Residues formulae for volumes and Ehrhart polynomials of convex polytopes.

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January 2001

Introduction

These notes on volumes and Ehrhart polynomials of convex polytopes are afterwards thoughts on lectures already delivered in Roma, by the second author in December 1999. The subject of these lectures was to explain Jeffrey-Kirwan residue formula for volumes of convex polytopes [J-K] and the residue formula for Ehrhart polynomials of rational polytopes [B-V 1, 2]. The main concept used in these formulae is the study of rational functions with poles on an arrangement of hyperplanes and the total residue of such a function, which can be computed recursively. Now, what about "concrete" polytopes ? The residue formula leads in principle to an answer, but to be carried out in practice, without further thinking, requires too many steps. Thus we have been fascinated by calculations of Chan-Robbins in [CR], and the conjectures of Chan-Robbins-Yuen [C-R-Y] on the volume of the a certain polytope (proved by Zeilberger [Z]) and of some of its faces, as well as by other examples given by Pitman-Stanley [Pi-S], Stanley [S]. All these examples are related with the root system A_n. Therefore we will study here in more details what can be said on residues formulae for this root system and subsets of this system.

Let us sketch briefly the content of these notes. Short bibliographical comments are also given at the end of each section. We first recall the definition of total residue in Section 1 and Jeffrey-Kirwan residue formula for volumes of general convex polytopes (and give a proof) in Section 2 and Appendix (Section 14). A formula for change of variables in total residues is obtained in Section 9. We go on considering flow polytopes (see Examples below, and Definition in Section 3), and defining a special class of flow polytopes: the cascade polytopes . In Section 4, we show that the Jeffrey-Kirwan residue formula for the volume of cascade polytopes is an iterated constant term formula. We then give a proof of a divisibility property between the volume of a cascade polytope and the volume of some particular face (this implies Conjecture 4 of [C-R-Y]) and a similar divisibility property for Ehrhart polynomial. We carry out the computation of the residue for the complete flow polytope, following the indications of Zeilberger [Z] . It thus leads to the known formula [Z] for the volume of the complete flow polytope. Our method is algebraic though combinatorists would probably prefer another type of proof. For example, Chan-Robbins-Yuen proved that their polytope P could be explicitly decomposed in a union of K elementary simplices (each of volume equal $\frac{1}{(\dim P)!}$). Later K was computed by Zeilberger. Here we apply directly the residue formula for the volume, and we do not use any simplicial decomposition.

In Section 8, we state a residue formula for Ehrhart polynomials of flow polytopes (the proof is not given here, but is similar to the one given for volumes; it is a reformulation of Khovanskii-Pukhlikov theorem [KP]).

In Section 10, as suggested by R. Stanley, we show that formulae for volumes of a family of flow polytopes can be transmuted very explicitly to formulae for Ehrhart polynomials, due to the formula of change of variables in residues. Sections 4, 5, 6 and 7 can be skipped if the reader is interested only in the nice formula of Section 10. Finally, we state a symmetry property that must satisfy the Kostant partition function.

Let us now state here more precisely the setting and the contents of these notes.

Let V be a r-dimensional real vector space and V^* its dual vector space. Let $\Phi = \{\alpha^1, \alpha^2, ..., \alpha^N\}$ be a sequence of non-zero linear forms on V^* all lying in an open half space (we do not assume the α^k to be distinct) and spanning V^*. We denote by Δ^+ the set $\{\Phi\}$ (we mean Δ^+ and Φ are the same sets, but Φ may have multiplicities), and by Δ the set $\Delta^+ \cup -\Delta^+$. The closed convex cone $C(\Delta^+)$ generated by Δ^+ is decomposed as a union of closure of big chambers $\mathfrak{c}$ as in [B-V 1]. For $a \in V^*$, we denote by $P_\Phi(a) \subset \mathbb{R}_+^N$ the convex polytope consisting of all solutions $(u_1, u_2, ..., u_N)$, in non negative real numbers u_k, of the equation

$$\sum_{k=1}^{N} u_k \alpha^k = a$$

and by $\operatorname{vol} P_\Phi(a)$ its volume. Any convex polytope can be realized in that way.

Consider the space R_Δ of rational functions of $x \in V_{\mathbb{C}}$ with poles on the union $\mathcal{H}_{\mathbb{C}}$ of the hyperplanes $\alpha^k(x) = 0$. A subset σ of Δ is called a **basis of** Δ, if the elements $\alpha \in \sigma$ form a basis of V^* . For such σ, set

$$f_\sigma(x) := \frac{1}{\prod_{\alpha\in\sigma} \alpha(x)}.$$

In appropriate coordinates $x_1, ..., x_r$, the function f_σ is simply $\frac{1}{x_1 x_2 \ldots x_r}$ and we denote by S_Δ the subspace of R_Δ spanned by such "simple" elements f_σ . The vector space S_Δ is contained in the homogeneous component of degree $-r$ of R_Δ and we have the direct sum decomposition

$$R_\Delta = S_\Delta \oplus (\sum_{k=1}^{r} \partial_k R_\Delta).$$

We call the projection map

$$Tres_\Delta : R_\Delta \longrightarrow S_\Delta$$

according to this decomposition the total residue map. This projection vanishes outside the homogeneous component of degree $-r$ of R_Δ. The total residue of a function is again a function. It consists of computing a rational function $f(x)$ up to derivatives, in other words, $f(x)dx$ and $(Tres_\Delta(f))(x)dx$ are top dimensional holomorphic forms representing the same cohomology class on $V_{\mathbb{C}} - \mathcal{H}_{\mathbb{C}}$.

