin <\Phi>$,* $k(\Phi)(a) = 0$ *if $a \notin <\Phi>$.*

Clearly, $\mathrm{vol}(\Phi, dx)(a) = 0$ if a is not in $C(\Phi)$ and $k(\Phi, a) = 0$ if a is not in $\mathbb{Z}\Phi \cap C(\Phi)$.

In the rest of this article, we will formulate many of our statements when Φ generates V, as we can always reduce to this case replacing eventually V by $<\Phi>$.

If $\Phi = [\phi_1, \phi_2, \ldots, \phi_r]$ consists of linearly independent vectors, then the set $P(\Phi, a)$ is just one point when $a \in C(\Phi)$ and is empty when a is not in the closed cone $C(\Phi)$. Thus the function $\mathrm{vol}(\Phi)(a, dx)$ is just the characteristic function of the closed cone $C(\Phi)$ multiplied by $|\det(\Phi)|^{-1}$ where the determinant is computed with respect to the Lebesque measure da. Similarly, the function $k(\Phi)(a)$ is the characteristic function of $C(\Phi) \cap \sum_{i=1}^r \mathbb{Z}\phi_i$.

Lemma 3 *Assume $\Phi = \Phi' \cup \{\phi\}$ where Φ' generates $<\Phi>$. Then*

$$\mathrm{vol}(\Phi, dx)(a) = \int_{t \geq 0} \mathrm{vol}(\Phi', dx)(a - t\phi) dt$$

for any $a \in V$.

Proof. Indeed, decompose $\Phi = [\phi, \Phi']$. Then

$$P(\Phi, a) = \{[t, t']; t \geq 0, t' \in P(\Phi', a - t\phi)\}.$$

The proof follows by Fubini.

By induction, we obtain the following corollary.

Corollary 4 *If Φ consists of linearly independent vectors, the function* $\mathrm{vol}(\Phi, dx)(a)$ *is continuous on $C(\Phi)$.*

If Φ generates V with $|\Phi| > \dim V$, then the function $\mathrm{vol}(\Phi, dx)(a)$ *is continuous on V.*

For an element γ in V, define the translation operator $\tau(\gamma)$ on functions $k(a)$ on V by the formula: if $a \in V$, then

$$(\tau(\gamma)k)(a) = k(a - \gamma).$$

The difference operator $D(\gamma) = 1 - \tau(\gamma)$ acts on functions $k(a)$ on V by the formula:

$$(D(\gamma)k)(a) = k(a) - k(a - \gamma).$$

The following lemma is obvious from the definition.

Lemma 5 *Let $\phi \in \Phi$ and $a \in \Gamma$. Then*

$$k(\Phi)(a) = \sum_{n=0}^{\infty} k(\Phi \setminus \{\phi\})(a - n\phi).$$

The following relation follows immediately.

Lemma 6 *Let $\phi \in \Phi$ and $a \in \Gamma$. Then*

$$(D(\phi)k(\Phi))(a) = k(\Phi \setminus \{\phi\})(a).$$

In particular, $(D(\phi)k(\Phi))(a)$ is equal to 0 if a is not in the subspace of V generated by $\Phi \setminus \{\phi\}$.

Lemma 7 *Assume Φ generates V. Let W be a hyperplane in V such that $W \cap C(\Phi)$ is a facet of $C(\Phi)$. Let Φ_0 be the sequence $\Phi \cap W$ which spans W. If $a \in W$, then $k(\Phi)(a) = k(\Phi_0)(a)$.*

Proof. As $W \cap C(\Phi)$ is a facet of $C(\Phi)$, if $a \in W \cap C(\Phi)$, any solution of $a = \sum_{i=1}^{N} y_i \phi_i$ with $y_i \geq 0$ will have $y_i = 0$ for $\phi_i \notin W$.

The following lemma is also obtained immediately from Fubini's theorem applied to the integral $\int_{\mathbb{R}^N_{\geq 0}} e^{-\langle \sum_{i=1}^{N} t_i \phi_i, x\rangle} dt_1 dt_2 \cdots dt_N$ decomposed along the fibers of the map $A : \mathbb{R}^N_{\geq 0} \to C(\Phi)$, or to the analogous discrete sum.

Lemma 8 *For x in the interior of $C(\Phi)^*$,*

$$\int_{C(\Phi)} \mathrm{vol}(\Phi, dx)(a) e^{-\langle a, x\rangle} da = \frac{1}{\prod_{\phi \in \Phi} \langle \phi, x\rangle},$$

$$\sum_{a \in C(\Phi) \cap \Gamma} k(\Phi)(a) e^{-\langle a, x\rangle} = \frac{1}{\prod_{\phi \in \Phi} 1 - e^{-\langle \phi, x\rangle}}.$$

2.2 Chambers and the qualitative behavior of partition functions

In this section, we assume that Φ generates V. For any subset ν of Φ, we denote by $C(\nu)$ the closed cone generated by ν. We denote by $C(\Phi)_{\mathrm{sing}}$ the union of the cones $C(\nu)$ where ν is any subset of Φ of cardinality strictly less than $r = \dim(V)$. By definition, the set $C(\Phi)_{\mathrm{reg}}$ of Φ-regular elements is the complement of $C(\Phi)_{\mathrm{sing}}$. A connected component of $C(\Phi)_{\mathrm{reg}}$ is called a chamber. We remark that, according to our definition, the exterior of $C(\Phi)$ is itself a chamber denoted by $\mathfrak{c}_{\mathrm{ext}}$. The chambers contained in $C(\Phi)$ will be called interior chambers. If $\mathfrak{c}$ is a chamber, and σ is a basis of V contained in Φ, then either $\mathfrak{c} \subset C(\sigma)$, or $\mathfrak{c} \cap C(\sigma) = \emptyset$, as the boundary of $C(\sigma)$ does not intersect $\mathfrak{c}$.

Let $\Phi' \subset \Phi$ be such that Φ' generates V. If $\mathfrak{c}$ is a chamber for Φ, there exists a unique chamber $\mathfrak{c}'$ for Φ' such that $\mathfrak{c} \subset \mathfrak{c}'$.

A wall of Φ is a (real) hyperplane generated by $r-1$ linearly independent elements of Φ. It is clear that the boundary of a chamber $\mathfrak{c}$ is contained in an union of walls.

We now define the notion of a quasi-polynomial function on the lattice Γ. Let Γ^* be the dual lattice of Γ. An element $x \in U$ gives rise to the exponential function $e_x(a) = e^{2i\pi\langle x,a\rangle}$ on Γ. Remark that the function $e_x(a)$ depends only of the class of x (still denoted by x) in the torus $T(\Gamma) := U/\Gamma^*$.

Let M be a positive integer. A quasi-polynomial function with period M on Γ is a function K on Γ of the form $K(a) = \sum_{x\in F} e_x(a)P_x(a)$ where F is a finite set of points of U such that $MF \subset \Gamma^*$ and P_x are polynomial functions on V. Then the restriction of the function K to cosets $h + M\Gamma$ of $\Gamma/M\Gamma$ coincide with the restriction to $h + M\Gamma$ of a polynomial function on V. If the degree of the polynomial $P_x(a)$ is less or equal to k for all $x \in F$, we say that K is a quasi-polynomial function of degree k and period M.

If $\Gamma = \mathbb{Z}$ and $\gamma \in \mathbb{C}^*$ is a M^{th} root of unity, the function $n \mapsto n^k\gamma^n$ is a quasi-polynomial function on $\mathbb{Z}$ of period M and degree k.

If C is an affine closed cone in V with non empty interior, a quasi-polynomial function on Γ vanishing on $\Gamma \cap C$ is identically equal to 0 on Γ.

If $\gamma \in \Gamma$, the difference operator $D(\gamma)k(a) = k(a) - k(a - \gamma)$ leaves the space of quasi-polynomial functions on Γ stable.

The following theorem is well known (see [4],[3],[9], [5]). See yet another proof in the forthcoming article [6].

Proposition 9 *Let $\mathfrak{c}$ be an interior chamber of $C(\Phi)$.*

- *There exists a unique homogeneous polynomial function $v(\Phi, dx, \mathfrak{c})$ of degree d on V such that, for $a \in \overline{\mathfrak{c}}$,*

$$\mathrm{vol}(\Phi, dx)(a) = v(\Phi, dx, \mathfrak{c})(a).$$

- *There exists a unique quasi-polynomial function $k(\Phi, \mathfrak{c})$ on Γ such that, for $a \in \overline{\mathfrak{c}} \cap \Gamma$,*

$$k(\Phi)(a) = k(\Phi, \mathfrak{c})(a).$$

Remark 10 *The sequence Φ is called unimodular if, for any subset σ of Φ forming a basis of V, the subset σ is a basis of $\mathbb{Z}\Phi$. In other words, we have $|\det(\sigma)| = 1$, where the determinant is computed using the volume da giving volume 1 to a fundamental domain for $\mathbb{Z}\Phi$. In this particular case, the function $k(\Phi, \mathfrak{c})$ is polynomial on $\mathbb{Z}\Phi$.*

In the next lemma, we list differential equations satisfied by the polynomial function $v(\Phi, dx, \mathfrak{c})$.

Lemma 11 *Let $\phi \in \Phi$. If $\Phi \backslash \{\phi\}$ does not generate V, then $\partial(\phi)v(\Phi, dx, \mathfrak{c}) = 0$.*

If $\Phi \setminus \{\phi\}$ generates V, let $\mathfrak{c}'$ be the chamber of $\Phi \setminus \{\phi\}$ containing $\mathfrak{c}$, then $\partial(\phi)v(\Phi, dx, \mathfrak{c}) = v(\Phi \setminus \{\phi\}, dx, \mathfrak{c}')$.

Proof. If $\Phi_0 = \Phi \setminus \{\phi\}$ is contained in a wall W, then $V = W \oplus \mathbb{R}\phi$, and it is immediate to see that an interior chamber $\mathfrak{c}$ for Φ is of the form $\mathfrak{c} = \mathfrak{c}_0 + \mathbb{R}_{>0}\phi$, where $\mathfrak{c}_0$ is a chamber for Φ_0. If $a = w + t\phi$ with $w \in W$ and $t > 0$, then $\mathrm{vol}(\Phi, dx, \mathfrak{c})(w + t\phi) = \mathrm{vol}(\Phi_0, dx_0, \mathfrak{c}_0)(w)$, with $dx_0 d\phi = dx$. This proves the first statement.

To prove the second statement, if $a \in \mathfrak{c}$, we use the following relation (as given in Lemma 3)

$$\mathrm{vol}(\Phi, dx)(a) - \mathrm{vol}(\Phi, dx)(a - \epsilon\phi) = \int_{t=0}^{\epsilon} \mathrm{vol}(\Phi_0, dx)(a - t\phi) dt.$$

Corollary 12 *Let $\Phi_0 \subset \Phi$ such that Φ_0 does not generate V. Then*

$$(\prod_{\phi \in \Phi \setminus \Phi_0} \partial(\phi)) v(\Phi, dx, \mathfrak{c}) = 0.$$

In the next lemma, we list difference equations satisfied by the quasi-polynomial function $k(\Phi, \mathfrak{c})$.

Lemma 13 *Let $\phi \in \Phi$. If $\Phi \backslash \{\phi\}$ does not generate V, then $D(\phi)k(\Phi, \mathfrak{c}) = 0$.*

If $\Phi \setminus \{\phi\}$ generates V, let $\mathfrak{c}'$ be the chamber of $\Phi \setminus \{\phi\}$ containing $\mathfrak{c}$, then $D(\phi)k(\Phi, \mathfrak{c}) = k(\Phi \setminus \{\phi\}, \mathfrak{c}')$

Proof. By Lemma 6, the function $k(\Phi)$ satisfies $D(\phi)k(\Phi) = k(\Phi \setminus \{\phi\})$. Considering this relation on an affine subcone S of $\overline{\mathfrak{c}}$ such that $S - \phi$ does not touch the boundary of $\mathfrak{c}$, we obtain the relations of the lemma.

3 Two polynomial functions

3.1 Residue formula

Let $\mathcal{L}$ be the space of Laurent series in one variable z:

$$\mathcal{L} := \{f(z) = \sum_{k \geq k_0} f_k z^k\}.$$

For $f \in \mathcal{L}$, we denote by $\mathrm{Res}_{z=0} f(z)$ the coefficient f_{-1} of z^{-1}. If g is a germ of meromorphic function at $z = 0$, then g gives rise to an element of $\mathcal{L}$ by considering the Laurent series at $z = 0$ and we still denote by $\mathrm{Res}_{z=0} g$ its residue at $z = 0$. If $g = \frac{d}{dz} f$, then $\mathrm{Res}_{z=0} g = 0$.

With the notation of Section 2.1, let E be a vector in U. It defines a hyperplane $W = \{a \in V | \langle a, E\rangle = 0\}$ in V.

Definition 14 *Let P be a polynomial function on V and let Ψ be a sequence of vectors not belonging to W. We define, for $a \in V$,*

- $\mathrm{Pol}(P, \Psi, E)(a) = \mathrm{Res}_{z=0} \left(P(\partial_x) \cdot \frac{e^{\langle a, x+zE\rangle}}{\prod_{\psi\in\Psi} \langle \psi, x+zE\rangle} \right)_{x=0}.$
- $\mathrm{Par}(P, \Psi, E)(a) = \mathrm{Res}_{z=0} \left(P(\partial_x) \cdot \frac{e^{\langle a, x+zE\rangle}}{\prod_{\psi\in\Psi} (1-e^{-\langle \psi, x+zE\rangle})} \right)_{x=0}.$

It is easy to see that $\mathrm{Pol}(P, \Psi, E)(a)$ as well as $\mathrm{Par}(P, \Psi, E)(a)$ are polynomial functions of $a \in V$.

