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PARTITION FUNCTION AND GENERALIZED DAHMEN-MICCHELLI SPACES

C. DE CONCINI, C. PROCESI, M. VERGNE

Abstract. This is the first of two papers on partition functions and the index theory of transversally elliptic operators. In this paper we only discuss algebraic and combinatorial issues related to partition functions. The applications to index theory will appear in [4].

Here we introduce a space of functions on a lattice which generalizes the space of quasi–polynomials satisfying the difference equations associated to cocircuits of a sequence of vectors X. This space $\mathcal{F}(X)$ contains the partition function $\mathcal{P}_X$. We prove a "localization formula" for any f in $\mathcal{F}(X)$. In particular, this implies that the partition function $\mathcal{P}_X$ is a quasi–polynomial on the sets $\mathfrak{c} - B(X)$ where $\mathfrak{c}$ is a big cell.

1. Introduction

Recall some notions. We take a lattice Γ in a vector space V and $X := [a_1, \dots, a_m]$ a list of non zero elements of Γ, spanning V as vector space. If X generates a pointed cone $C(X)$, the *partition function* $\mathcal{P}_X(\gamma)$ counts the number of ways in which a vector $\gamma \in \Gamma$ can be written as $\sum_{i=1}^m k_i a_i$ with $k_i \in \mathbb{N}$ non negative integers.

A quasi–polynomial is a function on Γ which coincides with a polynomial on each coset of some sublattice of finite index in Γ. A theorem [6],[10], generalizing the theory of the Ehrhart polynomials [5], shows that $\mathcal{P}_X(\gamma)$ is a quasi–polynomial on certain regions $\mathfrak{c} - B(X)$ where $B(X) := \{\sum_{i=1}^m t_i a_i,\ 0 \le t_i \le 1\}$ is the *zonotope* generated by X while $\mathfrak{c}$ denotes a *big cell*, that is a connected component of the complement in V of the *singular vectors* which are formed by the union of all cones $C(Y)$ for all the sublists Y of X which do not span V. The complement of $C(X)$ is a big cell. The other cells are inside $C(X)$ and are convex.

The quasi–polynomials describing the partition function belong to a remarkable finite dimensional space introduced and described by Dahmen–Micchelli [6] and which in this paper will be denoted by $DM(X)$. This is the space of solutions of a system of difference equations. In order to describe it, let us a call a subspace $\underline{s}$ of V *rational* if $\underline{s}$ is the span of a sublist of X. We need to recall that a *cocircuit* Y in X is a sublist of X such that $X \setminus Y$ does not span V and Y is minimal with this property. Thus Y is of the form $Y = X \setminus H$ where H is a rational hyperplane. Given

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$a \in \Gamma$, the *difference operator* ∇_a is the operator on functions defined by $\nabla_a(f)(b) := f(b) - f(b-a)$. For a list Y of vectors, we set $\nabla_Y := \prod_{a\in Y} \nabla_a$.

$$DM(X) := \{f \mid \nabla_Y f = 0, \text{ for every cocircuit } Y \text{ in } X\}.$$

It is easy to see that $DM(X)$ is finite dimensional and consists of quasi–polynomial functions (cf. [3]).

In this article, we introduce

$$\mathcal{F}(X) := \{f \mid \nabla_{X\setminus \underline{s}} f \text{ is supported on } \underline{s} \text{ for every proper rational subspace } \underline{s}\}.$$

Clearly $\mathcal{P}_X$ as well as $DM(X)$ are contained in $\mathcal{F}(X)$. The space $\mathcal{F}(X)$ is of interest even when X does not span a pointed cone and occurs in studying indices of transversally elliptic operators on a vector space.

The main result of this article is a "localization formula" for an element f in $\mathcal{F}(X)$. In particular, given a chamber $\mathfrak{c}$, our localization formula allows us to write explicitly the partition function $\mathcal{P}_X$ as a sum of a quasi-polynomial function $\mathcal{P}_X^{\mathfrak{c}} \in DM(X)$ and of other functions $f_{\underline{s}} \in \mathcal{F}(X)$ supported outside $\mathfrak{c} - B(X)$. This allows us to give a short proof of the quasi-polynomiality of $\mathcal{P}_X$ on the regions $\mathfrak{c} - B(X)$. Furthermore this decomposition implies Paradan's wall crossing formulae [8] for the quasi–polynomials $\mathcal{P}_X^{\mathfrak{c}}$. Our approach is strongly inspired by Paradan's localization formula in Hamiltonian geometry, but our methods here are elementary. We wish to thank Michel Duflo and Paul-Emile Paradan for several suggestions and corrections.

2. Special functions

2.1. **Basic notations.** Let Γ be a lattice and $E = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Consider the space $\mathcal{C}_E(\Gamma)$ of E valued functions on Γ. When $E = \mathbb{Z}$ we shall simply write $\mathcal{C}(\Gamma)$. We display such a function $f(\gamma)$ also as a formal series

$$\Theta(f) := \sum_{\gamma\in\Gamma} f(\gamma) e^{\gamma}.$$

Of course, under suitable convergence conditions, the series $\sum_{\gamma\in\Gamma} f(\gamma)e^{\gamma}$ is a function on the torus T whose character group is Γ, and it is the Laplace–Fourier transform of f. In fact the functions that we shall study are Fourier coefficients of some important generalized functions on T. This fact and the several implications for the index theory of transversally elliptic operators will be the subject of a subsequent paper [4].

The space $\mathcal{C}_E(\Gamma)$ is in an obvious way a module over the group algebra $E[\Gamma]$, multiplication by e^{λ} on the series $\Theta(f)$ corresponding to the translation operator τ_λ defined by

$$(\tau_\lambda f)(\gamma) := f(\gamma - \lambda)$$

on the function f. Thus $1 - e^{\lambda}$ corresponds to the difference operator ∇_λ.

We denote by δ_0 the function on Γ identically equal to 0 on Γ, except for $\delta_0(0) = 1$. Remark that the product of two formal series $\Theta(f_1)\Theta(f_2)$, whenever it is defined, corresponds to *convolution* $f_1 * f_2$ of the functions f_1 and f_2. The function δ_0 is the unit element.

Remark 2.2. Notice that, for a difference operator ∇_a acting on a convolution, we have:

$$\nabla_a(f_1 * f_2) = \nabla_a(f_1) * f_2 = f_1 * \nabla_a(f_2).$$

Let now $X := [a_1, \ldots, a_m]$ be a list of non zero elements of Γ and let $V := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be the real vector space generated by Γ. We assume that X generates the vector space V but we do not necessarily assume that X generates a pointed cone in V.

If X generates a pointed cone, then we can define

$$\Theta_X = \prod_{a \in X} \sum_{k=0}^{\infty} e^{ka}.$$

We write

$$\Theta_X = \sum_{\gamma \in \Gamma} \mathcal{P}_X(\gamma) e^{\gamma}$$

where $\mathcal{P}_X \in \mathcal{C}(\Gamma)$ is the partition function. " Morally", the series Θ_X is equal to $\prod_{a \in X} \frac{1}{1-e^a}$, but $\frac{1}{1-e^a}$ has to be understood as the geometric series expansion $\sum_{k=0}^{\infty} e^{ka}$.

Remark 2.3. We easily see that the partition function satisfies the difference equation $\nabla_X \mathcal{P}_X = \delta_0$. Clearly this equation has infinitely many solutions. The fact that $\mathcal{P}_X$ is uniquely determined by the recursion expressed by this equation comes from the further property of this solution of *having support in the cone* $C(X)$. We shall see other functions of the same type appearing in this paper.

Definition 2.4. i) A subspace of V generated by a subset of the elements of X will be called **rational (relative to X)**.

ii) Given a rational subspace $\underline{s}$, we denote by $\mathcal{C}(\Gamma, \underline{s})$ the set of elements in $\mathcal{C}(\Gamma)$ which have support in the lattice $\Gamma \cap \underline{s}$.

iii) Given a rational subspace $\underline{s}$, we set $M_{X \setminus \underline{s}} = \prod_{a \in X \setminus \underline{s}} (1 - e^a) \in \mathbb{Z}[\Gamma]$ and $\nabla_{X \setminus \underline{s}} := \prod_{a \in X \setminus \underline{s}} \nabla_a$ the corresponding operator.