We consider $a \in V^*$. Then

$$J_\Phi(a)(x) = Tres_\Delta(\frac{e^{\langle a,x\rangle}}{\prod_{k=1}^{N} \alpha^k(x)}) = \frac{1}{(N-r)!} Tres_\Delta(\frac{\langle a, x\rangle^{N-r}}{\prod_{k=1}^{N} \alpha^k(x)})$$

is an element of the vector space S_Δ.

We denote by

$$K_\Phi(a) = Tres_\Delta(\frac{e^{\langle a,x\rangle}}{\prod_{k=1}^N (1-e^{\langle \alpha^k,x\rangle})})$$

the "periodic" version of J_Φ. This is also an element of the vector space S_Δ.

Now, to each big chamber $\mathfrak{c}$ of the subdivision of $C(\Delta^+)$ is associated a linear form $f \to \langle\langle \mathfrak{c}, f\rangle\rangle$ on S_Δ. It takes value 1 or 0 on a normalized multiple of f_σ whether or not $\mathfrak{c}$ is contained in $C(\sigma)$. This is explained in Section 2 and in the Appendix (Section 14). Here are the two fundamental formulae we will use in these notes.

Formula 1: for $a \in \overline{\mathfrak{c}}$, we have

$$\operatorname{vol} P_\Phi(a) = \langle\langle \mathfrak{c}, J_\Phi(a)\rangle\rangle.$$

If Φ spans a lattice in V^* and if a belongs to this lattice, we denote by $k_\Phi(a)$ the number of all solutions $(u_1, u_2, ..., u_N)$, in non negative **integral** numbers u_k of the equation

$$\sum_{k=1}^N u_k \alpha^k = a.$$

Similarly (under simplifying assumptions that we do not state in the introduction), the following formula holds.

Formula 2: for $a \in \overline{\mathfrak{c}}$, we have

$$k_\Phi(a) = \langle\langle \mathfrak{c}, K_\Phi(a)\rangle\rangle.$$

In particular, as well known, the functions $a \mapsto \operatorname{vol} P_\Phi(a)$ and the Ehrhart function $a \mapsto k_\Phi(a)$ are polynomials on a big chamber $\mathfrak{c}$. (The notion of Minkowski sums and of big chambers are intimately related: if $\mathbf{a_1}, ..., \mathbf{a_m}$ are vectors in the closure of a big chamber $\mathfrak{c}$, then the polytope $P_\Phi(\mathbf{a_1} + \cdots + \mathbf{a_m})$ is isomorphic to the Minkowski sum of the polytopes $P_\Phi(\mathbf{a_k})$.)

This shows that calculation of volumes (or of Ehrhart polynomials) is done by an algebraic version of "integration": it consists of computing particular rational function $f(x)$ on $V_\mathbb{C}$ modulo derivatives. Furthermore this formula shows clearly that the S_Δ- valued **polynomial function** $a \to J_\Phi(a)$ determines entirely the **locally polynomial function** $a \mapsto \operatorname{vol} P_\Phi(a)$.

Thus the calculation of the volume of a flow polytope $P_\Phi(a), a \in \overline{\mathfrak{c}}$, is divided in TWO problems.

A) Compute the linear form $f \to \langle\langle \mathfrak{c}, f \rangle\rangle$.

B) Compute the function

$$\frac{e^{\langle a,x \rangle}}{\prod_{k=1}^N \alpha^k(x)}$$

up to derivatives.

We will now study some particular cases, where A) or B) can be solved.

The Chan-Robbins-Yuen polytope CRY_r consists of solutions $(x_{ij}) \geq 0$ where $1 \leq i < j \leq (r+1)$ of the linear equations:

- $$\sum_{i=2}^{r+1} x_{1,i} = 1,$$
- $$\sum_{i=1}^{r} x_{i,r+1} = 1,$$
- For $2 \leq j \leq r$,
$$\sum_{i=1}^{j-1} x_{ij} = \sum_{k=j+1}^{r+1} x_{jk}.$$