Lemma 15 *The functions* $\mathrm{Pol}(P, \Psi, E)$ *and* $\mathrm{Par}(P, \Psi, E)$ *depend only on the restriction p of P to W.*

Proof. If $p = 0$, then $P = EQ$ where Q is a polynomial function on V. Then

$$\begin{aligned} P(\partial_x)F(x + zE) &= \tfrac{d}{d\epsilon} Q(\partial_x)F(\epsilon E + x + zE)_{\epsilon=0} \\ &= \tfrac{d}{d\epsilon} Q(\partial_x)F(x + (z + \epsilon)E)_{\epsilon=0} \\ &= \tfrac{d}{dz} (Q(\partial_x)F(x + zE)) \end{aligned}$$

so that the residue $\mathrm{Res}_{z=0}$ vanishes on the function $z \mapsto P(\partial_x)F(x+zE)_{x=0}$.

We can then give the following definitions.

Definition 16 *Let p be a polynomial function on W. We define*

$$\mathrm{Pol}(p, \Psi, E) := \mathrm{Pol}(P, \Psi, E)$$

where P is any polynomial on V extending p.

We define

$$\mathrm{Par}(p, \Psi, E) := \mathrm{Par}(P, \Psi, E)$$

where P is any polynomial on V extending p.

In the following, given polynomials $p, q, \ldots$ on W, we denote by $P, Q, \ldots$ polynomials on V extending $p, q, \ldots$.

3.2 Some properties

Let us list some properties satisfied by the function $\mathrm{Pol}(p, \Psi, E)$.

We first remark that if we replace ψ in Ψ by $c\psi$ with $c \neq 0$, then $\mathrm{Pol}(p, \Psi, E)$ becomes $\frac{1}{c}\mathrm{Pol}(p, \Psi, E)$.

We now discuss how $\mathrm{Pol}(p, \Psi, E)$ transforms under the action of differentiation.

Proposition 17 *Let $\psi \in \Psi$. Then*

$$\partial(\psi)\mathrm{Pol}(p, \Psi, E) = \mathrm{Pol}(p, \Psi \setminus \{\psi\}, E).$$

Let $w \in W$. Then

$$\partial(w)\mathrm{Pol}(p, \Psi, E) = \mathrm{Pol}(\partial(w)p, \Psi, E).$$

Proof. The first formula follows immediately from the definition.

For the second part of the proposition, we will use the following lemma, which is implied by the relation $P(\partial_x)\langle x, w\rangle - \langle x, w\rangle P(\partial_x) = (\partial(w)P)(\partial_x)$.

Lemma 18 *For any function $J(x)$ of $x \in U$,*

$$\Big(P(\partial_x)\langle x, w\rangle J(x)\Big)_{x=0} = \Big((\partial(w)P)(\partial_x)J(x)\Big)_{x=0}.$$

Now, as $\langle w, E\rangle = 0$, for $J(x, z) = \frac{1}{\prod_{\psi\in\Psi}\langle\psi, x+zE\rangle}$, we have

$$\begin{aligned}\partial(w)\mathrm{Res}_{z=0}\left(P(\partial_x)e^{\langle a, x+zE\rangle}J(x,z)\right)_{x=0} &= \mathrm{Res}_{z=0}\left(P(\partial_x)\langle w, x\rangle e^{\langle a, x+zE\rangle}J(x,z)\right)_{x=0} \\ &= \mathrm{Res}_{z=0}\left((\partial(w)P)(\partial_x)e^{\langle a, x+zE\rangle}J(x,z)\right)_{x=0}.\end{aligned}$$

So we obtain the formula of the proposition.

Lemma 19

- *If $\Psi = \{\psi\}$, then for $w \in W$ and $t \in \mathbb{R}$,*

$$\mathrm{Pol}(p, \{\psi\}, E)(w + t\psi) = \mathrm{Par}(p, \{\psi\}, E)(w + t\psi) = \frac{p(w)}{\langle\psi, E\rangle}.$$

- *If $|\Psi| > 1$, then the restriction of $\mathrm{Pol}(p, \Psi, E)$ to W vanishes of order $|\Psi| - 1$.*

Proof. Let $U_0 = \{x | \langle \psi, x\rangle = 0\}$. We write $U = U_0 \oplus \mathbb{R}E$. The space $S(U_0)$ is isomorphic to the space of polynomial functions on W. We may choose P in $S(U_0)$. We write $x = x_0 + x_1 E$, with $x_0 \in U_0$. In these coordinates $\langle \psi, x + zE\rangle = (x_1 + z)\langle \psi, E\rangle$ is independent of x_0. So we can set $x_1 = 0$ in the formula

$$\mathrm{Res}_{z=0}\left(P(\partial_{x_0}) \cdot \frac{e^{\langle a, x+zE\rangle}}{(x_1 + z)\langle \psi, E\rangle}\right)_{x=0}$$

and the residue is computed for a function that have a simple pole at $z = 0$. The formula follows. The other points are also easy to prove.

Let us list some difference equations satisfied by the function $\mathrm{Par}(p, \Psi, E)$.

Proposition 20 *Let $\psi \in \Psi$. Then*

$$D(\psi)\mathrm{Par}(p, \Psi, E) = \mathrm{Par}(p, \Psi \setminus \{\psi\}, E).$$

Let $w \in W$. Then

$$\tau(w)\mathrm{Par}(p, \Psi, E) = \mathrm{Par}(\tau(w)p, \Psi, E).$$

Proof. The first formula follows immediately from the definition.

The translation operator $\tau(w)$ satisfies the relation

$$P(\partial_x)e^{-\langle w, x\rangle} = e^{-\langle w, x\rangle}(\tau(w)P)(\partial_x).$$

Thus, the second formula follows from the same argument as in the proof of the second item in Proposition 17.

4 Wall crossing formula for the volume

In this section, we give two formulae for the jump of the volume function across a wall. The first one uses convolutions of Heaviside distributions and is in the spirit of Paradan's formula ([8], Theorem 5.2) for the jump of the partition function. The second one is a one dimensional residue formula.

4.1 Inversion formula

We will need some formulae for Laplace transforms in dimension 1. For $z > 0$ and $k \geq 0$ an integer, we have

$$\frac{1}{z^{k+1}} = \int_0^\infty \frac{t^k}{k!} e^{-tz} dt. \tag{1}$$

Consider the Laplace transform

$$L(p)(z) = \int_{\mathbb{R}^+} e^{-tz} p(t) dt.$$

Assume that $p(t) = \sum_{ik} c_{ik} p_{i,k}(t)$ is a linear combination of the functions $p_{i,k}(t) = e^{-tx_i} \frac{t^k}{k!}$. We assume that $x_i > 0$. Then, the integral defining $L(p)$ is convergent. We have

$$L(p)(z) = \sum_{i,k} \frac{c_{ik}}{(z + x_i)^{k+1}}. \tag{2}$$

The following inversion formula is immediate to verify.

Lemma 21 *Let $R > 0$. Assume that $|x_i| < R$ for all i. Then we have*

$$p(t) = \frac{1}{2i\pi} \int_{|z|=R} L(p)(z) e^{tz} dz. \tag{3}$$

Reciprocally, if p is a continuous function on $\mathbb{R}$ such that $L(p)(z)$ is convergent and given by Formula (2), then p is given by Equation (3).

If $p(t) = \sum_k c_k \frac{t^k}{k!}$ is a polynomial (that is all the elements x_i are equal to 0), then $L(p)(z)$ is the Laurent polynomial $\sum_k c_k z^{-k-1}$, and the inversion formula above reads

$$p(t) = \mathrm{Res}_{z=0} L(p)(z) e^{tz}. \tag{4}$$

4.2 Convolution of measures

Let $E \in U$ be a non zero linear form on V and $W \subset V$ the corresponding hyperplane. Let $V^+ = \{a \in V | \langle a, E \rangle > 0\}$ and $V^- = \{a \in V | \langle a, E \rangle < 0\}$ denote the corresponding open half spaces. Let $\Delta^+ = [\alpha_1, \alpha_2, \ldots, \alpha_Q]$ be a sequence of vectors contained in V^+. Consider the span $< \Delta^+ >$ of Δ^+.

We choose a Lebesgue measure da on $< \Delta^+ >$ with dual measure dx on $< \Delta^+ >^*$. We define the continuous function $v(\Delta^+, dx)(a)$ on the cone $C(\Delta^+) \subset < \Delta^+ >$ such that, for $x \in C(\Delta^+)^*$, we have

$$\frac{1}{\prod_{\alpha \in \Delta^+} \langle \alpha, x \rangle} = \int_{C(\Delta^+)} v(\Delta^+, dx)(a) e^{-\langle a, x \rangle} da. \tag{5}$$

By Lemma 8, $v(\Delta^+, dx)(a) = \mathrm{vol}(\Delta^+, dx)(a)$.

We choose the Lebesgue measure $dw = da/dt$ on $W \cap < \Delta^+ >$ where $t = \langle a, E \rangle$. The measure dw determines a measure on all affine spaces $W \cap (a+ < \Delta^+ >)$.

Proposition 22 *Let p be a polynomial function on W. We define*

$$(p * v(\Delta^+, dx))(a) = \int_{W \cap (a+<\Delta^+>)} p(w) v(\Delta^+, dx)(a - w) dw.$$

Then, for $a \in V^+$, we have

$$(p * v(\Delta^+, dx))(a) = \mathrm{Pol}(p, \Delta^+, E)(a) \text{ for } a \in V^+.$$

Remark that $p * v(\Delta^+, dx)$ depends only on the choice of E. Indeed, $p * v(\Delta^+, dx)$ is the convolution of two functions, one of which depends on the measure dx, while the convolution depends on the measure dw. We see that finally this depends only of the choice of E.

We also remark that, for fixed $a \in V^+$, the integral defining $p * v(\Delta^+, dx)$ is in fact over the compact set $W \cap (a - C(\Delta^+))$ where $v(\Delta^+, dx)(a - w)$ is not equal to zero .

Proof. We decompose $W = W_0 \oplus W_1$, where $W_0 = W \cap < \Delta^+ >$. Then, we can write $a \in V$ as $a = tF + w_0 + w_1$, with $\langle F, E \rangle = 1$. If $p(w_0 + w_1) = p_0(w_0) p_1(w_1)$, we see that $(p * v(\Delta^+, dx))(tF + w_0 + w_1) = p_1(w_1)(p_0 * v(\Delta^+, dx))(tF + w_0)$. Hence, it is sufficient to prove the proposition in the case where Δ^+ generates V. Then

$$(p * v(\Delta^+, dx))(a) = \int_W p(w) v(\Delta^+, dx)(a - w) dw.$$

The polynomial nature of $(p * v(\Delta^+, dx))(a)$ is clear intuitively. In any case, we will prove the explicit formula of the proposition, which gives a polynomial formula for $(p * v(\Delta^+, dx))(a)$.

We need to compute, for $a \in V^+$, $I(a) := \int_W p(w) v(\Delta^+, dx)(a-w) dw$. This integral is over a compact subset of W. Let $P \in S(U)$ be a polynomial function on V extending p. We may write

$$I(a) = \left(P(\partial_x) \cdot \int_W v(\Delta^+, dx)(a-w) e^{\langle w,x \rangle} dw \right)_{x=0}.$$

Define

$$g_x(a) = \int_W v(\Delta^+, dx)(a-w) e^{-\langle a-w,x \rangle} dw.$$

Then $g_x(a)$ depends analytically on the variable $x \in U$, and we have

$$I(a) = \left(P(\partial_x) \cdot e^{\langle a,x \rangle} g_x(a) \right)|_{x=0}. \tag{6}$$

The function $a \mapsto g_x(a) = \int_W v(\Delta^+, dx)(a-w) e^{-\langle a-w,x \rangle} dw$ is a continuous function of a modulo W, that is, it is a continuous function of the variable $t = \langle a, E \rangle \geq 0$ when $a \in V^+$. We then write $g_x(t) = g_x(tF) = \int_W v(\Delta^+, dx)(tF - w) e^{-\langle tF-w,x \rangle} dw$.

To identify the function $g_x(t)$, we compute its Laplace transform in one variable. Let $z > 0$. If x is in $C(\Delta^+)^*$, the integral defining $L(g_x)$ is convergent and we have

$$\begin{aligned} L(g_x)(z) = \int_{t>0} e^{-tz} g_x(tF) dt &= \int_{V^+} e^{-\langle a,zE \rangle} v(\Delta^+, dx)(a) e^{-\langle a,x \rangle} da \\ &= \frac{1}{\prod_{\alpha \in \Delta^+} \langle \alpha, x+zE \rangle} \end{aligned}$$

by Formula (5). Here $\langle \alpha, x+zE \rangle = d_\alpha z + \langle \alpha, x \rangle$ with $d_\alpha = \langle \alpha, E \rangle > 0$ and $\langle \alpha, x \rangle > 0$.