With these notations, the space $DM_E(X)$ defined by Dahmen–Micchelli is formed by the set of functions $f \in \mathcal{C}_E(\Gamma)$ satisfying the system of difference equations $\nabla_{X \setminus \underline{s}} f = 0$ as $\underline{s}$ varies among all proper rational subspaces relative to X. It is easy to see that $DM_E(X)$ consists of quasi–polynomials.

It follows from their theory (see also [3]), that for each E, the space $DM_E(X)$ is a free $E-$module of dimension $\delta(X)$, the volume of the zonotope $B(X)$. In particular $DM_E(X) = E \otimes_{\mathbb{Z}} DM_{\mathbb{Z}}(X)$ for all E. Therefore from now on we shall work directly over $\mathbb{Z}$ and drop the subscript E.

Let Λ be the smallest sub–lattice of Γ for which each function of $DM(X)$ is a polynomial on its cosets. Λ is the intersection of all the sublattices of Γ generated by all the bases of V that one can extract from X (the least common multiple).

Given a rational subspace $\underline{s}$, we will identify the space $\mathcal{C}(\Gamma \cap \underline{s})$ with the subspace $\mathcal{C}(\Gamma, \underline{s})$ of $\mathcal{C}(\Gamma)$ by extending the functions with 0 outside $\underline{s}$.

2.5. **The special functions $\mathcal{P}_X^u$.** Let $U := V^*$ be the dual of V. Given a (finite) list Y of non zero vectors in V, we shall say that a vector $u \in U$ is $Y-$*regular* if it does not vanish on any of the vectors in Y, in other words if it is in one of the (open) faces of the hyperplane arrangement in U determined by Y.

The vectors X define a hyperplane arrangement in U and this decomposes U into *faces*. Given a rational subspace $\underline{s}$ and a vector $u \in U$ vanishing on $\underline{s}$ and regular for $X \setminus \underline{s}$, we divide the set $X \setminus \underline{s}$ into two parts A, B, where u is respectively positive and negative. These two sets A, B depend only upon the face F in which u lies. Let $V(u \geq 0)$ be the closed half space of vectors v where u is non negative. The half space $V(u \geq 0)$ contains $C(A, -B)$ the cone generated by the list $[A, -B]$. We denote this cone by $C(F, X)$. Furthermore the space $\underline{s}$ is determined by u and X, as $\underline{s}$ is the subspace generated by the elements $a \in X$ such that $\langle u, a\rangle = 0$.

We are going to consider the series Θ_X^u which is characterized by the following two properties:

Lemma 2.6. *There exists a unique element*

$$\Theta_X^u = \sum_\gamma \mathcal{P}_X^u(\gamma) e^\gamma$$

such that

i) $M_{X\setminus\underline{s}}\Theta_X^u = 1$, *equivalently* $\nabla_{X\setminus\underline{s}}\mathcal{P}_X^u = \delta_0$.

ii) $\mathcal{P}_X^u$ *is supported in* $-\sum_{b\in B} b + C(F, X)$.

Proof. Set:

$$\Theta_X^u = (-1)^{|B|} e^{-\sum_{b\in B} b} \prod_{a\in A} \Big(\sum_{k=0}^\infty e^{ka}\Big) \prod_{b\in B} \Big(\sum_{k=0}^\infty e^{-kb}\Big). \tag{1}$$

It is easily seen that this element satisfies the two properties and is unique. □

If $\underline{s} = V$, $u = 0$ and $\mathcal{P}_X^0 = \delta_0$.

Morally, $\Theta_X^u = \prod_{a\in X\setminus\underline{s}} \frac{1}{1-e^a} = \prod_{a\in A} \frac{1}{1-e^a} \prod_{b\in B} \frac{-e^{-b}}{1-e^{-b}}$. We indeed need to reverse the sign of some of the vectors in $X \setminus \underline{s}$ in order that the convolution product of the corresponding geometric series makes sense.

It is clear that Θ_X^u depends only of the face F of $\underline{s}^\perp$ where the regular element u lies.

Although a function $f \in \mathcal{C}(\Gamma, \underline{s})$ may have infinite support, we easily see that the convolution $\mathcal{P}_X^u * f$ is well defined. In fact we claim that, given any $\gamma \in \Gamma$, we can write $\gamma = \lambda + \mu$ with $\mu \in \underline{s} \cap \Gamma$, and $\lambda \in (-\sum_{b\in B} b + C(A, -B)) \cap \Gamma$ only in finitely many ways. This we see since $\langle u \,|\, \gamma\rangle = \langle u \,|\, \lambda\rangle$

and $\lambda = \sum_{a\in A} k_a a + \sum_{b\in B} h_b(-b)$ with $k_a \geq 0,\ h_b \geq 1$. Thus the equality $\langle u \,|\, \gamma\rangle = \sum_{a\in A} k_a \langle u \,|\, a\rangle + \sum_{b\in B} h_b\langle u \,|\, -b\rangle$ yields that the vector λ is in a bounded set, intersecting the lattice Γ in a finite set.

Choose two rational spaces $\underline{s}, \underline{t}$ and a vector $u \in U$ vanishing on $\underline{s}$ and regular for $X \setminus \underline{s}$. The restriction of u to $\underline{t}$ vanishes on $\underline{t} \cap \underline{s}$ and is regular for $(X \cap \underline{t}) \setminus \underline{s}$. To simplify notations, we still denote by u the restriction of u to $\underline{t}$. We have:

Proposition 2.7. *i)* $\nabla_{(X\setminus\underline{t})\setminus\underline{s}} \mathcal{P}^u_X = \mathcal{P}^u_{X\cap\underline{t}}$.

ii) For $g \in \mathcal{C}(\Gamma \cap \underline{s})$*:*

$$\nabla_{X\setminus\underline{t}}(\mathcal{P}^u_X * g) = \mathcal{P}^u_{X\cap\underline{t}} * (\nabla_{(X\cap\underline{s})\setminus(\underline{t}\cap\underline{s})} g). \tag{2}$$

Proof. i) From Equation (1), we see that the series associated to the function $\nabla_{(X\setminus\underline{t})\setminus\underline{s}} \mathcal{P}^u_X$ equals

$$\Theta^u_{X\cap\underline{t}} = (-1)^{|B\cap\underline{t}|} e^{-\sum_{b\in B\cap\underline{t}} b} \prod_{a\in A\cap\underline{t}} (\sum_{k=0}^{\infty} e^{ka}) \prod_{b\in B\cap\underline{t}} (\sum_{k=0}^{\infty} e^{-kb}).$$

ii) Let $g \in \mathcal{C}(\Gamma \cap \underline{s})$. Take any rational subspace $\underline{t}$, we have that $\nabla_{X\setminus\underline{t}} = \nabla_{(X\cap\underline{s})\setminus(\underline{t}\cap\underline{s})} \nabla_{(X\setminus\underline{t})\setminus\underline{s}}$, thus

$$\nabla_{X\setminus\underline{t}}(\mathcal{P}^u_X * g) = (\nabla_{(X\setminus\underline{t})\setminus\underline{s}} \mathcal{P}^u_X) * (\nabla_{(X\cap\underline{s})\setminus(\underline{t}\cap\underline{s})} g).$$

As $\nabla_{(X\setminus\underline{t})\setminus\underline{s}} \mathcal{P}^u_X = \mathcal{P}^u_{X\cap\underline{t}}$ from part *i)*, we obtain Formula (2), which is the mother of all other formulae of this article.

□

In particular, for $\underline{s} = \underline{t}$, the restriction of u to $\underline{t}$ is 0 and Formula (2) implies the following.

Proposition 2.8. *If* $f \in \mathcal{C}(\Gamma \cap \underline{s})$*, we have* $f = \nabla_{X\setminus\underline{s}}(\mathcal{P}^u_X * f)$*.*

3. A remarkable space

3.1. **The space $\mathcal{F}(X)$.** We let $\mathcal{S}_X$ denote the set of all rational subspaces relative to X.