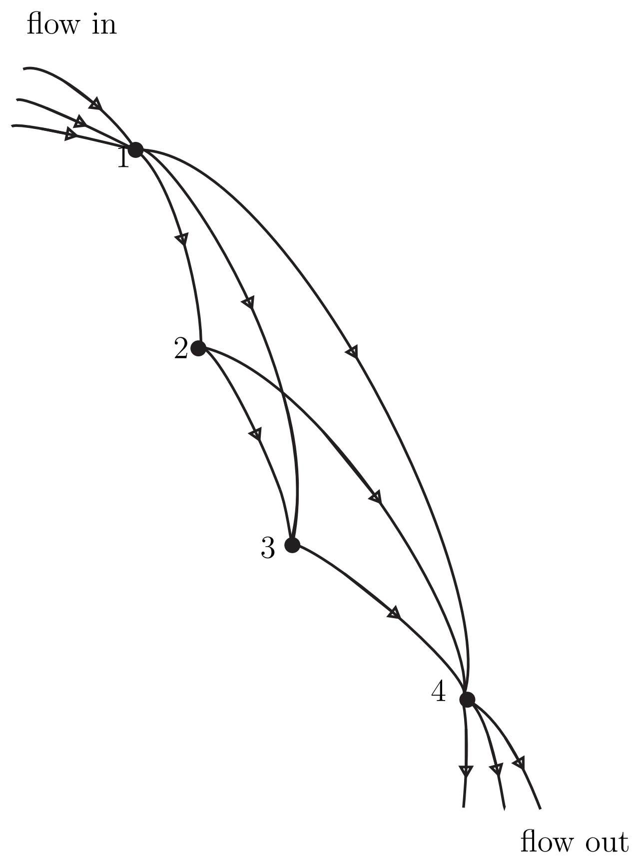


Figure 1: Graph for CRY_3.

The CRY polytope is also called the complete flow polytope. Indeed, consider a graph G with $r+1$ vertices $1, 2, 3, ..., r, r+1$ and edges $i \mapsto j$ $(1 \leq i < j \leq r+1)$, for all $i < j$.

We can imagine that the positive quantity x_{ij} is the quantity of liquid at time t in the branch $i \mapsto j$ of this cascade. Thus the linear equations above reflect the constant flow of the cascade (See Figure 1).

We can also consider other graphs with possibly multiple edges. The associated polytope will be called a flow polytope. See the precise definition in Section 3. For example, Pitman-Stanley polytope is the polytope consisting of solutions in non negative numbers of the inequations

$$y_{1,r+1} \leq a_1,$$

$$y_{1,r+1} + y_{2,r+1} \leq a_1 + a_2,$$

$$y_{1,r+1} + y_{2,r+1} + y_{3,r+1} \leq a_1 + a_2 + a_3,$$

$$...$$

where a_i are non negative real numbers.

It is associated to a flow graph, with $r+1$ vertices, and edges from i to $r+1$ and from i to $i+1$ and last edge $r \mapsto r+1$ of multiplicity 2.

We can imagine that a_1, a_2,...a_r are additional sources of water, so that the quantities of water $y_{i,r+1}$ in the streams $i \to r+1$ satisfies the inequalities stated above, with a leakage of $z_{i,i+1}$. (See g 2).

In the rest of these notes, we consider the positive root system

$$A_r^+ = \{(e^i - e^j), 1 \leq i < j \leq (r+1)\}.$$

The system A_r^+ spans the vector space

$$E_r = \{a = a_1 e^1 + a_2 e^2 + ... + a_{r+1} e^{r+1}, \sum_{i=1}^{r+1} a_i = 0\}.$$

The cone $C(A_r^+) \subset E_r$ generated by positive roots is the cone $a_1 \geq 0$, $a_1 + a_2 \geq 0$,..., $a_1 + a_2 + \cdots + a_r \geq 0, a_1 + a_2 + ... + a_{r+1} = 0$. We denote by $\mathfrak{c}^+$ the open set of $C(\Delta^+)$ defined by

$$\mathfrak{c}^+ = \{a \in C({A_r}^+) \text{ such that } a_i > 0,\ i = 1,..,r\}.$$

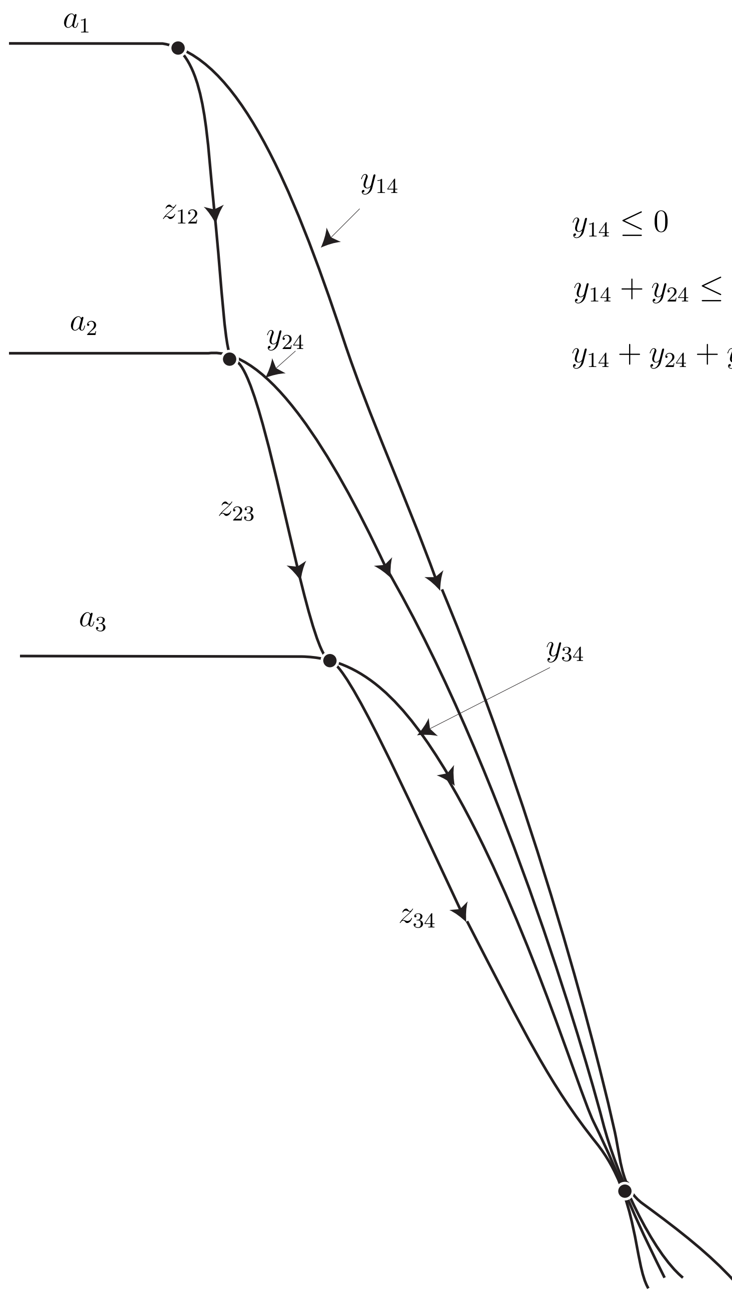
It is a big chamber of our subdivision, and will be called the nice chamber.