Thus, by partial fraction decomposition, $L(g_x)(z)$ is a function of the type given by Formula (2). By the inversion formula for the Laplace transform in one variable, we obtain that (for x small enough)

$$g_x(a) = \frac{1}{2i\pi} \int_{|z|=1} \frac{e^{\langle a,zE \rangle}}{\prod_{\alpha \in \Delta^+} \langle \alpha, x+zE \rangle} dz.$$

Thus Formula (6) becomes

$$\begin{aligned} I(a) &= P(\partial_x) \cdot \left(e^{\langle a,x \rangle} \frac{1}{2i\pi} \int_{|z|=1} \frac{e^{\langle a,zE \rangle}}{\prod_{\alpha \in \Delta^+} \langle \alpha, x+zE \rangle} dz \right)_{x=0} \\ &= \frac{1}{2i\pi} \int_{|z|=1} \left(P(\partial_x) \cdot \frac{e^{\langle a,x+zE \rangle}}{\prod_{\alpha \in \Delta^+} \langle \alpha, x+zE \rangle} \right)_{x=0} dz. \end{aligned}$$

The function in the integrand has a Laurent series at $z = 0$ with polynomial coefficients in a of the form $\sum_k g_k(a)z^k$. Thus we obtain

$$I(a) = \mathrm{Res}_{z=0} \left(P(\partial_x) \cdot \frac{e^{\langle a, x+zE \rangle}}{\prod_{\alpha \in \Delta^+} \langle \alpha, x + zE \rangle} \right)_{x=0}.$$

This shows that $I(a)$ coincide with the polynomial function $\mathrm{Pol}(p, \Delta^+, E)$ on V^+.

We also remark that, if p is homogeneous, $\mathrm{Pol}(p, \Delta^+, E)$ is homogenous in a of degree $|\Delta^+| - 1 + \deg(\mathrm{p})$.

4.3 The jump for the volume function

Let $\mathrm{vol}(\Phi, dx)$ be the locally polynomial function on the cone $C(\Phi)$ generated by Φ. Let W be a wall, determined by a vector $E \in U$. Let V^+ and V^- denote the corresponding open half spaces. Define $\Phi_0 = \Phi \cap W$; this is a sequence of vectors in W spanning W.

Let $\mathfrak{c}_1 \subset V^+$ and $\mathfrak{c}_2 \subset V^-$ be two chambers on two sides of W and adjacent. Here, we mean that $\overline{\mathfrak{c}_1} \cap \overline{\mathfrak{c}_2}$ has non empty relative interior in W. Thus $\overline{\mathfrak{c}_1} \cap \overline{\mathfrak{c}_2}$ is contained in the closure of a chamber $\mathfrak{c}_{12}$ of Φ_0. We choose the measure dw on W such that $da = dwdt$ with $t = \langle a, E \rangle$. We write

$$\Phi = [\Phi_0, \Phi^+, \Phi^-]$$

where $\Phi^+ = \Phi \cap V^+$ and $\Phi^- = \Phi \cap V^-$.

Let

$$R_+(\Phi) = [\phi | \phi \in \Phi^+] \cup [-\phi | \phi \in \Phi^-].$$

By construction, the sequence $R_+(\Phi)$ is contained in V^+.

Theorem 23 *Let $v_{12} = v(\Phi_0, dw, \mathfrak{c}_{12})$ be the polynomial function on W associated to the chamber $\mathfrak{c}_{12}$ of Φ_0. Then, if $\langle E, \mathfrak{c}_1 \rangle > 0$,*

$$v(\Phi, dx, \mathfrak{c}_1) - v(\Phi, dx, \mathfrak{c}_2) = \mathrm{Pol}(v_{12}, \Phi \setminus \Phi_0, E). \quad (7)$$

Remark 24 *We have*

$$\mathrm{Pol}(v_{12}, \Phi \setminus \Phi_0, E) = (-1)^{|\Phi^-|} \mathrm{Pol}(v_{12}, R_+(\Phi), E).$$

Thus, by results of the preceding section, the difference of the volume functions $v(\Phi, dx, \mathfrak{c}_1) - v(\Phi, dx, \mathfrak{c}_2)$ *coincides on* V^+*, up to sign, with the convolution of the* **polynomial measure** $v_{12}(w)dw$ *associated to the chamber* $\mathfrak{c}_{12}$ *and with the Heaviside distributions associated to the vectors* $\psi \in R_+(\Phi)$*. This is in the line of Paradan's description of the jump formula for partition functions ([8], Theorem 5.2) .*

Proof. Denote by $\mathrm{Leq}(\Phi)$ the left hand side and by $\mathrm{Req}(\Phi)$ the right hand side of Equation (7) above.

We will first verify the claim in the theorem when there is only one vector ϕ of Φ that does not lie in W. We can suppose that $\Phi^+ = \{\phi\}$ and that $\langle E, \phi\rangle = 1$. Then the chamber $\mathfrak{c}_1$ is equal to $\mathfrak{c}_{12} \times \mathbb{R}_{>0}\phi$, while $\mathfrak{c}_2$ is the exterior chamber. In this case, $v(\Phi, dx, \mathfrak{c}_1)(w + t\phi) = v(\Phi_0, (dw)^*, \mathfrak{c}_{12})(w) = v_{12}(w)$, while $v(\Phi, dx, \mathfrak{c}_2) = 0$. The equation (7) follows from the first item of Lemma 19.

If not, then $|\Phi| > \dim(V)$ and the function $\mathrm{vol}(\Phi, dx)$ is continuous on V by Corollary 4. Let ϕ be a vector in Φ that does not lie in W. We may assume that $\phi \in V^+$. Then the sequence $\Phi' = \Phi \setminus \{\phi\}$ will still span V. The intersection of Φ' with W is Φ_0. If $\mathfrak{c}'_1$ and $\mathfrak{c}'_2$ are the chambers of Φ' containing $\mathfrak{c}_1$ and $\mathfrak{c}_2$ respectively, they are adjacent with respect to W. As $\Phi' \cap W = \Phi_0$, the polynomial v'_{12} attached to $\mathfrak{c}_{12}$ and $\Phi' \cap W$ is equal to v_{12}. By Lemma 11, we have

$$\partial(\phi)(v(\Phi, \mathfrak{c}_1) - v(\Phi, \mathfrak{c}_2)) = v(\Phi', \mathfrak{c}'_1) - v(\Phi', \mathfrak{c}'_2).$$

By Proposition 17,

$$\partial(\phi)\mathrm{Pol}(v_{12}, \Phi \setminus \Phi_0, E) = \mathrm{Pol}(v_{12}, \Phi' \setminus \Phi_0, E).$$

By induction, we obtain $\partial(\phi)(\mathrm{Leq}(\Phi) - \mathrm{Req}(\Phi)) = 0$. Thus the polynomial function $\mathrm{Leq}(\Phi) - \mathrm{Req}(\Phi)$ is constant in the direction $\mathbb{R}\phi$. The left hand side vanishes on W, as the function $\mathrm{vol}(\Phi)$ is continuous on V. The right hand side also vanishes by Lemma 19. This establishes the theorem.

Consider a vector space V with basis $\{e_i,\ i : 1, \ldots, r\}$; we denote its dual basis by $\{e^i\}$. The set

$$\Phi(B_r) = \{e_i,\ 1 \leq i \leq r\} \cup \{e_i + e_j,\ 1 \leq i < j \leq r\} \cup \{e_i - e_j,\ 1 \leq i < j \leq r\}$$

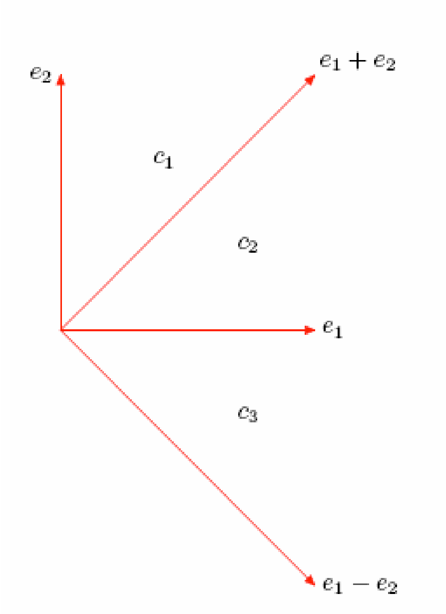

Figure 1: Chambers of B_2

is the set of positive roots for the system of type B_r and generates V. We will denote a vector $a \in V$ by $a = \sum_{i=1}^{r} a_i e_i$; it lies in $C(\Phi(B_r))$ if and only if $a_1 + \cdots + a_i \geq 0$ for all $i : 1, \ldots, r$. This will be our notation for this root system in subsequent examples.

Example 25 *We consider the root system of type B_2 (see Figure 1) with $\Phi = \{e_1, e_2, e_1 - e_2, e_1 + e_2\}$. We will calculate $v(\Phi, \mathfrak{c})$ for all the chambers using our formula in Theorem 23 iteratively starting from the exterior chamber.*

(i) Jump from the exterior chamber to $\mathfrak{c}_1$ ($W = \mathbb{R}e_2$): In this case $E = e^1$, $\Phi_0 = \{e_2\}$, $\Phi^+ = \{e_1 + e_2, e_1, e_1 - e_2\}$ and $\Phi^- = \emptyset$.

$$\begin{aligned}
v(\Phi, \mathfrak{c}_1)(a) - v(\Phi, \mathfrak{c}_{\text{ext}})(a) &= \text{Pol}(1, \Phi \setminus \Phi_0, E)(a) \\
&= \text{Res}_{z=0} \left(\frac{e^{\langle a, x + z e^1 \rangle}}{\prod_{\phi \in \Phi^+ \cup \Phi^-} \langle \phi, x + z e^1 \rangle} \right)_{x=0} \\
&= \text{Res}_{z=0} \left(\frac{e^{a_1(x_1+z) + a_2 x_2}}{(x_1+z+x_2)(x_1+z)(x_1+z-x_2)} \right)_{x=0} \\
v(\Phi, \mathfrak{c}_1)(a) &= \text{Res}_{z=0} \left(\frac{e^{a_1 z}}{z^3} \right) = \frac{1}{2} a_1^2.
\end{aligned}$$

(ii) Jump from $\mathfrak{c}_1$ to $\mathfrak{c}_2$ ($W = \mathbb{R}(e_1 + e_2)$): We have $E = e^1 - e^2$, $\Phi_0 =$

$\{e_1 + e_2\}$, $\Phi^+ = \{e_1, e_1 - e_2\}$ *and* $\Phi^- = \{e_2\}$.

$$\begin{aligned} v(\Phi, \mathfrak{c}_2)(a) - v(\Phi, \mathfrak{c}_1)(a) &= \mathrm{Pol}(1, \Phi \setminus \Phi_0, E)(a) \\ &= \mathrm{Res}_{z=0} \left(\frac{e^{a_1(x_1+z)+a_2(x_2-z)}}{(x_1+z)(x_1-x_2+2z)(x_2-z)} \right)_{x=0} \\ &= -\mathrm{Res}_{z=0} \left(\frac{e^{(a_1-a_2)z}}{2z^3} \right) = -\frac{1}{4}(a_1 - a_2)^2. \end{aligned}$$

Using (i), $v(\Phi, \mathfrak{c}_2)(a) = \frac{1}{2}a_1^2 - \frac{1}{4}(a_1 - a_2)^2 = \frac{1}{4}(a_1 + a_2)^2 - \frac{1}{2}a_2^2$.

(iii) Jump from $\mathfrak{c}_2$ *to* $\mathfrak{c}_3$ *(*$W = \mathbb{R}e_1$*): We have* $E = e^2$, $\Phi_0 = \{e_1\}$, $\Phi^+ = \{e_2, e_1 + e_2\}$ *and* $\Phi^- = \{e_1 - e_2\}$.

$$\begin{aligned} v(\Phi, \mathfrak{c}_2)(a) - v(\Phi, \mathfrak{c}_3)(a) &= \mathrm{Pol}(1, \Phi \setminus \Phi_0, E)(a) \\ &= \mathrm{Res}_{z=0} \left(\frac{e^{a_1x_1+a_2(x_2+z)}}{(x_2+z)(x_1+x_2+z)(x_1-x_2-z)} \right)_{x=0} = -\frac{1}{2}a_2^2. \end{aligned}$$

Using (ii), $v(\Phi, \mathfrak{c}_3)(a) = \frac{1}{2}a_1^2 - \frac{1}{4}(a_1 - a_2)^2 + \frac{1}{2}a_2^2 = \frac{1}{4}(a_1 + a_2)^2$.

5 Wall crossing formula for the partition function: unimodular case

In this section, we compute the jump $k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2)$ of the partition function $k(\Phi)$ across a wall. In order to outline the main ideas in the proof, we will first consider the case where Φ is unimodular (see Remark 10 for the definition). We give two formulae. The first one is the convolution formula of Paradan ([8], Theorem 5.2). The second one is a one dimensional residue formula.

5.1 Discrete convolution

Let $E \in U$ be a primitive element with respect to Γ^* so that $\langle E, \Gamma \rangle = \mathbb{Z}$. Let Ψ be a sequence of vectors in Γ such that $\langle \psi, E \rangle \neq 0$ for all $\psi \in \Psi$. Thus $\Psi = \Psi^+ \cup \Psi^-$ with $\Psi^+ = \Psi \cap V^+$ and $\Psi^- = \Psi \cap V^-$. Define

$$R_+(\Psi) = [\psi \| \psi \in \Psi^+] \cup [-\psi | \psi \in \Psi^-].$$

Let

$$W := \{a \in V \mid \langle a, E \rangle = 0\},$$

$$\Gamma_0 = \Gamma \cap W.$$

We choose $F \in \Gamma$ such that $\langle E, F\rangle = 1$. We thus have $\Gamma = \Gamma_0 \oplus \mathbb{Z}F$.

