Index of transversally elliptic operators

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1 Introduction

Let M be a compact manifold. The Atiyah-Singer formula for the index of an *elliptic* pseudo-differential operator P on M with *elliptic* symbol σ on $\mathbf{T}^*M$ involves integration over the non compact manifold $\mathbf{T}^*M$ of the Chern character $\mathrm{Ch}_c(\sigma)$ of σ multiplied by the square of the $\widehat{\mathrm{A}}$-genus of M:

$$\mathrm{index}(P) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2 \,\mathrm{Ch}_c(\sigma).$$

Here σ, the principal symbol of P, is a morphism of vector bundles on $\mathbf{T}^*M$ invertible outside the zero section of $\mathbf{T}^*M$ and the Chern character $\mathrm{Ch}_c(\sigma)$ is supported on a small neighborhood of M embedded in $\mathbf{T}^*M$ as the zero section. It is important that the representative of the Chern character $\mathrm{Ch}_c(\sigma)$ is compactly supported to perform integration.

Assume that a compact Lie group K (with Lie algebra $\mathfrak{k}$) acts on M. If the elliptic operator P is K-invariant, then $\mathrm{index}(P)$ is a smooth function on K. The equivariant index of P can be expressed similarly as the integral of the equivariant Chern character of σ multiplied by the square of the equivariant $\widehat{\mathrm{A}}$-genus of M: for $X \in \mathfrak{k}$ small enough,

$$\mathrm{index}(P)(\mathrm{e}^X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2(X) \,\mathrm{Ch}_c(\sigma)(X). \tag{1}$$

Here $\mathrm{Ch}_c(\sigma)(X)$ is a compactly supported closed equivariant differential form, that is a differential form on $\mathbf{T}^*M$ depending smoothly of $X \in \mathfrak{k}$, and closed for the equivariant differential D. The result of the integration determines a smooth function on a neighborhood of 1 in K and similar formulae can be given near any point of K. Formula (1) is a "delocalization" of the Atiyah-Bott-Segal-Singer formula, in the sense of Bismut [9].

The delocalized index formula (1) can be adapted to new cases such as:

- Index of transversally elliptic operators.
- L^2-index of some elliptic operators on some non-compact manifolds (Rossmann formula for discrete series [19]).

Indeed, in these two contexts, the index exists in the sense of generalized functions but cannot be always computed in terms of fixed point formulae. A "delocalized" formula will however continue to have a meaning, as we explain now for transversally elliptic operators.

The invariant operator P with symbol $\sigma(x, \xi)$ on $\mathbf{T}^*M$ is called transversally elliptic, if it is elliptic in the directions transversal to K-orbits. In this case, the operator P has again an index which is a generalized function on K. A very simple example of transversally elliptic operator is the operator 0 on $L^2(K)$: its index is the trace of the action of K in $L^2(K)$, that is the δ-function on K. At the opposite side, K-invariant elliptic operators are of course transversally elliptic, and index of such operators are smooth functions on K given by Formula (1).

Thus a cohomological formula must incorporate these two extreme cases. Such a cohomological formula was given in Berline-Vergne [7, 8]. We present here a new point of view, where the equivariant Chern character $\mathrm{Ch}_c(\sigma)(X)$ entering in Formula (1) is replaced by a Chern character with generalized coefficients, but still **compactly supported**. Let us briefly explain the construction.

We denote by $\mathbf{T}^*_K M$ the conormal bundle to the K-orbits. An element (x, ξ) is in $\mathbf{T}^*_K M$ if ξ vanishes on tangent vectors to K-orbits. Let $\mathrm{supp}(\sigma)$ be the support of the symbol σ of a transversally elliptic operator P. By definition, the intersection $\mathrm{supp}(\sigma) \cap \mathbf{T}^*_K(M)$ is compact. By Quillen super-connection construction, the Chern character $\mathrm{Ch}(\sigma)(X)$ is a closed equivariant differential form supported near the closed set $\mathrm{supp}(\sigma)$. Using the Liouville one form ω of $\mathbf{T}^*M$, we construct a closed equivariant form $\mathrm{Par}(\omega)$ supported near $\mathbf{T}^*_K M$. Outside $\mathbf{T}^*_K M$, one has indeed the equation $1 = D(\frac{\omega}{D\omega})$, where the inverse of the form $D\omega$ is defined by $-i\int_0^\infty e^{itD\omega}dt$, integral which is well defined in the generalized sense, that is tested against a smooth compactly supported density on $\mathfrak{k}$. Thus using a function χ equal to 1 on a small neighborhood of $\mathbf{T}^*_K M$, the closed equivariant form

$$\mathrm{Par}(\omega)(X) = \chi + d\chi \frac{\omega}{D\omega(X)}, \quad X \in \mathfrak{k},$$

is well defined, supported near $\mathbf{T}^*_K M$, and represents 1 in cohomology without support conditions. Remark that

$$\mathrm{Ch}_c(\sigma, \omega) := \mathrm{Ch}(\sigma)(X)\mathrm{Par}(\omega)(X)$$

is **compactly supported**. We prove that, for $X \in \mathfrak{k}$ small enough, we have

$$\text{(2)} \qquad \mathrm{index}(P)(\mathrm{e}^X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2(X)\,\mathrm{Ch}(\sigma)(X)\,\mathrm{Par}(\omega)(X).$$

This formula is thus entirely similar to the delocalized version of the Atiyah-Bott-Segal-Singer equivariant index theorem. We have just localized the formula for the index near $\mathbf{T}^*_K M$ with the help of the form $\mathrm{Par}(\omega)$, equal to 1 in cohomology, but supported near $\mathbf{T}^*_K M$.

When P is elliptic we can furthermore localize on the zeros of VX (the vector field on M produced by the action of X) and we obtain the Atiyah-Bott-Segal-Singer fixed point formulae for the equivariant index of P. However the main difference is that usually we cannot obtain fixed point formula for the index. For example, the index of a transversally elliptic operator P where K acts freely is a generalized function on K supported at the origin. Thus in this case the use of the form $\mathrm{Par}(\omega)$ is essential. Its role is clearly explained in the example of the 0 operator on S^1 given at the end of this introduction.

We need also to define the formula for the index at any point of $s \in K$, in function of integrals over $\mathbf{T}^*M(s)$, where $M(s)$ is the fixed point submanifold of M under the action of s. The compatibility properties (descent method) between the formulae at different points s are easy to prove, thanks to a localization formula adapted to this generalized setting.

In the Berline-Vergne cohomological formula for the index of P, the Chern character $\mathrm{Ch}_c(\sigma)(X)$ in Formula (1) was replaced by a Chern character $\mathrm{Ch}_{BV}(\sigma,\omega)(X)$ depending also of the Liouville one form ω. This Chern character $\mathrm{Ch}_{BV}(\sigma,\omega)$ is constructed for "good symbols" σ. It looks like a Gaussian in the transverse directions, and is oscillatory in the directions of the orbits. Our new point of view defines the **compactly supported** product class $\mathrm{Ch}(\sigma)\mathrm{Par}(\omega)$ in a straightforward way. We proved in [17] that the classes $\mathrm{Ch}_{BV}(\sigma,\omega)$ and $\mathrm{Ch}(\sigma)\mathrm{Par}(\omega)$ are equivalent in an appropriate cohomology space, so that our new cohomological formula gives the analytic index. However, in this article, we choose to prove directly the equality between the analytic index and the cohomological index, since we want to show that our formula in terms of the product class $\mathrm{Ch}(\sigma)\mathrm{Par}(\omega)$ is natural. We follow the same line than Atiyah-Singer: functoriality with respect to products and free actions. The compatibility with the free action reduces basically to the case of the zero operator on K, and the calculation is straightforward. The typical calculation is shown below. The multiplicativity property is more delicate, but is based on a general principle on multiplicativity of relative Chern characters that we proved in a preceding article [17]. Thus, following Atiyah-Singer [1], we are reduced to the case of S^1 acting on a vector space. The basic examples are then the pushed symbol with index $-\sum_{n=1}^{\infty} e^{in\theta}$ and the index of the tangential $\overline{\partial}$ operators on odd dimensional spheres. We include at the end a general formula due to Fitzpatrick [11] for contact manifolds.

Let us finally point out that there are many examples of transversally elliptic operators of great interest. The index of elliptic operators on orbifolds are best understood as indices of transversally elliptic operators on manifolds where a group K acts with finite stabilizers. The restriction to the maximal compact subgroup K of a representation of the discrete series of a real reductive group are indices of transversally elliptic operators [15]. More generally, there is a canonical transversally elliptic operator on any prequantized Hamiltonian manifold with proper moment map (under some mild assumptions) [15],[20]. For example, when a torus T acts on a Hermitian vector V with a proper moment map, then the partition function which computes the T-multiplicities on the polynomial algebra $\mathcal{S}(V^*)$ is equal to the index of a T-transversally elliptic operator on V. Furthermore, as already noticed in Atiyah-Singer, and systematically used in [14], transversally elliptic operators associated to symplectic vector spaces with proper moment maps and to cotangent manifolds $\mathbf{T}^*K$ are the local building pieces of any K-invariant elliptic operator.

Example 1.1 *Let us check the validity of (2) in the example of the zero operator* 0_{S^1} *acting on the circle group* S^1*. This operator is* S^1*-transversally elliptic and its index is equal to*

$$\delta_1(\mathrm{e}^{iX}) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{ikX}, \quad X \in \mathrm{Lie}(S^1) \simeq \mathbb{R}.$$

The principal symbol σ of 0_{S^1} is the zero morphism on the trivial bundle $S^1 \times \mathbb{C}$. Hence $\mathrm{Ch}(\sigma)(X) = 1$. The equivariant class $\widehat{\mathrm{A}}(S^1)^2(X)$ is also equal to 1. Thus the right hand side of (2) becomes

$$(2i\pi)^{-1} \int_{\mathbf{T}^* S^1} \mathrm{Par}(\omega)(X).$$

The cotangent bundle $\mathbf{T}^* S^1$ is parametrized by $(\mathrm{e}^{i\theta}, \xi) \in S^1 \times \mathbb{R}$. The Liouville 1-form is $\omega = -\xi d\theta$: the symplectic form $d\omega = d\theta \wedge d\xi$ gives the orientation of $\mathbf{T}^* S^1$. Since $VX = -X\frac{\partial}{\partial\theta}$, we have $D\omega(X) = d\theta \wedge d\xi - X\xi$.

Let $g \in \mathcal{C}^\infty(\mathbb{R})$ with compact support and equal to 1 in a neighborhood of 0. Then $\chi = g(\xi^2)$ is a function on $\mathbf{T}^* S^1$ which is supported in a neighborhood of $\mathbf{T}^*_{S^1} S^1 = \text{zero section}$. We look now at the equivariant form $\mathrm{Par}(\omega)(X) = \chi + d\chi \wedge (-i\omega) \int_0^\infty \mathrm{e}^{itD\omega(X)}\, dt$. We have

$$\begin{aligned} \mathrm{Par}(\omega)(\mathrm{e}^{i\theta}, \xi, X) &= g(\xi^2) + g'(\xi^2) 2\xi d\xi \wedge (i\xi d\theta) \int_0^\infty \mathrm{e}^{it(d\theta\wedge d\xi - X\xi)}\, dt \\ &= g(\xi^2) - i d\theta \wedge d(g(\xi^2)) \left(\int_0^\infty \mathrm{e}^{-itX\xi}\, \xi dt \right). \end{aligned}$$

If we make the change of variable $t\xi \to t$ in the integral $\int_0^\infty \mathrm{e}^{-itX\xi}\, \xi dt$ we get

$$\mathrm{Par}(\omega)(\mathrm{e}^{i\theta}, \xi, X) = \begin{cases} g(\xi^2) - i d\theta \wedge d(g(\xi^2)) \left(\int_0^\infty \mathrm{e}^{-itX}\, dt \right), & \text{if} \quad \xi \geq 0; \\ g(\xi^2) + i d\theta \wedge d(g(\xi^2)) \left(\int_{-\infty}^0 \mathrm{e}^{-itX}\, dt \right), & \text{if} \quad \xi \leq 0. \end{cases}$$

Finally, since $-\int_{\xi \geq 0} d(g(\xi^2)) = \int_{\xi \leq 0} d(g(\xi^2)) = 1$, we have

$$(2i\pi)^{-1} \int_{\mathbf{T}^* S^1} \mathrm{Par}(\omega)(X) = \int_{-\infty}^\infty \mathrm{e}^{-itX}\, dt.$$

The generalized function $\delta_0(X) = \int_{-\infty}^\infty \mathrm{e}^{-itX}\, dt$ satisfies

$$\int_{\mathrm{Lie}(S^1)} \delta_0(X) \varphi(X) dX = \mathrm{vol}(S^1, dX) \varphi(0)$$

for any function $\varphi \in \mathcal{C}^\infty(\mathrm{Lie}(S^1))$ with compact support. Here $\mathrm{vol}(S^1, dX) = \int_0^{2\pi} dX$ is also the volume of S^1 with the Haar measure compatible with dX.

Finally, we see that (2) corresponds to the following equality of generalized functions

$$\delta_1(\mathrm{e}^{iX}) = \delta_0(X),$$

which holds for $X \in \mathrm{Lie}(S^1)$ small enough.

2 The analytic index

2.1 Generalized functions

Let K be a compact Lie group. We denote by $\hat{K}$ the set of unitary irreducible representations of K. If $\tau \in \hat{K}$, we denote by V_τ the representation space of τ

and by $k \mapsto \mathrm{Tr}(k,\tau)$ its character. Let τ^* be the dual representation of K in V_τ^*.

We denote by $C^{-\infty}(K)$ the space of generalized functions on K and by $C^{-\infty}(K)^K$ the space of central invariant generalized functions. The space $C^\infty(K)$ of smooth functions on K is naturally a subspace of $C^{-\infty}(K)$. We will often use the notation $\Theta(k)$ to denote a generalized function Θ on K, although (in general) the value of Θ on a particular point k of K does not have a meaning. By definition, Θ is a linear form on the space of smooth densities on K. If dk is a Haar measure on K and $\Phi \in C^\infty(K)$, we denote by $\int_K \Theta(k)\Phi(k)dk$ the value of Θ on the density Φdk.

Any invariant generalized function on K is expressed as $\Theta(k) = \sum_{\tau\in\hat{K}} n_\tau \mathrm{Tr}(k,\tau)$ where the Fourier coefficients n_τ have at most of polynomial growth.

2.2 Symbols and pseudo-differential operators

Let M be a compact manifold with a smooth action of a compact Lie group K. We consider the closed subset $\mathbf{T}^*_K M$ of the cotangent bundle $\mathbf{T}^*M$, union of the spaces $(\mathbf{T}^*_K M)_x, x \in M$, where $(\mathbf{T}^*_K M)_x \subset \mathbf{T}^*_x M$ is the orthogonal of the tangent space at x to the orbit $K\cdot x$. Let $\mathcal{E}^\pm$ be two K-equivariant complex vector bundles over M. We denote by $\Gamma(M,\mathcal{E}^\pm)$ the space of smooth sections of $\mathcal{E}^\pm$. Let $P : \Gamma(M,\mathcal{E}^+) \to \Gamma(M,\mathcal{E}^-)$ be a K-invariant pseudo-differential operator of order m. Let $p : \mathbf{T}^*M \to M$ be the natural projection. The principal symbol $\sigma(P)$ of P is a bundle map $p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ which is homogeneous of degree m, defined over $\mathbf{T}^*M \setminus M$.

The operator P is elliptic if its principal symbol $\sigma(P)(x,\xi)$ is invertible for all $(x,\xi) \in \mathbf{T}^*M$ such that $\xi \neq 0$. The operator P is said to be K-transversally elliptic if its principal symbol $\sigma(P)(x,\xi)$ is invertible for all $(x,\xi) \in \mathbf{T}^*_K M$ such that $\xi \neq 0$.

Using a K- invariant function χ on $\mathbf{T}^*M$ identically equal to 1 in a neighborhood of M and compactly supported, then $\sigma_P(x,\xi) := (1-\chi(x,\xi))\sigma(P)(x,\xi)$ is a morphism from $p^*\mathcal{E}^+$ to $p^*\mathcal{E}^-$ defined on the whole space $\mathbf{T}^*M$ and which is *almost homogeneous*: $\sigma_P(x,t\xi) = t^m\sigma_P(x,\xi)$ for $t > 1$ and ξ large enough. We consider the support of the morphism σ_P,

$$\mathrm{supp}(\sigma_P) := \{(x,\xi) \in \mathbf{T}^*M \mid \sigma_P(x,\xi) \text{ is not invertible}\}$$

which is a closed K-invariant subset of $\mathbf{T}^*M$.

When P is elliptic, then $\mathrm{supp}(\sigma_P)$ is compact, and the morphism σ_P gives rise to a $\mathbf{K}^0$-theory class $[\sigma_P] \in \mathbf{K}^0_K(\mathbf{T}^*M)$ which does not depend of the choice of χ. Similarly, when P is K-transversally elliptic, then $\mathrm{supp}(\sigma_P)\cap\mathbf{T}^*_K M$ is compact and the morphism σ_P gives rise to a $\mathbf{K}^0$-theory class $[\sigma_P|_{\mathbf{T}^*_K M}] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$ which does not depend of the choice of χ.

Recall the definition of the K-equivariant index of a pseudo-differential operator P which is K-transversally elliptic. Let us choose a K-invariant metric on M and K-invariant Hermitian structures on $\mathcal{E}^\pm$. Then the adjoint P^* of P is also a K-transversally elliptic pseudo-differential operator.

If P is elliptic, its kernel $\ker P := \{s \in \Gamma(M, \mathcal{E}^+) |\ Ps = 0\}$ is finite dimensional, and the K-equivariant index of P is the central function $\mathrm{index}^K(P)(k) = \mathrm{Tr}(k, \ker P) - \mathrm{Tr}(k, \ker P^*)$.