Definition 3.2. We define the space of interest for this article by:

$$\mathcal{F}(X) := \{f \in \mathcal{C}(\Gamma) \,|\, \nabla_{X\setminus\underline{s}} f \in \mathcal{C}(\Gamma, \underline{s}),\ \text{for all } \underline{s} \in \mathcal{S}_X\}. \tag{3}$$

One of the equations (corresponding to $\underline{s} = \{0\}$) that must satisfy $\Theta(f)$ when $f \in \mathcal{F}(X)$ is the relation $\prod_{a\in X}(1-e^a)\Theta(f) = c$, where c is a constant. This equation was the motivation for introducing the space $\mathcal{F}(X)$. Indeed the first important fact on this space is the following:

Lemma 3.3. *i) If u is regular for X then $\mathcal{P}^u_X$ lies in $\mathcal{F}(X)$.*

ii) The space $DM(X)$ is contained in $\mathcal{F}(X)$.

Proof. i) Indeed, $\nabla_{X\setminus\underline{s}} \mathcal{P}^u_X = \mathcal{P}^u_{X\cap\underline{s}} \in \mathcal{C}(\Gamma, s)$.

ii) Is clear from the definitions. □

In particular if X generates a pointed cone, then the partition function $\mathcal{P}_X$ lies in $\mathcal{F}(X)$.

Example 3.4. Let us give a simple example. Let $\Gamma = \mathbb{Z}\omega$, and $X = [2\omega, -\omega]$. Then it is easy to see that $\mathcal{F}(X)$ is a free $\mathbb{Z}$ module of dimension 4, with corresponding basis

$$\theta_1 = \sum_{n\in\mathbb{Z}} e^{n\omega}, \qquad \theta_2 = \sum_{n\in\mathbb{Z}} n e^{n\omega},$$

$$\theta_3 = \sum_{n\in\mathbb{Z}} (\frac{n}{2} + \frac{1-(-1)^n}{4}) e^{n\omega}, \qquad \theta_4 = \sum_{n\geq 0} (\frac{n}{2} + \frac{1-(-1)^n}{4}) e^{n\omega}.$$

Here $\theta_1, \theta_2, \theta_3$ are a $\mathbb{Z}$ basis of $DM(X)$.

In fact, there is a much more precise statement of which Lemma 3.3 is a very special case and which will be the object of Theorem 3.8.

3.5. **Some properties of $\mathcal{F}(X)$.** Let $\underline{s}$ be a rational subspace and $u \in \underline{s}^\perp$ be a $X \setminus \underline{s}$ regular element.

Proposition 3.6. *i) $\nabla_{X\setminus\underline{s}}$ maps $\mathcal{F}(X)$ to $\mathcal{F}(X\cap\underline{s})$.*

ii) The map $g \mapsto \mathcal{P}_X^u * g$ gives an injection from $\mathcal{F}(X\cap\underline{s})$ to $\mathcal{F}(X)$.

iii) If $g \in DM(X\cap\underline{s})$, then $\nabla_{X\setminus\underline{t}}(\mathcal{P}_X^u * g) = 0$, for any rational subspace $\underline{t}$ such that $\underline{t}\cap\underline{s} \neq \underline{s}$.

Proof. i) If $f \in \mathcal{F}(X)$, we have $\nabla_{X\setminus\underline{s}} f \in \mathcal{F}(X\cap\underline{s})$. In fact take a rational subspace $\underline{t}$ of $\underline{s}$, we have that $\nabla_{(X\cap\underline{s})\setminus\underline{t}}\nabla_{X\setminus\underline{s}} f = \nabla_{X\setminus\underline{t}} f \in \mathcal{C}(\Gamma\cap\underline{t})$.

ii) If $g \in \mathcal{F}(X\cap\underline{s})$, then $\nabla_{(X\cap\underline{s})\setminus(\underline{t}\cap\underline{s})} g \in \mathcal{C}(\Gamma\cap\underline{t}\cap\underline{s})$, hence Formula (2) in Proposition 2.7 shows that $\nabla_{X\setminus\underline{t}}(\mathcal{P}_X^u * g) \in \mathcal{C}(\Gamma,\underline{t})$, so that $\mathcal{P}_X^u * g \in \mathcal{F}(X)$ as desired.

iii) Similarly, if $g \in \mathcal{D}M(X\cap\underline{s})$, Formula (2) in Proposition 2.7 implies the third assertion of our proposition.

□

Proposition 3.6 allows us to associate to a rational space $\underline{s}$ and a regular vector u vanishing on $\underline{s}$, the operator

$$\Pi_X^{\underline{s},u} : f \mapsto \mathcal{P}_X^u * (\nabla_{X\setminus\underline{s}} f)$$

on $\mathcal{F}(X)$. The operator $\Pi_X^{\underline{s},u}$ is a projector.

3.7. **The main theorem.** Choose for every rational space $\underline{s}$, a vector $u_{\underline{s}} \in U$ vanishing on $\underline{s}$ and regular for $X \setminus \underline{s}$. The following theorem is the main theorem of this section.

Theorem 3.8. *Then:*

$$\mathcal{F}(X) = \oplus_{\underline{s}\in\mathcal{S}_X} \mathcal{P}_X^{u_{\underline{s}}} * DM(X\cap\underline{s}). \tag{4}$$

Proof. Denote by $\mathcal{S}_X^i$, the subset of subspaces $\underline{s} \in \mathcal{S}_X$ of dimension i. Consider $\nabla_{X\setminus\underline{s}}$ as an operator on $\mathcal{F}(X)$ with values in $\mathcal{C}(\Gamma)$. Define the spaces

$$K_i := \cap_{\underline{t}\in\mathcal{S}_X^{i-1}}\mathrm{Ker}\nabla_{\mathrm{X}\setminus\underline{\mathrm{t}}}.$$

Notice that by definition $K_{\{0\}} = \mathcal{F}(X)$, that $K_{\dim V}$ is the space $DM(X)$ and that $K_{i+1} \subseteq K_i$.

Lemma 3.9. *Let* $\underline{s} \in \mathcal{S}_X^i$.

i) The image of $\nabla_{X\setminus\underline{s}}$ *restricted to* K_i *is contained in the space* $DM(X\cap\underline{s})$.

ii) If f *is in* $DM(X\cap\underline{s})$, *then* $\mathcal{P}_X^u * f \in K_i$.

Proof. *i)* First we know, by the definition of $\mathcal{F}(X)$, that $\nabla_{X\setminus\underline{s}}K_i$ is contained in the space $\mathcal{C}(\Gamma,\underline{s})$. Let $\underline{t}$ be a rational hyperplane of $\underline{s}$, so that $\underline{t}$ is of dimension $i-1$. By construction, we have that for every $f \in K_i$

$$0 = \prod_{a\in X\setminus\underline{t}} \nabla_a f = \prod_{a\in(X\cap\underline{s})\setminus\underline{t}} \nabla_a\nabla_{X\setminus\underline{s}}f.$$

This means that $\nabla_{X\setminus\underline{s}}f$ satisfies the difference equations given by the cocircuits of $X\cap\underline{s}$.

ii) Follows from the third item of Proposition 3.6.

∎

Consider the map $\mathbf{\Delta}_i : K_i \to \oplus_{\underline{s}\in\mathcal{S}_X^i} DM(X\cap\underline{s})$ given by

$$\mathbf{\Delta}_i f = \oplus_{\underline{s}\in\mathcal{S}_X^i}\nabla_{X\setminus\underline{s}}f$$

and the map $\mathbf{P}_i : \oplus_{\underline{s}\in\mathcal{S}_X^i} DM(X\cap\underline{s}) \to K_i$ given by

$$\mathbf{P}_i(\oplus g_{\underline{s}}) = \sum \mathcal{P}_X^{u_{\underline{s}}} * g_{\underline{s}}.$$

Lemma 3.10. *The sequence*

$$0 \longrightarrow K_{i+1} \longrightarrow K_i \xrightarrow{\mathbf{\Delta}_i} \oplus_{\underline{s}\in\mathcal{S}_X^i} DM(X\cap\underline{s}) \longrightarrow 0$$

is exact. Furthermore, the map $\mathbf{P_i}$ *provides a splitting of this exact sequence:* $\mathbf{\Delta}_i\mathbf{P_i} = \mathrm{Id}$.