Let $\Gamma_{\geq 0}$ be the set of elements $a \in \Gamma$ such that $\langle a, E\rangle \geq 0$. Let us define the function $K^+(\Psi)$ on $\Gamma_{\geq 0}$ such that we have, for $x \in C(R_+(\Psi))^*$,

$$\prod_{\psi\in\Psi} \frac{1}{1-e^{-\langle\psi,x\rangle}} = \sum_{a\in\Gamma_{\geq 0}} K^+(\Psi)(a)e^{-\langle a,x\rangle}, \tag{8}$$

that is we have written

$$1/(1-e^{-\psi}) = \sum_{n\geq 0} e^{-n\psi}, \text{ if } \psi \in \Psi^+$$

and

$$1/(1-e^{-\psi}) = -e^{\psi}/(1-e^{\psi}) = -\sum_{n>0} e^{n\psi} \text{ if } \psi \in \Psi^-.$$

Let $\kappa_- = \sum_{\psi\in\Psi^-}\psi$ so that $\langle\kappa_-, E\rangle = \sum_{\psi\in\Psi^-}\langle\psi, E\rangle$ is a strictly negative number if and only Ψ^- is non empty. Then we have

$$K^+(\Psi)(a) = (-1)^{|\Psi_-|}k(R_+(\Psi))(a-\kappa_-),$$

where $k(R_+(\Psi))$ is the partition function of the system $R_+(\Psi)$.

The function $K^+(\Psi)$ is supported on the pointed cone $-\kappa_- + C(R_+(\Psi))$. In particular the value $K^+(\Psi)(0)$ is 1 if Ψ^- is empty, or 0 if Ψ^- is not empty.

Let q be a polynomial function on Γ_0. Define for $a \in \Gamma$

$$C(q,\Psi,E)(a) := \sum_{w\in\Gamma_0} q(w)K^+(\Psi)(a-w).$$

The sum is over the finite set $\Gamma_0 \cap (a - C(R_+(\Psi)))$.

Theorem 26 *Assume Ψ^+ is non empty. Assume that, for any $\psi \in \Psi$, we have $\langle\psi, E\rangle = \pm 1$. Let q be a polynomial function on Γ_0. Then, for $a \in \Gamma_{\geq 0}$,*

$$C(q,\Psi,E)(a) = \operatorname{Par}(q,\Psi,E)(a).$$

Proof. We need to compute, for $a \in \Gamma_{\geq 0}$,

$$S(a) := \sum_{w\in\Gamma_0} q(w)K^+(\Psi)(a-w).$$

This sum is over a finite set.

Let $Q \in S(U)$ be a polynomial function on V extending q. We may write

$$S(a) = \left(Q(\partial_x) \cdot \sum_{w \in \Gamma_0} K^+(\Psi)(a-w) e^{\langle w, x\rangle} \right)_{x=0}.$$

Define

$$G_x(a) = \sum_{w \in \Gamma_0} K^+(\Psi)(a-w) e^{-\langle a-w, x\rangle}.$$

Then $G_x(a)$ depends in an analytic way of the variable $x \in U$, and we have

$$S(a) = \left(Q(\partial_x) \cdot e^{\langle a, x\rangle} G_x(a) \right)|_{x=0}. \tag{9}$$

The function $a \mapsto G_x(a) = \sum_{w \in \Gamma_0} K^+(\Psi)(a-w) e^{-\langle a-w, x\rangle}$ is a function on $\Gamma/\Gamma_0 = \mathbb{Z}F$. To identify the function $G_x(nF)$, we compute its discrete Laplace transform in one variable. Let x be in $C(R_+(\Psi))^*$. Write $u = e^{-z}$. We compute

$$\begin{array}{ll} L_{\mathrm{dis}}(G_x)(u) & = \sum_{n \geq 0} G_x(nF) u^n \\ & = \sum_{n \geq 0} G_x(nF) e^{-nz} \\ & = \sum_{n,w} K^+(\Psi)(nF-w) e^{-\langle nF-w, x\rangle} e^{-\langle nF, zE\rangle} \\ & = \sum_{a \in \Gamma} K^+(\Psi)(a) e^{-\langle a, x\rangle} e^{-\langle a, zE\rangle} \\ & = \frac{1}{\prod_{\psi \in \Psi}(1-e^{-\langle \psi, x+zE\rangle})} \\ & = \frac{1}{\prod_{\psi \in \Psi}(1-e^{-\langle \psi, x\rangle} u^{\langle \psi, E\rangle})}. \end{array}$$

In the fourth equality, we have written any element $a \in \Gamma_{\geq 0}$ as $a = nF - w$, with $n \geq 0$ and $w \in \Gamma_0$. The next equality is by the definition of the function $K^+(\Psi)(a)$. Furthermore, we see that the sum is convergent when $|u| < 1$.

As $L_{\mathrm{dis}}(G_x)(u) = \sum_{n \geq 0} G_x(nF) u^n$, Cauchy formula reads

$$\begin{array}{ll} G_x(nF) & = \frac{1}{2i\pi} \int_{|u|=\epsilon} u^{-n} L_{\mathrm{dis}}(G_x)(u) \frac{du}{u} \\ & = \frac{1}{2i\pi} \int_{|u|=\epsilon} \frac{u^{-n}}{\prod_{\psi \in \Psi}(1-e^{-\langle \psi, x\rangle} u^{\langle \psi, E\rangle})} \frac{du}{u}. \end{array}$$

Thus we obtain for $n = \langle a, E\rangle$,

$$S(a) = \left(Q(\partial_x) \cdot e^{\langle a, x\rangle} \frac{1}{2i\pi} \int_{|u|=\epsilon} \frac{u^{-n}}{\prod_{\psi \in \Psi}(1-e^{-\langle \psi, x\rangle} u^{\langle \psi, E\rangle})} \frac{du}{u} \right)_{x=0}$$

$$= \frac{1}{2i\pi}\int_{|u|=\epsilon} u^{-n}\left(Q(\partial_x)\cdot\frac{e^{\langle a,x\rangle}}{\prod_{\psi\in\Psi}(1-e^{-\langle\psi,x\rangle}u^{\langle\psi,E\rangle})}\right)_{x=0}\frac{du}{u}.$$

As at least one of the $\langle\psi, E\rangle$ is positive and $n \geq 0$, it is easy to see that the function under the integrand has no pole at $u = \infty$. As $\langle\psi, E\rangle = \pm 1$, its poles are obtained for $u = 0$ and $u = 1$. The integral on $|u| = \epsilon$ computes the residue at $u = 0$. We use the residue theorem so that $-S(a)$ can also be computed as the residue for $u = 1$. We use the coordinate $u = e^{-z}$ near $u = 1$, and we obtain

$$S(a) = \mathrm{Res}_{z=0}\left(Q(\partial_x)\cdot\frac{e^{\langle a,x+zE\rangle}}{\prod_{\psi\in\Psi}(1-e^{-\langle\psi,x+zE\rangle})}\right)_{x=0}$$

which establishes the formula in the theorem.

Finally, we compute the restriction to $W\cap\Gamma$ of $\mathrm{Par}(q,\Psi,E)$.

Lemma 27
- *If* $|\Psi^-| = \emptyset$*, then the restriction of* $\mathrm{Par}(q,\Psi,E)$ *to* W *is equal to* q*.*
- *If* $|\Psi^-| > 0$*, then the restriction of* $\mathrm{Par}(q,\Psi,E)$ *to* W *vanishes.*

Proof. The sum formula gives $\mathrm{Par}(q,\Psi,E)(w) = K^+(\Psi)(0)q(w)$. Recall that $K^+(\Psi)(0)$ vanishes as soon as $|\Psi^-| > 0$; it is equal to 1 if $\Psi^- = \emptyset$.

5.2 The jump for the partition function

Let Φ be a sequence of vectors spanning the lattice Γ. In this section we assume that Φ is unimodular and that $\Gamma = \mathbb{Z}\Phi$.

Let $k(\Phi)(a)$ be the partition function. Then $k(\Phi)(a)$ coincides with a polynomial function on each chamber. We consider, as in Section 4.3, two adjacent chambers $\mathfrak{c}_1$ and $\mathfrak{c}_2$ separated by a wall W. As before, Φ_0 denotes $W \cap \Phi$; it is also a unimodular system for the lattice $\Gamma \cap W$. Let $k_{12} = k(\Phi_0, \mathfrak{c}_{12})$ be the polynomial function on $W \cap \Gamma$ associated to the chamber $\mathfrak{c}_{12}$ of Φ_0. Consider the sequence $\Psi = \Phi \setminus \Phi_0$. We choose $E \in U$ such that Ψ^+ is non empty.

As the system Φ is assumed to be unimodular, the integers $d_\phi = \langle\phi, E\rangle$ are equal to ± 1 for any $\phi \in \Phi$ not in W.

Theorem 28 *Let $k_{12} = k(\Phi_0, \mathfrak{c}_{12})$ be the polynomial function on W associated to the chamber $\mathfrak{c}_{12}$. Then, if $\langle E, \mathfrak{c}_1\rangle > 0$, we have*

$$k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2) = \mathrm{Par}(k_{12}, \Phi \setminus \Phi_0, E). \tag{10}$$

Remark 29 *By Theorem 26, the function $\mathrm{Par}(k_{12}, \Phi\setminus\Phi_0, E)$ coincide, up to sign, on $\Gamma_{\geq 0}$ with the discrete convolution $\sum_{w\in W\cap\Gamma} k_{12}(w)k(R_+(\Psi))(a-\kappa_- - w)$ of the polynomial function $k_{12}(w)$ on W by the partition function (shifted) $k(R_+(\Psi))$. Thus, our residue formula for $k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2)$ coincide with Paradan's formula ([8], Theorem 5.2) for the jump of the partition function.*

Proof. Denote by $\mathrm{Leq}(\Phi)$ the left hand side and by $\mathrm{Req}(\Phi)$ the right hand side of Equation (10) above.

As the system is unimodular, the lattice Γ is generated by Γ_0 and any ϕ not in W. So in a parallel way to the proof of the jump for the volume function, it would be sufficient to verify that $D(\phi)(\mathrm{Leq}(\Phi)-\mathrm{Req}(\Phi)) = 0$ for some $\phi \in \Phi$ not in W, and that $(\mathrm{Leq}(\Phi)-\mathrm{Req}(\Phi))$ vanishes on W. However, in order that our proof adapts without change to the non unimodular case, we will check $D(\phi)(\mathrm{Leq}(\Phi) - \mathrm{Req}(\Phi)) = 0$ for any $\phi \in \Phi$.

Let us first verify Equation (10) above when there is only one vector ϕ of Φ not in W. We can suppose that $\Psi^+ = \{\phi\}$ and Ψ^- is empty. In this case the wall W is a facet of the cone $C(\Phi)$. The chamber $\mathfrak{c}_1$ is equal to $\mathfrak{c}_{12}\times\mathbb{R}_{>0}\phi$, while $\mathfrak{c}_2$ is the exterior chamber. It is easy to see that $k(\Phi, \mathfrak{c}_1)(w + t\phi) = k_{12}(w)$, whereas $k(\Phi, \mathfrak{c}_2) = 0$. The equation follows from the first item of Lemma 19.

Suppose that this is not the case. Let ϕ be in Φ, and denote $\Phi' = \Phi\setminus\{\phi\}$. We study the difference equations satisfied by $\mathrm{Leq}(\Phi)$ and $\mathrm{Req}(\Phi)$.

We have several cases to consider.

- ϕ is not in W.

  Then the sequence $\Phi' = \Phi \setminus \{\phi\}$ spans V and W is a wall for Φ'. The intersection of Φ' with W is Φ_0. Let $\mathfrak{c}_1'$ and $\mathfrak{c}_2'$ be the chambers for Φ' containing $\mathfrak{c}_1, \mathfrak{c}_2$. Then, they are adjacent with respect to W. The chamber $\mathfrak{c}_{12}$ remains the same. By Lemma 13, we have

  $$D(\phi)(k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2)) = k(\Phi', \mathfrak{c}_1') - k(\Phi', \mathfrak{c}_2').$$

  By Proposition 20,

  $$D(\phi)\mathrm{Par}(k_{12}, \Phi \setminus \Phi_0, E) = \mathrm{Par}(k_{12}, \Phi' \setminus \Phi_0, E).$$

  By induction, we obtain $D(\phi)(\mathrm{Leq}(\Phi) - \mathrm{Req}(\Phi)) = 0$.

- ϕ is in W and $\Phi_0' = \Phi_0 \setminus \{\phi\}$ span W.

  Then the sequence $\Phi' = \Phi \setminus \{\phi\}$ spans V, and W is a wall for the system Φ'. Let $\mathfrak{c}_1'$ and $\mathfrak{c}_2'$ be the chambers for Φ' containing $\mathfrak{c}_1, \mathfrak{c}_2$. Then, they are adjacent with respect to W. By Lemma 13, we have

  $$D(\phi)(k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2)) = k(\Phi', \mathfrak{c}_1') - k(\Phi', \mathfrak{c}_2').$$

  Let $\mathfrak{c}_{12}'$ be the chamber for the sequence Φ_0' containing $\mathfrak{c}_{12}$. The sequence $\Phi \setminus \Phi_0$ is equal to $\Phi' \setminus \Phi_0'$. By Proposition 20, we obtain

  $$\begin{aligned} D(\phi)\mathrm{Par}(k(\Phi_0, \mathfrak{c}_{12}), \Phi \setminus \Phi_0, E) &= \mathrm{Par}(k(\Phi_0', \mathfrak{c}_{12}'), \Phi \setminus \Phi_0, E) \\ &= \mathrm{Par}(k(\Phi_0', \mathfrak{c}_{12}'), \Phi' \setminus \Phi_0', E). \end{aligned}$$

  So by induction, we conclude that $D(\phi)(\mathrm{Leq}(\Phi) - \mathrm{Req}(\Phi)) = 0$ again.

- ϕ is in W and Φ_0' does not span W. Then W is not a wall for the sequence Φ'.