If P is K-transversally elliptic, its kernel $\ker P$ is not finite dimensional, but it has finite multiplicities: the vector space $\ker P$ is an admissible K-representation. Let us explain this notion. For any irreducible representation $\tau \in \hat{K}$, the multiplicity $m_\tau(P) := \hom_K(V_\tau, \ker P)$ is finite, and $\tau \mapsto m_\tau(P)$ has at most a polynomial growth. We define then a central invariant generalized function on K by setting

$$\mathrm{Tr}(k, \ker P) := \sum_{\mu \in \hat{K}} m_\tau(P)\, \mathrm{Tr}(k, \tau).$$

Definition 2.1 *The K-equivariant index of a K-transversally elliptic pseudo-differential operator P is the generalized function*

$$\mathrm{index}^K(P)(k) = \mathrm{Tr}(k, \ker P) - \mathrm{Tr}(k, \ker P^*).$$

We recall

Theorem 2.2 (Atiyah-Singer) • *The K-equivariant index of a K-invariant elliptic pseudo-differential operator P depends only of $[\sigma_P] \in \mathbf{K}^0_K(\mathbf{T}^*M)$.*

• *The K-equivariant index of a K-transversally elliptic pseudo-differential operator P depends only of $[\sigma_P|_{\mathbf{T}^*_K M}] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$.*

• *Each element in $\mathbf{K}^0_K(\mathbf{T}^*_K M)$ is represented by the class $[\sigma_P|_{\mathbf{T}^*_K M}]$ of a K-transversally elliptic pseudo differential operator P of order m. Similarly, each element in $\mathbf{K}^0_K(\mathbf{T}^*M)$ is represented by the class $[\sigma_P]$ of a K-invariant elliptic pseudo differential operator P of order m.*

Thus we can define

$$\mathrm{index}^{K,M}_a : \mathbf{K}^0_K(\mathbf{T}^*_K M) \to C^{-\infty}(K)^K \tag{3}$$

by setting, for P a K-transversally elliptic pseudo-differential operator of order m, $\mathrm{index}^{K,M}_a([\sigma_P|_{\mathbf{T}^*_K M}]) = \mathrm{index}^K(P)$. Similarly, we can define in the elliptic setting

$$\mathrm{index}^{K,M}_a : \mathbf{K}^0_K(\mathbf{T}^*M) \to \mathcal{C}^{\infty}(K)^K. \tag{4}$$

Note that we have a natural restriction map $\mathbf{K}^0_K(\mathbf{T}^*M) \to \mathbf{K}^0_K(\mathbf{T}^*_K M)$ which make the following diagram

$$\begin{tikzcd} \mathbf{K}^0_K(\mathbf{T}^*M) \arrow[r] \arrow[d, "\mathrm{index}^{K,M}_a"'] & \mathbf{K}^0_K(\mathbf{T}^*_K M) \arrow[d, "\mathrm{index}^{K,M}_a"] \\ \mathcal{C}^{\infty}(K)^K \arrow[r] & \mathcal{C}^{-\infty}(K)^K \end{tikzcd} \tag{5}$$

commutative.

Let $R(K)$ be the representation ring of K. Using the trace, we will consider $R(K)$ as a sub-ring of $\mathcal{C}^\infty(K)^K$. The map (3) and (4) are homomorphism of $R(K)$-modules and will be called the analytic indices.

Remark 2.3 *In order to simplify the notations we will make no distinction between an element $V \in R(K)$, and its trace function $k \to \mathrm{Tr}(k, V)$ which belongs to $\mathcal{C}^\infty(K)^K$. For example the constant function 1 on K is identified with the trivial representation of K.*

Let H be a compact Lie group acting on M and commuting with the action of K. Then the space $\mathbf{T}^*_K M$ is provided with an action of $K \times H$. If $[\sigma] \in \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M)$, we can associate to $[\sigma]$ a virtual trace class representation of $K \times H$. Indeed, we can choose as representative of $[\sigma]$ the symbol of a H-invariant and K-transversally elliptic operator P. Then $\ker P - \ker P^*$ is a trace class representation of $K \times H$. Thus we can define a $R(K\times H)$-homomorphism:

$$\mathrm{index}_a^{K,H,M} : \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M) \to C^{-\infty}(K \times H)^{K\times H}.$$

Obviously $\mathbf{T}^*_{K\times H}(M)$ is contained in $\mathbf{T}^*_K(M)$ so we have a restriction morphism $r : \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M) \to \mathbf{K}^0_{K\times H}(\mathbf{T}^*_{K\times H} M)$. We see that

$$\mathrm{index}_a^{K,H,M} = \mathrm{index}_a^{K\times H,M} \circ r.$$

However, it is easy to see that for a H-invariant and K-transversally elliptic symbol $[\sigma]$, $\mathrm{index}_a^{K,H,M}([\sigma])$ belongs to $C^\infty(H, C^{-\infty}(K))$. In particular we can restrict $\mathrm{index}_a^{K,H,M}([\sigma])$ to $H' \times K$, for a subgroup H' of H. We can also multiply $\mathrm{index}_a^{K,H,M}([\sigma])(g, h)$ by generalized functions $\Psi(h)$ on H.

2.3 Functoriality property of the analytic index

We have defined for any compact $K \times H$-manifold M a $R(K \times H)$-morphism

$$\mathrm{index}_a^{K,H,M} : \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M) \to \mathcal{C}^\infty(H, C^{-\infty}(K))^{K\times H}.$$

Let us recall some basic properties of the analytic index map:

- [**N1**] If $M = \{\text{point}\}$, then $\mathrm{index}_a^{K,M}$ is the trace map $R(K) \hookrightarrow \mathcal{C}^\infty(K)^K$.
- [**Diff**] Compatibility with diffeomorphisms: if $f : M_1 \to M_2$ is a $K \times H$-diffeomorphism then $\mathrm{index}_a^{K,H,M_1} \circ f^*$ is equal to $\mathrm{index}_a^{K,H,M_2}$.
- [**Morph**] If $\phi : H' \to H$ is a Lie group morphism, we have $\phi^* \circ \mathrm{index}_a^{K,H,M} = \mathrm{index}_a^{K,H',M}$.

2.3.1 Excision

Let U be a non-compact K-manifold. Lemma 3.6 of [1] tell us that, for any open K-embedding $j : U \hookrightarrow M$ into a compact manifold, we have a pushforward map $j_* : \mathbf{K}^0_K(\mathbf{T}^*_K U) \to \mathbf{K}^0_K(\mathbf{T}^*_K M)$.

Let us rephrase Theorem 3.7 of [1].

Theorem 2.4 (Excision property) *The composition*

$$\mathbf{K}^0_K(\mathbf{T}^*_K U) \xrightarrow{j_*} \mathbf{K}^0_K(\mathbf{T}^*_K M) \overset{\mathrm{index}_a^{K,M}}{\longrightarrow} \mathcal{C}^{-\infty}(K)^K$$

is independent of the choice of $j : U \hookrightarrow M$*: we denote this map* $\mathrm{index}_a^{K,U}$.

Note that a *relatively compact* K-invariant open subset U of a K-manifold admits an open K-embedding $j : U \hookrightarrow M$ into a compact K-manifold. So the index map $\mathrm{index}_a^{K,U}$ is defined in this case. An important example is when $U \to N$ is a K-equivariant vector bundle over a compact manifold N : we can imbed U as an open subset of the real projective bundle $\mathbb{P}(U \oplus \mathbb{R})$.

2.3.2 Exterior product

Let us recall the multiplicative property of the analytic index for the product of manifolds. Consider a compact Lie group K_2 acting on two manifolds M_1 and M_2, and assume that another compact Lie group K_1 acts on M_1 commuting with the action of K_2.

The external product of complexes on $\mathbf{T}^*M_1$ and $\mathbf{T}^*M_2$ induces a multiplication (see [1]):

$$\odot_{\mathrm{ext}} : \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}^*_{K_1} M_1) \times \mathbf{K}^0_{K_2}(\mathbf{T}^*_{K_2} M_2) \longrightarrow \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}^*_{K_1\times K_2}(M_1 \times M_2)).$$

Let us recall the definition of this external product. For $k = 1, 2$, we consider equivariant morphisms[1] $\sigma_k : \mathcal{E}_k^+ \to \mathcal{E}_k^-$ on $\mathbf{T}^*M_k$. We consider the equivariant morphism on $\mathbf{T}^*(M_1 \times M_2)$

$$\sigma_1 \odot_{\mathrm{ext}} \sigma_2 : \mathcal{E}_1^+ \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^- \otimes \mathcal{E}_2^- \longrightarrow \mathcal{E}_1^- \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^+ \otimes \mathcal{E}_2^-$$

defined by

$$\sigma_1 \odot_{\mathrm{ext}} \sigma_2 = \begin{pmatrix} \sigma_1 \otimes \mathrm{Id} & -\mathrm{Id} \otimes \sigma_2^* \\ \mathrm{Id} \otimes \sigma_2 & \sigma_1^* \otimes \mathrm{Id} \end{pmatrix}. \tag{6}$$

We see that the set $\mathrm{supp}(\sigma_1 \odot \sigma_2) \subset \mathbf{T}^*M_1 \times \mathbf{T}^*M_2$ is equal to $\mathrm{supp}(\sigma_1) \times \mathrm{supp}(\sigma_2)$.

We suppose now that the morphisms σ_k are respectively K_k-transversally elliptic. Since $\mathbf{T}^*_{K_1\times K_2}(M_1 \times M_2) \neq \mathbf{T}^*_{K_1} M_1 \times \mathbf{T}^*_{K_2} M_2$, the morphism $\sigma_1 \odot_{\mathrm{ext}} \sigma_2$ is not necessarily $K_1 \times K_2$-transversally elliptic. Nevertheless, if σ_2 is taken *almost homogeneous* of order $m = 0$, then the morphism $\sigma_1 \odot_{\mathrm{ext}} \sigma_2$ is $K_1 \times K_2$-transversally elliptic (see Lemma 4.9 in [17]). So the exterior product $a_1 \odot_{\mathrm{ext}} a_2$ is the $\mathbf{K}^0$-theory class defined by $\sigma_1 \odot_{\mathrm{ext}} \sigma_2$, where $a_k = [\sigma_k]$ and σ_2 is taken almost homogeneous of order $m = 0$.

Theorem 2.5 (Multiplicative property) *For any* $[\sigma_1] \in \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}^*_{K_1} M_1)$ *and any* $[\sigma_2] \in \mathbf{K}^0_{K_2}(\mathbf{T}^*_{K_2} M_2)$ *we have*

$$\mathrm{index}_a^{K_1\times K_2, M_1\times M_2}([\sigma_1] \odot_{\mathrm{ext}} [\sigma_2]) = \mathrm{index}_a^{K_1, K_2, M_1}([\sigma_1])\, \mathrm{index}_a^{K_2, M_2}([\sigma_2]).$$

[1] In order to simplify the notation, we do not make the distinctions between vector bundles on $\mathbf{T}^*M$ and on M.

2.3.3 Free action

Let K and G be two compact Lie groups. Let P be a compact manifold provided with an action of $K \times G$. We assume that the action of K is free. Then the manifold $M := P/K$ is provided with an action of G and the quotient map $q : P \to M$ is G-equivariant. Note that we have the natural identification of $\mathbf{T}^*_K P$ with $q^*\mathbf{T}^*M$, hence $(\mathbf{T}^*_K P)/K \simeq \mathbf{T}^*M$ and more generally

$$(\mathbf{T}^*_{K\times G}P)/K \simeq \mathbf{T}^*_G M.$$

This isomorphism induces an isomorphism

$$Q^* : \mathbf{K}^0_G(\mathbf{T}^*_G M) \to \mathbf{K}^0_{K\times G}(\mathbf{T}^*_{K\times G}P).$$

Let $\mathcal{E}^\pm$ be two G-equivariant complex vector bundles on M and $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a G-transversally elliptic symbol. For any irreducible representation (τ, V_τ) of K, we form the G-equivariant complex vector bundle $\mathcal{V}_\tau := P \times_K V_\tau$ on M. We consider the morphism

$$\sigma_\tau := \sigma \otimes \mathrm{Id}_{V_\tau} : p^*(\mathcal{E}^+ \otimes \mathcal{V}_\tau) \to p^*(\mathcal{E}^- \otimes \mathcal{V}_\tau)$$

which is G-transversally elliptic.

Theorem 2.6 (Free action property) *We have the following equality in $\mathcal{C}^{-\infty}(K\times G)^{K\times G}$: for $(k,g) \in K \times G$*

$$\mathrm{index}_a^{K\times G,P}(Q^*[\sigma])(k,g) = \sum_{\tau\in\hat{K}} \mathrm{Tr}(k,\tau)\,\mathrm{index}_a^{G,M}([\sigma_{\tau^*}])(g).$$

2.4 Basic examples

2.4.1 Bott symbols

Let W be a Hermitian vector space. For any $v \in W$, we consider on the $\mathbb{Z}_2$-graded vector space $\wedge W$ the following odd operators: the exterior multiplication $\mathrm{m}(v)$ and the contraction $\iota(v)$. The contraction $\iota(v)$ is an odd derivation of $\wedge W$ such that $\iota(v)w = (w, v)$ for $w \in \wedge^1 W = W$.

The Clifford action of W on $\wedge W$ is defined by the formula

$$\mathbf{c}(v) = \mathrm{m}(v) - \iota(v) \tag{7}$$

Then $\mathbf{c}(v)$ is an odd operator on $\wedge W$ such that $\mathbf{c}(v)^2 = -\|v\|^2 Id$. So $\mathbf{c}(v)$ is invertible when $v \neq 0$.

Consider the trivial vector bundles $\mathcal{E}^\pm := W \times \wedge^\pm W$ over W with fiber $\wedge^\pm W$. The Bott morphism $\mathrm{Bott}(W) : \mathcal{E}^+ \to \mathcal{E}^-$ is defined by

$$\mathrm{Bott}(W)(v, w) = (v, \mathbf{c}(v)w). \tag{8}$$

Consider now an Euclidean vector space V. Then its complexification $V_\mathbb{C}$ is an Hermitian vector space. The cotangent bundle $\mathbf{T}^*V$ is identified with

$V_{\mathbb{C}}$: we associate to the covector $\xi \in \mathbf{T}^*_v V$ the element $v + i\hat{\xi} \in V_{\mathbb{C}}$, where $\xi \in V^* \to \hat{\xi} \in V$ is the identification given by the Euclidean structure.

Then $\mathrm{Bott}(V_{\mathbb{C}})$ defines an elliptic symbol on V which is equivariant relatively to the action of the orthogonal group $O(V)$.

Proposition 2.7 *We have* [**N2**]*:* $\qquad \mathrm{index}_a^{O(V),V}(\mathrm{Bott}(V_{\mathbb{C}})) = 1.$

Remark 2.8 *If V and W are two euclidean vector spaces we see that the symbol* $\mathrm{Bott}((V \times W)_{\mathbb{C}})$ *is equal to the product* $\mathrm{Bott}(V_{\mathbb{C}}) \odot \mathrm{Bott}(W_{\mathbb{C}})$. *Then for* $(g,h) \in O(V) \times O(W)$*, the multiplicative property tells us that*

$$\mathrm{index}_a^{O(V\times W),V\times W}(\mathrm{Bott}((V \times W)_{\mathbb{C}}))(g,h)$$

is equal to the product $\mathrm{index}_a^{O(V),V}(\mathrm{Bott}(V_{\mathbb{C}}))(g)\, \mathrm{index}_a^{O(W),W}(\mathrm{Bott}(W_{\mathbb{C}}))(h)$.

For any $g \in O(V)$, the vector space V decomposes as an orthogonal sum $\oplus_i V_i$ of g-stables subspaces, where either $\dim V_i = 1$ *and g acts on V_i as ± 1, or* $\dim V_i = 2$ *and g acts on V_i as a rotation.*

Hence [**N2**] *is satisfied for any euclidean vector space if one check it for the cases:*

- $V = \mathbb{R}$ *with the action of the group* $O(V) = \mathbb{Z}_2$,
- $V = \mathbb{R}^2$ *with the action of the group* $SO(V) = S^1$.

2.4.2 Atiyah symbol

In the following example, we denote, for any integer k, by $\mathbb{C}_{[k]}$ the vector space $\mathbb{C}$ with the action of the circle group S^1 given by : $t \cdot z = t^k z$.

The Atiyah symbol is the S^1-equivariant morphism on $N = \mathbf{T}^*\mathbb{C}_{[1]} \simeq \mathbb{C}_{[1]} \times \mathbb{C}_{[1]}$

$$\begin{array}{rcl} \sigma_{\mathrm{At}} : N \times \mathbb{C}_{[0]} & \longrightarrow & N \times \mathbb{C}_{[1]} \\ (\xi, v) & \longmapsto & (\xi, \sigma_{\mathrm{At}}(\xi) v) \end{array}$$

defined by $\sigma_{\mathrm{At}}(\xi) = \xi_2 + i\xi_1$ for $\xi = (\xi_1, \xi_2) \in \mathbf{T}^*\mathbb{C}_{[1]}$.

The symbol σ_{At} is not elliptic since $\mathrm{supp}(\sigma_{\mathrm{At}}) = \{\xi_1 = i\xi_2\} \subset \mathbb{C}^2$ is not compact. But $\mathbf{T}^*_{S^1}\mathbb{C}_{[1]} = \{(\xi_1,\xi_2) \mid \mathrm{Im}\,(\xi_1\overline{\xi_2}) = 0\}$ and $\mathrm{supp}(\sigma_{\mathrm{At}}) \cap \mathbf{T}^*_{S^1}\mathbb{C}_{[1]} = \{(0,0)\}$: the symbol σ_{At} is S^1-transversally elliptic.

Proposition 2.9 *We have* [**N3**]*:* $\qquad \mathrm{index}_a^{S^1,\mathbb{C}}(\sigma_{\mathrm{At}})(t) = -\sum_{k=1}^{\infty} t^k.$

2.5 Unicity of the index

Suppose that for any compact Lie groups K and H, and any compact $K \times H$-manifold M, we have a map of $R(K \times H)$-modules:

$$\mathbb{I}^{K,H,M} : \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M) \to \mathcal{C}^{\infty}(H, C^{-\infty}(K))^{K\times H}.$$

Theorem 2.10 *Suppose that the maps* $\mathbb{I}^-$ *satisfy*

- *the normalization conditions* [**N1**], [**N2**] *and* [**N3**],
- *the functorial properties* **Diff** *and* **Morph**,
- *the "excision property", the "multiplicative property" and the "free action property".*

Then $\mathbb{I}^-$ *coincides with the analytic index map* index_a^-.

3 The cohomological index

Let N be a manifold, and let $\mathcal{A}(N)$ be the algebra of differential forms on N. We denote by $\mathcal{A}_c(N)$ the subalgebra of compactly supported differential forms. We will consider on $\mathcal{A}(N)$ and $\mathcal{A}_c(N)$ the $\mathbb{Z}_2$-grading in even or odd differential forms.

Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. We suppose that the manifold N is provided with an action of K. We denote $X \mapsto VX$ the corresponding morphism from $\mathfrak{k}$ into the Lie algebra of vectors fields on N: for $n \in N$, $V_nX := \frac{d}{d\epsilon}\exp(-\epsilon X)\cdot n|_{\epsilon=0}$.

Let $\mathcal{A}^\infty(\mathfrak{k}, N)$ be the $\mathbb{Z}_2$-graded algebra of equivariant smooth functions $\alpha : \mathfrak{k} \to \mathcal{A}(N)$. Its $\mathbb{Z}_2$-grading is the grading induced by the exterior degree. Let $D = d - \iota(VX)$ be the equivariant differential: $(D\alpha)(X) = d(\alpha(X)) - \iota(VX)\alpha(X)$. Here the operator $\iota(VX)$ is the contraction of a differential form by the vector field VX. Let $\mathcal{H}^\infty(\mathfrak{k}, N) := \mathrm{Ker}D/\mathrm{Im}D$ be the equivariant cohomology algebra with C^∞-coefficients. It is a module over the algebra $\mathcal{C}^\infty(\mathfrak{k})^K$ of K-invariant C^∞-functions on $\mathfrak{k}$.