Thus the functions $a \mapsto \operatorname{vol} P_\Phi(a)$ and the Ehrhart function $a \mapsto k_\Phi(a)$ on the chamber $\mathfrak{c}^+$ are the mixed volumes and the mixed lattice point enumerators of the fountain polytopes $P_k = P_\Phi(e^k - e^{r+1})$.

Let Φ be a sequence of N elements of A_r^+ spanning E_r. By definition, a flow polytope is a polytope isomorphic to a polytope $P_\Phi(a)$, where a is an element of $C(A_r^+)$ and a cascade polytope the special class of flow polytopes $P_\Phi(a)$, where a is constrained to be in the nice chamber of $C(A_r^+)$. The flow graph associated to Φ has m_{ij} edges from i to j, if $(e^i - e^j)$ has multiplicity $m_{ij} \geq 0$ in Φ. In fact, any polytope associated to a smooth toric variety (i.e. Delzant polytopes and limits) can be realized as a flow polytope (Szenes: personal communication).

The set of simple roots

$$\Pi = \{(e^1 - e^2), (e^2 - e^3), ..., (e^{r-1} - e^r), (e^r - e^{r+1})\}$$



$$y_{14} \leq 0$$

$$y_{14} + y_{24} \leq a_1 + a_2$$

$$y_{14} + y_{24} + y_{34} \leq a_1 + a_2 + a_3$$

Figure 2: Graph for $\pi_3(a)$.

is a basis of A_r^+ and the element

$$f_\Pi = \frac{1}{(e^1-e^2)(e^2-e^3)\cdots(e^{r-1}-e^r)(e^r-e^{r+1})}$$

is an element of S_{A_r}. The dimension of S_{A_r} is $r!$ and a particularly nice basis of S_{A_r} is given by the elements $w \cdot f_\Pi$ where $w \in \Sigma_r$= permutation group of r elements, acts by permutation on the set $\{1,2,...r\}$. By THEOREM 19, the linear form $f \mapsto \langle\langle \mathfrak{c}^+, f\rangle\rangle$ is the "simplest" possible: the iterated residue $Ires_{x=0}$. This is defined as follows: a function in R_{A_r} is identified with rational function $f(x_1,...,x_r)$ (setting $e^{r+1}=0$) with poles on the hyperplanes $x_i = x_j$, or $x_i = 0$, and we have

$$\langle\langle \mathfrak{c}^+, f\rangle\rangle = Ires_{x=0} f = Res_{x_1=0} Res_{x_2=0} ... Res_{x_r=0} f(x_1,x_2,...,x_r).$$

Thus the volume of a cascade polytope is given by an iterated residue formula. But problem B still remains to be solved.

When a is a root of the system A_{r+1}, the calculation of the total residue of the particular rational function

$$J_\Phi(a)(x) = \frac{\langle a, x\rangle^{r(r+1)/2}}{\prod_{\alpha\in A_{r+1}^+}\langle \alpha, x\rangle}$$

(as well as very similar examples) will be done in Section 6. Let $a = e^0 - e^{r+1}$. Setting as before $e^{r+1}=0$,

$$J_\Phi(a)(x_0,...,x_r) = \frac{x_0^{\frac{r(r+1)}{2}}}{x_0 x_1....x_r \prod_{0\le i<j\le r}(x_i-x_j)}.$$

The "integration" of this function is very familiar in the context of Selberg integral ([Se]). However the usual factor $\prod_{0\le i<j\le r}(x_i-x_j)$ here is in the denominator and "Integration " means that we explicitly compute

$$\frac{1}{x_1....x_r \prod_{0\le i<j\le r}(x_i-x_j)}$$

modulo partial derivatives $\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_r}$, following Aomoto [A]. In THEOREM 26, we obtain the formula

$$Tres_{A_{r+1}^+}\left(\frac{(e^0-e^{r+1})^{r(r+1)/2}}{\prod_{0\leq i<j\leq r+1}(e^i-e^j)}\right)$$

$$=\prod_{i=1}^{r-1}\frac{(2i)!}{i!(i+1)!}\sum_{w\in\Sigma_r}\epsilon(w)w\cdot\frac{1}{(e^0-e^1)(e^1-e^2)\cdots(e^{r-1}-e^r)(e^r-e^{r+1})}$$

where Σ_r acts by permutations on $\{1,2,...,r\}$, . Similar formulae are also obtained. As pointed out by Zeilberger [Z], these calculations are mere reformulations of Morris Identity [M]. The total residue formula replaces here the iterated constant term. It would be interesting to generalize them to other root systems.