  It follows from the description given in Proposition 9 of the regular behavior of functions on chambers that $k(\Phi', \mathfrak{c}_1') - k(\Phi', \mathfrak{c}_2') = 0$. Thus, by Lemma 13,

  $$D(\phi)(k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2)) = k(\Phi', \mathfrak{c}_1') - k(\Phi', \mathfrak{c}_2') = 0.$$

  Similarly, the function $k(\Phi_0, \mathfrak{c}_{12})$ satisfies $D(\phi)k(\Phi_0, \mathfrak{c}_{12}) = 0$. As

  $$D(\phi)\mathrm{Par}(k(\Phi_0, \mathfrak{c}_{12}), \Phi \setminus \Phi_0, E) = \mathrm{Par}(D(\phi)k(\Phi_0, \mathfrak{c}_{12}), \Phi \setminus \Phi_0, E),$$

  we obtain $D(\phi)(\mathrm{Leq}(\Phi)) = 0 = D(\phi)(\mathrm{Req}(\Phi))$.

We conclude that $D(\phi)(\mathrm{Leq}(\Phi) - \mathrm{Req}(\Phi)) = 0$ for any $\phi \in \Phi$ so that $\mathrm{Leq}(\Phi) - \mathrm{Req}(\Phi)$ is constant on $\Gamma = \mathbb{Z}\Phi$. It is thus sufficient to verify that $\mathrm{Leq}(\Phi) - \mathrm{Req}(\Phi)$ vanishes on W. If Φ^+ and Φ^- are both non empty, both chambers $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are interior chambers. So, $\mathrm{Leq}(\Phi)$ vanishes on W. By Lemma 27, $\mathrm{Req}(\Phi)$ also vanishes on W. If Φ^- is empty, the wall W is a facet of $C(\Phi)$. The restriction of the function $k(\Phi, \mathfrak{c})$ to a facet is the function $k(\Phi_0, \mathfrak{c}_{12})$. Thus $\mathrm{Leq}(\Phi)$ restricts to k_{12} on W by Lemma 7. This is the same for $\mathrm{Req}(\Phi)$ by Lemma 27. Thus we established the theorem.

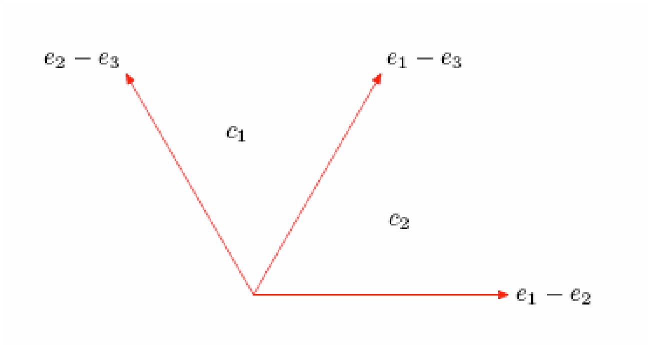

Figure 2: Chambers of A_2

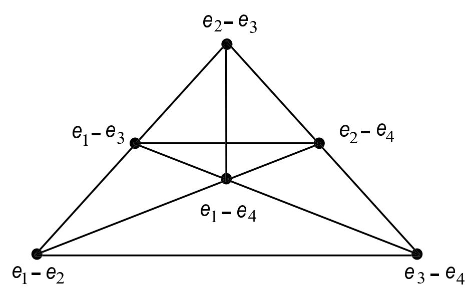

Figure 3: Chambers of A_3

We now give some examples of jumps in partition functions for the root system of type A_r (which is unimodular).

Let Y be an $(r+1)$ dimensional vector space with basis $\{e_i, i : 1, \dots, r+1\}$; we denote its dual basis by $\{e^i\}$. Let V denote the vector space generated by the set of positive roots

$$\Phi(A_r) = \{e_i - e_j : \ 1 \leq i < j \leq r+1\}$$

of A_r. Then, V is a hyperplane in Y formed by points $v = \sum_{i=1}^{r+1} v_i e_i \in Y$ satisfying $\sum_{i=1}^{r+1} v_i = 0$. Using the explicit isomorphism $p : \mathbb{R}^r \to V$ defined by $(a_1, \dots, a_r) \mapsto a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1}$, we write $a \in V$ as $a = \sum_{i=1}^{r} a_i (e_i - e_{r+1})$. Under p^*, the vector $e_i - e_{r+1}$ determines the linear function x_i in $U = V^* \sim \mathbb{R}^r$. The vector $a \in V$ lies in $C(\Phi(A_r))$ if and only if $a_1 + \cdots + a_i \geq 0$ for all $i : 1, \dots, r$. This will be our notation for subsequent examples concerning A_r.

Example 30 *We consider the root system of type A_2 with $\Phi = \{e_1 - e_2, e_2 - e_3, e_1 - e_3\}$ (see Figure 2). The cone $C(\Phi)$ is comprised of two chambers $\mathfrak{c}_1 = C(\{e_1 - e_3, e_2 - e_3\})$ and $\mathfrak{c}_2 = C(\{e_1 - e_3, e_1 - e_2\})$. We will calculate both $k(\Phi, \mathfrak{c}_1)$ and $k(\Phi, \mathfrak{c}_2)$ using our formula in Theorem 28 iteratively starting from an exterior chamber.*

(i) Jump from the exterior chamber to $\mathfrak{c}_1$: In this case, $E = e^1$, $\Phi_0 = \{e_2 - e_3\}$, $\Phi^+ = \{e_1 - e_3, e_1 - e_2\}$ and $\Phi^- = \emptyset$.

$$\begin{aligned} k(\Phi, \mathfrak{c}_1)(a) - k(\Phi, \mathfrak{c}_{\text{ext}})(a) &= \operatorname{Par}(1, \Phi \setminus \Phi_0, e^1)(a) \\ k(\Phi, \mathfrak{c}_1)(a) &= \operatorname{Res}_{z=0} \left(\tfrac{e^{a_1 x_1 + a_2 x_2 + z a_1}}{(1-e^{-x_1 - z})(1 - e^{-x_1 + x_2 - z})} \right)_{x=0} \\ &= \operatorname{Res}_{z=0} \tfrac{e^{z a_1}}{(1-e^{-z})^2} = 1 + a_1. \end{aligned}$$

(ii) Jump from $\mathfrak{c}_1$ to $\mathfrak{c}_2$: We have $E = e^2$, $\Phi_0 = \{e_1 - e_3\}$, $\Phi^- = \{e_1 - e_2\}$ and $\Phi^+ = \{e_2 - e_3\}$.

$$\begin{aligned} k(\Phi, \mathfrak{c}_1)(a) - k(\Phi, \mathfrak{c}_2)(a) &= \operatorname{Par}(1, \Phi \setminus \Phi_0, e^2)(a) \\ &= \operatorname{Res}_{z=0} \left(\tfrac{e^{a_1 x_1 + a_2 x_2 + z a_2}}{(1-e^{-x_2 - z})(1 - e^{-x_1 + x_2 - z})} \right)_{x=0} \\ &= \operatorname{Res}_{z=0} \tfrac{e^{z a_2}}{(1-e^{-z})(1-e^{z})} = -a_2. \end{aligned}$$

Then, $k(\Phi, \mathfrak{c}_2)(a) = 1 + a_1 + a_2$.

Example 31 *We now consider the root system of type A_3 (see Figure 3 which depicts the 7 chambers of A_3 via the intersection of the ray $\mathbb{R}^+\alpha$ of each root α with the plane $3a_1 + 2a_2 + a_3 = 1$).*

We will calculate the jump in the partition function from $\mathfrak{c}_1 := C(\{e_1 - e_2, e_1 - e_3, e_1 - e_4\})$ to $\mathfrak{c}_2 := C(\{e_1 - e_2, e_3 - e_4, e_1 - e_4\})$. In this case, $E = e^3$, $\Phi_0 = \{e_1 - e_4, e_2 - e_4, e_1 - e_2\}$, $\Phi^+ = \{e_3 - e_4\}$ and $\Phi^- = \{e_1 - e_3, e_2 - e_3\}$. Notice that k_{12} is the partition function corresponding to the chamber $C(\{e_1 - e_4, e_1 - e_2\})$ of the copy of A_2 in A_3 having the set Φ_0 as its set of positive roots. Using the final calculation in part (ii) of Example 30 (e_4 here plays the role of e_3 in that example), $k_{12}(a) = a_1 + a_2 + 1$. Then, by Theorem 28,

$$\begin{aligned} k(\Phi, \mathfrak{c}_2)(a) - k(\Phi, \mathfrak{c}_1)(a) &= \operatorname{Par}(k_{12}, \Phi \setminus \Phi_0, E)(a) \\ &= \operatorname{Res}_{z=0} \left((\partial_{x_1} + \partial_{x_2} + 1) \cdot \tfrac{e^{a_1 x_1 + a_2 x_2 + z a_3}}{(1-e^{-z})(1-e^{-x_1 + z})(1 - e^{-x_2 + z}))} \right)_{x=0} \\ &= \tfrac{1}{6} a_3 (a_3 - 1)(2a_3 + 3a_2 + 3a_1 + 5). \end{aligned}$$

5.3 Khovanskii-Pukhlikov differential operator

We recall that $\Gamma = \mathbb{Z}\Phi$. We normalize the measure dx in order that it gives volume 1 to a fundamental domain for Γ^*, and we write $v(\Phi, dx, \mathfrak{c})$ simply as $v(\Phi, \mathfrak{c})$.

We recall the relation between the function $k(\Phi, \mathfrak{c})$ and $v(\Phi, \mathfrak{c})$. Define $\mathrm{Todd}(z)$ as the expansion of

$$\frac{z}{1 - e^{-z}} = 1 + \frac{1}{2}z + \frac{1}{12}z^2 + \cdots$$

in power series in z. For $\phi \in \Phi$, $\mathrm{Todd}(\partial(\phi))$ is a differential operator of infinite order with constant coefficients. If p is a polynomial function on V, $\mathrm{Todd}(\partial(\phi))p$ is well defined and is a polynomial on V. We denote by $\mathrm{Todd}(\Phi, \partial)$ the operator defined on polynomial functions on V by

$$\mathrm{Todd}(\Phi, \partial) = \prod_{\phi \in \Phi} \mathrm{Todd}(\partial(\phi)).$$

The operator $\mathrm{Todd}(\Phi, \partial)$ transforms a polynomial function into a polynomial function on Γ.

The following result has been proven in Dahmen-Micchelli [4].

Theorem 32 *Let $\mathfrak{c}$ be a chamber. Then*

$$k(\Phi, \mathfrak{c})(a) = \mathrm{Todd}(\Phi, \partial) \cdot v(\Phi, \mathfrak{c}).$$

Here we give yet another proof of this theorem, by verifying that our explicit formula for the jumps are related by the Todd operator.

Let W be a wall of Φ determined by E. Assume that Φ^+ is non empty. Let $\mathrm{Todd}(\Phi_0, \partial)$ be the Todd operator related to the sequence $\Phi_0 = \Phi \cap W$ which is also unimodular.

Proposition 33 *Let p be a polynomial function on W. Then*

$$\mathrm{Todd}(\Phi, \partial)\mathrm{Pol}(p, \Phi \setminus \Phi_0, E) = \mathrm{Par}(\mathrm{Todd}(\Phi_0, \partial)P, \Phi \setminus \Phi_0, E).$$

Proof. We have

$$\mathrm{Todd}(\Phi, \partial) = \mathrm{Todd}(\Phi \setminus \Phi_0, \partial)\mathrm{Todd}(\Phi_0, \partial).$$

By Proposition 17

$$\mathrm{Todd}(\Phi_0,\partial)\mathrm{Pol}(p,\Phi\setminus\Phi_0,E)=\mathrm{Pol}(\mathrm{Todd}(\Phi_0,\partial)p,\Phi\setminus\Phi_0,E).$$

Let $q=\mathrm{Todd}(\Phi_0,\partial)p$. Then we apply $\mathrm{Todd}(\Phi\setminus\Phi_0,\partial)$ to

$$\mathrm{Pol}(q,\Phi\setminus\Phi_0,E)=\mathrm{Res}_{z=0}\left(Q(\partial_x)\cdot\frac{e^{\langle a,x+zE\rangle}}{\prod_{\phi\in\Phi\setminus\Phi_0}\langle\phi,x+zE\rangle}\right)_{x=0}.$$

We obtain

$$\mathrm{Todd}(\Phi\setminus\Phi_0,\partial)\mathrm{Pol}(q,\Phi\setminus\Phi_0,E)(a)=\mathrm{Par}(q,\Phi\setminus\Phi_0,E).$$

We now prove Theorem 32 by induction. We assume that $\mathrm{Todd}(\Phi_0,\partial)v(\Phi_0,\mathfrak{c}_{12})=k(\Phi_0,\mathfrak{c}_{12})$. We then obtain from Proposition 33:

$$\begin{aligned}\mathrm{Todd}(\Phi,\partial)(v(\Phi,\mathfrak{c}_1)-v(\Phi,\mathfrak{c}_2)) &=\mathrm{Todd}(\Phi,\partial)\mathrm{Pol}(v(\Phi_0,\mathfrak{c}_{12}),\Phi\setminus\Phi_0,E)\\ &=\mathrm{Par}(\mathrm{Todd}(\Phi_0,\partial)v(\Phi_0,\mathfrak{c}_{12}),\Phi\setminus\Phi_0,E)\\ &=\mathrm{Par}(k(\Phi_0,\mathfrak{c}_{12}),\Phi\setminus\Phi_0,E)\\ &=k(\Phi,\mathfrak{c}_1)-k(\Phi,\mathfrak{c}_2).\end{aligned}$$

Starting from the exterior chamber where $k(\Phi,\mathfrak{c}_{\mathrm{ext}})=\mathrm{Todd}(\Phi,\partial)\cdot v(\Phi,\mathfrak{c}_{\mathrm{ext}})=0$, we obtain by jumping over the walls that $k(\Phi,\mathfrak{c})=\mathrm{Todd}(\Phi,\partial)\cdot v(\Phi,\mathfrak{c})$ for any chamber.