The sub-algebra $\mathcal{A}_c^\infty(\mathfrak{k}, N) \subset \mathcal{A}^\infty(\mathfrak{k}, N)$ of equivariant differential forms with compact support is defined as follows : $\alpha \in \mathcal{A}_c^\infty(\mathfrak{k}, N)$ if there exists a compact subset $\mathcal{K}_\alpha \subset N$ such that the differential form $\alpha(X) \in \mathcal{A}(N)$ is supported on $\mathcal{K}_\alpha$ for any $X \in \mathfrak{k}$. We denote $\mathcal{H}_c^\infty(\mathfrak{k}, N)$ the corresponding algebra of cohomology: it is a $\mathbb{Z}_2$-graded algebra.

Let $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ be the space of *generalized* equivariant differential forms. An element $\alpha \in \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ is, by definition, a $\mathcal{C}^{-\infty}$-equivariant map $\alpha : \mathfrak{k} \to \mathcal{A}(N)$. The value taken by α on a smooth compactly supported density $Q(X)dX$ on $\mathfrak{k}$ is denoted by $\int_{\mathfrak{k}} \alpha(X)Q(X)dX \in \mathcal{A}(N)$. We have $\mathcal{A}^\infty(\mathfrak{k}, N) \subset \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ and we can extend the differential D to $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ [12]. We denote by $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ the corresponding cohomology space. Note that $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ is a module over $\mathcal{A}^\infty(\mathfrak{k}, N)$ under the wedge product, hence the cohomology space $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ is a module over $\mathcal{H}^\infty(\mathfrak{k}, N)$.

The sub-space $\mathcal{A}_c^{-\infty}(\mathfrak{k}, N) \subset \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ of generalized equivariant differential forms with compact support is defined as follows : $\alpha \in \mathcal{A}_c^{-\infty}(\mathfrak{k}, N)$ if there exits a compact subset $\mathcal{K}_\alpha \subset N$ such that the differential form $\int_{\mathfrak{k}} \alpha(X)Q(X)dX \in \mathcal{A}(N)$ is supported on $\mathcal{K}_\alpha$ for any compactly supported density $Q(X)dX$. We denote $\mathcal{H}_c^{-\infty}(\mathfrak{k}, N)$ the corresponding space of cohomology. The $\mathbb{Z}_2$-grading on $\mathcal{A}(N)$ induces a $\mathbb{Z}_2$-grading on the cohomology spaces $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ and $\mathcal{H}_c^{-\infty}(\mathfrak{k}, N)$.

If $\mathcal{U}$ is a K-invariant open subset of $\mathfrak{k}$, ones defines also $\mathcal{H}^{-\infty}(\mathcal{U}, N)$ and $\mathcal{H}_c^{-\infty}(\mathcal{U}, N)$. If N is equipped with a K-invariant orientation, the integration over N defines a morphism

$$\int_N : \mathcal{H}_c^{-\infty}(\mathcal{U}, N) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U})^K.$$

3.1 Restrictions of generalized functions

Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. For any $s \in K$ (resp. $S \in \mathfrak{k}$), we denote $K(s)$ (resp. $K(S)$) the stabilizer subgroup: the corresponding Lie algebra is denoted $\mathfrak{k}(s)$ (resp. $\mathfrak{k}(S)$).

For any $s \in K$, we consider a (small) open $K(s)$-invariant neighborhood $\mathcal{U}_s$ of 0 in $\mathfrak{k}(s)$ such that the map $[k, Y] \mapsto kse^Y k^{-1}$ is an open embedding of $K \times_{K(s)} \mathcal{U}_s$ on an open neighborhood of the conjugacy class $K \cdot s := \{ksk^{-1},\ k \in K\} \simeq K/K(s)$.

Similarly, for any $S \in \mathfrak{k}$, we consider a (small) open $K(S)$-invariant neighborhood $\mathcal{U}_S$ of 0 in $\mathfrak{k}(S)$ such that the map $[k, Y] \mapsto \mathrm{Ad}(k)(S + Y)$ is a open embedding of $K \times_{K(S)} \mathcal{U}_S$ on an open neighborhood of the adjoint orbit $K \cdot S \simeq K/K(S)$.

Note that the map $Y \mapsto [1, Y]$ realizes $\mathcal{U}_s$ (resp. $\mathcal{U}_S$) as a $K(s)$-invariant sub-manifold of $K \times_{K(s)} \mathcal{U}_s$ (resp. $K \times_{K(S)} \mathcal{U}_S$).

Let Θ be a generalized function on K invariant by conjugation. For any $s \in K$, Θ defines a K-invariant generalized function on $K \times_{K(s)} \mathcal{U}_s \hookrightarrow K$ which admits a restriction to the submanifold $\mathcal{U}_s$ that we denote

$$\Theta|_s \in \mathcal{C}^{-\infty}(\mathcal{U}_s)^{K(s)}.$$

If Θ is smooth we have $\Theta|_s(Y) = \Theta(s\,\mathrm{e}^Y)$.

Similarly, let θ be a K-invariant generalized function on $\mathfrak{k}$. For any $S \in \mathfrak{k}$, θ defines a K-invariant generalized function on $K \times_{K(S)} \mathcal{U}_S \hookrightarrow \mathfrak{k}$ which admits a restriction to the submanifold $\mathcal{U}_S$ that we denote

$$\theta|_S \in \mathcal{C}^{-\infty}(\mathcal{U}_S)^{K(S)}.$$

If θ is smooth we have $\theta|_S(Y) = \Theta(S + Y)$.

We have $K(se^S) = K(s) \cap K(S)$ for any $S \in \mathcal{U}_s$. Let $\Theta|_s \in \mathcal{C}^{-\infty}(\mathcal{U}_s)^{K(s)}$ be the restriction of a generalized function $\Theta \in \mathcal{C}^{-\infty}(K)^K$. For any $S \in \mathcal{U}_s$, the generalized function $\Theta|_s$ admits a restriction $(\Theta|_s)|_S$ which is a $K(s\,\mathrm{e}^S)$-invariant generalized function defined in a neighborhood of 0 in $\mathfrak{k}(s) \cap \mathfrak{k}(S) = \mathfrak{k}(s\,\mathrm{e}^S)$.

Lemma 3.1 *Let* $\Theta \in \mathcal{C}^{-\infty}(K)^K$.

• *For* $s \in K$, *and* $S \in \mathcal{U}_s$, *we have the following equality of generalized functions defined in a neighborhood of* 0 *in* $\mathfrak{k}(s\,\mathrm{e}^S)$

$$(\Theta|_s)|_S = \Theta|_{s\,\mathrm{e}^S}. \tag{9}$$

• *Let* $s, k \in K$. *We have the following equality of generalized functions defined in a neighborhood of* 0 *in* $\mathfrak{k}(s)$

$$\Theta|_s = \Theta|_{ksk^{-1}} \circ \mathrm{Ad}(k). \tag{10}$$

When $\Theta \in \mathcal{C}^\infty(K)^K$ is smooth, condition (9) is easy to check: for $Y \in \mathfrak{k}(s\,\mathrm{e}^S)$, we have

$$(\Theta|_s)|_S(Y) = \Theta|_s(S+Y) = \Theta(s\,\mathrm{e}^{S+Y}) = \Theta(s\,\mathrm{e}^S\,\mathrm{e}^Y) = \Theta|_{s\,\mathrm{e}^S}(Y).$$

We have the following

Theorem 3.2 *Let K be a compact Lie group. Consider a family of generalized function $\theta_s \in \mathcal{C}^{-\infty}(\mathcal{U}_s)^{K(s)}$. We assume that the following conditions are verified.*

• **Invariance***: for any k and $s \in K$, we have the following equality of generalized functions defined in a neighborhood of* 0 *in* $\mathfrak{k}(s)$

$$\theta_s = \theta_{ksk^{-1}} \circ \mathrm{Ad}(k).$$

• **Compatibility***: for every $s \in K$ and $S \in U_s$, we have the following equality of generalized functions defined in a neighborhood of* 0 *in* $\mathfrak{k}(s\,\mathrm{e}^S)$

$$\theta_s|_S = \theta_{s\,\mathrm{e}^S}.$$

Then there exists a unique generalized function $\Theta \in C^{-\infty}(K)^K$ such that, for any $s \in K$, the equality $\Theta|_s = \theta_s$ holds in $\mathcal{C}^{-\infty}(\mathcal{U}_s)^{K(s)}$.

3.2 Integration of bouquet of equivariant forms

Let K be a compact Lie group acting on a compact manifold M. We are interested in the central functions on K that can be defined by integrating equivariant forms on $\mathbf{T}^*M$.

Let ω be the Liouville 1-form on $\mathbf{T}^*M$. For any $s \in K$, we denote $M(s)$ the fixed points set $\{x \in M \mid sx = x\}$. As K is compact, $M(s)$ is a submanifold of M, and $\mathbf{T}^*(M(s)) = (\mathbf{T}^*M)(s)$. The cotangent bundle $\mathbf{T}^*M(s)$ is a symplectic submanifold of $\mathbf{T}^*M$ and the restriction $\omega|_{\mathbf{T}^*M(s)}$ is equal to the Liouville one form ω_s on $\mathbf{T}^*M(s)$. The manifolds $\mathbf{T}^*M(s)$ are oriented by their symplectic form $d\omega_s$.

For any $s \in K$, the tangent bundle $\mathbf{T}M$, when restricted to $M(s)$, decomposes as

$$\mathbf{T}M|_{M(s)} = \mathbf{T}M(s) \oplus \mathcal{N}.$$

Let $\underline{s}$ be the linear action induced by s on the bundle $\mathbf{T}M|_{M(s)}$: here $\mathbf{T}M(s)$ is the kernel of $\underline{s} - \mathrm{Id}$, and the normal bundle $\mathcal{N}$ is equal to the image of $\underline{s} - \mathrm{Id}$.

Let ∇ be a K-equivariant connection on the the tangent bundle $\mathbf{T}M$. It induces $K(s)$-equivariant connections : $\nabla^{0,s}$ on the bundle $\mathbf{T}M(s)$ and $\nabla^{1,s}$ on the bundle $\mathcal{N}$. For $i = 0, 1$, we consider the equivariant curvature $R_i(Y), Y \in \mathfrak{k}(s)$ of the connections $\nabla^{i,s}$. We will use the following equivariant forms

Definition 3.3 *We consider the following smooth closed $K(s)$-equivariant forms on $M(s)$:*

$$\begin{aligned} \widehat{\mathrm{A}}(M(s))^2(Y) &:= \det\left(\frac{R_0(Y)}{\mathrm{e}^{R_0(Y)/2} - \mathrm{e}^{-R_0(Y)/2}}\right), \\ \mathrm{D}_s(\mathcal{N})(Y) &:= \det\left(1 - \underline{s}\,\mathrm{e}^{R_1(Y)}\right). \end{aligned}$$

The form $\widehat{\mathrm{A}}(M(s))^2(Y)$ is defined for Y in a (small) neighborhood $\mathcal{U}_s$ of $0 \in \mathfrak{k}(s)$.

If the manifold $M(s)$ admits a $K(s)$-invariant orientation, one can define the square root of $\widehat{\mathrm{A}}(M(s))^2$: it is the equivariant $\widehat{\mathrm{A}}$-genus of the manifold $M(s)$.

The manifold $M(s)$ has several connected components C_i. We denote by $\dim M(s)$ the locally constant function on $M(s)$ equal to $\dim C_i$ on C_i. In the formulas of the cohomological index, we will use the following closed equivariant form on $M(s)$.

Definition 3.4 *We consider the smooth closed equivariant form on $M(s)$*

$$\Lambda_s(Y) := (2i\pi)^{-\dim M(s)} \, \frac{\widehat{\mathrm{A}}(M(s))^2(Y)}{\mathrm{D}_s(\mathcal{N})(Y)}$$

which is defined for Y in a (small) neighborhood $\mathcal{U}_s$ of $0 \in \mathfrak{k}(s)$.

Here $\mathcal{U}_s$ is a small $K(s)$-invariant neighborhood of 0 in $\mathfrak{k}(s)$. It is chosen so that : $\mathrm{ad}(k)\mathcal{U}_s = \mathcal{U}_{ksk^{-1}}$, and $M(s) \cap M(S) = M(s\,\mathrm{e}^S)$ for any $s \in K$ and any $S \in \mathcal{U}_s$. For any $s \in K$ and any $S \in \mathcal{U}_s$, let $\mathcal{N}_{(s,S)}$ be the normal bundle of $M(s\,\mathrm{e}^S) = M(s) \cap M(S)$ in $M(s)$. Let $R(Z)$ be the $K(s\,\mathrm{e}^S)$-equivariant curvature of an invariant euclidean connection on $\mathcal{N}_{(s,S)}$. Let

$$\mathrm{Eul}\left(\mathcal{N}_{(s,S)}\right)(Z) := (-2\pi)^{-\mathrm{rank}\,\mathcal{N}_{(s,S)}/2} \, \det{}_o^{1/2}(R(Z))$$

be its $K(s\,\mathrm{e}^S)$-equivariant Euler form. Recall that S induces a complex structure on the bundle $\mathcal{N}_{(s,S)}$. The square root $\det{}_o^{1/2}$ is computed using the orientation o defined by this complex structure.

Note that the diffeomorphism $k : \mathbf{T}^*M(s) \to \mathbf{T}^*M(ksk^{-1})$ induces a map $k : \mathcal{A}_c^\infty(\mathcal{U}_s, \mathbf{T}^*M(s)) \to \mathcal{A}_c^\infty(\mathcal{U}_{ksk^{-1}}, \mathbf{T}^*M(ksk^{-1}))$. It is easy to check that the family $\Lambda_s \in \mathcal{A}^\infty(\mathcal{U}_s, M(s))$ satisfies :

$$k \cdot \Lambda_s = \Lambda_{ksk^{-1}} \quad \text{in} \quad \mathcal{H}^\infty(\mathcal{U}_s, M(s)), \tag{11}$$

$$\Lambda_{s\,\mathrm{e}^S}(Z) = (-1)^r \frac{\Lambda_s|_{M(s\,\mathrm{e}^S)}}{\mathrm{Eul}(\mathcal{N}_{(s,S)})^2}(S+Z) \quad \text{in} \quad \mathcal{H}^\infty(\mathcal{U}', M(s\,\mathrm{e}^S)), \tag{12}$$

where $\mathcal{U}' \subset \mathfrak{k}(s\,\mathrm{e}^S)$ is a small invariant neighborhood of 0, and $r = \frac{1}{2}\mathrm{rank}_{\mathbb{R}}\,\mathcal{N}_{(s,S)}$.

Let $\gamma_s \in \mathcal{A}_c^\infty(\mathcal{U}(s), \mathbf{T}^*M(s))$ be a family of closed equivariant forms with compact support. We look now at the family of *smooth* invariant functions

$$\theta(\gamma)_s(Y) = \int_{\mathbf{T}^*M(s)} \Lambda_s(Y)\gamma_s(Y), \quad Y \in \mathcal{U}_s.$$

Lemma 3.5 *The family $\theta(\gamma)_s$ defines an invariant function $\Theta(\gamma) \in C^\infty(K)^K$ if*

$$k \cdot \gamma_s = \gamma_{ksk^{-1}} \quad \text{in} \quad \mathcal{H}_c^\infty(\mathcal{U}_s, \mathbf{T}^*M(s)),$$

and

$$\gamma_{s\,\mathrm{e}^S}(Z) = \gamma_s|_{\mathbf{T}^*M(s\,\mathrm{e}^S)}(S+Z) \quad \text{in} \quad \mathcal{H}_c^\infty(\mathcal{U}', \mathbf{T}^*M(s\,\mathrm{e}^S)),$$

where $\mathcal{U}' \subset \mathfrak{k}(s\,\mathrm{e}^S)$ is a small invariant neighborhood of 0.

Proof. The proof, that can be found in [10] and [7, 8], follows directly from the localization formula in equivariant cohomology. Note that the square $\mathrm{Eul}(\mathcal{N}_{(s,S)})^2$ is equal to the equivariant Euler form of the normal bundle of $\mathbf{T}^*M(s\,\mathrm{e}^S)$ in $\mathbf{T}^*M(s)$.

In this article, the equivariant forms γ_s that we use are the Chern forms attached to a transversally elliptic symbol : since they have *generalized coefficients*, we will need an extension of Lemma 3.5 in this case.

3.3 The Chern character with support

Let M be a K-manifold. Let $p : \mathbf{T}^*M \to M$ be the projection.

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a Hermitian K-equivariant super-vector bundle over M. Let $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a K-equivariant elliptic symbol: in this section we do not impose any conditions of ellipticity on σ. Recall that $\mathrm{supp}(\sigma) \subset \mathbf{T}^*M$ is the set where σ is not invertible.

Choose a K-invariant super-connection $\mathbb{A}$ on $p^*\mathcal{E}$, without 0 exterior degree term. As in [18, 16], we deform $\mathbb{A}$ with the help of σ : we consider the family of super-connections

$$\mathbb{A}^\sigma(t) = \mathbb{A} + it\, v_\sigma, \ t \in \mathbb{R},$$

on $\mathcal{E}$ where $v_\sigma = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}$ is an odd endomorphism of $\mathcal{E}$ defined with the help of the Hermitian structure. Let $\mathbf{F}(\sigma, \mathbb{A}, t)(X), X \in \mathfrak{k}$, be the equivariant curvature of $\mathbb{A}^\sigma(t)$.

We denote by $\mathbf{F}(X), X \in \mathfrak{k}$, the equivariant curvature of $\mathbb{A}$: we have $\mathbf{F}(X) = \mathbb{A}^2 + \mu^{\mathbb{A}}(X)$ where $\mu^{\mathbb{A}}(X) \in \mathcal{A}(\mathbf{T}^*M, \mathrm{End}(p^*\mathcal{E}))$ is the moment of $\mathbb{A}$ [6]. Then $\mathbf{F}(\sigma, \mathbb{A}, t)(X) \in \mathcal{A}(\mathbf{T}^*M, \mathrm{End}(p^*\mathcal{E}))^+$ is given by:

$$\mathbf{F}(\sigma, \omega, \mathbb{A}, t)(X) = -t^2 v_\sigma + it[\mathbb{A}, v_\sigma] + \mathbf{F}(X). \tag{13}$$

Let $t \in \mathbb{R}$. Consider the K-equivariant forms on $\mathbf{T}^*M$:

$$\begin{aligned}
\mathrm{Ch}(\mathbb{A})(X) &= \mathrm{Str}(\mathrm{e}^{\mathbf{F}(X)}),\\
\mathrm{Ch}(\mathbb{A},t)(X) &= \mathrm{Str}\left(\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)}\right),\\
\eta(\sigma,\mathbb{A},t)(X) &= -i\,\mathrm{Str}\left(v_\sigma\,\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)}\right),\\
\beta(\sigma,\mathbb{A},t)(X) &= \int_0^t \eta(\sigma,\mathbb{A},r)(X)dr.
\end{aligned}$$

The forms $\mathrm{Ch}(\mathbb{A})$, $\mathrm{Ch}(\mathbb{A},t)$ and $\beta(\sigma,\mathbb{A},t)$ are equivariant forms on $\mathbf{T}^*M$ with C^∞-coefficients. We have on $\mathbf{T}^*M$ the relation $D(\beta(\sigma,\mathbb{A},t)) = \mathrm{Ch}(\mathbb{A}) - \mathrm{Ch}(\mathbb{A},t)$.