The Chan-Robbins polytope is described as a face of the polytope of doubly stochastic matrices in Section 7, as Chan-Robbins-Yuen originally described it. We apply then THEOREM 27, to prove a formula for the volume of CRY_n , and PROPOSITION 23 for the volume of a particular face of it.

In Section 9, we prove a formula of Change of variables in total residue.

As a corollary, the S_Δ- valued polynomial function $K_\Phi(a)$ for Φ a sequence of N elements of A_r^+ is deduced from the function $J_\Phi(a)$, in a way which will be explained now. Define

$$t_j^\Phi=\sum_{k=j+1}^{r+1}m_{jk}-1,$$

$$s_j^\Phi=1-\sum_{k=1}^{j-1}m_{kj}.$$

We write the vector valued polynomial J_Φ, which is homogeneous of degree $(N-r)$,

$$J_\Phi(a_1e^1+a_2e^2+...+a_re^r-(a_1+a_2+...+a_r)e^{r+1})$$

$$=\sum_{i_1+i_2+\cdots+i_r=N-r}\frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!}\cdots\frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!}f_\Phi(\mathbf{i})$$

where $f_\Phi(\mathbf{i})$ are elements of S_Δ.

Then we obtain THEOREM 46:

$$K_\Phi(a_1e^1+a_2e^2+...+a_re^r-(a_1+a_2+...+a_r)e^{r+1})$$

$$= \sum_{|\mathbf{i}|=N-r} \binom{a_1+t_1^\Phi}{i_1}\binom{a_2+t_2^\Phi}{i_2}\cdots\binom{a_{r-1}+t_{r-1}^\Phi}{i_{r-1}}\binom{a_r+t_r^\Phi}{i_r} f_\Phi(\mathbf{i}).$$

Applying FORMULAE (1) and (2) for the volume and Ehrhart polynomials, we see that the Ehrhart polynomial on a big chamber $\mathfrak{c}$ is immediately deduced from the polynomial function $\operatorname{vol} P_\Phi(a)$ (i.e; its highest degree component) by replacing the monomial $\frac{a_k^{i_k}}{i_k!}$ by the function $\binom{a_k+t_k^\Phi}{i_k}$ (with same leading term). The function $a \mapsto \binom{a}{k}$ is more adequate than the monomial $a^k/k!$ in the integral context, as it takes integral values on integers.

THEOREM 46 is a generalization of B.V. Lidskii formula for Kostant partition function [Li]. It was also proven by Stanley for general flow polytopes. R. Stanley suggested to look for a proof via residues.

In Section 11, we apply the "nice formula" to a polytope considered by Pitman and Stanley [Pi-S].

Finally, in Section 13, using the action of the full Weyl group Σ_{r+1} on S_{A_r}, we list $r-1$ symmetries properties of the Kostant polynomial

$$k(A_r^+,\mathfrak{c}^+)(a_1(e^1-e^{r+1})+\cdots+a_{r-2}(e^{r-2}-e^{r+1})+a_{r-1}(e^{r-1}-e^{r+1})+a_r(e^r-e^{r+1})).$$

(This is the mixed lattice point enumerator $N(\sum a_k P_k)$ of the Chan-Robbins-Yuen polytopes $P_k = P_{A_r^+}(e^k - e^{r+1})$, $1 \leq k \leq r$.) The function $k(A_r^+,\mathfrak{c}^+)(a)$ is a function of $(r-1)$ variables $k^+(a_1, a_2, \ldots\ldots, a_{r-1})$. This function satisfies for example, the properties:

- for any values $x_1, \ldots, x_r$, we have

$$k^+(x_1,\ldots., x_{r-2}, -(x_1+\ldots+x_{r-2}+x_{r-1}+x_r+2)) =$$
$$-k^+(x_1,x_2,\ldots,x_{r-2},x_{r-1}) - k^+(x_1,\ldots,x_{r-2},x_r-1).$$

- for any values $x_1, \ldots, x_r$, we have

$$k^+(x_1,x_2,\ldots,x_{r-3},-(x_1+x_2+\ldots+x_r+3),x_{r-1}) =$$
$$-k^+(x_1,x_2,\ldots,x_{r-3},x_{r-2},x_r-1)+k^+(x_1,x_2,\ldots,x_{r-3},x_{r-1}-1,x_{r-2}+1)$$
$$-k^+(x_1,x_2,\ldots,x_{r-3},x_r-2,x_{r-2}+1)+k^+(x_1,x_2,\ldots,x_{r-3},x_{r-1}-1,x_r-1).$$

We do hope to have shown in these notes that the space S_Δ, well known in the study of hyperplanes arrangements, is providing an efficient tool in the computation of Volumes and Ehrhart polynomials of polytopes. However, we also learnt in writing these notes how difficult and exciting are "concrete polytopes".