6 Wall crossing formula for the partition function: general case

In this section, we compute the jump $k(\Phi,\mathfrak{c}_1)-k(\Phi,\mathfrak{c}_2)$ of the partition function $k(\Phi)$ across a wall when Φ is an arbitrary system.

6.1 A particular quasi-polynomial function

Let W be a hyperplane of V determined by a primitive vector E, and $\Gamma_0=W\cap\Gamma$. We denote by T the torus V^*/Γ^* and T_0 the torus W^*/Γ_0^*. The restriction map $V^*\to W^*$ induces a surjective homomorphism $r:T\to T_0$. The kernel of r is isomorphic to $\mathbb{R}/\mathbb{Z}$.

Let Q be a quasi-polynomial function on Γ. We may write $Q(a) = \sum_{y\in T} e_y(a)Q_y(a)$ where $y \in V^*$ give rise to an element of finite order in V^*/Γ^*, still denoted by y. The set of elements $g \in T$ such that $r(g) = r(y)$ is isomorphic to $\mathbb{R}/\mathbb{Z}$.

Definition 34 *Let $Q(a) = \sum_{y\in T} e_y(a)Q_y(a)$ be a quasi-polynomial function on Γ and let Ψ be a sequence of vectors not belonging to W. We define, for $a \in \Gamma$,*

$$\mathrm{Para}(Q, \Psi, E)(a) = \sum_{y\in T} \sum_{g\in T|r(g)=r(y)} \mathrm{Res}_{z=0} \left(Q_y(\partial_x) \cdot \frac{e^{\langle a,x+2i\pi g+zE\rangle}}{\prod_{\psi\in\Psi}(1 - e^{-\langle \psi,x+2i\pi g+zE\rangle})} \right)_{x=0}.$$

Remark 35 *The definition may look strange, as we sum a priori on the infinite set $r(g) = r(y)$. However, in order that the function*

$$\left(Q_y(\partial_x) \frac{e^{\langle a,x+2i\pi g+zE\rangle}}{\prod_{\psi\in\Psi}(1 - e^{-\langle \psi,x+2i\pi g+zE\rangle})} \right)_{x=0}$$

has a pole at $z = 0$, we see that there must exist a ψ in Ψ such that $e^{2i\pi\langle\psi,g\rangle} = 1$. As $r(g)$ is fixed, this leaves a finite number of possibilities for g. More concretely, if y is given, any g such that $r(g) = r(y)$ is of the form $g = y+GE$, and G must satisfy $e^{2i\pi G\langle\psi,E\rangle} = e^{-2i\pi\langle y,\psi\rangle}$ for some $\psi \in \Psi$. Furthermore, we see that if the integers $\langle\psi, E\rangle$ are equal to ± 1 for all $\psi \in \Psi$, and if Q is polynomial (so that $y = 0$ on the above equation), then $\mathrm{Para}(Q, \Psi, E)$ *is equal to* $\mathrm{Par}(Q, \Psi, E)$.

It is easy to see that $\mathrm{Para}(Q, \Psi, E)(a)$ is a quasi-polynomial function of $a \in V$. Furthermore, using the same argument as in the proof of Lemma 15, we obtain the following.

Lemma 36 *The quasi-polynomial function* $\mathrm{Para}(Q, \Psi, E)$ *depends only on the restriction q of Q to Γ_0.*

Choose a primitive vector F such that $\Gamma = \Gamma_0 \oplus \mathbb{Z}F$. Then we see that any quasi-polynomial function q on Γ_0 extends to a quasi-polynomial function Q on Γ.

Definition 37 *Let q be a quasi-polynomial function on Γ_0. We define*

$$\mathrm{Para}(q,\Psi,E) := \mathrm{Para}(Q,\Psi,E),$$

where Q is any quasi-polynomial function on Γ extending q.

Remark 38 *Let q be a quasi-polynomial function on Γ_0; we may write $q(w) = \sum_{y\in T_0} e_y(w)q_y(w)$. Let Q_y denote any extension of the polynomial function q_y on V. Then, while calculating $\mathrm{Para}(q,\Psi,E)$, we are in fact summing over $g \in T$ such that $r(g) = y$.*

Proposition 39 *Let $\psi \in \Psi$. Then*

$$D(\psi)\mathrm{Para}(q,\Psi,E) = \mathrm{Para}(q,\Psi \setminus \{\psi\},E).$$

Let $w \in \Gamma_0$. Then,

$$D(w)\mathrm{Para}(q,\Psi,E) = \mathrm{Para}(D(w)q,\Psi,E).$$

Proof. The first formula is immediate from the definition. For the second formula, if $r(g) = r(y)$ and if $w \in \Gamma_0$, then

$$e^{\langle a-w,x+2i\pi g+zE\rangle} = e^{-\langle w,x\rangle}e_y(-w)e^{\langle a,x+2i\pi g+zE\rangle},$$

and the result follows as in the proof of Proposition 20.

6.2 Discrete convolution

We take the same notations as in Section 5.1. However here the system Ψ is arbitrary. We define, as before, the function $K^+(\Psi)$ on $\Gamma_{\geq 0}$ by the equation

$$\prod_{\psi\in\Psi} \frac{1}{1-e^{-\langle\psi,x\rangle}} = \sum_{a\in\Gamma_{\geq 0}} K^+(\Psi)(a)e^{-\langle a,x\rangle}. \tag{11}$$

Let q be a quasi-polynomial function on Γ_0. Define for $a \in \Gamma_{\geq 0}$

$$C(q,\Psi,E)(a) := \sum_{w\in\Gamma_0} q(w)K^+(\Psi)(a-w).$$

Theorem 40 *Let q be a quasi-polynomial function on Γ_0. Assume that Ψ^+ is non empty. Then, for $a \in \Gamma_{\geq 0}$,*

$$C(q, \Psi, E)(a) = \mathrm{Para}(q, \Psi, E)(a).$$

Proof. We need to compute, for $a \in \Gamma_{\geq 0}$,

$$S(a) := \sum_{w \in \Gamma_0} q(w) K^+(\Psi)(a - w).$$

This sum is over a finite set.

Let $Q(a) = \sum_{y \in T} e_y(a) Q_y(a)$ be any quasi-polynomial function on Γ extending q. We may write

$$S(a) = \sum_{y \in T} \left(Q_y(\partial_x) \cdot \sum_{w \in \Gamma_0} K^+(\Psi)(a - w) e^{\langle w, x + 2i\pi y \rangle} \right)_{x=0}.$$

Define

$$G_{x,y}(a) = \sum_{w \in \Gamma_0} K^+(\Psi)(a - w) e^{-\langle a - w, x + 2i\pi y \rangle}.$$

Then $G_{x,y}(a)$ depends in an analytic way of the variable $x \in U$, and we have

$$S(a) = \sum_{y \in T} \left(Q_y(\partial_x) \cdot e^{\langle a, x + 2i\pi y \rangle} G_{x,y}(a) \right)_{x=0}. \tag{12}$$

The function $a \mapsto G_{x,y}(a) = \sum_{w \in \Gamma_0} K^+(\Psi)(a - w) e^{-\langle a - w, x + 2i\pi y \rangle}$ is a function on $\Gamma/\Gamma_0 = \mathbb{Z}F$. To identify the function $G_{x,y}(nF)$, with $n = \langle a, E \rangle$, we compute its discrete Laplace transform in one variable. With the same proof as the proof in Theorem 26, we obtain that for x in the dual cone to $C(R_+(\Psi))$, $L_{\mathrm{dis}}(G_{x,y})(u) = \sum_{n \geq 0} G_{x,y}(nF) u^n$ is convergent for $|u| < 1$ and we obtain

$$G_{x,y}(a) = \frac{1}{2i\pi} \int_{|u|=\epsilon} \frac{u^{-n}}{\prod_{\psi \in \Psi} (1 - e^{-\langle \psi, x + 2i\pi y \rangle} u^{\langle \psi, E \rangle})} \frac{du}{u}$$

where $n = \langle a, E \rangle$.

Thus Formula (12) becomes

$$S(a) = \sum_{y \in T} \left(Q_y(\partial_x) \cdot e^{\langle a, x + 2i\pi y \rangle} \frac{1}{2i\pi} \int_{|u|=\epsilon} \frac{u^{-n}}{\prod_{\psi \in \Psi} (1 - e^{-\langle \psi, x + 2i\pi y \rangle} u^{\langle \psi, E \rangle})} \frac{du}{u} \right)_{x=0}.$$

Let

$$F_y(u) = \left(Q_y(\partial_x) \cdot e^{\langle a, x+2i\pi y\rangle} \frac{u^{-n}}{\prod_{\psi\in\Psi}(1 - e^{-\langle \psi, x+2i\pi y\rangle} u^{\langle \psi, E\rangle})} \right)_{x=0}.$$

As at least one of the $\langle \psi, E\rangle$ is strictly positive and $n \geq 0$, the function $F_y(u)$ has no pole at ∞. The integral over $|u| = \epsilon$ computes the residue of $F_y(u)$ at $u = 0$. All other poles are such that $u^{\langle \psi, E\rangle} = e^{\langle \psi, 2i\pi y\rangle}$ for some $\psi \in \Psi$, so they are roots of unity $\zeta = e^{2i\pi G}$ with $G \in \mathbb{R}/\mathbb{Z}$. Any element $g \in T$ with $r(g) = r(y)$ is of the form $g = y + GE$ with some G. We obtain

$$S(a) = -\sum_{y\in T} \sum_{G\in\mathbb{R}/\mathbb{Z}} \mathrm{Res}_{u=e^{2i\pi G}} F_y(u).$$

We write $u = e^{2i\pi G} e^{-z}$ in the neighborhood of $e^{2i\pi G}$ and we obtain the formula of the theorem.

Similarly, we compute the restriction of $\mathrm{Para}(q, \Psi, E)$ to $W \cap \Gamma$.

Lemma 41
- *If* $|\Psi^-| = \emptyset$, *then the restriction of* $\mathrm{Para}(q, \Psi, E)$ *to* W *is equal to* q.
- *If* $|\Psi^-| > 0$, *then the restriction of* $\mathrm{Para}(q, \Psi, E)$ *to* W *vanishes.*

Proof. The sum formula gives $\mathrm{Para}(q, \Psi, E)(w) = K^+(\Psi)(0)q(w)$ and $K^+(\Psi)(0)$ vanishes as soon as $|\Psi^-| > 0$.

6.3 The jump for the partition function

Let Φ be a sequence of vectors spanning the lattice Γ. Let $k(\Phi)(a)$ be the partition function given by quasi-polynomial functions on chambers. We consider as in Section 5.2 two adjacent chambers $\mathfrak{c}_1$ and $\mathfrak{c}_2$ separated by a wall W. As before, Φ_0 denotes $W \cap \Phi$. Let $k_{12} = k(\Phi_0, \mathfrak{c}_{12})$ be the quasi-polynomial function on $W \cap \Gamma$ associated to the chamber $\mathfrak{c}_{12}$ of Φ_0. Consider the sequence $\Psi = \Phi \setminus \Phi_0$. We choose $E \in U$ such that Ψ^+ is non empty. In the preceding section, we have associated a quasi-polynomial function $\mathrm{Para}(k_{12}, \Phi \setminus \Phi_0, E)$ on Γ to $\Phi \setminus \Phi_0$, E and k_{12} . We recall that

$$\mathrm{Para}(k_{12}, \Phi \setminus \Phi_0, E)(a) = \sum_{w\in W\cap\Gamma} k_{12}(w) K^+(\Phi \setminus \Phi_0)(a - w).$$

Theorem 42 *Let $k_{12} = k(\Phi_0, \mathfrak{c}_{12})$ be the quasi-polynomial function on Γ_0 associated to the chamber $\mathfrak{c}_{12}$. Then, if $\langle E, \mathfrak{c}_1\rangle > 0$, we have*

$$k(\Phi, \mathfrak{c}_1) - k(\Phi, \mathfrak{c}_2) = \mathrm{Para}(k_{12}, \Phi \setminus \Phi_0, E). \tag{13}$$

Proof. The proof is exactly the same as in the proof of Theorem 28 corresponding to the unimodular case.