We show in [16] that the equivariant forms $\mathrm{Ch}(\mathbb{A},t)$ and $\eta(\sigma,\mathbb{A},t)$ tends to zero exponentially fast on the open subset $\mathbf{T}^*M \setminus \mathrm{supp}(\sigma)$, when t goes to infinity. Hence the integral

$$\beta(\sigma,\mathbb{A})(X) = \int_0^\infty \eta(\sigma,\mathbb{A},t)(X)dt$$

defines an equivariant form with $\mathcal{C}^\infty$-coefficients on $\mathbf{T}^*M \setminus \mathrm{supp}(\sigma)$, and we have $D\beta(\sigma,\mathbb{A}) = \mathrm{Ch}(\mathbb{A})$ on $\mathbf{T}^*M \setminus \mathrm{supp}(\sigma)$.

We will now define the Chern character with support of σ. For any invariant open neighborhood U of $\mathrm{supp}(\sigma)$, we consider the algebra $\mathcal{A}_U(\mathbf{T}^*M)$ of differential forms on $\mathbf{T}^*M$ which are supported in U. Let $\mathcal{A}^\infty_U(\mathfrak{k},\mathbf{T}^*M)$ be the vector space of equivariant differential forms $\alpha : \mathfrak{k} \to \mathcal{A}_U(\mathbf{T}^*M)$ which are supported in U : $\mathcal{A}^\infty_U(\mathfrak{k},\mathbf{T}^*M)$ is a subspace of $\mathcal{A}^\infty(\mathfrak{k},\mathbf{T}^*M)$ which is stable under the derivative D. Let $\mathcal{H}^\infty_U(\mathfrak{k},\mathbf{T}^*M)$ be the corresponding cohomology space.

The following proposition follows easily:

Proposition 3.6 *Let U be a K-invariant open neighborhood of* $\mathrm{supp}(\sigma)$. *Let $\chi \in \mathcal{C}^\infty(\mathbf{T}^*M)$ be a K-invariant function, with support contained in U and equal to* 1 *in a neighborhood of* $\mathrm{supp}(\sigma)$. *The equivariant differential form on* $\mathbf{T}^*M$

$$c(\sigma,\mathbb{A},\chi) = \chi\,\mathrm{Ch}(\mathbb{A}) + d\chi\,\beta(\sigma,\mathbb{A})$$

is equivariantly closed and supported in U. Its cohomology class $\mathrm{Ch}_U(\sigma)$ *in* $\mathcal{H}^\infty_U(\mathfrak{k},\mathbf{T}^*M)$ *does not depend on the choice of* $(\mathbb{A},\chi)$, *nor on the Hermitian structure on $\mathcal{E}$.*

Definition 3.7 *We define the "Chern character with support"* $\mathrm{Ch}_{\mathrm{sup}}(\sigma)$ *as the collection* $(\mathrm{Ch}_U(\sigma))_U$, *where U runs over K-invariant open neighborhood of* $\mathrm{supp}(\sigma)$.

In practice, the Chern character with support $\mathrm{Ch}_{\mathrm{sup}}(\sigma)$ will be identified with a class $\mathrm{Ch}_U(\sigma) \in \mathcal{H}^\infty_U(\mathfrak{k},\mathbf{T}^*M)$, where U is a "sufficiently" small neighborhood of $\mathrm{supp}(\sigma)$.

When σ is elliptic, we can choose $\chi \in \mathcal{C}^\infty(\mathbf{T}^*M)^K$ with compact support, and we denote

$$\mathrm{Ch}_c(\sigma) \in \mathcal{H}^\infty_c(\mathfrak{k}, \mathbf{T}^*M) \tag{14}$$

the class defined by the equivariant form with *compact support* $c(\sigma, \mathbb{A}, \chi)$.

We introduce now the Bouquet of Chern characters with support.

Let $s \in K$. Then the action of s on $\mathcal{E}|_{M(s)}$ is given by $s^{\mathcal{E}}$, an even endomorphism of $\mathcal{E}|_{M(s)}$. The restriction of ω to $\mathbf{T}^*M(s)$ is the canonical 1-form ω_s of $\mathbf{T}^*M(s)$.

The super-connection $\mathbb{A}+itv_\sigma$ restricts to a super-connection on $p^*\mathcal{E}|\mathbf{T}^*M(s)$. Its curvature $\mathbf{F}(\sigma,\mathbb{A},t)$ restricted to $\mathbf{T}^*M(s) = N(s)$ gives an element of $\mathcal{A}(N(s), \mathrm{End}(p^*\mathcal{E}|_{N(s)}))$. To avoid further notations, if χ is a function on M, we still denote by χ its restriction to $M(s)$, by σ the restriction of σ to $\mathbf{T}^*M(s)$, by $\mathbf{F}(\sigma,\mathbb{A},t)$ the restriction of $\mathbf{F}(\sigma,\mathbb{A},t)$ to $\mathbf{T}^*M(s)$, etc....

For $Y \in \mathfrak{k}(s)$, we introduce the following $K(s)$-equivariant forms on $\mathbf{T}^*M(s)$:

$$\begin{aligned}
\mathrm{Ch}_s(\mathbb{A})(Y) &= \mathrm{Str}(s^{\mathcal{E}}\, \mathrm{e}^{\mathbf{F}(Y)}),\\
\eta_s(\sigma,\mathbb{A},t)(Y) &= -i\, \mathrm{Str}\left(v_\sigma\, s^{\mathcal{E}}\ \mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(Y)}\right),\\
\beta_s(\sigma,\mathbb{A})(Y) &= \int_0^\infty \eta_s(\sigma,,\mathbb{A},t)(Y)dt.
\end{aligned}$$

Then $\beta_s(\sigma,\omega,\mathbb{A})$ is well defined $K(s)$-equivariant form with $\mathcal{C}^\infty$-coefficients on $\mathbf{T}^*M(s) \setminus \mathrm{supp}(\sigma) \cap \mathbf{T}^*M(s)$. We have similarly $d\beta_s(\sigma,\mathbb{A}) = \mathrm{Ch}_s(\mathbb{A})$ outside $\mathrm{supp}(\sigma) \cap \mathbf{T}^*M(s)$.

The bouquet of Chern characters $(\mathrm{Ch}_{\sup}(\sigma,s))_{s\in K}$ can be constructed as follows.

Proposition 3.8 *Let U be a $K(s)$-invariant open neighborhood of $\mathrm{supp}(\sigma) \cap \mathbf{T}^*M(s)$ in $\mathbf{T}^*M(s)$. Let $\chi \in \mathcal{C}^\infty(\mathbf{T}^*M(s))$ be a $K(s)$-invariant function, with support contained in U and equal to 1 in a neighborhood of $\mathrm{supp}(\sigma) \cap \mathbf{T}^*M(s)$. The equivariant differential form on $\mathbf{T}^*M(s)$*

$$c_s(\sigma,\mathbb{A},\chi)(Y) = \chi\, \mathrm{Ch}_s(\mathbb{A})(Y) + d\chi\, \beta_s(\sigma,\mathbb{A})(Y), \quad Y \in \mathfrak{k}(s),$$

is equivariantly closed and supported in U. Its cohomology class $\mathrm{Ch}_U(\sigma,s)$ in $\mathcal{H}^\infty_U(\mathfrak{k}, \mathbf{T}^*M(s))$ does not depend on the choice of $(\mathbb{A},\chi)$, nor on the Hermitian structure on $\mathcal{E}$.

We define the "Chern character with support" $\mathrm{Ch}_{\sup}(\sigma,s)$ as the collection $(\mathrm{Ch}_U(\sigma,s))_U$, where U runs over the $K(s)$-invariant open neighborhood of $\mathrm{supp}(\sigma) \cap \mathbf{T}^*M(s)$ in $\mathbf{T}^*M(s)$.

Lemma 3.9 *Let $s \in K$ and $S \in K(s)$. Then for all $Y \in \mathfrak{k}(s) \cap \mathfrak{k}(S)$, one has*

$$c_{s\,\mathrm{e}^S}(\sigma,\mathbb{A},\chi)(Y) = c_s(\sigma,\mathbb{A},\chi)(S+Y)|_{N(s)\cap N(S)}.$$

Proof. Let $N = \mathbf{T}^*M$. We have to compare the following forms on $N(s) \cap N(S)$

$$\begin{aligned}
\mathrm{Ch}_{s\,\mathrm{e}^S}(\mathbb{A})(Y) &= \mathrm{Str}(s^{\mathcal{E}}\,\mathrm{e}^{S^{\mathcal{E}}}\,\mathrm{e}^{\mathbf{F}(Y)}),\\
\mathrm{Ch}_s(\mathbb{A})(S+Y) &= \mathrm{Str}(s^{\mathcal{E}}\,\mathrm{e}^{\mathbf{F}(S+Y)}),
\end{aligned}$$

as well as the following forms

$$\begin{aligned}
\eta_{s\,\mathrm{e}^S}(\sigma,\omega,\mathbb{A},t)(Y) &= -i\,\mathrm{Str}\left(v_\sigma\, s^{\mathcal{E}}\,\mathrm{e}^{S^{\mathcal{E}}}\,\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(Y)}\right),\\
\eta_s(\sigma,\mathbb{A},t)(S+Y) &= -i\,\mathrm{Str}\left(v_\sigma\, s^{\mathcal{E}}\,\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(S+Y)}\right).
\end{aligned}$$

For $S \in \mathfrak{k}$, the equivariant curvature $\mathbf{F}(\sigma,\mathbb{A},t)(S+Y)$ on $N(S)$ is equal to $S^{\mathcal{E}} + \mathbf{F}(\sigma,\mathbb{A},t)(Y)$ as the vector field VS vanishes on $N(S)$. Furthermore, above $N(s) \cap N(S)$, the endomorphism $\mathbf{F}(\sigma,\mathbb{A},t)(Y)$ commutes with $S^{\mathcal{E}}$, for $Y \in \mathfrak{k}(S) \cap \mathfrak{k}(s)$. Thus the result follows.

Then, for any open neighborhood U of $\mathrm{supp}(\sigma)$, the family $(\mathrm{Ch}_U(\sigma,s))_{s\in K}$ forms a bouquet of cohomology classes in the sense of [10].

3.4 The Chern character of a transversally elliptic symbol

We keep the same notations than in the previous sections. We denote by ω the Liouville form on $\mathbf{T}^*M$. In local coordinates (q,p) then $\omega = -\sum_a p_a dq_a$. The two-form $\Omega = d\omega = \sum_a dq_a \wedge dp_a$ gives a symplectic structure to $\mathbf{T}^*M$. The orientation of $\mathbf{T}^*M$ is the orientation determined by the symplectic structure (our convention for the canonical one-form ω differs from [7], but the symplectic form Ω is the same).

The moment map for the action of K on $(\mathbf{T}^*M,\Omega)$ is the map $f_\omega : \mathbf{T}^*M \to \mathfrak{k}^*$ defined by $\langle f_\omega(x,\xi), X\rangle = \langle \xi, V_x X\rangle$: we have $D\omega(X) = \Omega + \langle f_\omega, X\rangle$.

Remark that $\mathbf{T}^*_K M$ is the set of zeroes of f_ω. Recall how to associate to the 1-form ω a K-equivariant form $\mathrm{Par}(\omega)$ with generalized coefficients supported near $\mathbf{T}^*_K M$.

On the complement of $\mathbf{T}^*_K M$, the K-equivariant form

$$\beta(\omega) = -i\omega \int_0^\infty \mathrm{e}^{itD\omega}\,dt \tag{15}$$

is well defined as a K-equivariant form with generalized coefficients, and it is obvious to check that $D\beta(\omega) = 1$ outside $\mathbf{T}^*_K M$.

Definition 3.10 *Let U' be a K-invariant open neighborhood of $\mathbf{T}^*_K M$. Let $\chi' \in \mathcal{C}^\infty(\mathbf{T}^*M)$ be a K-invariant function, with support contained in U' and equal to 1 in a neighborhood of $\mathbf{T}^*_K M$. The equivariant differential form on $\mathbf{T}^*M$*

$$\mathrm{Par}(\omega,\chi') = \chi' + d\chi'\,\beta(\omega)$$

is closed, with generalized coefficients, and supported in U'*. Its cohomology class* $\mathrm{Par}_{U'}(\omega)$ *in* $\mathcal{H}^{-\infty}_{U'}(\mathfrak{k}, \mathbf{T}^*M)$ *does not depend on the choice of* χ'*.*

We will denote $\mathrm{Par}(\omega)$ *the collection* $(\mathrm{Par}_{U'}(\omega))_{U'}$*.*

It is immediate to verify that

$$\mathrm{Par}(\omega, \chi') = 1 + D\Big((\chi' - 1)\beta(\omega)\Big). \tag{16}$$

Thus, if we do not impose support conditions, the K-equivariant form $\mathrm{Par}(\omega, \chi')$ represents 1 in $\mathcal{H}^{-\infty}(\mathfrak{k}, \mathbf{T}^*M)$.

We consider now a K-transversally elliptic symbol σ on M. We have the Chern character $\mathrm{Ch}_{\mathrm{sup}}(\sigma)$ which is an equivariant form with $\mathcal{C}^{\infty}$-coefficients which is supported near $\mathrm{supp}(\sigma)$, and the equivariant form $\mathrm{Par}(\omega)$ with $\mathcal{C}^{-\infty}$-coefficients which is supported near $\mathbf{T}^*_K M$. Since $\mathrm{supp}(\sigma) \cap \mathbf{T}^*_K M$ is compact, the product

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma) \wedge \mathrm{Par}(\omega)$$

defines an equivariant form **with compact support** with $\mathcal{C}^{-\infty}$-coefficients. We summarize the preceding discussion by the

Theorem 3.11 *Let* σ *be a* K*-transversally elliptic symbol. Let* U, U' *be respectively* K*-invariant open neighborhoods of* $\mathrm{supp}(\sigma)$ *and* $\mathbf{T}^*_K M$ *such that* $\overline{U \cap U'}$ *is compact. The product*

$$\mathrm{Ch}_U(\sigma) \wedge \mathrm{Par}_{U'}(\omega)$$

defines a compactly supported class in $\mathcal{H}^{-\infty}_c(\mathfrak{k}, \mathbf{T}^*M)$ *which depends uniquely of* $[\sigma|_{\mathbf{T}^*_K M}] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$ *: this equivariant class is denoted* $\mathrm{Ch}_{\mathrm{c}}(\sigma, \omega)$*.*

We will use the notation

$$\mathrm{Ch}_c(\sigma, \omega) = \mathrm{Ch}_{\mathrm{sup}}(\sigma) \wedge \mathrm{Par}(\omega)$$

which summarizes the fact that the class with compact support $\mathrm{Ch}_{\mathrm{c}}(\sigma, \omega)$ is represented by the product

$$c(\sigma, \mathbb{A}, \chi) \wedge \mathrm{Par}(\omega, \chi') \tag{17}$$

where $\chi, \chi' \in \mathcal{C}^{\infty}(\mathbf{T}^*M)^K$ are equal to 1 respectively in a neighborhood of $\mathrm{supp}(\sigma)$ and $\mathbf{T}^*_K M$, and furthermore the product $\chi\chi'$ is compactly supported.

Remark 3.12 *If* σ *is elliptic, one can take* χ *with compact support, and* $\chi' = 1$ *on* $\mathbf{T}^*M$ *in Equation (17). We see then that*

$$\mathrm{Ch}_{\mathrm{c}}(\sigma, \omega) = \mathrm{Ch}_{\mathrm{c}}(\sigma) \quad \text{in} \quad \mathcal{H}^{-\infty}_c(\mathfrak{k}, \mathbf{T}^*M).$$

Let $s \in K$. Similarly, we denote by $\mathrm{Par}(\omega_s, \chi')$ the closed $K(s)$-equivariant form on $\mathbf{T}^*M(s)$ associated to the canonical 1-form $\omega_s = \omega|\mathbf{T}^*M(s)$ and a function $\chi' \in \mathcal{C}^\infty(\mathbf{T}^*M(s))^{K(s)}$ equal to 1 in a neighborhood of $\mathbf{T}^*_{K(s)}M(s)$. For any $K(s)$-invariant neighborhood $U' \subset \mathbf{T}^*M(s)$ of $\mathbf{T}^*_{K(s)}M(s)$, we denote

$$\mathrm{Par}_{U'}(\omega_s) \in \mathcal{H}^{-\infty}_{U'}(\mathfrak{k}(s), \mathbf{T}^*M(s))$$

the class defined by $\mathrm{Par}(\omega_s, \chi')$ when χ' is supported in U'. We denote $\mathrm{Par}(\omega_s)$ the collection $(\mathrm{Par}_{U'}(\omega, s))_{U'}$.

We defined, in Section 3.3, the family of Chern classes $(\mathrm{Ch}_{\mathrm{sup}}(\sigma, s))_{s\in K}$ for any K-invariant symbol. We now define a family $(\mathrm{Ch_c}(\sigma, \omega, s))_{s\in K}$ with compact support and $\mathcal{C}^{-\infty}$-coefficients when σ is a K-transversally elliptic symbol on M. Note that the restriction $\sigma|\mathbf{T}^*M(s)$ of a K-transversally elliptic symbol on M is a $K(s)$-transversally elliptic symbol on $M(s)$.

Proposition 3.13 *Let σ be a K-transversally elliptic symbol. Let $s \in K$.*

Let $U, U' \subset \mathbf{T}^*M(s)$ be respectively $K(s)$-invariant open neighborhoods of $\mathrm{supp}(\sigma|\mathbf{T}^*M(s))$ and $\mathbf{T}^*_{K(s)}M$ such that $\overline{U \cap U'}$ is compact. The product

$$\mathrm{Ch}_U(\sigma, s) \wedge \mathrm{Par}_{U'}(\omega_s)$$

defines a compactly supported class in $\mathcal{H}^{-\infty}_c(\mathfrak{k}(s), \mathbf{T}^*M(s))$ which depends uniquely of $[\sigma|_{\mathbf{T}^*_{K(s)}M(s)}]$. The notation

$$\mathrm{Ch}_c(\sigma, \omega, s) = \mathrm{Ch}_{\mathrm{sup}}(\sigma, s) \wedge \mathrm{Par}(\omega_s)$$

summarizes the fact that the class with compact support $\mathrm{Ch_c}(\sigma, \omega, s)$ is represented by $c_s(\sigma, \mathbb{A}, \chi) \wedge \mathrm{Par}(\omega_s, \chi')$ where $\chi, \chi' \in \mathcal{C}^\infty(\mathbf{T}^*M(s))^{K(s)}$ are chosen so that $\chi\chi'$ is compactly supported.

3.5 Definition of the cohomological index

Let K be a compact Lie group and let M be a compact K-manifold. The aim of this section is to define the cohomological index

$$\mathrm{index}_c^{K,M} : \mathbf{K}^0_K(\mathbf{T}^*_K M) \to \mathcal{C}^{-\infty}(K)^K.$$

For any $[\sigma] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$, the generalized function $\mathrm{index}_c^{K,M}([\sigma])$ will be described through their restrictions $\mathrm{index}_c^{K,M}([\sigma])|_s, s \in K$ (see Section 3.1).