Proof. By definition, K_{i+1} is the kernel of $\mathbf{\Delta}_i$, thus we only need to show that $\mathbf{\Delta}_i\mathbf{P_i} = \mathrm{Id}$. If $\underline{s},\underline{t}$ are two distinct subspaces of $\mathcal{S}_X^i$, $\underline{s}\cap\underline{t}$ is a proper subspace of $\underline{t}$. Item *iii)* of Proposition 3.6 says that for $g \in DM(X\cap\underline{s})$, $\nabla_{X\setminus\underline{t}}(\mathcal{P}_X^{u_{\underline{s}}} * g) = 0$. Thus given a family $g_{\underline{s}} \in DM(X\cap\underline{s})$, the function $f = \sum_{\underline{t}\in\mathcal{S}_X^i}\mathcal{P}_X^{u_{\underline{t}}} * g_{\underline{t}}$ is such that $\nabla_{X\setminus\underline{s}}f = g_{\underline{s}}$. This proves our claim that $\mathbf{\Delta}_i\mathbf{P_i} = \mathrm{Id}$. ∎

Putting together these facts, Theorem 3.8 follows.

∎

A collection $\mathbf{u} = (u_{\underline{s}})$ of elements $u_{\underline{s}} \in U$ vanishing on $\underline{s}$ and regular for $X \setminus \underline{s}$ will be called a *regular collection* (for X). Given a regular collection $\mathbf{u}$, we can write an element $f \in \mathcal{F}(X)$ as

$$f = \sum_{\underline{s} \in \mathcal{S}_X} f_{\underline{s}}$$

with $f_{\underline{s}} \in \mathcal{P}_X^{u_{\underline{s}}} * DM(X \cap \underline{s})$. This expression for f will be called the $\mathbf{u}$ decomposition of f. In this decomposition, the component f_V is in $DM(X)$.

The space $\mathcal{P}_X^{u_{\underline{s}}} * DM(X \cap \underline{s})$ will be referred to as *the $u_{\underline{s}}$-component* of $\mathcal{F}(X)$.

From Lemma 3.10 it follows that the operator $\mathrm{Id} - \mathbf{P_i}\boldsymbol{\Delta}_i$ projects K_i to K_{i+1} with kernel $\oplus_{\underline{s} \in \mathcal{S}_X^i} \mathcal{P}_X^{u_{\underline{s}}} * DM(X \cap \underline{s})$ (this operator depends of $\mathbf{u}$). Thus the ordered product

$$\boldsymbol{\Pi_i}^{\mathbf{u}} := (\mathrm{Id} - \mathbf{P_{i-1}}\boldsymbol{\Delta}_{i-1})(\mathrm{Id} - \mathbf{P_{i-2}}\boldsymbol{\Delta}_{i-2}) \dots (\mathrm{Id} - \mathbf{P_0}\boldsymbol{\Delta}_0)$$

projects $\mathcal{F}(X)$ to K_i; therefore, we have

Proposition 3.11. *Let $\mathbf{u}$ be a regular family and $\underline{s}$ a rational subspace of dimension i. The operator*

$$P_{\underline{s}}^{\mathbf{u}} = \Pi_X^{\underline{s}, u_{\underline{s}}}(\mathrm{Id} - \mathbf{P_{i-1}}\boldsymbol{\Delta}_{i-1})(\mathrm{Id} - \mathbf{P_{i-2}}\boldsymbol{\Delta}_{i-2}) \dots (\mathrm{Id} - \mathbf{P_0}\boldsymbol{\Delta}_0) = \Pi_X^{\underline{s}, u_{\underline{s}}} \boldsymbol{\Pi_i}^{\mathbf{u}}$$

is the projector of $\mathcal{F}(X)$ to the $u_{\underline{s}}$-component $\mathcal{P}_X^{u_{\underline{s}}} * DM(X \cap \underline{s})$ of $\mathcal{F}(X)$.

In particular the operator

$$P_V := (\mathrm{Id} - \mathbf{P}_{\dim(\mathbf{V})-\mathbf{1}}\boldsymbol{\Delta}_{\dim(V)-1})(\mathrm{Id} - \mathbf{P}_{\dim(\mathbf{V})-\mathbf{2}}\boldsymbol{\Delta}_{\dim(V)-2}) \dots (\mathrm{Id} - \mathbf{P_0}\boldsymbol{\Delta}_0)$$

is the projector $\mathcal{F}(X) \to DM(X)$ associated to the direct sum decomposition:

$$\mathcal{F}(X) = DM(X) \oplus \Big(\oplus_{\underline{s} \in \mathcal{S}_X | \underline{s} \neq V} \mathcal{P}_X^{u_{\underline{s}}} * DM(X \cap \underline{s}) \Big).$$

Let $\mathbf{u} = (u_{\underline{t}})$ be a X regular collection. If $\underline{s}$ is a rational subspace, the collection $(u_{\underline{t}}|_{\underline{s}})$, with $\underline{t} \in \mathcal{S}_{X \cap \underline{s}}$ is a $X \cap \underline{s}$ regular collection. We still denote it by $\mathbf{u}$ in the next proposition. The proof of this proposition is skipped, as it is very similar to preceding proofs.

Proposition 3.12. *Let $\underline{s}$ be a rational subspace. Let $f \in \mathcal{F}(X)$ and $f = \sum_{\underline{t} \in \mathcal{S}(X)} f_{\underline{t}}$ be the $\mathbf{u}$ decomposition of f and $\nabla_{X \setminus \underline{s}} f = \sum_{\underline{t} \in \mathcal{S}_{X \cap \underline{s}}} g_{\underline{t}}$ be the $\mathbf{u}$ decomposition of $\nabla_{X \setminus \underline{s}} f$, then*

- $\nabla_{X \setminus s} f_{\underline{t}} = 0$ *if* $\underline{t} \notin \mathcal{S}_{X \cap \underline{s}}$,
- $\nabla_{X \setminus \underline{s}} f_{\underline{t}} = g_{\underline{t}}$ *if* $\underline{t} \in \mathcal{S}_{X \cap \underline{s}}$.

Remark 3.13. It follows from the previous theorems and the properties of $DM(X)$ that, for every E, we could define a space $\mathcal{F}_E(X)$ of E valued functions as in Definition 3.2 and we have $\mathcal{F}_E(X) = E \otimes_{\mathbb{Z}} \mathcal{F}(X)$.

3.14. **Localization theorem.** Let $B(X) := \{\sum_{i=1}^{m} t_i a_i,\ 0 \le t_i \le 1\}$ be the *zonotope* generated by X, and let τ denote a *tope*, that is a connected component of the complement in V of the union of the hyperplanes generated by subsets of X. We will show that for any element $f \in \mathcal{F}(X)$, the function $f(\gamma)$ coincides with a quasi–polynomial on the sets $(\tau - B(X)) \cap \Gamma$ (we simply say f is a quasi–polynomial on $\tau - B(X)$).

In order to do this remark that, given $x_0 \in \tau$ and a proper rational space $\underline{s}$, it is possible to choose a regular element $u_{\underline{s}}$ such that $u_{\underline{s}}(x_0) < 0$, as the projection of x_0 on $V/\underline{s}$ is not zero.

Let $f \in \mathcal{F}(X)$ and let τ be a tope. Let $x_0 \in \tau$ and let $\mathbf{u}$ be a regular collection such that $u_{\underline{s}}(x_0) < 0$ for every $\underline{s} \neq V$. Our previous claim then follows from the explicit construction below.

Proposition 3.15 (Localization theorem). *Let $f = \sum f_{\underline{s}}$ be the $\mathbf{u}$-decomposition of f. Then the component f_V of this decomposition is a quasi–polynomial function in $DM(X)$ such that $f = f_V$ on $\tau - B(X)$.*

Remark 3.16. A choice of $\mathbf{u} = (u_{\underline{s}})$ negative on x_0 has the effect of pushing the supports of the elements $f_{\underline{s}}$ ($\underline{s} \neq V$) away from the neighborhood τ of x_0. See Figures 1, 2 which describe the $\mathbf{u}$ decomposition of the partition function $\mathcal{P}_X$ for $X := [a, b, c]$ with $a := \omega_1, b := \omega_2, c := \omega_1 + \omega_2$ in the lattice $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Thus the content of Proposition 3.15 is very similar to Paradan's localization theorem [9].