Example 43 *We consider the root system of type B_2 (see Figure 1). We will calculate $k(\Phi, \mathfrak{c})$ for all chambers $\mathfrak{c}$ using our formula in Theorem 42 iteratively starting from an exterior chamber.*

(i) Jump from the exterior chamber to $\mathfrak{c}_1$: We have $E = e^1$, $\Phi_0 = \{e_2\}$, $\Phi^+ = \{e_1 + e_2, e_1, e_1 - e_2\}$ and $\Phi^- = \emptyset$. With the notation of Definition 14, $Q = 1$(hence $y = 0$) and $\langle \phi, E\rangle = \pm 1$ for all $\phi \in \Phi \setminus \Phi_0$. Then, by Remark 35, $\mathrm{Para}(1, \Phi \setminus \Phi_0, E) = \mathrm{Par}(1, \Phi \setminus \Phi_0, E)$. We get

$$\begin{aligned} k(\Phi, \mathfrak{c}_1)(a) - k(\Phi, \mathfrak{c}_{\mathrm{ext}})(a) &= \mathrm{Res}_{z=0} \left(\frac{e^{\langle a, x+ze^1\rangle}}{(1-e^{-(x_1+x_2+z)})(1-e^{-(x_1+z)})(1-e^{-(x_1-x_2+z)})} \right)_{x=0} \\ k(\Phi, \mathfrak{c}_1)(a) &= \mathrm{Res}_{z=0} \left(\frac{e^{a_1 z}}{(1-e^{-z})^3} \right) = \frac{1}{2}(a_1+2)(a_1+1). \end{aligned}$$

(ii) Jump from $\mathfrak{c}_1$ to $\mathfrak{c}_2$: We have $E = e^1 - e^2$, $\Phi_0 = \{e_1 + e_2\}$, $\Phi^+ = \{e_1, e_1 - e_2\}$ and $\Phi^- = \{e_2\}$. We also have $Q = k_{12} = 1$, thus $y = 0$. Then, the set of feasible $g \in T$ giving a nontrivial residue at $z = 0$ for a summand in $\mathrm{Para}(1, \Phi \setminus \Phi_0, E)$ and satisfying $r(g) = 0$ is $\{0, \frac{e^1-e^2}{2}\}$. By Theorem 42,

$$\begin{aligned} k(\Phi, \mathfrak{c}_2)(a) - k(\Phi, \mathfrak{c}_1)(a) &= \mathrm{Para}(1, \Phi \setminus \Phi_0, E)(a) \\ &= \sum_{g=0, g=\frac{e^1-e^2}{2}} \mathrm{Res}_{z=0} \left(\frac{e^{\langle a, x+2i\pi g+zE\rangle}}{\prod_{\phi\in\Phi\setminus\Phi_0}(1-e^{-\langle \phi, x+2i\pi g+zE\rangle})} \right)_{x=0} \\ &= \mathrm{Res}_{z=0}\left(\frac{e^{(a_1-a_2)z}}{(1-e^{-z})(1-e^{-2z})(1-e^{z})} \right) + \mathrm{Res}_{z=0}\left(\frac{(-1)^{a_1+a_2} e^{(a_1-a_2)z}}{(1+e^{-z})(1-e^{-2z})(1+e^{z})} \right) \\ &= (-1)^{a_1+a_2}\frac{1}{8} - \frac{1}{8}(2a_2^2 - 4a_1a_2 + 2a_1^2 + 1 - 4a_2 + 4a_1) \\ &= \begin{cases} -\frac{1}{4}(a_2 - a_1)(a_2 - a_1 - 2) & \text{if } a_1 + a_2 \text{ even,} \\ -\frac{1}{4}(a_2 - a_1 - 1)^2 & \text{if } a_1 + a_2 \text{ odd} \end{cases} \\ &= -\frac{1}{4}\left(a_2 - a_1 - 1 + \left(\frac{1+(-1)^{a_1+a_2}}{2}\right)\right)\left(a_2 - a_1 - 1 - \left(\frac{1+(-1)^{a_1+a_2}}{2}\right)\right). \end{aligned}$$

Using (i), we get

$$\begin{aligned} k(\Phi, \mathfrak{c}_2)(a) &= \frac{1}{2}(a_1+2)(a_1+1) + (-1)^{a_1+a_2}\frac{1}{8} - \frac{1}{8}(2a_2^2 - 4a_1a_2 + 2a_1^2 + 1 - 4a_2 + 4a_1) \\ &= \frac{1}{4}a_1^2 + \frac{1}{2}a_1a_2 - \frac{1}{4}a_2^2 + a_1 + \frac{1}{2}a_2 + \frac{7}{8} + (-1)^{a_1+a_2}\frac{1}{8}. \end{aligned}$$

(iii) Jump from $\mathfrak{c}_2$ *to* $\mathfrak{c}_3$*: We have* $E = e^2$*,* $\Phi_0 = \{e_1\}$*,* $\Phi^+ = \{e_2, e_1 + e_2\}$ *and* $\Phi^- = \{e_1 - e_2\}$*. We again have* $Q = k_{23} = 1$ *and* $\langle \phi, E\rangle = \pm 1$ *for all* $\phi \in \Phi \setminus \Phi_0$*. Thus,* $y = g = 0$ *and (as in part (i)) we can use the formula for the unimodular case:*

$$\begin{aligned} k(\Phi,\mathfrak{c}_2)(a) - k(\Phi,\mathfrak{c}_3)(a) &= \mathrm{Res}_{z=0}\left(\frac{e^{\langle a, x+zE\rangle}}{\prod_{\phi\in\Phi\setminus\Phi_0}(1-e^{-\langle \phi, x+zE\rangle})}\right)_{x=0} \\ &= \mathrm{Res}_{z=0}\left(\frac{e^{a_2 z}}{(1-e^{-z})(1-e^{-z})(1-e^{z})}\right) = -\tfrac{1}{2}a_2(a_2+1). \end{aligned}$$

Then, $k(\Phi,\mathfrak{c}_3)(a) = \frac{1}{4}a_1^2 + \frac{1}{2}a_1a_2 + \frac{1}{4}a_2^2 + a_1 + a_2 + \frac{7}{8} + (-1)^{a_1+a_2}\frac{1}{8}$.

6.4 Generalized Khovanskii-Pukhlikov differential operator

Here Φ is a general sequence (not necessarily unimodular). We assume again that $\mathbb{Z}\Phi = \Gamma$. We choose the measure dx giving volume 1 to U/Γ^*. We write $v(\Phi, \mathfrak{c}, dx) = v(\Phi, \mathfrak{c})$.

For the complex number ζ, define $\mathrm{Todd}(\zeta, z)$ as the expansion of

$$\frac{z}{1-\zeta^{-1}e^{-z}}$$

into a power series in z. If $\zeta \neq 1$,

$$\frac{\mathrm{Todd}(\zeta,z)}{z} = \frac{1}{1-\zeta^{-1}e^{-z}} \tag{14}$$

is analytic at $z = 0$.

For $\phi \in \Phi$, $\mathrm{Todd}(\zeta, \partial(\phi))$ is a differential operator of infinite order with constant coefficients. If p is a polynomial function on V, $\mathrm{Todd}(\zeta, \partial(\phi))p$ is well defined and is a polynomial on V.

For $g \in T = U/\Gamma^*$, define the *Todd operator* (a series of differential operators with constant coefficients) by

$$\mathrm{Todd}(g, \Phi, \partial) := \prod_{k=1}^{N} \mathrm{Todd}(e_g(\phi_k), \partial(\phi_k)),$$

where $e_g(\phi_k) := e^{2i\pi\langle g, \phi_k\rangle}$.

If G is a finite subset of T, we denote by $\text{Todd}(G, \Phi, \partial)$ the operator defined on polynomial functions $v(a)$ on V by

$$(\text{Todd}(G, \Phi, \partial)v)(a) = \sum_{g \in G} e_g(a)(\text{Todd}(g, \Phi, \partial)v)(a).$$

Let $g \in T$, and define $\Phi(g) = \{\phi \in \Phi \,|\, e_g(\phi) = 1\}$. If $\Phi(g)$ do not generate V, it follows from Corollary 12 that $(\prod_{\phi \notin \Phi(g)} \partial(\phi))v(\Phi, \mathfrak{c}) = 0$. Thus $\text{Todd}(g, \Phi, \partial)v(\Phi, \mathfrak{c}) = 0$. Indeed $\text{Todd}(g, \Phi, \partial)$ is divisible by $(\prod_{\phi \notin \Phi(g)} \partial(\phi))$ as follows from Equation (14) above.

Definition 44 *Define*

$$G(\Phi) = \{g \in T \,|\, <\Phi(g)>= V\}.$$

The set $G(\Phi)$ is finite. Indeed, if $g \in G(\Phi)$, there must exists a basis σ of V extracted from Φ such that $e_\phi(g) = 1$ for all $\phi \in \sigma$, and this gives a finite set of solutions. If Φ is unimodular, then $G(\Phi)$ is reduced to the identity element.

The following result has been proven in [3].

Theorem 45 *Let $\mathfrak{c}$ be a chamber. Then*

$$k(\Phi, \mathfrak{c})(a) = \text{Todd}(G(\Phi), \Phi, \partial) \cdot v(\Phi, \mathfrak{c}).$$

Here we will give yet another proof of this theorem, by verifying that the explicit formula for the jumps are related by the Todd operator.

For the proof, it is easier to sum over 'all elements' t of T. If v is a polynomial function on V such that

(15) $\quad \text{Todd}(t, \Phi, \partial) \cdot v = 0$ except for a finite number of elements t,

we may define

$$\text{Todd}(T, \Phi, \partial)v(a) = \sum_{t \in T} e_t(a)\text{Todd}(t, \Phi, \partial) \cdot v,$$

being understood that we only sum over the finite subset of $t \in T$ such that $\text{Todd}(t, \Phi, \partial) \cdot v \neq 0$. With this definition, for $v = v(\Phi, \mathfrak{c})$, then

$$\text{Todd}(T, \Phi, \partial)v = \text{Todd}(G(\Phi), \Phi, \partial)v.$$

To prove Theorem 45, we follow the same scheme of proof as in Theorem 32.

Let W be a wall and let $\mathfrak{c}_0$ be a chamber of the wall for the sequence $\Phi_0 = \Phi \cap W$. We only need to prove

Theorem 46

$$\text{Todd}(T,\Phi,\partial)\text{Pol}(v(\Phi_0,\mathfrak{c}_0),\Phi\setminus\Phi_0,E)=\text{Para}(k(\Phi_0,\mathfrak{c}_0),\Phi\setminus\Phi_0,E). \tag{16}$$

Proof. It is easy to see that the function $v:=\text{Pol}(v(\Phi_0,\mathfrak{c}_0),\Phi\setminus\Phi_0,E)$ satisfies the hypothesis (15) above. Let V_0 be a polynomial function on V extending $v_0=v(\Phi_0,\mathfrak{c}_0)$. Then $(\text{Todd}(T,\Phi,\partial)v)(a)$ is equal to

$$\sum_{t_0\in T_0}\sum_{t\in T\,|\,r(t)=t_0} e_t(a)(\text{Todd}(t_0,\Phi_0,\partial)\text{Todd}(t,\Phi\setminus\Phi_0,\partial)\cdot v)(a),$$

where T_0 denotes the torus W^*/Γ_0^*. We have

$$e_t(a)(\text{Todd}(t,\Phi\backslash\Phi_0,\partial)v)(a)=\text{Res}_{z=0}\left(V_0(\partial_x)\cdot\frac{e^{\langle a,x+2i\pi t+zE\rangle}}{\prod_{\phi\in\Phi\backslash\Phi_0}(1-e^{-\langle\phi,x+2i\pi t+zE\rangle})}\right)_{x=0},$$

so that $(\text{Todd}(T,\Phi,\partial)v)(a)$ is equal to

$$=\sum_{t_0\in T_0}\sum_{t\in T\,|r(t)=t_0}\text{Res}_{z=0}\left((\text{Todd}(t_0,\Phi_0,\partial)V_0)(\partial_x)\frac{e^{\langle a,x+2i\pi t+zE\rangle}}{\prod_{\phi\in\Phi\backslash\Phi_0}(1-e^{-\langle\phi,x+2i\pi t+zE\rangle})}\right)_{x=0}.$$

By induction hypothesis, a quasi-polynomial function extending $k_0(\Phi_0,\mathfrak{c}_0)$ is

$$\sum_{t_0\in T_0} e_{t_0}(a)(\text{Todd}(t_0,\Phi_0,\partial)V_0)(a)$$

where we denote again by t_0 any element of T such that $r(t)=t_0$. Thus the last formula is exactly the definition of $\text{Para}(k_0(\Phi_0,\mathfrak{c}_0),\Phi\setminus\Phi_0,E)(a)$.

The rest of the proof is identical to the proof of Theorem 32.

7 Some examples

In this section we give further examples of jumps in partition and volume for various root systems.

Example 47 *We will calculate the jump in volume from* $\mathfrak{c}_1:=C(\{e_1,e_1-e_2,e_1-e_3\}$ *to* $\mathfrak{c}_2:=C(\{e_1,e_1-e_2,e_1+e_3\})$ *of* B_3 *(see Figure 4 where chambers*

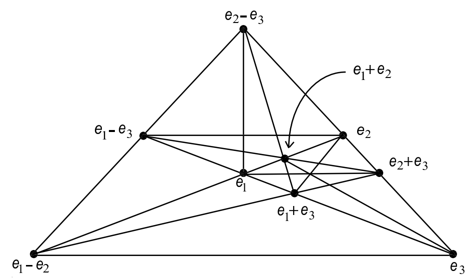

Figure 4: Chambers of B_3

of B_3 are depicted via the intersection of the ray $\mathbb{R}^+\alpha$ of each root with the plane $3a_1 + 2a_2 + a_3 = 1$).