In Subsection 3.2, we have introduce for any $s \in K$, the closed equivariant form on $M(s)$

$$\Lambda_s(Y) := (2i\pi)^{-\dim M(s)} \frac{\widehat{\mathrm{A}}(M(s))^2(Y)}{\mathrm{D}_s(\mathcal{N})(Y)}.$$

We wish to prove first the following theorem.

Theorem 3.14 *Let σ be a K-transversally elliptic symbol. There exists a unique invariant generalized function* $\mathrm{index}_c^{K,M}(\sigma)$ *on K satisfying the following equations. Let $s \in K$. For every $Y \in \mathfrak{k}(s)$ sufficiently small,*

$$\mathrm{index}_c^{K,M}(\sigma)|_s(Y) = \int_{\mathbf{T}^* M(s)} \Lambda_s(Y)\ \mathrm{Ch}_{\sup}(\sigma, s)(Y)\, \mathrm{Par}(\omega_s)(Y). \tag{18}$$

As $\mathrm{Ch_c}(\sigma, \omega, s) = \mathrm{Ch}_{\sup}(\sigma, s)(Y)\, \mathrm{Par}(\omega_s)(Y)$ is **compactly supported**, the integral (18) of equivariant differential forms with generalized coefficients defines a generalized function on a neighborhood of zero in $\mathfrak{k}(s)$. However, we need to prove that the different local formulae match together. The proof of this theorem occupies the rest of this subsection. Once this theorem is proved, we can make the following definition.

Definition 3.15 *Let σ be a K-transversally elliptic symbol. The cohomological index of σ is the invariant generalized function* $\mathrm{index}_c^{K,M}(\sigma)$ *on K satisfying Equation (18). We also rewrite the formula for the cohomological index as*

$$\mathrm{index}_c^{K,M}(\sigma)|_s(Y) = \int_{\mathbf{T}^* M(s)} \Lambda_s(Y)\ \mathrm{Ch_c}(\sigma, \omega, s)(Y). \tag{19}$$

In particular, when $s = e$, Equation (18) becomes

$$\mathrm{index}_c^{K,M}(\sigma)(e^X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^* M} \widehat{\mathrm{A}}(M)^2(X)\, \mathrm{Ch}_{\sup}(\sigma)(X)\, \mathrm{Par}(\omega)(X). \tag{20}$$

Remark 3.16 *In (18), (20) and (19) we take for the integration the symplectic orientation on the cotangent bundles.*

Let us now prove Theorem 3.14.

Proof. The right hand side of (18) defines a $K(s)$-invariant generalized function $\theta_s(Y)$ on a neighborhood $\mathcal{U}_s$ of 0 in $\mathfrak{k}(s)$. Following Theorem 3.2, the family $(\theta_s)_{s\in K}$ defines an invariant generalized function on K, if the *invariance condition* and the *compatibility condition* are satisfied. The invariance condition is easy to check. We will now prove the compatibility condition.

Let $s \in K$, and $S \in \mathcal{U}_s$. We have to check that the restriction $\theta_s|_S$ coincides with $\theta_{s\,\mathrm{e}^S}$ in a neighborhood of 0 in $\mathfrak{k}(s) \cap \mathfrak{k}(S) = \mathfrak{k}(s\,\mathrm{e}^S)$. We conduct the proof only for s equal to the identity e, as the proof for s general is entirely similar.

We have to compute the restriction at $\theta_e|_S$ of the generalized invariant function

$$\theta_e(X) := \int_{\mathbf{T}^* M} \Lambda_e(X)\ \mathrm{Ch}_{\sup}(\sigma)(X)\, \mathrm{Par}(\omega)(X), \quad X \in \mathcal{U}_e.$$

For this purpose, we choose a particular representant of the class $\mathrm{Ch}_{\sup}(\sigma)\mathrm{Par}(\omega)$ in $\mathcal{H}_c^{-\infty}(\mathfrak{k}, \mathbf{T}^* M)$. Since this class only depends of $[\sigma|_{\mathbf{T}^*_K M}] \in \mathbf{K}^0_K(\mathbf{T}^*_K M)$, we choose a transversally elliptic symbol σ_h which is almost homogeneous of degree 0 and such that $[\sigma_h|_{\mathbf{T}^*_K M}] = [\sigma|_{\mathbf{T}^*_K M}]$.

One can show [17] that the moment map $f_\omega : \mathbf{T}^*M \to \mathfrak{k}^*$ is *proper* when restricted to the support $\mathrm{supp}(\sigma_h)$. We represent $\mathrm{Ch}_{\sup}(\sigma_h)$ by the form $c(\sigma_h, \mathbb{A}, \chi_\sigma)$ where χ_σ is a function on $\mathbf{T}^*M$ such that $\mathrm{support}(\chi_\sigma) \cap \{\|f_\omega\|^2 \leq 1\}$ is compact. For this choice of χ_σ, the equivariant form $\alpha(X) = \Lambda_e(X)\, c(\sigma_h, \mathbb{A}, \chi_\sigma)(X)$ is thus such that $\mathrm{support}(\alpha) \cap \{\|f_\omega\|^2 \leq 1\}$ is compact. It is defined for X small enough. Multiplying by a smooth invariant function of X with small compact support and equal to 1 in a neighborhood of 0, we may find $\alpha(X)$ defined for all $X \in \mathfrak{k}$ and which coincide with $\Lambda_e(X)\, c(\sigma_h, \mathbb{A}, \chi_\sigma)(X)$ for X small enough.

We choose χ supported in $\{\|f_\omega\|^2 < 1\}$ and equal to 1 on $\{\|f_\omega\|^2 \leq \epsilon\}$, and define $\mathrm{Par}(\omega)$ with this choice of χ. Then $\alpha(X)\mathrm{Par}(\omega)(X)$ is compactly supported.

We will now prove the following result:

Proposition 3.17 *Let $\alpha(X)$ be a closed equivariant form with C^∞-coefficients on $N = \mathbf{T}^*M$ such that $\{\|f_\omega\|^2 \leq 1\} \cap \mathrm{support}(\alpha)$ is compact. Define the generalized function $\theta \in C^{-\infty}(\mathfrak{k})^K$ by*

$$\theta(X) := \int_N \alpha(X)\mathrm{Par}(\omega)(X). \tag{21}$$

Then, the restriction $\theta|_S$ is given, for $Y = Z - S$ sufficiently close to 0 by

$$\theta|_S(Y) = (-1)^r \int_{N(S)} \frac{\alpha|_{N(S)}(Z)}{\mathrm{Eul}(\mathcal{N}_S)^2(Z)} \mathrm{Par}(\omega_S)(Z). \tag{22}$$

Here $\mathcal{N}_S$ denotes the normal bundle of $M(S)$ in M, and $r = \frac{1}{2}(\dim M - \dim M(S))$.

Remark 3.18 *The integral (22) is defined using the symplectic orientation $o(\omega_S)$ on $N(S) = \mathbf{T}^*M(S)$. The linear action of S on the normal bundle $\mathcal{N}'_S$ of $N(S)$ in N induces a complex structure J_S : let $o(J_S)$ be corresponding orientation of the fibers of $\mathcal{N}'_S$. We have then on $N(S)$ the orientation $o(S)$ such that $o(\omega) = o(S)o(J_S)$. One can check that $(-1)^r$ is the quotient between $o(S)$ and $o(\omega_S)$.*

Let us apply the last proposition to the form $\alpha(X) = \Lambda_e(X)\, \mathrm{Ch}_{\sup}(\sigma_h)(X)$. If we use (12) and Lemma 3.9, we see that $(-1)^r \frac{\alpha|_{N(S)}}{\mathrm{Eul}(\mathcal{N}_S)^2}(S+Y)$ is equal to $\Lambda_{\mathrm{e}^S}(Y)\, \mathrm{Ch}_{\sup}(\sigma_h, \mathrm{e}^S)(Y)$. Hence Proposition 3.17 tells us that the the restriction of $\theta_e|_S$ is equal to θ_{e^S} : Theorem 3.14 is proved.

Proof. We now concentrate on the proof of Proposition 3.17.

Remark that if α is compactly supported, we can get rid of the forms $\mathrm{Par}(\omega)$ and $\mathrm{Par}(\omega_S)$ in the integrals (21) and (22), since they are equal to 1 in cohomology. In this case, the proposition is just the localization formula, as $\mathrm{Eul}(\mathcal{N}_S)^2$ is the Euler class of the normal bundle of $\mathbf{T}^*M(S)$ in $\mathbf{T}^*M$.

The proof will follow the same scheme as the usual localization formula (see [6]) and will use the fact that $\alpha|_{\mathfrak{k}(S)}$ is exact outside the set of zeroes of S. To extend the proof of the localization formula in our setting, we have to bypass the fact that the restriction of $\mathrm{Par}(\omega)$ to $\mathfrak{k}(S)$ has no meaning, since $\mathrm{Par}(\omega)$ is an equivariant form with generalized coefficients. However, we will use in a crucial way the fact that the closed equivariant form $\mathrm{Par}(\omega)$ is the limit of smooth equivariant forms

$$\mathrm{Par}^T(\omega)(X) = \chi + d\chi \int_0^T (-i\omega\, \mathrm{e}^{itD\omega(X)})dt.$$

Here $D\left(\mathrm{Par}^T(\omega)\right) = d\chi\, \mathrm{e}^{iTD\omega}$ tends to zero as T goes to infinity.

Let $\mathrm{Par}^T(\omega)(Z)$ be the restriction of $\mathrm{Par}^T(\omega)(X)$ to $\mathfrak{k}(S)$. We write $f_\omega = f_\omega^S + f_\omega^{\mathfrak{q}}$ relative to the $K(S)$-invariant decomposition $\mathfrak{k}^* = \mathfrak{k}(S)^* \oplus \mathfrak{q}$. Then the family of $K(S)$-equivariant forms

$$\mathrm{e}^{itD\omega(Z)} = \mathrm{e}^{itd\omega}\, \mathrm{e}^{it\langle f_\omega^S, Z\rangle}$$

tends to 0 outside $\{f_\omega^S = 0\}$, as t goes to ∞. Since $d\chi$ can be non-zero on the subset $\{f_\omega^S = 0\}$, the family of $\mathfrak{k}(S)$-equivariant forms $\mathrm{Par}^T(\omega)(Z)$ does not have a limit when $T \to \infty$ in general.

Consider the sub-manifold $N(S) := \mathbf{T}^*M(S)$ of $N := \mathbf{T}^*M$. Note that $f_\omega^{\mathfrak{q}}$ vanishes on $N(S)$. Let $\mathcal{V}$ be an invariant tubular neighborhood of $N(S)$ which is contained in $\{\|f_\omega^{\mathfrak{q}}\|^2 \leq \frac{\epsilon}{2}\}$. We are interested in the restriction $\mathrm{Par}^T(\omega)|_{\mathcal{V}}(Z)$ to $\mathcal{V}$. Since the function χ is equal to 1 on $\{\|f_\omega\|^2 \leq \epsilon\}$, we see that $d\chi|_{\mathcal{V}}$ is equal to zero in the neighborhood $\mathcal{V} \cap \{\|f_\omega^S\|^2 \leq \frac{\epsilon}{2}\}$ of $\mathcal{V} \cap \{f_\omega^S = 0\}$. Hence the limit

$$\mathrm{Par}(\omega)|_{\mathcal{V}}(Z) = \lim_{T\to\infty} \mathrm{Par}^T(\omega)|_{\mathcal{V}}(Z), \quad Z \in \mathfrak{k}(S), \tag{23}$$

defines a $K(S)$-equivariant form with generalized coefficients on $\mathcal{V}$. Note that the restriction of $\mathrm{Par}(\omega)|_{\mathcal{V}}$ to $N(S) \subset \mathcal{V}$ is the $K(S)$-equivariant form $\mathrm{Par}(\omega_S)$ associated to the Liouville 1-form ω_S on $\mathbf{T}^*M(S)$.

The generalized function $\theta \in \mathcal{C}^{-\infty}(\mathfrak{k})^K$ is the limit, as T goes to infinity, of the family of smooth functions

$$\theta^T(X) := \int_N \alpha(X)\mathrm{Par}^T(\omega)(X).$$

Here the equivariant forms $\alpha^T = \alpha\, \mathrm{Par}^T(\omega)$ stay supported in the fixed compact set $\mathcal{K} := \{\|f_\omega\|^2 \leq 1\} \cap \mathrm{support}(\alpha)$.

The proof will be completed if we show that the family of smooth functions $\theta^T(Z), Z \in \mathfrak{k}(S)$, converge to the generalized function

$$\theta'(Z) := (-1)^r \int_{N(S)} \frac{\alpha(Z)}{\mathrm{Eul}(\mathcal{N}_S)^2(Z)} \mathrm{Par}(\omega_S)(Z),$$

as T goes to infinity, and when Z varies in a small neighborhood of S in $\mathfrak{k}(S)$.

Let U be a relatively compact invariant neighborhood of $\mathcal{K}$ in N. Let $\chi' \in \mathcal{C}^\infty(U)^{K(S)}$ be such that χ' is supported in $\mathcal{V} \cap U$, and $\chi' = 1$ in a neighborhood of $U(S) = N(S) \cap U$. Here $\mathcal{V}$ is a tubular neighborhood of $N(S)$ satisfying the conditions for the existence of the limit (23).

Choose a K-invariant metric $\langle -, - \rangle$ on $\mathbf{T}N$. Let λ be the $K(S)$-invariant one form on N defined by $\lambda = \langle VS, - \rangle$. Note that $D(\lambda)(S) = d\lambda - \|VS\|^2$ is invertible outside $N(S)$. One sees that

$$\mathrm{P}_{\chi'}(Z) = \chi' + d\chi' \frac{\lambda}{D\lambda(Z)}$$

is a $K(S)$-equivariant form on U for Z in a small neighborhood of S. The following equation of $K(S)$-equivariant forms on U is immediate to verify:

$$1 = \mathrm{P}_{\chi'} + D\left((1-\chi')\frac{\lambda}{D\lambda}\right). \tag{24}$$

Since the $K(S)$-equivariant forms

$$\alpha^T(Z) := \alpha(Z)\mathrm{Par}^T(\omega)(Z)$$

are supported in U, one can multiply (24) by α^T. We have then the following relations between *compactly supported* $K(S)$-equivariant forms on N:

$$\begin{aligned} \alpha^T &= \mathrm{P}_{\chi'}\alpha^T + D\left((1-\chi')\frac{\lambda}{D\lambda}\right)\alpha^T \\ &= \mathrm{P}_{\chi'}\alpha^T + D\left((1-\chi')\frac{\lambda}{D\lambda}\alpha^T\right) + (1-\chi')\frac{\lambda}{D\lambda}D(\alpha^T). \end{aligned}$$

According to this equation, we divide the function $\theta^T(Z)$ in two parts

$$\theta^T(Z) = A^T(Z) + B^T(Z), \quad \text{for} \quad Z - S \quad \text{small}$$

with

$$A^T(Z) = \int_N \mathrm{P}_{\chi'}(Z)\alpha^T(Z)$$

and

$$B^T(Z) = \int_N (1-\chi')\frac{\lambda}{D\lambda(Z)} D\alpha^T(Z) = \int_N (1-\chi')\frac{\lambda}{D\lambda(Z)}\alpha(Z) d\chi\, \mathrm{e}^{iTD\omega(Z)}.$$

Let $p : \mathcal{V} \to N(S)$ be the projection, and let $i : N(S) \to \mathcal{V}$ be the inclusion. Since the form $\mathrm{P}_{\chi'}(Z)$ is supported in $\mathcal{V}$, the family of smooth equivariant forms $\mathrm{P}_{\chi'}(Z)\alpha(Z)\mathrm{Par}^T(\omega)(Z)$ converges to

$$\mathrm{P}_{\chi'}(Z)\alpha(Z)\mathrm{Par}(\omega)|_{\mathcal{V}}(Z)$$

as T goes to ∞, by our previous computation of the limit (23). Hence the functions $A^T(Z)$ converge to

$$\begin{aligned}
\int_{\mathcal{V}} \mathrm{P}_{\chi'}(Z)\alpha(Z)\mathrm{Par}(\omega)|_{\mathcal{V}}(Z) &= \int_{\mathcal{V}} \mathrm{P}_{\chi'}(Z)\ p^*\circ i^*\left(\alpha\,\mathrm{Par}(\omega)|_{\mathcal{V}}\right)(Z) \quad [1] \\
&= \int_{N(S),o(S)} p_*(\mathrm{P}_{\chi'})(Z)\ \alpha|_{N(S)}(Z)\,\mathrm{Par}(\omega_S)(Z) \quad [2] \\
&= \int_{N(S),o(S)} \frac{\alpha|_{N(S)}(Z)}{\mathrm{Eul}(\mathcal{N}_S)^2(Z)}\mathrm{Par}(\omega_S)(Z) \quad [3] \\
&= (-1)^r \int_{N(S)} \frac{\alpha|_{N(S)}(Z)}{\mathrm{Eul}(\mathcal{N}_S)^2(Z)}\mathrm{Par}(\omega_S)(Z). \quad [4]
\end{aligned}$$

Points [1] and [2] are due to the fact that $\alpha(Z)\mathrm{Par}(\omega)|_{\mathcal{V}}(Z)$ is equal to $p^*\circ i^*\left(\alpha\,\mathrm{Par}(\omega)|_{\mathcal{V}}\right)(Z)$ in $\mathcal{H}^{-\infty}(\mathfrak{k}(S),\mathcal{V})$ and that $\mathrm{P}_{\chi'}$ has a compact support relatively to the fibers of p (here p_* denotes the integration along the fibers). For point [3], we use then that $p_*(\mathrm{P}_{\chi'})$ multiplied by the Euler class[2] of $\mathcal{V}$ is equal to the restriction of $\mathrm{P}_{\chi'}$ to $N(S)$, which is identically equal to 1. In [4], we use the symplectic orientation for the integration.

Let us show that the integral $\int_{\mathfrak{k}(S)} B^T(Z)\varphi(Z)dZ$ tends to 0, as T goes to infinity, for any $\varphi \in \mathcal{C}^\infty(\mathfrak{k}(S))^{K(S)}$ supported in a small neighborhood of S. As $\det_{\mathfrak{k}/\mathfrak{k}(S)}(Z)$ does not vanish when $Z-S$ remains small enough, it is enough to show that

$$I(T) := \int_{N\times\mathfrak{k}(S)} (1-\chi')\frac{\lambda}{D\lambda(Z)} D\alpha^T(Z)\varphi(Z)\det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ$$

tends to 0, as T goes to infinity. We have

$$I(T) := \int_{N\times\mathfrak{k}(S)} \mathrm{e}^{iTD\omega(Z)}\,\eta(Z)\det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ$$

where $\eta(Z) = (\chi'-1)\frac{\lambda}{D\lambda(Z)}\alpha(Z)d\chi\varphi(Z)$ is a compactly supported $K(S)$-equivariant form on N with $\mathcal{C}^\infty$-coefficients, which is defined for all $Z\in\mathfrak{k}(S)$. Furthermore we have $\eta(Z)=0$ for Z outside a small neighborhood of S and

$$\mathrm{support}(\eta)\cap\left\{f_\omega = 0\right\} = \emptyset.$$

There exists a K-equivariant form $\Gamma:\mathfrak{k}\to\mathcal{A}(N)$ such that $\Gamma(Z)=\eta(Z)$ for any $Z-S$ small in $\mathfrak{k}(S)$. Indeed we define $\Gamma(X) = k\cdot\eta(Z)$ for any choice of k, Z such that $k\cdot Z = X$. Here X varies in a (small) neighborhood of $K\cdot S$. As $\eta(Z)$ is zero when Z is not near S, the map $X\mapsto\Gamma(X)$ is supported on a compact neighborhood of $K\cdot S$ in $\mathfrak{k}$. We see also that

$$\mathrm{support}(\Gamma)\cap\left\{f_\omega = 0\right\} = \emptyset. \tag{25}$$

[2] The Euler form of the vector bundle $\mathcal{V}\to N(S)$ is equal to the square of the Euler form of the normal bundle $\mathcal{N}_S$ of $M(S)$ in M.