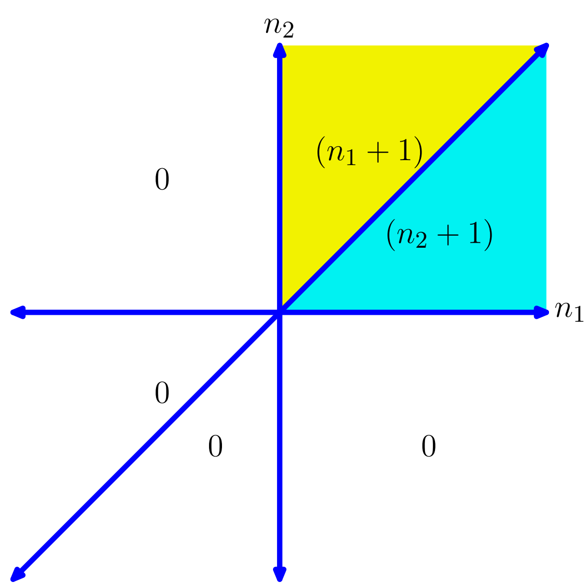

FIGURE 1. The partition function of $X := [a, b, c]$

Proof. Write $f = \sum_{\underline{s} \in \mathcal{S}_X} f_{\underline{s}}$ with $f_{\underline{s}} = \mathcal{P}_X^{u_{\underline{s}}} * k_{\underline{s}}$ where $k_{\underline{s}} \in DM(X \cap \underline{s})$. Let $\underline{s}$ be a proper rational space. Denote by $F_{\underline{s}}$ the face of $\underline{s}^{\perp}$ where $u_{\underline{s}}$

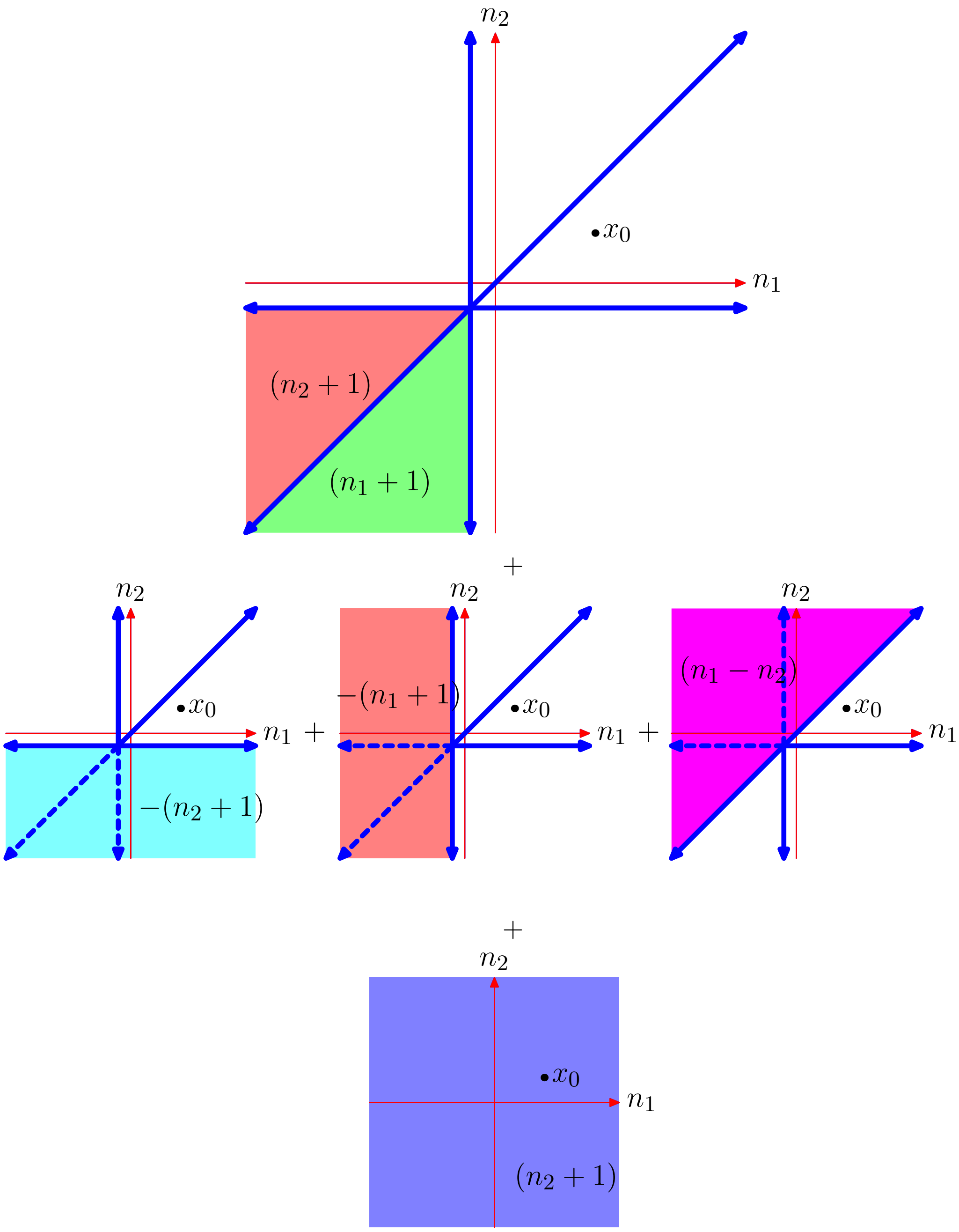

FIGURE 2. **u** decomposition of the partition function of $X := [a, b, c]$ for **u** negative on x_0

belongs. In the notation of Lemma 2.6, the support of $f_{\underline{s}}$ is contained in the polyhedron $\underline{s} - \sum_{b \in B} b + C(F_{\underline{s}}, X) \subset \underline{s} + C(F_{\underline{s}}, X)$. This last polyhedron is convex and, by construction, it has a boundary limited by hyperplanes which are rational with respect to X. Thus either $\tau \subset \underline{s} + C(F_{\underline{s}}, X)$ or $\tau \cap (\underline{s} + C(F_{\underline{s}}, X)) = \emptyset$. As $u_{\underline{s}} \geq 0$ on $\underline{s} + C(F_{\underline{s}}, X)$ and $u_{\underline{s}}(x_0) < 0$, it follows that τ is not a subset of $\underline{s} + C(F_{\underline{s}}, X)$, so that $\tau \cap (\underline{s} + C(F_{\underline{s}}, X)) = \emptyset$.

In fact we claim that $\tau - B(X)$ does not intersect the support $\underline{s} - \sum_{b\in B} b + C(F_{\underline{s}}, X)$ of $f_{\underline{s}}$. Indeed, otherwise we would have an equation $v - \sum_{x\in X} t_x x = s + \sum_{a\in A} k_a a + \sum_{b\in B} h_b(-b)$ with $v \in \tau$, $0 \leq t_x \leq 1$, $k_a \geq 0$, $h_b \geq 1$. This would imply that $v \in \underline{s} + C(A, -B)$ a contradiction. Thus f coincides with the quasi–polynomial f_V on $\tau - B(X)$. $\square$

One should remark that a quasi–polynomial is completely determined by the values that it takes on $\tau - B(X)$ thus f_V is independent on the construction.

Definition 3.17. We shall denote by f^τ the quasi–polynomial coinciding with f on $\tau - B(X)$.

Let us remark that the open subsets $\tau - B(X)$ cover V, when τ runs over the topes of V (with possible overlapping). Thus the element $f \in \mathcal{F}(X)$ is entirely determined by the quasi–polynomials f^τ.

Example 3.18. In Figure 3, for each tope τ, the set of integral points in $\tau - B(X)$ is contained in one of the affine **closed** cones limited by thick lines. We are showing in color the convex envelop of the integral points in $\tau - B(X)$ and *not* the larger open set $\tau - B(X)$.

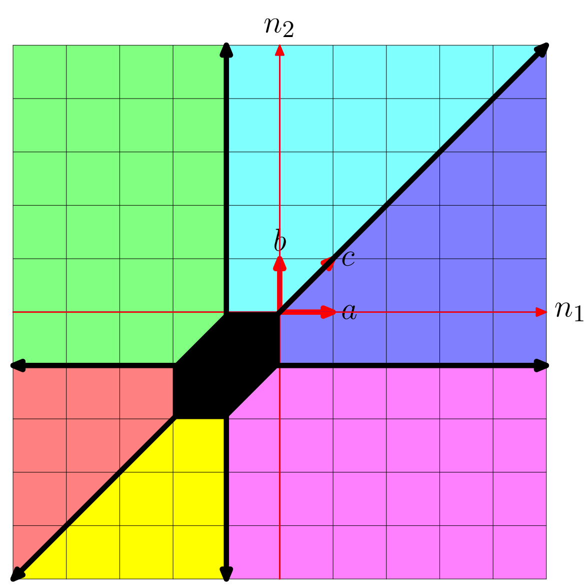

FIGURE 3. Translated topes of X. The zonotope $-B(X)$ is in black.