1 Total residues

Let V be a r-dimensional real vector space and V^* its dual vector space. Let $\Delta \subset V^*$ be a finite subset of non-zero linear forms. We assume Δ symmetric: $\Delta = -\Delta$. Each $\alpha \in \Delta$ determines a linear form on V and a complex hyperplane $\{x \in V_{\mathbb{C}}; \alpha(x) = 0\}$ in $V_{\mathbb{C}}$. Consider the hyperplane arrangement

$$\mathcal{H}_{\mathbb{C}} = \bigcup_{\alpha \in \Delta} \{x \in V_{\mathbb{C}}, \alpha(x) = 0\}.$$

The ring R_Δ of rational functions with poles on $\mathcal{H}_{\mathbb{C}}$ is the ring $\Delta^{-1}S(V^*)$ generated by the ring $S(V^*)$ of polynomial functions on V, together with inverses of the linear functions $\alpha \in \Delta$.

A subset σ of Δ is called a **basis of** Δ, if the elements $\alpha \in \sigma$ form a basis of V^*. We denote by $\mathcal{B}(\Delta)$ the set of bases of Δ. For σ a basis of Δ, set

$$f_\sigma(x) := \frac{1}{\prod_{\alpha \in \sigma} \alpha(x)}.$$

Definition 1 *The subspace S_Δ of R_Δ spanned by the elements f_σ, $\sigma \in \mathcal{B}(\Delta)$, will be called the space of simple elements of R_Δ:*

$$S_\Delta = \sum_{\sigma \in \mathcal{B}(\Delta)} \mathbb{R} f_\sigma.$$

The vector space S_Δ is contained in the homogeneous component of degree $-r$ of R_Δ. If Δ does not span V^*, then the set $\mathcal{B}(\Delta)$ is empty and $S_\Delta = 0$.

In general, elements f_σ are not linearly independent.

Example 1.

Let V be a 2-dimensional vector space with basis e_1, e_2. Let Δ be the set

$$\Delta = \{\pm e^1, \pm e^2, \pm(e^1 - e^2)\}.$$

Then we have the linear relation

$$\frac{1}{e^1 e^2} = \frac{1}{e^2(e^1 - e^2)} - \frac{1}{e^1(e^1 - e^2)}$$

between elements f_{σ_1},f_{σ_2}, f_{σ_3} with $\sigma_1 = \{e^1, e^2\}$, $\sigma_2 = \{e^1, (e^1 - e^2)\}$ and $\sigma_3 = \{e^2, (e^1 - e^2)\}$ bases of Δ.

We let elements v of V act on R_Δ by differentiation:

$$(\partial(v)f)(x) := \frac{d}{d\epsilon} f(x+\epsilon v)|_{\epsilon=0}.$$

Then the following holds ([B-V 2], Proposition 7.)

Theorem 2

$$R_\Delta = \partial(V)R_\Delta \oplus S_\Delta.$$

As a corollary of this decomposition, we can define the projection map

$$Tres_\Delta : R_\Delta \to S_\Delta.$$

The projection $(Tres_\Delta f)(x)$ of a function $f(x)$ is again a function of x that we called the **total residue** of f. By definition, this function can be expressed as a linear combination of the simple fractions $f_\sigma(x)$. **The main property of the map $Tres_\Delta$ is that it vanishes on derivatives**. If $f \in R_\Delta = \frac{P}{\prod_k \alpha^k}$ ($P \in S(V^*), \alpha^k \in \Delta$) has a denominator product of linear forms $\alpha^k \in \Delta$ which do not generate V^*, then it is easy to see that f is a derivative and the total residue of f is equal to 0.

Example 2: Let us do a computation. Let V be with basis e_0, e_1, e_2 and let

$$\Delta = \{\pm e^1, \pm e^2, \pm(e^0 - e^1), \pm(e^0 - e^2), \pm(e^1 - e^2)\}.$$

We write $x \in V$ as $x = x_0e_0 + x_1e_1 + x_2e_2$. The reason, for writing x_i instead of x^i is in order not to misinterpret an upper index with a power. Consider the following function W_3 of R_Δ:

$$W_3(x_0, x_1, x_2) = \frac{x_0^2}{(x_0 - x_1)(x_0 - x_2)(x_1 - x_2)x_1x_2}.$$

Then W_3 is homogeneous of degree -3. To compute the total residue of $W_3(x_0, x_1, x_2)$, we write x_0 as a linear combination of linear forms in the denominator of W_3, in order to reduce the degree of the denominator. For example, writing $x_0^2 = ((x_0 - x_1) + x_1)((x_0 - x_2) + x_2)$, we obtain

$$W_3(x_0, x_1, x_2) = \frac{1}{(x_1 - x_2)x_1x_2} + \frac{1}{(x_0 - x_2)(x_1 - x_2)x_1}$$
$$+\frac{1}{(x_0 - x_1)(x_1 - x_2)x_2} + \frac{1}{(x_0 - x_1)(x_0 - x_2)(x_1 - x_2)}.$$