Condition (25) implies that the integral $J(T) := \int_{\mathfrak{k}\times N} \mathrm{e}^{iTD\omega(X)}\,\Gamma(X)dX$ goes to 0, as T goes to infinity. But $I(T) = J(T)$. Indeed, write $X = k\cdot Z$ and apply Weyl integration formula. We obtain

$$\begin{aligned} J(T) &= \int_{\mathfrak{k}(S)} \Big(\int_{K\times N} \mathrm{e}^{iTD\omega(k\cdot Z)}\,\Gamma(k\cdot Z)dk\Big) \det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ \\ &= \int_{\mathfrak{k}(S)} \int_{K\times N} k\cdot \Big(\mathrm{e}^{iTD\omega(Z)}\,\eta(Z)\Big)dk\ \det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ. \end{aligned}$$

Integration on the K-manifold N is invariant by diffeomorphisms, thus

$$J(T) = \int_{\mathfrak{k}(S)} \int_N \mathrm{e}^{iTD\omega(Z)}\,\eta(Z) \det{}_{\mathfrak{k}/\mathfrak{k}(S)}(Z)dZ = I(T).$$

We have shown that the family of smooth function $B^T(Z)$ goes to 0, as T goes to infinity. The proof of Proposition 3.17 is then completed.

Let H be a compact Lie group acting on M and commuting with the action of K. Then the space $\mathbf{T}^*_K M$ is provided with an action of $K\times H$.

Lemma 3.19 *If* $[\sigma] \in \mathbf{K}^0_{K\times H}(\mathbf{T}^*_K M)$*, then the cohomological index* $\mathrm{index}_c^{K,H,M}(\sigma) \in C^{-\infty}(K\times H)^{K\times H}$ *is smooth relatively to* H.

Proof. We have to prove that for any $s = (s_1, s_2) \in K\times H$, the generalized function

$$\mathrm{index}_c^{K,H,M}(\sigma)|_s(Y_1, Y_2)$$

which is defined for (Y_1, Y_2) in a neighborhood of 0 in $\mathfrak{k}(s_1)\times\mathfrak{h}(s_2)$, is smooth relatively to the parameter $Y_2 \in \mathfrak{h}(s_2)$. We check it for $s = e$.

We have

$$\mathrm{index}_c^{K,H,M}(\sigma)|_e(X,Y) = \int_{\mathbf{T}^*M} \Lambda_e(X,Y)\,\mathrm{Ch}_{\sup}(\sigma)(X,Y)\mathrm{Par}(\omega)(X,Y) \tag{26}$$

for $(X,Y) \in \mathfrak{k}\times\mathfrak{h}$ in a neighborhood of 0. The equivariant class with compact support $\mathrm{Ch}_{\sup}(\sigma)\mathrm{Par}(\omega)$ is represented by the product $c(\sigma,\mathbb{A},\chi)\mathrm{Par}(\omega,\chi')$ where (χ,χ') is chosen so that $\chi = 1$ in a neighborhood of $\mathrm{supp}(\sigma)$, $\chi' = 1$ in a neighborhood of $\mathbf{T}^*_{K\times H}M$, and $\chi\chi'$ is compactly supported.

Since σ is K-transversally elliptic, the set $\mathrm{supp}(\sigma)\cap\mathbf{T}^*_K M$ is compact. Hence we can choose (χ,χ') so that $\chi' = 1$ in a neighborhood of $\mathbf{T}^*_K M$ and $\chi\chi'$ is compactly supported. It easy to check that the equivariant form $\mathrm{Par}(\omega,\chi')(X,Y)$ is then smooth relatively to the parameter $Y\in\mathfrak{h}$. This show that the right hand side of (26) is smooth relatively to the parameter $Y \in \mathfrak{h}$.

Remark 3.20 *We will denote* $\mathrm{Ch}^1_c(\sigma,\omega)(X,Y)$ *the* $K\times H$*-equivariant form defined by the product* $c(\sigma,\mathbb{A},\chi)\mathrm{Par}(\omega,\chi')$ *where* (χ,χ') *is chosen so that* $\chi = 1$ *in a neighborhood of* $\mathrm{supp}(\sigma)$, $\chi' = 1$ *in a neighborhood of* $\mathbf{T}^*_K M$*, and* $\chi\chi'$ *is compactly supported. The equivariant form* $\mathrm{Ch}^1_c(\sigma,\omega)(X,Y)$ *is compactly supported and is smooth relatively to* $Y\in\mathfrak{h}$.

4 The cohomological index coincides with the analytic one

In this section, we now prove that the cohomological index is equal to the analytical index. The main difficulty in the proof of this result in Berline-Vergne [7, 8] was to prove that their formulae were defining generalized functions which, moreover, were compatible with each other. The heart of this new proof is the fact the Chern character with compact support is multiplicative. Thus we rely heavily here on the results of [17], so that the proof is now easy.

Theorem 4.1 *The analytic index of a transversally elliptic operator P on a K-manifold M is equal to* $\mathrm{index}_c^{K,M}([\sigma_P])$.

To prove that the cohomological index is equal to the analytic index, following Atiyah-Singer algorithm, we need only to verify that the cohomological index satisfies the properties that we listed of the analytic index:

- Invariance by diffeomorphism : **Diff**,
- Functorial with respect to subgroups : **Morph**,
- Excision property,
- Free action properties,
- Multiplicative properties,
- Normalization conditions [**N1**], [**N2**] and [**N3**].

The invariance by diffeomorphism, the functoriality with respect to subgroups and the excision property are obviously satisfied by $\mathrm{index}_c^{K,M}$.

4.1 Free action

We now prove that the cohomological index satisfies the free action property. We consider the setting of Subsection 2.3.3. The action of K on the bundle $\mathbf{T}^*_K P$ is free and the quotient $\mathbf{T}^*_K P/K$ admit a canonical identification with $\mathbf{T}^* M$. Then we still denote by

$$q : \mathbf{T}^*_K P \to \mathbf{T}^* M$$

the quotient map by K: it is a G-equivariant map such that $q^{-1}(\mathbf{T}^*_G M) = \mathbf{T}^*_{K\times G} P$.

We choose a G-invariant connection θ for the principal fibration $q : P \to M$ of group K. With the help of this connection, we have a direct sum decomposition

$$\mathbf{T}^* P = \mathbf{T}^*_K P \oplus P \times \mathfrak{k}^*.$$

Let $\pi_1 : \mathbf{T}^*P \to \mathbf{T}^*_K P$ and $\pi_2 : \mathbf{T}^*P \to P \times \mathfrak{k}^*$ be the projections on each factors. Let

$$Q : \mathbf{T}^*P \to \mathbf{T}^*M$$

be the map $q \circ \pi_1$.

Let σ be a G-transversally elliptic morphism on $\mathbf{T}^*M$. Its pull-back $Q^*\sigma$ is then a $K \times G$-transversally elliptic morphism on $\mathbf{T}^*P$: we have $\mathrm{supp}(Q^*\sigma) = Q^{-1}(\mathrm{supp}(\sigma))$ and then $\mathrm{supp}(Q^*\sigma) \cap \mathbf{T}^*_{K\times G}P = q^{-1}(\mathrm{supp}(\sigma) \cap \mathbf{T}^*_G M)$ is compact.

Theorem 4.2 *Let $P \to M$ be a principal fibration with a free right action of K, provided with a left action of G. Consider a class $[\sigma] \in \mathbf{K}^0_G(\mathbf{T}^*_G M)$ and its pull-back by Q : $[Q^*\sigma] \in \mathbf{K}^0_{K\times G}(\mathbf{T}^*_{K\times G}P)$. Then we have the equality of generalized functions : for $(k,g) \in K \times G$*

$$\mathrm{index}_c^{K\times G, P}([Q^*\sigma])(k,g) = \sum_{\tau \in \hat{K}} \mathrm{Tr}(k,\tau)\, \mathrm{index}_c^{G,M}([\sigma_{\tau^*}])(g).$$

The rest of this section is devoted to the proof. We have to check that for any $(s,s') \in K \times G$ we have the following equality of generalized functions defined in a neighborhood of $\mathfrak{k}(s) \times \mathfrak{g}(s')$:

$$\mathrm{index}_c^{K\times G,P}([Q^*\sigma])|_{(s,s')}(X,Y) = \sum_{\tau\in\hat{K}} \mathrm{Tr}(s\,\mathrm{e}^X, \tau)\, \mathrm{index}_c^{G,M}([\sigma_{\tau^*}])|_{s'}(Y). \tag{27}$$

We conduct the proof of (27) only for $(s,s') = (1,1)$. This proof can be adapted to the general case by using the same arguments as Berline-Vergne [8].

First, we analyze the left hand side of (27) at $(s,s') = (1,1)$.

We consider the $K \times G$-invariant one form $\nu = \langle \xi, \theta \rangle$ on $P \times \mathfrak{k}^*$: here $\theta \in \mathcal{A}^1(P) \otimes \mathfrak{k}$ is our connection form, and ξ is the variable in $\mathfrak{k}^*$. We have

$$D\nu(X,Y) = d\nu + \langle \xi, \mu(Y) - X \rangle, \quad X \in \mathfrak{k}, \quad Y \in \mathfrak{g}. \tag{28}$$

where $\mu(Y) = -\theta(VY) \in C^\infty(P) \otimes \mathfrak{k}$.

We associate to ν the $K \times G$-equivariant form with *generalized coefficients* $\beta(-\nu)(X,Y) = i\nu \int_0^\infty \mathrm{e}^{-itD\nu(X,Y)}\, dt$, $(X,Y) \in \mathfrak{k} \times \mathfrak{g}$, which is defined on the open subset $P \times \mathfrak{k}^* \setminus \{0\}$. One checks that $\beta(-\nu)(X,Y)$ is smooth relatively to the variable $Y \in \mathfrak{g}$. Let $\chi_{\mathfrak{k}^*} \in \mathcal{C}^\infty(\mathfrak{k}^*)^K$ be a function with compact support and equal to 1 near 0. Then

$$\mathrm{Par}(-\nu)(X,Y) := \chi_{\mathfrak{k}^*} + d\chi_{\mathfrak{k}^*}\beta(-\nu)(X,Y) \tag{29}$$

is a closed equivariant form on $P \times \mathfrak{k}^*$, with compact support, and which is smooth relatively to the variable $Y \in \mathfrak{g}$.

Let σ be a G-transversally elliptic morphism on $\mathbf{T}^*M$. Its pull-back $Q^*\sigma$ is then a $K \times G$-transversally elliptic morphism on $\mathbf{T}^*P$. Let ω_P and ω_M be the Liouville 1-forms on $\mathbf{T}^*P$ and $\mathbf{T}^*M$ respectively. We have defined the equivariant Chern class with compact support $\mathrm{Ch}_c(\sigma, \omega_M) \in \mathcal{H}_c^{-\infty}(\mathfrak{g}, \mathbf{T}^*M)$ and $\mathrm{Ch}_c(Q^*\sigma, \omega_P) \in \mathcal{H}_c^{-\infty}(\mathfrak{k} \times \mathfrak{g}, \mathbf{T}^*P)$.

Proposition 4.3 *We have the following equality*

$$\mathrm{Ch}_c(Q^*\sigma, \omega_P)(X,Y) = Q^*\Big(\mathrm{Ch}_c(\sigma,\omega_M)\Big)(Y) \wedge \pi_2^*\Big(\mathrm{Par}(-\nu)\Big)(X,Y)$$

in $\mathcal{H}_c^{-\infty}(\mathfrak{k}\times\mathfrak{g}, \mathbf{T}^*P)$. *Note that the product on the right hand side is well defined since* $\mathrm{Par}(-\nu)(X,Y)$ *is smooth relatively to the variable* $Y \in \mathfrak{g}$.

Proof. The proof which is done in [17] follows from the relation

$$\omega_P = Q^*(\omega_M) - \pi_2^*(\nu). \tag{30}$$

We now analyze the term

$$\mathrm{index}_c^{K\times G,P}([Q^*\sigma])|_{(1,1)}(X,Y) = (2i\pi)^{-\dim P}\int_{\mathbf{T}^*P} \widehat{\mathrm{A}}(P)^2\,\mathrm{Ch}_c(Q^*\sigma,\omega_P)(X,Y).$$

An easy computation gives that $\widehat{\mathrm{A}}(P)^2(X,Y) = j_{\mathfrak{k}}(X)^{-1}q^*\,\widehat{\mathrm{A}}(M)^2(Y)$, with $j_{\mathfrak{k}}(X) = \det_{\mathfrak{k}}\left(\frac{\mathrm{e}^{\mathrm{ad}(X)/2}-\mathrm{e}^{-\mathrm{ad}(X)/2}}{\mathrm{ad}(X)}\right)$. If we use Proposition 4.3, we see that

$$\begin{aligned}
&\mathrm{index}_c^{K\times G,P}([Q^*\sigma])|_{(1,1)}(X,Y)\\
&= \frac{(2i\pi)^{-\dim P}}{j_{\mathfrak{k}}(X)}\int_{\mathbf{T}^*P}\pi_1^*\circ q^*\Big(\widehat{\mathrm{A}}(M)^2\,\mathrm{Ch}_c(\sigma,\omega_M)\Big)(Y)\wedge \pi_2^*\mathrm{Par}(-\nu)(X,Y)\\
(31) &= \frac{(2i\pi)^{-\dim P}}{j_{\mathfrak{k}}(X)}\int_{\mathbf{T}^*_K P} q^*\left(\widehat{\mathrm{A}}(M)^2\,\mathrm{Ch}_c(\sigma,\omega_M)\right)(Y)\wedge \int_{\mathfrak{k}^*}\mathrm{Par}(-\nu)(X,Y).
\end{aligned}$$

Let us compute the integral $\int_{\mathfrak{k}^*}\mathrm{Par}(-\nu)(X,Y)$.

We choose a K-invariant scalar product on $\mathfrak{k}$ and an orthonormal basis $E^1,\ldots,E^r$ of $\mathfrak{k}$, with dual basis $E_1,\ldots,E_r$: we write $X = \sum_k X_k E^k$ for $X\in\mathfrak{k}$, and $\xi = \sum_k \xi_k E_k$ for $\xi\in\mathfrak{k}^*$. Let $\theta_k = \langle E_k,\theta\rangle$ be the 1-forms on P associated to the connection one form. Let $\mathrm{vol}(K, dX^o)$ be the volume of K computed with the Haar measure compatible with the volume form $dX^o = dX_1\ldots dX_r$.

We have $d\nu = \sum_k \xi_k d\theta_k + d\xi_k\theta_k$, and (30) gives that

$$(d\omega_P)^{\dim P} = q^*\left(d\omega_M\right)^{\dim M}\wedge\theta_r\cdots\theta_1\wedge\pi_2^*\left(d\xi_1\cdots d\xi_r\right).$$

So, in the integral (31), the vector space $\mathfrak{k}^*$ is oriented by the volume form $d\xi^o = d\xi_1\cdots d\xi_r$, and $\mathbf{T}^*_K P$ is oriented by $q^*\left(d\omega_M\right)^{\dim M}\wedge\theta_r\cdots\theta_1$.

Let $\Theta = d\theta + \frac{1}{2}[\theta,\theta]\in\mathcal{A}^2(P)\otimes\mathfrak{k}$ be the curvature of θ. The equivariant curvature of θ is

$$\Theta(Y) = \mu(Y) + \Theta.$$

Then $\Theta(Y)\in\mathcal{A}(P)\otimes\mathfrak{k}$ is horizontal, and the element $\Theta\in\mathcal{A}^2(P)\otimes\mathfrak{k}$ is nilpotent. If φ is a $\mathcal{C}^\infty$ function on $\mathfrak{k}$, then $\varphi(\Theta(Y))$ is computed via the Taylor series expansion at $\mu(Y)(p)$ and $\varphi(\Theta(Y))$ is a horizontal form on P which depends smoothly and G-equivariantly of $Y\in\mathfrak{g}$. When $\varphi\in\mathcal{C}^\infty(\mathfrak{k})$ is K-invariant, the form $\varphi(\Theta(Y))$ is basic : hence we can look at it as a differential form on M which depends smoothly and G-equivariantly of $Y\in\mathfrak{g}$.

Definition 4.4 *Let $\delta(X - \Theta(Y))$ be the $K \times G$-equivariant form on P defined by the relation*

$$\int_{\mathfrak{k}\times\mathfrak{g}} \delta(X - \Theta(Y))\varphi(X,Y)dXdY := \mathrm{vol}(K,dX)\int_{\mathfrak{g}} \varphi(\Theta(Y),Y)dY,$$

for any $\varphi \in \mathcal{C}^\infty(\mathfrak{k}\times\mathfrak{g})$ with compact support. Here $\mathrm{vol}(K,dX)$ is the volume of K computed with the Haar measure compatible with dX.

One sees that $\delta(X-\Theta(Y))$ is a $K\times G$-equivariant form on P which depends smoothly of the variable $Y \in \mathfrak{g}$.

Lemma 4.5 *Let $\mathfrak{k}^*$ be oriented by the volume form $d\xi^o = d\xi_1\cdots d\xi_r$. Then*

$$\int_{\mathfrak{k}^*} \mathrm{Par}(-\nu)(X,Y) = (2i\pi)^{\dim K}\delta(X-\Theta(Y))\frac{\theta_r\cdots\theta_1}{\mathrm{vol}(K,dX^o)}.$$

Proof. Take $\chi_{\mathfrak{k}^*}(\xi) = g(\|\xi\|^2)$ where $g \in \mathcal{C}_c^\infty(\mathbb{R})$ is equal to 1 in a neighborhood of 0. Let $\varphi \in \mathcal{C}_c^\infty(\mathfrak{k})$ and let $\widehat{\varphi}(\xi) = \int_{\mathfrak{k}} \mathrm{e}^{i\langle\xi,X\rangle}\,\varphi(X)dX^o$ be its Fourier transform relatively to dX^o.