3.19. **Wall crossing formula.** We first develop a general formula describing how the functions f^τ change when crossing a wall. Then we apply this to the partition function $\mathcal{P}_X$ and deduce that it is a quasi–polynomial on $\mathfrak{c} - B(X)$, where $\mathfrak{c}$ is a big cell.

Let H be a rational hyperplane, and let $u \in U$ be an equation of the hyperplane.

Lemma 3.20. *If $q \in DM(X \cap H)$, then $w := \mathcal{P}_X^u * q - \mathcal{P}_X^{-u} * q$ is an element of $DM(X)$.*

Remark 3.21. In [1], a one-dimensional residue formula is given for w allowing us to compute it.

Proof. If $\underline{t} \in \mathcal{S}_X$ is different from H, $\nabla_{X \setminus \underline{t}}(\mathcal{P}_X^u * q) = \nabla_{X \setminus \underline{t}}(\mathcal{P}_X^{-u} * q) = 0$, as follows from Proposition 3.6 *iii)*. If $\underline{s} = H$, then $\nabla_{X \setminus H}(\mathcal{P}_X^u * q - \mathcal{P}_X^{-u} * q) = q - q = 0$. □

Assume that τ_1, τ_2 are two adjacent topes namely $\overline{\tau}_1 \cap \overline{\tau}_2$ spans a hyperplane H. The hyperplane H is a rational subspace. Let τ_{12} be the unique tope for $X \cap H$ such that $\overline{\tau}_1 \cap \overline{\tau}_2 \subset \overline{\tau_{12}}$ (see Figure 4).

Example 3.22. Let C be the cone generated by the vectors $a := \omega_3 + \omega_1$, $b := \omega_3 + \omega_2$, $c := \omega_3 - \omega_1$, $d := \omega_3 - \omega_2$ in a 3-dimensional space $V := \mathbb{R}\omega_1 \oplus \mathbb{R}\omega_2 \oplus \mathbb{R}\omega_3$. Figure 4 represents the section of C cut by the affine hyperplane containing a, b, c, d. We consider $X := [a, b, c, d]$.

We show in section, on the left of the picture, two topes τ_1, τ_2 adjacent along the hyperplane H generated by b, d and, on the right, the tope τ_{12}. The list $X \cap H$ is $[b, d]$. The closure of the tope τ_{12} is "twice bigger " than $\overline{\tau}_1 \cap \overline{\tau}_2$.

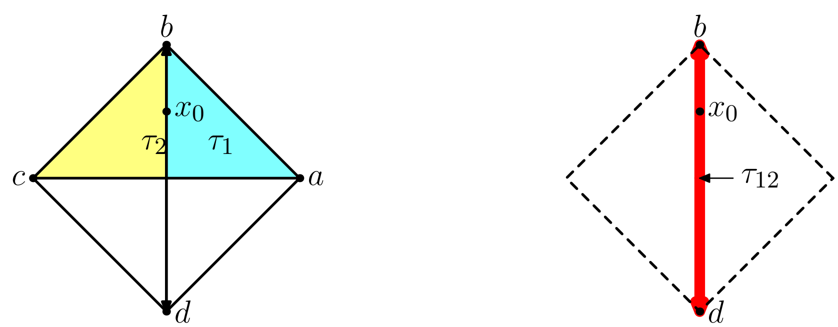

FIGURE 4. Two adjacent topes of $X := [a, b, c, d]$

Let $f \in \mathcal{F}(X)$. The function $\nabla_{X \setminus H} f$ is an element of $\mathcal{F}(H \cap X)$, thus by Proposition 3.15, there exists a quasi–polynomial $(\nabla_{X \setminus H} f)^{\tau_{12}}$ on H such that $\nabla_{X \setminus H} f$ agrees with $(\nabla_{X \setminus H} f)^{\tau_{12}}$ on τ_{12}.

Theorem 3.23. *Let $\tau_1, \tau_2, H, \tau_{12}$ be as before and $f \in \mathcal{F}(X)$. Take $u \in U$ vanishing on H such that $\langle u, \tau_1 \rangle > 0$ and so $\langle u, \tau_2 \rangle < 0$. Then*

$$f^{\tau_1} - f^{\tau_2} = \mathcal{P}_X^u * (\nabla_{X \setminus H} f)^{\tau_{12}} - \mathcal{P}_X^{-u} * (\nabla_{X \setminus H} f)^{\tau_{12}}. \tag{5}$$

Proof. Let x_0 be a point in the relative interior of $\overline{\tau}_1 \cap \overline{\tau}_2$ in H. Then x_0 does not belong to any X-rational hyperplane different from H (see Figure 4). Therefore we can choose a regular element $u_{\underline{s}}$ for every $\underline{s}$ different from H, V such that $u_{\underline{s}}$ is negative on x_0.

Consider the regular collection $\mathbf{u^1} = (u^1_{\underline{s}})$ where $u^1_{\underline{s}} = u_{\underline{s}}$ for $\underline{s} \neq H$ and $u^1_H = -u$ and the regular collection $\mathbf{u^2} = (u^2_{\underline{s}})$ where $u^2_{\underline{s}} = u_{\underline{s}}$ for $\underline{s} \neq H$ and $u^2_H = u$.

For $i = 1, 2$ let $f = f^i_V + f^i_H + \sum_{\underline{s} \neq H, V} f^i_{\underline{s}}$ be the $\mathbf{u^i}$ decomposition of f.

We write $f^1_H = \mathcal{P}_X^{-u} * q^{(1)}$ with $q^{(1)} \in DM(X \cap H)$. The family $\mathbf{u^1}$ takes a negative value at any point x_1 of τ_1 sufficiently close to x_0, thus by Proposition 3.15, the component f^1_V is equal to f^{τ^1}. By Proposition 3.12,

$$\nabla_{X \setminus H} f = q^{(1)} + \sum_{\underline{s} \subset H, \underline{s} \neq H} \nabla_{X \setminus H} f^1_{\underline{s}}$$

is the $\mathbf{u^1}$ decomposition of $\nabla_{X \setminus H} f$ so that $q^{(1)} = (\nabla_{X \setminus H} f)^{\tau_{12}}$ by Proposition 3.15. We thus have $f^1_V = f^{\tau_1}$ and $f^1_H = \mathcal{P}_X^{-u} * (\nabla_{X \setminus H} f)^{\tau_{12}}$.

Similarly, $f^2_V = f^{\tau_2}$ and $f^2_H = \mathcal{P}_X^{u} * (\nabla_{X \setminus H} f)^{\tau_{12}}$.

Now from Proposition 3.11, when $\dim(\underline{s}) = i$,

$$f^1_{\underline{s}} = \Pi_X^{\underline{s}, u^1_{\underline{s}}} \mathbf{\Pi_i^{u^1}} f, \qquad f^2_{\underline{s}} = \Pi_X^{\underline{s}, u^2_{\underline{s}}} \mathbf{\Pi_i^{u^2}} f,$$

and, for any $i < \dim V$, the operators $\mathbf{\Pi_i^{u^1}}$ and $\mathbf{\Pi_i^{u^2}}$ are equal. Thus $f^1_{\underline{s}} = f^2_{\underline{s}}$ for $\underline{s} \neq V, H$. So we obtain $f^1_V + f^1_H = f^2_V + f^2_H$, and our formula. □

Consider now the case where X spans a pointed cone. Let us interpret Formula (5) in the case in which $f = \mathcal{P}_X$. We know that for a given tope τ, $\mathcal{P}_X$ agrees with a quasi–polynomial $\mathcal{P}_X^\tau$ on $\tau - B(X)$. Recall that $\nabla_{X \setminus H}(\mathcal{P}_X) = \mathcal{P}_{X \cap H}$ as we have seen in Lemma 3.3. It follows that given two adjacent topes τ_1, τ_2 as above, $(\nabla_{X \setminus H} f)^{\tau_{12}}$ equals $(\mathcal{P}_{X \cap H})^{\tau_{12}}$ (extended by zero outside H). So we deduce the identity

$$\mathcal{P}_X^{\tau_1} - \mathcal{P}_X^{\tau_2} = (\mathcal{P}_X^u - \mathcal{P}_X^{-u}) * \mathcal{P}_{X \cap H}^{\tau_{12}}. \tag{6}$$

This is Paradan's formula ([8], Theorem 5.2).