The first and last fractions have denominators with linearly dependent forms, so that their total residue is zero and we obtain:

$$Tres_\Delta(W_3(x)) = \frac{1}{(x_0 - x_1)(x_1 - x_2)x_2} + \frac{1}{(x_0 - x_2)(x_1 - x_2)x_1}.$$

(More precisely, in the direct sum decomposition,

$$R_\Delta = S_\Delta \oplus (\partial_{x_0} R_\Delta + \partial_{x_1} R_\Delta + \partial_{x_2} R_\Delta),$$

we have,

$$W_3(x_0, x_1, x_2) - Tres_\Delta(W_3(x)) = U_3(x_0, x_1, x_2),$$

with

$$U_3(x_0, x_1, x_2) = -\partial_{x_1} \frac{x_0 - 2x_2}{x_2(x_0 - x_2)(x_1 - x_2)} - \partial_{x_2} \frac{x_0 - 2x_1}{x_1(x_0 - x_1)(x_1 - x_2)}.)$$

If V is one dimensional, and $\Delta = \{\pm e^1\}$, then R_Δ is the ring of Laurent series

$$L = \{f(x) = \sum_{k \geq -q} a_k x^k\}.$$

The total residue of a function $f(x) \in L$ is the function $\frac{a_{-1}}{x}$. The usual residue denoted $Res_{x=0} f$ is the constant a_{-1}. The constant term a_0 of the Laurent series $f(x)$ is denoted by $CT_{x=0} f$. We have $CT_{x=0} x f(x) = Res_{x=0} f(x)$.

We will also use the following obvious properties.

Lemma 3

- *Assume $\Gamma \subset \Delta$ is a subset of Δ. Then*

$$R_\Gamma \subset R_\Delta,$$

$$S_\Gamma \subset S_\Delta.$$

Furthermore, if $f \in R_\Gamma$, then $Tres_\Delta(f)$ belongs to S_Γ and

$$Tres_\Gamma(f) = Tres_\Delta(f).$$

- *Assume that $V = V_1 \oplus V_2$ and $\Delta = \Delta_1 \cup \Delta_2$, with $\Delta_i \subset V_i$, then*

$$R_\Delta = R_{\Delta_1} \otimes R_{\Delta_2},$$

$$S_\Delta = S_{\Delta_1} \otimes S_{\Delta_2},$$

$$Tres_\Delta = Tres_{\Delta_1} \otimes Tres_{\Delta_2}.$$

Consider the vector space $\hat{S}(V^*)$ of formal power series on V. Define $\hat{R}_\Delta = \Delta^{-1}\hat{S}(V^*)$. As the total residue vanishes outside the homogeneous component of degree $-r$ of R_Δ, the map $Tres_\Delta$ extends as a map

$$Tres_\Delta : \hat{R}_\Delta \to S_\Delta.$$

Consider the open set $U_\Delta = V_\mathbb{C} - \mathcal{H}_\mathbb{C}$, complement of the union of hyperplanes $\{\alpha = 0\}$. Choose a basis e^i of V^* . This gives coordinates x^i on $V_\mathbb{C}$. Let $dx = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^r$. Let $f \in R_\Delta$. Consider the r- holomorphic differential form $f(x)dx$, defined on the open set U_Δ. Then $(f - Tres_\Delta(f))dx$ is an exact form on this open set : by definition of the total residue, the function $f - Tres_\Delta(f)$ is in the span of $\partial_k R_\Delta$, so $(f - Tres_\Delta(f))dx$ is the differential of some $(r-1)$- holomorphic differential form $\sum_{k=1}^r f^k dx^1 \wedge dx^2 \wedge \widehat{dx^k} \wedge \cdots \wedge dx^r$.

Bibliographical remarks.

The space S_Δ is isomorphic to the highest degree component of the Orlik-Solomon algebra $H^*(U_\Delta, \mathbb{C})$, described in [O-T] by generators and relations. The definition of the total residue is given in [B-V 2]. It formalizes notions introduced in Jeffrey-Kirwan [J-K]. Proposition 2 is proven in [B-V 2].

2 Jeffrey-Kirwan residue formula for volumes of convex polytopes

Now the vector space V is equipped with a fixed Lebesgue measure dx. We denote by da the dual measure on V^*. If σ is a basis of Δ, we denote by $\mathrm{vol}(\sigma)$ the volume of the parallelepiped $\oplus_{\alpha\in\sigma}[0,1]\alpha$, for our Lebesgue measure da.

We consider $\mathbb{R}^N$ with basis $(w^1, \ldots, w^N)$ and corresponding Lebesgue measure dw. Let p be a surjective linear map from $\mathbb{R}^N$ to the vector space V^*. Then the vector space $\mathrm{Ker}(p) = p^{-1}(0)$ is of dimension $d := N - r$. It is equipped with a Lebesgue measure dm quotient of dw by da. For $a \in V^*$, $p^{-1}(a)$ is an affine space parallel to $\mathrm{Ker}(p)$, thus also equipped with the Lebesgue measure dm. Volumes of subsets of $p^{-1}(a)$ are computed for this measure.