To compute the integral over the fiber $\mathfrak{k}^*$ of $\mathrm{Par}(-\nu)(X,Y)$, only the highest exterior degree term in $d\xi$ will contribute to the integral. This term comes only from the term $d\chi_{\mathfrak{k}^*}\beta(-\nu)(X,Y)$ in $\mathrm{Par}(-\nu)(X,Y) := \chi_{\mathfrak{k}^*} + d\chi_{\mathfrak{k}^*}\beta(-\nu)(X,Y)$. We compute

$$\begin{aligned}
\int_{\mathfrak{k}}\left(\int_{\mathfrak{k}^*}\mathrm{Par}(-\nu)(X,Y)\right)\varphi(X)dX^o &= \int_{\mathfrak{k}^*}\left(\int_{\mathfrak{k}}\mathrm{Par}(-\nu)(X,Y)\varphi(X)dX^o\right)\\
&= \int_{\mathfrak{k}^*} d\chi_{\mathfrak{k}^*}(i\nu)\left(\int_0^\infty \mathrm{e}^{-it(d\nu+\langle\xi,\mu(Y)\rangle)}\,\widehat{\varphi}(t\xi)dt\right)\\
&= \int_0^\infty \underbrace{\left(\int_{\mathfrak{k}^*} d\chi_{\mathfrak{k}^*}(i\nu)\,\mathrm{e}^{-it(d\nu+\langle\xi,\mu(Y)\rangle)}\,\widehat{\varphi}(t\xi)\right)}_{I(t)}dt.
\end{aligned}$$

Since $d\nu = \sum_k \xi_k d\theta_k + d\xi_k\theta_k$, the differential form $d\chi_{\mathfrak{k}^*}(i\nu)\,\mathrm{e}^{-itd\nu}$ is equal to

$$2i\,g'(\|\xi\|^2)(\sum_j \xi_j d\xi_j)(\sum_k \xi_k\theta_k)\prod_l(1-itd\xi_l\theta_l)\,\mathrm{e}^{-it\langle\xi,d\theta\rangle},$$

and its component $[d\chi_{\mathfrak{k}^*}(i\nu)\,\mathrm{e}^{-itd\nu}]_{\max}$ of highest exterior degree in $d\xi$ is

$$\begin{aligned}
[d\chi_{\mathfrak{k}^*}(i\nu)\,\mathrm{e}^{-itd\nu}]_{\max} &= -2(-i)^r t^{r-1}g'(\|\xi\|^2)\|\xi\|^2\prod_j(d\xi_j\wedge\theta_j)\,\mathrm{e}^{-it\langle\xi,d\theta\rangle}\\
&= -2(i)^r t^{r-1}\,\theta_r\cdots\theta_1\,g'(\|\xi\|^2)\|\xi\|^2\,\mathrm{e}^{-it\langle\xi,d\theta\rangle}\,d\xi^o.
\end{aligned}$$

So for $t>0$ we have

$$\begin{aligned} I(t) &= -2(i)^r t^{r-1}\theta_r\cdots\theta_1\left(\int_{\mathfrak{k}^*} g'(\|\xi\|^2)\|\xi\|^2\,\mathrm{e}^{-it\langle\xi,d\theta+\mu(Y)\rangle}\,\widehat{\varphi}(t\xi)d\xi^o\right)\\ &= (i)^r\theta_r\cdots\theta_1\left(\int_{\mathfrak{k}^*}\left[(-2g'(\tfrac{\|\xi\|^2}{t^2})\tfrac{\|\xi\|^2}{t^3}\right]\mathrm{e}^{-i\langle\xi,d\theta+\mu(Y)\rangle}\,\widehat{\varphi}(\xi)d\xi^o\right)\\ &= (i)^r\theta_r\cdots\theta_1\frac{d}{dt}\left(\int_{\mathfrak{k}^*} g(\tfrac{\|\xi\|^2}{t^2})\,\mathrm{e}^{-i\langle\xi,d\theta+\mu(Y)\rangle}\,\widehat{\varphi}(\xi)d\xi^o\right).\end{aligned}$$

Finally $\int_{\mathfrak{k}}\left(\int_{\mathfrak{k}^*}\mathrm{Par}(-\nu)(X,Y)\right)\varphi(X)dX^o$ is equal to

$$\begin{aligned}\int_0^\infty I(t)dt &= (i)^r\theta_r\cdots\theta_1\left(\int_{\mathfrak{k}^*}\mathrm{e}^{-i\langle\xi,d\theta+\mu(Y)\rangle}\,\widehat{\varphi}(\xi)d\xi^o\right)\\ &= (2i\pi)^r\ \theta_r\cdots\theta_1\ \varphi(d\theta+\mu(Y))\\ &= (2i\pi)^r\ \theta_r\cdots\theta_1\ \varphi(\Theta+\mu(Y))\\ &= (2i\pi)^r\left(\int_{\mathfrak{k}}\delta(X-\Theta(Y))\varphi(X)dX^o\right)\frac{\theta_r\cdots\theta_1}{\mathrm{vol}(K,dX^o)}.\end{aligned}$$

The last lemma shows that $\mathrm{index}_c^{K\times G,P}([Q^*\sigma])|_{(1,1)}(X,Y)$ is equal to

$$\frac{(2i\pi)^{-\dim M}}{j_{\mathfrak{k}}(X)}\int_{\mathbf{T}^*_K P} q^*\left(\widehat{\mathrm{A}}(M)^2\,\mathrm{Ch}_c(\sigma,\omega_M)\right)(Y)\,\delta(X-\Theta(Y))\frac{\theta_r\cdots\theta_1}{\mathrm{vol}(K,dX^o)}$$

$$(32)\qquad = \frac{(2i\pi)^{-\dim M}}{j_{\mathfrak{k}}(X)}\int_{\mathbf{T}^*M}\widehat{\mathrm{A}}(M)^2(Y)\,\mathrm{Ch}_c(\sigma,\omega_M)(Y)\,\delta_o(X-\Theta(Y)).$$

Here $\delta_o(X-\Theta(Y))$ denotes the closed $K\times G$-equivariant form M defined by the relation

$$\int_{\mathfrak{k}}\delta_o(X-\Theta(Y))\varphi(X)dX = \mathrm{vol}(K,dX)\overline{\varphi}(\Theta(Y))$$

for any $\varphi\in\mathcal{C}_c^\infty(\mathfrak{k})$. Here $\overline{\varphi}(X) := \mathrm{vol}(K,dk)^{-1}\int_K\varphi(kX)dk$ is the K-invariant function obtained by averaging φ.

Now we analyze the right hand side of (27) at $(s,s')=(1,1)$. Here the Chern class $\mathrm{Ch}_{\mathrm{sup}}(\sigma_{\tau^*})(Y)$ is equal to $\mathrm{Ch}_{\mathrm{sup}}(\sigma)(Y)\,\mathrm{Ch}(\mathcal{V}_{\tau^*})(Y)$ where the equivariant Chern character $\mathrm{Ch}(\mathcal{V}_{\tau^*})(Y)$ is represented by $\mathrm{Tr}(\mathrm{e}^{\Theta(Y)},\tau^*)$. Hence $\mathrm{Ch_c}(\sigma_{\tau^*},\omega_M)(Y)=\mathrm{Ch_c}(\sigma,\omega_M)(Y)\,\mathrm{Tr}(\mathrm{e}^{\Theta(Y)},\tau^*)$. So the generalized function $\sum_{\tau\in\hat{K}}\mathrm{Tr}(\mathrm{e}^X,\tau)\,\mathrm{index}_c^{G,M}([\sigma_{\tau^*}])|_1(Y)$ is equal to

$$(33)\qquad (2i\pi)^{-\dim M}\int_{\mathbf{T}^*M}\widehat{\mathrm{A}}(M)^2(Y)\,\mathrm{Ch}_c(\sigma,\omega_M)(Y)\,\Xi(X,\Theta(Y))$$

where $\Xi(X,X')$ is a generalized function on a neighborhood of 0 in $\mathfrak{k}\times\mathfrak{k}$ defined by the relation $\Xi(X,X')=\sum_{\tau\in\hat{K}}\mathrm{Tr}(\mathrm{e}^X,\tau)\,\mathrm{Tr}(\mathrm{e}^{X'},\tau^*)$.

The Schur orthogonality relation shows that

$$\Xi(X,X') = j_{\mathfrak{k}}(X)^{-1}\delta_o(X-X').$$

In other words, $\Xi(X,X')$ is smooth relatively to X' and for any $\varphi \in \mathcal{C}^\infty(\mathfrak{k})^K$ which is supported in a small neighborhood of 0, we have $\mathrm{vol}(K,dX)\varphi(X') = \int_{\mathfrak{k}} \Xi(X,X') j_{\mathfrak{k}}(X)\varphi(X)dX$.

Finally, we have proved that the generalized functions (32) and (33) coincides: the proof of (27) is then completed for $(s,s')=(1,1)$.

4.2 Multiplicative property

We consider the setting of Subsection 2.3.2. We will check that the cohomological index satisfies the *Mutiplicative property* (see Theorem 2.5).

Let M_1 be a compact $K_1\times K_2$-manifold, and let M_2 be a K_2-manifold. We consider the product $M := M_1 \times M_2$ with the action of $K := K_1 \times K_2$.

Theorem 4.6 (Multiplicative property) *For any* $[\sigma_1] \in \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}^*_{K_1} M_1)$ *and any* $[\sigma_2] \in \mathbf{K}^0_{K_2}(\mathbf{T}^*_{K_2} M_2)$ *we have*

$$\mathrm{index}_c^{K,M}([\sigma_1]\odot_{\mathrm{ext}}[\sigma_2]) = \mathrm{index}_c^{K_1,K_2,M_1}([\sigma_1])\,\mathrm{index}_c^{K_2,M_2}([\sigma_2]). \tag{34}$$

The product on the right hand side of (34) is well defined since $\mathrm{index}_c^{K_1,K_2,M_1}([\sigma_1])$ *is a generalized function on* $K_1\times K_2$ *which is smooth relatively to* K_2 *(see Lemma 3.19).*

Proof. Let σ_1 be a morphism on $\mathbf{T}^*M_1$, which is $K_1\times K_2$-equivariant and K_1-transversally elliptic. Let σ_2 be a morphism on $\mathbf{T}^*M_2$, which is K_2-transversally elliptic. The morphism σ_2 can be chosen so that it is almost homogeneous of degree 0. Then the product $\sigma := \sigma_1 \odot_{\mathrm{ext}} \sigma_2$ is K-transversally elliptic morphism on $\mathbf{T}^*M$, and $[\sigma] = [\sigma_1]\odot_{\mathrm{ext}}[\sigma_2]$ in $\mathbf{K}^0_K(\mathbf{T}^*_K M)$.

We have to show that for any $s=(s_1,s_2)\in K_1\times K_2$, we have

$$\begin{aligned}\mathrm{index}_c^{K,M}([\sigma])|_s(Y_1,Y_2) &= \\ &\mathrm{index}_c^{K_1,K_2,M_1}([\sigma_1])|_{s_1}(Y_1,Y_2)\,\mathrm{index}_c^{K_2,M_2}([\sigma_2])|_{s_2}(Y_2)\end{aligned} \tag{35}$$

for (Y_1,Y_2) in a neighborhood of 0 in $\mathfrak{k}_1(s_1)\times\mathfrak{k}_2(s_2)$. We conduct the proof only for s equal to the identity e, as the proof for s general is entirely similar.

For $k=1,2$, let $\pi_k : \mathbf{T}^*M \to \mathbf{T}^*M_k$ be the projection. The Liouville one form ω on $\mathbf{T}^*(M_1\times M_2)$ is equal to $\pi_1^*\omega_1 + \pi_2^*\omega_2$, where ω_k is the Liouville one form on $\mathbf{T}^*M_k$.

We have three index formulas:

$$\begin{aligned}
\mathrm{index}_c^{K,M}([\sigma])|_e(X_1,X_2) &:= (2i\pi)^{-\dim M}\int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2\,\mathrm{Ch_c}(\sigma,\omega)(X_1,X_2),\\
\mathrm{index}_c^{K,M_1}([\sigma_1])|_e(X_1,X_2) &:= (2i\pi)^{-\dim M_1}\int_{\mathbf{T}^*M_1} \widehat{\mathrm{A}}(M_1)^2\,\mathrm{Ch}^1_{\mathrm{c}}(\sigma_1,\omega_1)(X_1,X_2),\\
\mathrm{index}_c^{K_2,M_2}([\sigma_2])|_e(X_2) &:= (2i\pi)^{-\dim M_2}\int_{\mathbf{T}^*M_2} \widehat{\mathrm{A}}(M_2)^2\,\mathrm{Ch_c}(\sigma_2,\omega_2)(X_2).
\end{aligned}$$

Following Remark 3.20, $\mathrm{Ch}^1_{\mathrm{c}}(\sigma_1,\omega_1)(X_1,X_2)$ denotes a closed equivariant form with compact support which represents the class $\mathrm{Ch}_{\mathrm{c}}(\sigma_1,\omega_1)$, and which is *smooth* relatively to $X_2 \in \mathfrak{k}_2$.

It is immediate to check that $\widehat{\mathrm{A}}(M)^2(X_1,X_2) = \widehat{\mathrm{A}}(M_1)^2(X_1,X_2)\,\widehat{\mathrm{A}}(M_2)^2(X_2)$. Hence Equality (35) follows from the following identity in $\mathcal{H}^{-\infty}_c(\mathfrak{k}_1\times\mathfrak{k}_2,\mathbf{T}^*M)$ that we proved in [17]:

$$\pi_1^*\,\mathrm{Ch}^1_{\mathrm{c}}(\sigma_1,\omega_1)(X_1,X_2)\wedge\pi_2^*\,\mathrm{Ch}_{\mathrm{c}}(\sigma_2,\omega_2)(X_2) = \mathrm{Ch}_{\mathrm{c}}(\sigma,\omega)(X_1,X_2).$$

4.3 Normalization conditions

4.3.1 Atiyah symbol

Let $V := \mathbb{C}_{[1]}$ be equipped with the canonical action of S^1. The Atiyah symbol σ_{At} was introduced in Subsection 2.4.2 : it is a S^1-transversally elliptic symbol on V. It is the first basic example of a "pushed" symbol (see Subsection 5.1).

We consider on V the euclidean metric $(v,w) = \Re(v\overline{w})$: it gives at any $v\in V$ identifications $\mathbf{T}_vV \simeq \mathbf{T}^*_vV \simeq \mathbb{C}_{[1]}$. So in this example we will make no distinction between vectors fields and 1-forms on V. Let $\kappa(\xi_1) = i\xi_1$ be the vector field on V associated to the action of S^1 : $\kappa = -VX$ where $X = i \in \mathrm{Lie}(S^1)$.

Let σ_V be the symbol on the complex vector space V: at any $(\xi_1,\xi_2)\in\mathbf{T}^*V$, $\sigma_V(\xi_1,\xi_2) : \wedge^0V\to\wedge^1V$ acts by multiplication by ξ_2. We see then that

$$\sigma_{\mathrm{At}}(\xi_1,\xi_2) = \sigma_V(\xi_1,\xi_2+\kappa(\xi_1)).$$

The symbol σ_{At} is obtained by "pushing" the symbol σ by the vector field κ.

We can attached to the one form κ, the equivariant form $\mathrm{Par}(\kappa)$ which is defined on V, and localized near $\{\kappa=0\}=\{0\}\subset V$. Since the support of σ_V is the zero section, the equivariant Chern character $\mathrm{Ch}_{\mathrm{sup}}(\sigma_V)$ is an equivariant form on $\mathbf{T}^*V$ which is compactly supported in the fibers of $p:\mathbf{T}^*V\to V$. Then the product $\mathrm{Ch}_{\mathrm{sup}}(\sigma_V)p^*\mathrm{Par}(\kappa)$ defines an equivariant form with compact support on $\mathbf{T}^*V$.

Here we will use the relation (see Proposition 5.5)

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma_{\mathrm{At}})\mathrm{Par}(\omega) = \mathrm{Ch}_{\mathrm{sup}}(\sigma_V)\,p^*\mathrm{Par}(\kappa)\quad\text{in}\quad \mathcal{H}^{-\infty}_c(\mathfrak{k},\mathbf{T}^*V). \tag{36}$$

Using (36), we now compute the cohomological index of the Atiyah symbol.

Proposition 4.7 *We have*

$$[\mathbf{N3}]\qquad \mathrm{index}^{S^1,V}_c([a])(\mathrm{e}^{i\theta}) = -\sum_{n=1}^{\infty}\mathrm{e}^{in\theta}\,.$$

Proof. We first prove the equality above when $s=\mathrm{e}^{i\theta}$ is not equal to 1. Then, near s, the generalized function $-\sum_{n=1}^{\infty}\mathrm{e}^{in\theta}$ is analytic and given by $-\frac{s}{1-s}$.

Now, at a point $s \in S^1$ different from 1, the fixed point set $V(s)$ is $\{0\}$. The character $\mathrm{Ch}_s(\mathcal{E})$ is $(1-s)$, and the form $D_s(\mathcal{N})$ is $(1-s)(1-s^{-1})$. Thus

$$\mathrm{index}_c^{S^1,V}(s) = \frac{(1-s)}{(1-s)(1-s^{-1})} = -\frac{s}{1-s}.$$

This shows the equality of both members in Proposition 4.7 on the open set $s \neq 1$ of S^1.

We now compute near $s = 1$. Thanks to Formula (36) we have

$$\mathrm{index}_c^{S^1,V}(\sigma_{\mathrm{At}})|_1(\theta) = (2i\pi)^{-2} \int_{\mathbf{T}^*V} \widehat{\mathrm{A}}(V)^2(\theta)\ \mathrm{Ch}_{\mathrm{sup}}(\sigma_V)(\theta)\, p^*\mathrm{Par}(\kappa)(\theta).$$

The Chern character with support $\mathrm{Ch}_{\mathrm{sup}}(\sigma_V)(\theta)$ is proportional to the S^1-equivariant Thom form of the real vector bundle $\mathbf{T}^*V \to V$. More precisely, calculation already done in [16] shows that

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma_V)(\theta) = (2i\pi)\frac{\mathrm{e}^{i\theta} - 1}{i\theta}\mathrm{Thom}(\mathbf{T}^*V)(\theta).$$

However the symplectic orientation on $\mathbf{T}^*V \simeq \mathbb{C}^2$ is the opposite of the orientation given by its complex structure. Now

$$\widehat{\mathrm{A}}(V)^2(\theta) = \frac{(i\theta)(-i\theta)}{(1-\mathrm{e}^{i\theta})(1-\mathrm{e}^{-i\theta})}.$$

Thus we obtain

$$\mathrm{index}_c^{S^1,V}(\sigma_{\mathrm{At}})|_1(\theta) = \frac{-i\theta}{(1-\mathrm{e}^{-i\theta})}\frac{1}{2i\pi}\int_V \mathrm{Par}(\kappa)(\theta).$$

As $\frac{(1-\mathrm{e}^{-i\theta})}{-i\theta} = -\int_{-1}^0 \mathrm{e}^{ix\theta}\, dx$, we see that $\frac{(1-\mathrm{e}^{-i\theta})}{-i\theta}(-\sum_{n=1}^\infty \mathrm{e}^{in\theta}) = \int_0^\infty \mathrm{e}^{ix\theta}\, dx$. It remains to show

$$\frac{1}{2i\pi}\int_V \mathrm{Par}(\kappa)(\theta) = \int_0^\infty \mathrm{e}^{ir\theta}\, dr. \tag{37}$$

We have $D\kappa(\theta) = \theta(x^2+y^2) + 2dx \wedge dy$. Take a function g on $\mathbb{R}$ with compact support and equal to 1 on a neighborhood of 0. Let $\chi = g(x^2+y^2)$. Then

$$\begin{aligned}
\mathrm{Par}(\kappa)(\theta) &= \chi - id\chi \wedge \kappa \int_0^\infty \mathrm{e}^{itD\kappa(\theta)}\, dt \\
&= g(x^2+y^2) - 2ig'(x^2+y^2)dx \wedge dy \int_0^\infty (x^2+y^2)\,\mathrm{e}^{i\theta t(x^2+y^2)}\, dt \\
&= g(x^2+y^2) - 2ig'(x^2+y^2)dx \wedge dy \int_0^\infty \mathrm{e}^{i\theta t}\, dt.
\end{aligned}$$

Finally we obtain (37) since $\int_V -2ig'(x^2+y^2)dx \wedge dy = 2i\pi$. This completes the proof.