We consider the copy B_2 in B_3 having the set $\{e_1, e_2, e_1+e_2, e_1-e_2\}$ as its set of positive roots. This particular jump is over the chamber $C(\{e_1, e_1-e_2\})$ of this B_2. We have $E = e^3$, $\Phi^+ = \{e_3, e_1+e_3, e_2+e_3\}$ and $\Phi^- = \{e_1-e_3, e_2-e_3\}$. Using part (iii) of Example 25, $v_{12}(a) = \frac{1}{4}(a_1 + a_2)^2$. By Theorem 23,

$$\begin{aligned}&v(\Phi,\mathfrak{c}_2)(a) - v(\Phi,\mathfrak{c}_1)(a) = \mathrm{Pol}(v_{12}, \Phi \setminus \Phi_0, E)(a)\\ &= \mathrm{Res}_{z=0}\left(v_{12}(\partial_x)\cdot \frac{e^{a_1x_1+a_2x_2+a_3(z+x_3)}}{(x_3+z)(x_1+x_3+z)(x_2+x_3+z)(x_1-x_3-z)(x_2-x_3-z)}\right)_{x=0}\\ &= \tfrac{1}{1440}a_3^4(30a_1a_2 + 15a_1^2 + 15a_2^2 + 2a_3^2).\end{aligned}$$

Example 48 *We will calculate the jump in volume from $\mathfrak{c}_2 = C(\{e_1, e_1 - e_2, e_1 + e_3\})$ to $\mathfrak{c}_3 := C(\{e_1, e_1 + e_3, e_2 - e_3\}) \cap C(\{e_1, e_1 + e_3, e_2 + e_3\})$ of B_3 (see Figure 4). We consider the copy B_2 in B_3 having the set $\{e_1, e_3, e_1 + e_3, e_1 - e_3\}$ as its set of positive roots. This particular jump is over chamber $C(\{e_1, e_1+e_3\})$ of this B_2. We have $E = e^2$, $\Phi^+ = \{e_2, e_2+e_3, e_2-e_3, e_1+e_2\}$ and $\Phi^- = \{e_1-e_2\}$. Using part (ii) of Example 25, $v_{23}(a) = \frac{1}{4}(a_1+a_3)^2 - \frac{1}{2}a_3^2$. By Theorem 23,*

$$\begin{aligned}v(\Phi,\mathfrak{c}_3)(a) - v(\Phi,\mathfrak{c}_2)(a) &= \mathrm{Pol}(v_{23}, \Phi \setminus \Phi_0, E)(a)\\ &= \mathrm{Res}_{z=0}\left(v_{23}(\partial_x)\cdot \frac{e^{a_1x_1+a_2z+a_3x_3}}{z(x_3+z)(-x_3+z)(x_1+z)(x_1-z)}\right)_{x=0}\\ &= -\tfrac{1}{96}a_2^4(a_1^2 + 2a_1a_3 - a_3^2)\end{aligned}$$

Example 49 *We will calculate the jump in the partition function from $\mathfrak{c}_1 = C(\{e_1, e_1 - e_2, e_1 - e_3\}$ to $\mathfrak{c}_2 = C(\{e_1, e_1 - e_2, e_1 + e_3\})$ of B_3 (see Figure 4). This particular jump is over the chamber $C(\{e_1, e_1 - e_2\})$ of the copy of B_2*

in B_3 *having positive roots* $\Phi_0 = \{e_1, e_2, e_1+e_2, e_1-e_2\}$. *We have* $E = e^3$, $\Phi^+ = \{e_3, e_1+e_3, e_2+e_3\}$, $\Phi^- = \{e_1-e_3, e_2-e_3\}$.

Using part (iii) of Example 43, $k_{12}(a) = \frac{1}{4}(a_1+a_2)^2 + a_1 + a_2 + \frac{7}{8} + (-1)^{a_1+a_2}\frac{1}{8}$. *Then, with the notation of Section 6, we have* $y = 0$ *or* $y = \frac{e^1+e^2}{2}$; *correspondingly* $Q_0(a) = \frac{1}{4}(a_1+a_2)^2 + a_1 + a_2 + \frac{7}{8}$ *and* $Q_{\frac{e^1+e^2}{2}}(a) = \frac{1}{8}$. *For* $y = 0$, *the only feasible* $g \in T$ *giving a nontrivial residue at* $z = 0$ *for a summand in* $\mathrm{Para}(k_{12}, \Phi \setminus \Phi_0, E)$ *and satisfying* $r(g) = 0$ *is* $g = 0$. *On the other hand, for* $y = \frac{e^1+e^2}{2}$ *the feasible set of* g *giving a nontrivial residue at* $z = 0$ *for a summand in* $\mathrm{Para}(k_{12}, \Phi \setminus \Phi_0, E)$ *and satisfying* $r(g) = \frac{e^1+e^2}{2}$ *is* $\{\frac{e^1+e^2}{2}, \frac{e^1+e^2+e^3}{2}\}$. *Then, by Theorem 42,*

$$
\begin{aligned}
&k(\Phi,\mathfrak{c}_2)(a) - k(\Phi,\mathfrak{c}_1)(a) = \mathrm{Para}(k_{12}, \Phi\setminus\Phi_0, E)(a)\\
&= \mathrm{Res}_{z=0}\left(Q_0(\partial_x)\cdot \frac{e^{\langle a, x+ze^3\rangle}}{\prod_{\phi\in\Phi^+\cup\Phi^-}(1-e^{-\langle\phi, x+ze^3\rangle})}\right)_{x=0}\\
&+\mathrm{Res}_{z=0}\left(Q_{\frac{e^1+e^2}{2}}(\partial_x)\cdot \frac{e^{\langle a, x+i\pi(e^1+e^2)+ze^3\rangle}}{\prod_{\phi\in\Phi^+\cup\Phi^-}(1-e^{-\langle\phi, x+i\pi(e^1+e^2)+ze^3\rangle})}\right)_{x=0}\\
&+\mathrm{Res}_{z=0}\left(Q_{\frac{e^1+e^2}{2}}(\partial_x)\cdot \frac{e^{\langle a, x+i\pi(e^1+e^2+e^3)+ze^3\rangle}}{\prod_{\phi\in\Phi^+\cup\Phi^-}(1-e^{-\langle\phi, x+i\pi(e^1+e^2+e^3)+ze^3\rangle})}\right)_{x=0}\\
&= \mathrm{Res}_{z=0}\left(Q_0(\partial_x)\cdot \frac{e^{a_1x_1+a_2x_2+a_3z}}{(1-e^{-z})(1-e^{-(x_1+z)})(1-e^{-(x_2+z)})(1-e^{-(x_1-z)})(1-e^{-(x_2-z)})}\right)_{x=0}\\
&+\tfrac{1}{8}\mathrm{Res}_{z=0}\left(\frac{(-1)^{a_1+a_2}e^{a_3z}}{(1-e^{-z})(1+e^{-z})^2(1+e^{z})^2}\right) + \tfrac{1}{8}\mathrm{Res}_{z=0}\left(\frac{(-1)^{a_1+a_2+a_3}e^{a_3z}}{(1+e^{-z})(1-e^{-z})^2(1-e^{z})^2}\right)\\
&= \tfrac{1}{2880}a_3(a_3-1)(a_3+2)(a_3+1)(4a_3^2+4a_3+30a_1^2+60a_2a_1+30a_2^2+441+240a_1+240a_2)\\
&+(-1)^{a_1+a_2}\tfrac{1}{128} + (-1)^{a_1+a_2+a_3}\tfrac{1}{384}(2a_3+1)(2a_3^2+2a_3-3).
\end{aligned}
$$

Let $\gamma_3 := \frac{1-(-1)^{a_3}}{2}$ *and* $\gamma_{12} := \frac{1-(-1)^{a_1+a_2}}{2}$. *Then, after some calculation, we can factor* $k(\Phi,\mathfrak{c}_2)(a) - k(\Phi,\mathfrak{c}_1)(a)$ *as:*

$$
\frac{1}{2880}(a_3-\gamma_3)(a_3+2-\gamma_3)\cdot((1-\gamma_3)(f_1-30(1-\gamma_{12})(1-2a_3)) + \gamma_3(f_2-30(1-\gamma_{12})(3+2a_3))),
$$

where

$$
\begin{aligned}
f_1 = \;& 4a_3^4+4a_3^3+30a_1^2a_3^2+60a_2a_1a_3^2+240a_2a_3^2+30a_2^2a_3^2+437a_3^2+240a_3^2a_1\\
&-34a_3-426-60a_2a_1-30a_1^2-240a_2-30a_2^2-240a_1\\
f_2 = \;& 4a_3^4+12a_3^3+30a_1^2a_3^2+60a_2a_1a_3^2+240a_2a_3^2+30a_2^2a_3^2+449a_3^2+240a_3^2a_1\\
&+60a_2^2a_3+60a_1^2a_3+480a_3a_2+120a_2a_1a_3+912a_3+480a_1a_3+45
\end{aligned}
$$

Example 50 *Let* $\mathfrak{c}_{\text{nice}}$ *denote the interior of the cone generated by the roots* $\{e_i - e_{r+1},\ 1 \le i \le r\}$ *of* A_r. *With the notation of Section 5.2 ,* $a =$

$\sum_{i=1}^{r} a_i(e_i - e_{r+1})$ *is in* $\mathfrak{c}_{\rm nice}$ *if and only if* $a_i > 0$ *for all* $1 \leq i \leq r$. *Then, the copy of* A_{r-1} *(with positive roots* $\{e_i - e_j : 2 \leq i < j \leq r+1\}$*) in* A_r *can be thought as the hyperplane* W *with* $\mathfrak{c}_1 = c_{\rm ext}$, $\mathfrak{c}_2 = \mathfrak{c}_{\rm nice}(A_r)$ *and* $\mathfrak{c}_{12} = \mathfrak{c}_{\rm nice}(A_{r-1})$. *Together with the fact that* $k(A_{r-1}, \mathfrak{c}_{\rm nice})(a)$ *is independent of* a_{r-1}, *we have by Theorem 28,*

$$\begin{aligned} k(A_r, \mathfrak{c}_{\rm nice})(a) &= {\rm Par}(k(A_{r-1}, \mathfrak{c}_{\rm nice}), \{e_1 - e_2, \ldots, e_1 - e_{r+1}\}, e^1)(a) \\ &= {\rm Res}_{z=0}\left(k(A_{r-1}, \mathfrak{c}_{\rm nice})(\partial_{x_2}, \ldots, \partial_{x_{r-1}}) \cdot \frac{e^{a_1 z + a_2 x_2 + \cdots + a_{r-1}x_{r-1}}}{(1-e^{x_2 - z})\cdots(1-e^{x_{r-1}-z})(1-e^{-z})^2} \right)_{x=0}. \end{aligned}$$

For example, using $k(A_2, \mathfrak{c}_{\rm nice})(a) = a_1 + 1$ *(part (i) of Example 30),*

$$\begin{aligned} k(A_3, \mathfrak{c}_{\rm nice})(a) &= {\rm Res}_{z=0}\left((\partial_{x_2} + 1) \cdot \frac{e^{a_1 z + a_2 x_2}}{(1-e^{-z})^2(1-e^{x_2 - z})} \right)_{x=0} \\ &= \tfrac{1}{6}(a_1+2)(a_1+1)(a_1+3a_2+3). \end{aligned}$$

We can iteratively calculate,

$$\begin{aligned} k(A_4, \mathfrak{c}_{\rm nice})(a) &= {\rm Res}_{z=0}\left(\tfrac{1}{6}(\partial_{x_2}+2)(\partial_{x_2}+1)(\partial_{x_2}+3\partial_{x_3}+3) \cdot \frac{e^{a_1 z + a_2 x_2 + a_3 x_3}}{(1-e^{-z})^2(1-e^{x_2-z})(1-e^{x_3-z})} \right)_{x=0} \\ &= \tfrac{1}{360}(a_1+3)(a_1+2)(a_1+1)(a_1+3+a_2+3a_3) \\ &\quad .(a_1^2 + 9a_1 + 5a_1a_2 + 10a_2^2 + 20 + 30a_2). \end{aligned}$$

In a similar fashion, using Theorem 23, we can calculate $v(A_n, c_{\rm nice})$ iteratively. The computation of $v(A_7, c_{\rm nice})$ took 12 seconds. The result is too big to be written here.

Recall that Baldoni-Beck-Cochet-Vergne [1] can compute individual numbers $k(A_n)(a)$ for a fixed, for $n = 10$ in less than 30 minutes. The full polynomial $k(A_n, c_{\rm nice})$ is computed in 7 minutes when $n = 7$ and 30 minutes when $n = 8$ on a 1, 13GHz computer. The method of Baldoni-Beck-Cochet-Vergne uses an arbitrary order on roots. The method of calculation which follows from wall crossing formulae seems less efficient, but it may give some light on the best order strategy and the complexity of calculations.

References

[1] **Baldoni V., Beck M., Cochet C. and Vergne M.** *Volume computation for polytopes and partition functions for classical root systems*, Discrete and Computational Geometry, vol.35 (2006) 551-595.

[2] **Baldoni V., de Loera J. and Vergne M.** *Counting Integer Flows in Networks*, Foundations of Computational Mathematics 4 (2004) 277-314.

[3] **Brion M. and Vergne M.**, *Residue formulae, vector partition functions and lattice points in rational polytopes*, J. Amer. Math Soc. **10** no. 4 (1997), 797–833.

[4] **Dahmen W. and Micchelli C.**, *The number of solutions to linear Diophantine equations and multivariate splines*, Trans. Amer. Math. Soc. **308** (1988), 509-532.

[5] **De Concini C. and Procesi C.**, *Topics in hyperplane arrangements, polytopes and box splines* To appear (available on the personal web page of C. Procesi).

[6] **De Concini C., Procesi C. and Vergne M.**, *Partition functions and Dahmen-Miccelli quasi-polynomials*, In preparation.

[7] **Khovanskii A.G. and Pukhlikov A.V.**, *The Riemann-Roch theorem for integrals and sums of quasi-polynomials on virtual polytopes (Russian)*, Algebra i Analiz **4** (1992), 188-216; transl. in St. Petersburg Math. J. **4** (1993), 789-812.

[8] **Paradan P-E.**, *Jump formulas in Hamiltonian Geometry*, arXiv 0411306.

[9] **Szenes A. and Vergne M.**, *Residue formulae for vector partitions and Euler-MacLaurin sums*, Advances in Applied Mathematics **30** (2003), 295-342.