Example 3.24. Assume $X = [a, b, c]$ as in Remark 3.16. We write $v \in V$ as $v = v_1 \omega_1 + v_2 \omega_2$. Let $\tau_1 = \{v_1 > v_2 > 0\}$, $\tau_2 = \{0 < v_1 < v_2\}$. Then one easily (see Figure 1) sees that

$$\mathcal{P}_X^{\tau_1} = (n_2 + 1), \qquad \mathcal{P}_X^{\tau_2} = (n_1 + 1), \qquad \mathcal{P}_{X \cap H}^{\tau_{12}} = 1.$$

Equality (6) is equivalent to the following identity of series which is easily checked:

$$\sum_{n_1,n_2}(n_2-n_1)x_1^{n_1}x_2^{n_2} = (-\sum_{n_1\geq 0, n_2<0} x_1^{n_1}x_2^{n_2} + \sum_{n_1<0, n_2\geq 0} x_1^{n_1}x_1^{n_2})(\sum_h x_1^h x_2^h).$$

Recall that a big cell is a connected component of the complement in V of the *singular vectors* which are formed by the union of all cones $C(Y)$ for all the sublists Y of X which do not span V. A big cell is usually larger than a tope. See Figure 5 which shows a section of a cone in dimension 3 generated by 3 independent vectors a, b, c. Here $X = [a, b, c, a+b+c]$. On the drawing, the dots a,b,c,d represents the intersection of the section with the half lines $\mathbb{R}^+a$, $\mathbb{R}^+b$, $\mathbb{R}^+c$, $\mathbb{R}^+d$.

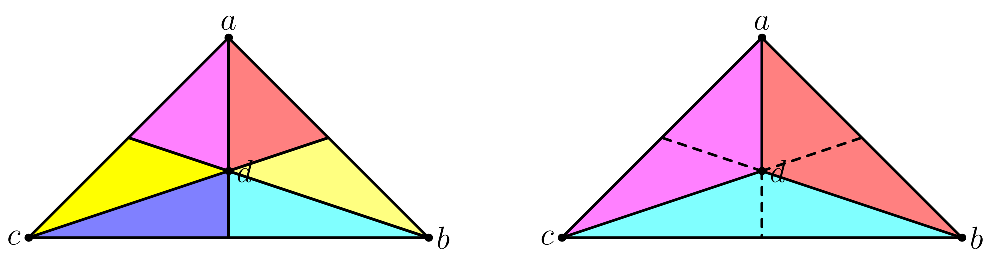

FIGURE 5. Topes and cells for $X := [a, b, c, d := a+b+c]$

Let us now consider a big cell $\mathfrak{c}$. We need

Lemma 3.25. *Given a big cell $\mathfrak{c}$, let $\tau_1, \dots, \tau_k$ be all the topes contained in $\mathfrak{c}$. Then:*

$$\mathfrak{c} - B(X) = \cup_{i=1}^k(\tau_i - B(X)).$$

Proof. Notice that $\cup_{i=1}^k \tau_i$ is dense in $\mathfrak{c}$. Given $v \in \mathfrak{c} - B(X)$, $v + B(X)$ has non empty interior and thus its non empty intersection with the open set $\mathfrak{c}$ has non empty interior. It follows that $v + B(X)$ meets $\cup_{i=1}^k \tau_i$ proving our claim. □

Now in order to prove the statement for big cells, we need to see what happens when we cross a wall between two adjacent topes.

Theorem 3.26. *On $\mathfrak{c} - B(X)$, the partition function $\mathcal{P}_X$ agrees with a quasi–polynomial $f^{\mathfrak{c}} \in DM(X)$.*

Proof. By Lemma 3.25, it suffices to show that given two adjacent topes τ_1, τ_2 in $\mathfrak{c}$, $\mathcal{P}_X^{\tau_1} = \mathcal{P}_X^{\tau_2}$.

But now notice that the positive cone spanned by $X \cap H$, support of $\mathcal{P}_{X\cap H}$, is formed of singular vectors and therefore it is disjoint from $\mathfrak{c}$ by definition of big cells. Therefore $\mathcal{P}_{X\cap H}$ vanishes on τ_{12}. Thus $\mathcal{P}_{X\cap H}^{\tau_{12}} = 0$ and our claim follows from identity 6. □

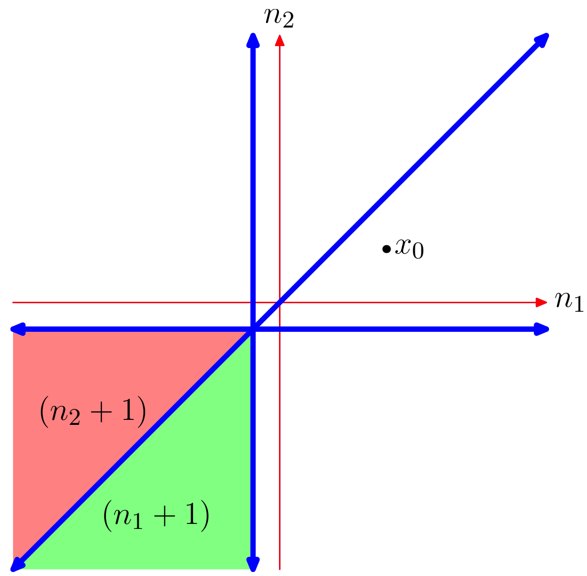

FIGURE 6. The function f

This theorem was proven [6] by Dahmen-Micchelli for topes, and by Szenes-Vergne [10] for cells. In many cases, the sets $\mathfrak{c} - B(X)$ are the maximal domains of quasi-polynomiality for $\mathcal{P}_X$.

Remark 3.27. If $\mathfrak{c}$ is a big cell contained in the cone $C(X)$, the open set $\mathfrak{c} - B(X)$ contains $\bar{\mathfrak{c}}$ so that the quasi–polynomial $f^{\mathfrak{c}}$ coincides with $\mathcal{P}_X$ on $\bar{\mathfrak{c}}$.

This is usually not so for $f \in \mathcal{F}(X)$ and a tope τ : the function f does not usually coincide with f^τ on $\overline{\tau}$. Figure 6 describes the function $f := -\mathcal{P}_X^u$ in $\mathcal{F}(X)$ with $X := [a, b, c]$ as in Remark 3.16 and u strictly negative on a, b. We see for example that f is equal to 0 on the set $n_1 = 0, n_2 \leq 0$ which is in the closure of the tope $\tau_3 := \{v_2 < v_1 < 0\}$ while the quasi–polynomial $f^{\tau_3} = (n_1 + 1)$ takes the value -1 there.

4. A SECOND REMARKABLE SPACE

4.1. **A decomposition formula.** In this section we want to present the analogue for distributions, the proofs are essentially the same or simpler than in the previous case, so we skip them. We shall freely use the notations of the previous sections.

Let V be a finite dimensional vector space, consider the space $\mathcal{D}(V)$ of distributions on V. We denote by δ_0 the delta distribution on V.

$\mathcal{D}(V)$ is in an obvious way a module over the algebra of distributions with compact support, under convolution. Let now $X := [a_1, \dots, a_m]$ be a list of non–zero elements of V.

Definition 4.2. i) Given a rational subspace $\underline{s}$, we denote by $\mathcal{D}(V, \underline{s})$ the set of elements in $\mathcal{D}(V)$ which vanish on all test functions vanishing on $\underline{s}$.

ii) Given a vector $a \neq 0$, we denote by ∂_a the directional derivative associated to a. For a list Y of non–zero vectors, we denote by $\partial_Y := \prod_{a \in Y} \partial_a$.

The restriction map $C_c^\infty(V) \to C_c^\infty(\underline{s})$ on test functions induces, by duality, an identification between the space of distributions on $\underline{s}$ and the space $\mathcal{D}(V, \underline{s})$.

If X spans V, we have the space defined by Dahmen–Micchelli, which we denote $D(X)$, and which is formed by the set of distributions $f \in \mathcal{D}(V)$ satisfying the system of differential equations $\partial_Y f = 0$ as Y varies among all cocircuits of X. It is easy to see that an element of $D(X)$ is a polynomial density $P(x)dx$ on V.