4.3.2 Bott symbols

We will check here that the cohomological index satisfies the condition [**N2**]: $\mathrm{index}_c^{O(V),V}(\mathrm{Bott}(V_{\mathbb{C}})) = 1$, for any euclidean vector space V.

We have explain in Remark 2.8 that it sufficient to prove [**N2**] for the cases:
- $V = \mathbb{R}$ with the action of the group $O(V) = \mathbb{Z}/2\mathbb{Z}$,
- $V = \mathbb{R}^2$ with the action of the group $SO(V) = S^1$.

Let $V = \mathbb{R}$ with the multiplicative action of $\mathbb{Z}_2$. We have to check that $\mathrm{index}_c^{\mathbb{R},\mathbb{Z}_2}(\mathrm{Bott}(\mathbb{C}))(\epsilon) = 1$ for $\epsilon \in \mathbb{Z}_2$. When $\epsilon = 1$, we have

$$\mathrm{index}_c^{\mathbb{R},\mathbb{Z}_2}(\mathrm{Bott}(\mathbb{C}))(1) = (2i\pi)^{-1} \int_{\mathbf{T}^*\mathbb{R}} \widehat{\mathrm{A}}(\mathbb{R})^2 \,\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C})).$$

Here $\widehat{\mathrm{A}}(\mathbb{R})^2 = 1$. We have proved in [16] that the class $\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C})) \in \mathcal{H}_c^2(\mathbf{T}^*\mathbb{R})$ is equal to $2i\pi$ times the Thom form of the oriented vector space of $\mathbb{R}^2 \simeq \mathbf{T}^*\mathbb{R}$. Hence $\mathrm{index}_c^{\mathbb{R},\mathbb{Z}_2}(\mathrm{Bott}(\mathbb{C}))(1) = 1$. When $\epsilon = -1$, the space $\mathbf{T}^*\mathbb{R}(\epsilon)$ is reduced to a point. We see that $\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C}), \epsilon) = 2$, $D_\epsilon(\mathcal{N}) = \det(1-\epsilon) = 2$. Then $\mathrm{index}_c^{\mathbb{R},\mathbb{Z}_2}(\mathrm{Bott}(\mathbb{C}))(1) = \frac{\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C}),\epsilon)}{D_\epsilon(\mathcal{N})} = 1$.

Let $V = \mathbb{R}^2$ with the rotation action of S^1. Like before $\mathrm{index}_c^{\mathbb{R}^2,S^1}(\mathrm{Bott}(\mathbb{C}^2))(1)$ is equal to 1 since the Chern class $\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C}^2))$ is equal to $(2i\pi)^2$ times the Thom form of the oriented vector space of $\mathbb{R}^4 \simeq \mathbf{T}^*\mathbb{R}^2$. When $\mathrm{e}^{i\theta} \neq 1$, the space $\mathbf{T}^*\mathbb{R}^2(\mathrm{e}^{i\theta})$ is reduced to a point. We see that $\mathrm{Ch_c}(\mathrm{Bott}(\mathbb{C}), \mathrm{e}^{i\theta}) = D_{\mathrm{e}^{i\theta}}(\mathcal{N}) = 2(1-\cos(\theta))$. Then $\mathrm{index}_c^{\mathbb{R}^2,S^1}(\mathrm{Bott}(\mathbb{C}^2))(\mathrm{e}^{i\theta}) = 1$.

5 Examples

5.1 Pushed symbols

Let M be a K-manifold and $N = \mathbf{T}^*M$. Let $\mathcal{E}^\pm \to M$ be two K-equivariant complex vector bundles on M and $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a K-equivariant symbol which is supposed to be **invertible exactly outside the zero section** : the set $\mathrm{supp}(\sigma)$ coincides with the zero section of $\mathbf{T}^*M$.

If M is compact, σ defines an elliptic symbol on $\mathbf{T}^*M$, thus a fortiori a transversally elliptic symbol.

Here we assume M non compact. Following Atiyah's strategy [1], we can "push" the symbol σ outside the zero section, if we dispose of a K-invariant real one-form κ on M. This construction provides radically new transversally elliptic symbols. We recall some definitions of [17]:

Definition 5.1 *Let κ be a real K-invariant one-form on M. Define $f_\kappa : M \to \mathfrak{k}^*$ by $\langle f_\kappa(x), X\rangle = \langle \kappa(x), V_x X\rangle$. We define the subset C_κ of M by $C_\kappa = f_\kappa^{-1}(0)$. We call C_κ the critical set of κ.*

We define the symbol $\sigma(\kappa)$ on M by

$$\sigma(\kappa)(x,\xi) = \sigma(x, \xi + \kappa(x)), \quad \text{for} \quad (x,\xi) \in \mathbf{T}^*M.$$

Thus $\sigma(\kappa)$ is not invertible at (x,ξ) if and only if $\xi = -\kappa(x)$, and then $(x,\xi) \in \mathrm{supp}(\sigma(\kappa)) \cap \mathbf{T}^*_K M$ if $\xi = -\kappa(x)$ and $\langle \kappa(x), V_x X\rangle = 0$ for all $X \in \mathfrak{k}$. Thus

$$\mathrm{supp}(\sigma(\kappa)) \cap \mathbf{T}^*_K M = \{(x, -\kappa(x)) \mid x \in C_\kappa\}.$$

If C_κ is compact, then the morphism $\sigma(\kappa)$ is transversally elliptic.

Using a K-invariant metric on $\mathbf{T}M$, we can associate to a K-invariant vector field $\mathcal{K}$ on M a K-invariant real one-form.

Example 5.2 *Let $S \in \mathfrak{k}$ be a central element of $\mathfrak{k}$ such that the set of zeroes of VS is compact. Then the associated form $\kappa_S(\bullet) = \langle VS, \bullet\rangle$ is a K-invariant real one-form such that C_{κ_S} is compact. Indeed the value of f_{κ_S} on S is $\|VS\|^2$, so that the set C_{κ_S} coincides with the fixed point set $M(S)$.*

Definition 5.3 *If κ is a K-invariant real one-form on M such that C_κ is compact, the transversally elliptic symbol*

$$\sigma(\kappa)(x,\xi) = \sigma(x, \xi + \kappa(x))$$

is called the pushed symbol of σ by κ.

Example 5.4 *The Atiyah symbol is a pushed symbol defined on $M = \mathbb{R}^2$ (see Subsection 4.3.1).*

We construct as in (15) the K-equivariant differential form

$$\beta(\kappa)(X) = -i\kappa \wedge \int_0^\infty \mathrm{e}^{itD\kappa(X)}\, dt$$

which is defined on $M \setminus C_\kappa$. We choose a compactly supported function χ_κ on M identically 1 near C_κ. Then the K-equivariant form

$$\mathrm{Par}(\kappa)(X) = \chi_\kappa + d\chi_\kappa \beta(\kappa)(X)$$

defined a class in $\mathcal{H}_c^{-\infty}(\mathfrak{k}, M)$.

The K-equivariant form $\mathrm{Par}(\kappa)$ is congruent to 1 in cohomology without support conditions. Indeed one verify that $\mathrm{Par}(\kappa) = 1 + D((\chi_\kappa - 1)\beta(\kappa))$.

Let $p : \mathbf{T}^*M \to M$ be the projection. We can multiply the K-equivariant form $p^*\mathrm{Par}(\kappa)(X)$ with $C^{-\infty}$-coefficients by the K-equivariant form $\mathrm{Ch}_{\mathrm{sup}}(\sigma)(X)$. In this way, we obtain a K-equivariant form with compact support on $\mathbf{T}^*M$.

Proposition 5.5 *The K-equivariant form $\mathrm{Ch}_{\mathrm{sup}}(\sigma)p^*\mathrm{Par}(\kappa)$ represents the class $\mathrm{Ch}_{\mathrm{c}}(\sigma(\kappa), \omega)$ in $\mathcal{H}_c^{-\infty}(\mathfrak{k}, \mathbf{T}^*M)$.*

Proof. By definition the class $\mathrm{Ch}_{\mathrm{c}}(\sigma(\kappa), \omega)$ is represented by the product $\mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))\mathrm{Par}(\omega)$ We first prove $\mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))\mathrm{Par}(\omega) = \mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))\mathrm{Par}(p^*\kappa)$ in $\mathcal{H}_c^{-\infty}(\mathfrak{k}, \mathbf{T}^*M)$.

Indeed if $(x,\xi) \in \mathrm{supp}(\sigma(\kappa))$, then $\xi = -\kappa(x)$. Thus $\langle \omega(x,\xi), v\rangle = -\langle \xi, p_* v\rangle = \langle \kappa(x), p_* v\rangle$ where v is any tangent vector at $(x,\xi) \in \mathbf{T}^*M$. So the 1-forms ω and $p^*\kappa$ coincides on the support of $\sigma(\kappa)$. Thus $\mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))\mathrm{Par}(\omega) = \mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))p^*\mathrm{Par}(\kappa)$ as consequence of ([17], Corollary 3.12).

Let us prove now that $\mathrm{Ch}_{\mathrm{sup}}(\sigma(\kappa))p^*\mathrm{Par}(\kappa) = \mathrm{Ch}_{\mathrm{sup}}(\sigma)p^*\mathrm{Par}(\kappa)$. Consider the family of symbols on M defined by $\sigma_t(x,\xi) = \sigma(x, \xi + t\kappa(x))$ for $t \in [0,1]$: we have $\sigma_0 = \sigma$ and $\sigma_1 = \sigma(\kappa)$.

On a compact neighborhood $\mathcal{U}$ of C_κ, the support of σ_t stays in the compact set $\{(x,\xi) : x \in \mathcal{U}, \xi = -t\kappa(x)\}$ when t varies between 0 and 1. It follows from ([17], Theorem 3.11) that all the classes $\mathrm{Ch}_{\mathrm{sup}}(\sigma_t)p^*\mathrm{Par}(\kappa),\ t \in [0,1]$ coincides in $\mathcal{H}_c^{-\infty}(\mathfrak{k}, \mathbf{T}^*M)$.

Similarly for any $s \in K$, we consider the restriction κ_s of the form κ to $M(s)$. We finally obtain the following formula:

Theorem 5.6 *For any $s \in K$ and $X \in \mathfrak{k}(s)$ small, the cohomological index $\mathrm{index}_c^{K,M}(\sigma(\kappa))|_s(Y)$ is given on $\mathfrak{k}(s)$ by the integral formula:*

$$\int_{\mathbf{T}^*M(s)} \Lambda_s(Y)\,\mathrm{Par}(\kappa_s)(Y)\ \mathrm{Ch}_{\mathrm{sup}}(\sigma, s)(Y).$$

In particular, when $s = 1$ we get

$$\mathrm{index}_c^{K,M}(\sigma(\kappa))|_e(X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2(X)\,\mathrm{Par}(\kappa)(X)\ \mathrm{Ch}_{\mathrm{sup}}(\sigma)(X).$$

An interesting situation is when the manifold M is oriented, and is equipped with a K-invariant Spin structure. Let $\mathcal{S}_M \to M$ be the corresponding spinor bundle. We consider the K-invariant symbol $\sigma_{spin} : p^*\mathcal{S}_M^+ \to p^*\mathcal{S}_M^-$. It support is exactly the zero section of the cotangent bundle. For any invariant 1-form κ on M such that C_κ is compact we consider the transversally elliptic symbol $\sigma_{\mathrm{Spin}}(\kappa)$.

We have proved in [16] that

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma_{spin})(X) = (2i\pi)^{\dim M}\hat{A}(M)^{-1}(X)\,\mathrm{Thom}(\mathbf{T}^*M)(X).$$

Hence Theorem 5.6 tell us that

$$\mathrm{index}_c^{K,M}(\sigma_{spin}(\kappa))|_e(X) = \int_M \widehat{\mathrm{A}}(M)(X)\,\mathrm{Par}(\kappa)(X).$$

5.2 Contact manifolds

The following geometric example is taken from [11].

Let M be a compact manifold of dimension $2n+1$. Suppose that M carries a *contact* 1-form α ; that is, $E = \ker(\alpha)$ is a hyperplane distribution of $\mathbf{T}M$, and the restriction of the 2-form $d\alpha$ to E is symplectic. The existence of a Reeb vector field $\mathbf{Y}$ on M gives canonical decompositions $\mathbf{T}M = E \oplus \mathbb{R}\mathbf{Y}$ and $\mathbf{T}^*M = E^* \oplus E^0$ with $E^0 = \mathbb{R}\alpha$.

Let J be a K-invariant complex structure on the bundle E which is compatible with the symplectic structure $d\alpha$. We equipped the bundle E^* with the complex structure J^* defined by $J^*(\xi) := \xi \circ J$ for any cotangent vector ξ. We note that the complex bundle (E^*, J^*) is the complex dual of the vector bundle (E, J).

We consider the $\mathbb{Z}_2$-graded complex vector bundle $\mathcal{E} := \wedge_{J^*} E^*$. The Clifford action defines a bundle map $\mathbf{c} : E^* \to \mathrm{End}_{\mathbb{C}}(\mathcal{E})$. We consider now the symbol on M

$$\sigma_b : p^*(\mathcal{E}^+) \to p^*(\mathcal{E}^-)$$

defined by $\sigma_b(x, \xi) = \mathbf{c}(\xi')$ where ξ' is the projection of $\xi \in \mathbf{T}^*M$ on E^*.

We see that the support of σ_b is equal to $E^0 \subset \mathbf{T}^*M$: σ_b is not an elliptic symbol.

Let K be a compact Lie group acting on M, which leaves α invariant. Then E, E^* are K-equivariant complex vector bundles, and the complex struture J can be chosen K-invariant. The morphism σ_b is then K-equivariant.

We suppose for the rest of this section that

$$E^0 \cap \mathbf{T}^*_K M = \text{ zero section of } \ \mathbf{T}^*M \tag{38}$$

It means that for any $x \in M$, the map $f_\alpha(x) : X \mapsto \alpha_x(V_x X)$ is not the zero map. Under this hypothesis the symbol σ_b is transversally elliptic.

Under the hypothesis (38), we can define the following closed equivariant form on M with $\mathcal{C}^{-\infty}$-coefficients

$$\mathcal{J}_\alpha(X) := \alpha \int_{\mathbb{R}} \mathrm{e}^{itD\alpha(X)}\, dt.$$

For any $\varphi \in \mathcal{C}^\infty_c(\mathfrak{k})$, the expression $\int_{\mathfrak{k}} \mathcal{J}_\alpha(X)\varphi(X)dX := \alpha \int_{\mathbb{R}} \mathrm{e}^{itd\alpha}\, \widehat{\varphi}(tf_\alpha)dt$ is a well defined differential form on M since the map $f_\alpha : M \to \mathfrak{k}^*$ as an empty 0-level set.

Let $\mathrm{Todd}(E)(X)$ be the equivariant Todd class of the complex vector bundle (E, J). We have the following

Theorem 5.7 ([11]) *For any $X \in \mathfrak{k}$ sufficiently small,*

$$\mathrm{index}^{K,M}_c(\sigma_b)(\mathrm{e}^X) = (2i\pi)^{-n} \int_M \mathrm{Todd}(E)(X)\, \mathcal{J}_\alpha(X).$$

Proof. Consider the equivariant form with compact support $\mathrm{Ch_{sup}}(\sigma_b)\mathrm{Par}(\omega)$. The Chern form $\mathrm{Ch_{sup}}(\sigma_b)$ attached to the complex vector bundle E^* is computed in [17] as follows. Let $\mathrm{Thom}(E^*)(X)$ be the equivariant Thom form, and let $\mathrm{Todd}(E^*)(X)$ be the equivariant Todd form. We have proved in [17], that

$$\mathrm{Ch_{sup}}(\sigma_b) = (2i\pi)^n\, \mathrm{Todd}(E^*)(X)^{-1}\mathrm{Thom}(E^*)(X). \tag{39}$$

Let $[\mathbb{R}]$ be the trivial vector bundle over M. We work trought the isomorphism $E^* \oplus [\mathbb{R}] \simeq \mathbf{T}^*M$ who sends (x, ξ, t) to $(x, \xi + t\alpha(x))$. We consider the invariant 1-form λ on $E^* \oplus [\mathbb{R}] \simeq \mathbf{T}^*M$ defined by

$$\lambda = -\underline{t}\; p^*(\alpha)$$

Here $p : E^* \oplus [\mathbb{R}] \to M$ is the projection, and $\underline{t}$ denotes the function that sends (x, ξ, t) to t.

It is easy to check that the form λ and the Liouville one form ω are equal on the support of σ_b. Thus

$$\mathrm{Ch}_{\sup}(\sigma_b)\mathrm{Par}(\omega) = \mathrm{Ch}_{\sup}(\sigma_b)\mathrm{Par}(\lambda), \quad \text{in} \quad \mathcal{H}_c^{-\infty}(\mathfrak{k}, \mathbf{T}^*M),$$

as consequence of ([17], Corollary 3.12). We have then

$$\mathrm{index}_c^{K,M}(\sigma_b)(\mathrm{e}^X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2(X)\, \mathrm{Ch}_{\sup}(\sigma_b)(X)\mathrm{Par}(\lambda)(X).$$

The integral of $\mathrm{Ch}_{\sup}(\sigma_b)(X)\mathrm{Par}(\lambda)(X)$ on the fibers of $\mathbf{T}^*M$ is then equal to the product

$$\left(\int_{E^*\, fiber} \mathrm{Ch}_{\sup}(\sigma_b)(X)\right)\left(\int_{\mathbb{R}} \mathrm{Par}(\lambda)(X)\right)$$

If we uses (39), we see that the integral $\int_{E^*\, fiber} \mathrm{Ch}_{\sup}(\sigma_b)(X)$ is equal to $(2i\pi)^n \mathrm{Todd}(E^*)(X)^{-1}$. A small computation gives that $\int_{\mathbb{R}} \mathrm{Par}(\lambda)(X)$ is equal to $(2i\pi)\mathcal{J}_\alpha(X)$. The proof is now completed since $\widehat{\mathrm{A}}(M)^2(X)\mathrm{Todd}(E^*)(X)^{-1} = \mathrm{Todd}(E)(X)$.

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