Assume that X spans a pointed cone. Recall the definition of the *multivariate spline* T_X, it is a tempered distribution defined by:

$$\langle T_X \,|\, f \rangle = \int_0^\infty \dots \int_0^\infty f(\sum_{i=1}^m t_i a_i) dt_1 \dots dt_m. \tag{7}$$

If W is the span of X and if we choose a Lebesgue measure dx on W, we may interpret T_X as a function on W supported in the cone $C(X)$ by writing $\langle T_X \,|\, f \rangle = \int_W f(x) T_X(x) dx$.

If Y is a sublist of X one has that $\partial_{X \setminus Y} T_X = T_Y$.

We next define the vector space of interest for this section:

$$G(X) := \{f \in \mathcal{D}(V) \,|\, \partial_{X \setminus \underline{s}} f \in \mathcal{D}(V, \underline{s}), \text{ for all } \underline{s} \in \mathcal{S}_X\}. \tag{8}$$

Lemma 4.3. *i) If X generates a pointed cone, the multivariate spline T_X lies in $G(X)$.*

ii) The space $D(X)$ is contained in $G(X)$.

Proof. *i)* $\partial_{X \setminus \underline{s}} T_X = T_{X \cap \underline{s}} \in \mathcal{D}(V, \underline{s})$.

ii) is clear from the definition.

□

As for the partition functions, this lemma is a very special case of Theorem 4.5 which follows.

Given a rational subspace $\underline{s}$, choose a regular vector u in $\underline{s}^\perp$. Divide as before the set $X \setminus \underline{s}$ into two parts A, B, where u is respectively positive and negative.

We want to define an element $T_X^u \in \mathcal{D}(V)$ which is characterized by the following two properties:

Lemma 4.4. *There exists a unique element T_X^u characterized by the properties $\partial_{X \setminus \underline{s}} T_X^u = \delta_0$ and T_X^u is supported in $C([A, -B])$.*

Proof. Set:
$$T_X^u = (-1)^{|B|} T_{[A,-B]}.$$
It is easily seen that this element satisfies the two properties. The uniqueness is also clear.

□

Identify the space of Dahmen–Micchelli $D(X \cap \underline{s})$ with a subspace of $\mathcal{D}(V, \underline{s})$. Although a distribution $f \in \mathcal{D}(V, \underline{s})$ may have non compact support, we easily see that the convolution product $T_X^u * f$ is well defined. In fact, given any $\gamma \in V$, we can write $\gamma = \lambda + \mu$ with $\mu \in \underline{s}$, and $\lambda \in C(A, -B)$ only in a bounded polytope.

The analog of the "mother formula" (2) of Proposition 2.7 is the following formula.

For $g \in \mathcal{D}(V, \underline{s})$:

$$\partial_{X \setminus \underline{t}}(T_X^u * g) = T_{X \cap \underline{t}}^u * (\partial_{(X \cap \underline{s}) \setminus (\underline{t} \cap \underline{s})} g). \tag{9}$$

Following the same scheme of proof as for Theorem 3.8, the following theorem follows:

Theorem 4.5. *Choose for every rational space $\underline{s}$, a vector $u_{\underline{s}} \in U$ vanishing on $\underline{s}$ and regular for $X \setminus \underline{s}$. Then:*
$$G(X) = \oplus_{\underline{s} \in \mathcal{S}_X} T_X^{u_{\underline{s}}} * D(X \cap \underline{s}).$$

We associate to a rational space $\underline{s}$ and a regular vector u vanishing on $\underline{s}$, the operator on $G(X)$ defined by
$$\pi_X^{\underline{s},u} : f \mapsto T_X^u * (\partial_{X \setminus \underline{s}} f).$$
This is well defined. Indeed $\partial_{X \setminus \underline{s}} f$ is supported on $\underline{s}$ so that the convolution is well defined. We see that $\pi_X^{\underline{s},u}$ maps $G(X)$ to $G(X)$ and that it is a projector.

Given a regular collection $\mathbf{u}$, we can write an element $f \in G(X)$ as
$$f = \sum_{\underline{s} \in \mathcal{S}_X} f_{\underline{s}}$$
with $f_{\underline{s}} \in T_X^{u_{\underline{s}}} * D(X \cap \underline{s})$. This expression for f will be called the $\mathbf{u}$ decomposition of f. In this decomposition, the component f_V is in $D(X)$.

The space $T_X^{u_{\underline{s}}} * D(X \cap \underline{s})$ will be referred to as *the $u_{\underline{s}}$-component* of $G(X)$.

Let $\mathbf{u}$ be a regular family. One can write in the same way as in Proposition 3.11 the explicit projectors to the various components.

4.6. **Polynomials.** Let $f \in G(X)$ and let τ be a tope. Let $x_0 \in \tau$ and let $\mathbf{u}$ be a regular collection such that $u_{\underline{s}}(x_0) < 0$ for any $\underline{s} \neq V$.

Proposition 4.7 (Localization theorem)**.** *Let $f = \sum f_{\underline{s}}$ be the $\mathbf{u}$-decomposition of f. Then the component f_V of this decomposition is a polynomial density in $D(X)$ such that $f = f_V$ on τ.*

Proof. Write $f = \sum_{\underline{s}\in\mathcal{S}_X} f_{\underline{s}}$ with $f_{\underline{s}} = T_X^{u_{\underline{s}}} * k_{\underline{s}}$ where $k_{\underline{s}} \in D(X \cap \underline{s})$. The distribution $f_{\underline{s}} = T_X^{u_{\underline{s}}} * k_{\underline{s}}$ is supported on $\underline{s} + C(F_{\underline{s}}, X)$. As in Proposition 3.15 we know that $\tau \cap (\underline{s} + C(F_{\underline{s}}, X)) = \emptyset$. □

Remark 4.8. Thus the distribution f is a locally polynomial density on V. In particular this is a tempered distribution. Here the distribution f coincides with the polynomial density f_V only on τ and not on the bigger open set $\tau - B(X)$. This extension property is replaced the regularity property that f is of class C^{r-1}, where r is the minimum of the cardinality of the cocircuits of X (see [3]).

We shall denote by f^τ the polynomial density in $D(X)$ coinciding with f on the tope τ.

Let H be a rational hyperplane, and let $u \in U$ be the equation of the hyperplane.

Lemma 4.9. *If $q \in D(X \cap H)$, then $w := T_X^u * q - T_X^{-u} * q$ is an element of $D(X)$.*

Let us use the notations $\tau_1, \tau_2, H, \tau_{12}$ as in Theorem 3.23. Let $f \in G(X)$. The distribution $\partial_{X\setminus H} f$ is an element of $G(H \cap X)$, thus by Proposition 4.7, there exists a polynomial density $(\partial_{X\setminus H} f)^{\tau_{12}} \in D(X \cap H)$ on H such that $\partial_{X\setminus H} f$ agrees with $(\partial_{X\setminus H} f)^{\tau_{12}}$ on τ_{12}.

Theorem 4.10. *Take $u \in U$ vanishing on H such that $\langle u, \tau_1\rangle > 0$. Then*

$$f^{\tau_1} - f^{\tau_2} = T_X^u * (\partial_{X\setminus H} f)^{\tau_{12}} - T_X^{-u} * (\partial_{X\setminus H} f)^{\tau_{12}}. \tag{10}$$

When X spans a pointed cone, we interpret Formula (10) for $f = T_X$. On a given tope τ, T_X agrees with a polynomial density T_X^τ.

Since $\partial_{X\setminus H}(T_X) = T_{X\cap H}$ we deduce, the identity

$$T_X^{\tau_1} - T_X^{\tau_2} = (T_X^u - T_X^{-u}) * T_{X\cap H}^{\tau_{12}}. \tag{11}$$

Now the statement for big cells:

Theorem 4.11. *On $\mathfrak{c}$, the multivariate spline T_X agrees with a polynomial density in $D(X)$.*

Remark 4.12. Once Theorems 3.26 and 4.11 have been proven, it is easy to deduce from them that the generalized Khovanskii- Pukhlikov formula relating volumes and number of points holds [2]. Indeed, one can prove it easily sufficiently far from the walls (cf. [3]).

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