We set $p(w^k) := \alpha^k$ for $1 \leq k \leq N$, and

$$\Phi = (\alpha^1, \ldots, \alpha^N).$$

We have thus, for $u_i \in \mathbb{R}$,

$$p(u_1 w^1 + u_2 w^2 + ... + u_N w^N) = u_1 \alpha^1 + u_2 \alpha^2 + \cdots + u_N \alpha^N.$$

The N elements α^k of the sequence Φ **need not to be distinct**. We denote by C_N the closed convex cone in $\mathbb{R}^N$ generated by $w^1, \ldots, w^N$, and we set $C(\Phi) := p(C_N)$, the cone generated by $(\alpha^1, \ldots, \alpha^N)$.

We assume that $p^{-1}(0) \cap C_N = \{0\}$. Then 0 is not in the convex hull of the vectors α^k and $C(\Phi)$ is an acute cone.

Definition 4 *Let $a \in V^*$. We define*

$$P_\Phi(a) := p^{-1}(a) \cap C_N \ .$$

We immediately see that the set $P_\Phi(a)$ is the convex polytope consisting of all solutions $(u_1, u_2, ..., u_N)$, in non negative real numbers u_k, of the equation

$$\sum_{k=1}^{N} u_k \alpha^k = a.$$

In particular, the polytope $P_\Phi(a)$ is empty if a is not in the cone $C(\Phi)$.

Remark: any convex polytope can be realized canonically as a polytope $P_\Phi(a)$, see for example [B-V 1, Section 4.1].

The following lemma is obvious.

Lemma 5 *For any invertible transformation w of V^*, then*

$$P_{w\Phi}(w \cdot a) = P_\Phi(a).$$

For a in the interior of $C(\Phi)$, the dimension of $P_\Phi(a)$ is $N-r$. The function $a \mapsto \operatorname{vol} P_\Phi(a)$ is a continuous function on $C(\Phi)$, homogeneous of degree $N-r$. This function is locally polynomial. Let us recall the description of regions where we are sure that this function is given by a polynomial formula.

We consider the set $\{\Phi\} \subset V^*$ consisting of elements of Φ: we mean Φ and $\{\Phi\}$ are the same sets, but Φ may have multiplicities. Let Δ^+ be a finite subset of V^* containing the set Φ and such that the cone $C(\Delta^+)$ is acute. Usually, we take $\Delta^+ = \{\Phi\}$. We define

$$\Delta = \Delta^+ \cup -\Delta^+.$$

For any subset ν^+ of Δ^+, we denote by $C(\nu^+)$ the closed cone generated by ν^+. We denote by $C(\Delta^+)_{sing}$ the union of the cones $C(\nu^+)$ where ν^+ is any subset of Δ^+ of cardinal strictly less than $r = \dim(V)$. By definition, the set $C(\Delta^+)_{reg}$ of Δ^+-regular elements is the complement of $C(\Delta^+)_{sing}$. A connected component of $C(\Delta^+)_{reg}$ is called a big chamber. If $\mathfrak{c}$ is a big chamber, and σ^+ a basis of Δ^+, then either $\mathfrak{c} \subset C(\sigma^+)$, or $\mathfrak{c} \cap C(\sigma^+) = \emptyset$, as the boundary of $C(\sigma^+)$ does not intersect $\mathfrak{c}$. Thus the closure of the big chamber $\mathfrak{c}$ is the intersection of the simplicial cones $C(\sigma^+)$, σ^+ a basis of Δ^+, containing $\mathfrak{c}$.

A wall of Δ is a (real) hyperplane generated by $r-1$ linearly independent elements of Δ. We denote by $\mathcal{H}^*$ the union of walls. A small chamber of $C(\Delta^+)$ is a connected component of $C(\Delta^+) - \mathcal{H}^*$. Clearly $C(\Delta^+) - \mathcal{H}^*$ is contained in $C(\Delta^+)_{reg}$ and each small chamber is contained in a big chamber. In the appendix, we describe big and small chambers $\mathfrak{c}$ when Δ^+ is the positive root system of A_n, for $n = 2, 3$. See Figure 4 in Section 15.

Remark. When a varies in a big chamber $\mathfrak{c}$, the combinatorial nature of $P_\Phi(a)$ remains the same, and the family of polytopes $P_\Phi(a)$ have parallel facets. The notion of Minkowski sums and of big chambers are intimately related: if $\mathbf{a_1}, ..., \mathbf{a_m}$ are vectors in the closure of a big chamber $\mathfrak{c}$, then the polytope $P_\Phi(\mathbf{a_1}+\cdots+\mathbf{a_m})$ is isomorphic to the Minkowski sum of the polytopes $P_\Phi(\mathbf{a_k})$. This follows for example of ([B-V 1 Section 3.1])

The function $a \mapsto \operatorname{vol} P_\Phi(a)$ is given by a polynomial formula, when a varies in a big chamber $\mathfrak{c}$. Let us recall this formula.

The Jeffrey-Kirwan residue ([J-K]) associate to a big chamber $\mathfrak{c}$ of $C(\Delta^+)$ a linear form $f \mapsto \langle\langle \mathfrak{c}, f\rangle\rangle$ on the vector space S_Δ of simple fractions. To determine the linear map $f \mapsto \langle\langle \mathfrak{c}, f\rangle\rangle$, it is enough to determine it on the generating set f_σ, with σ a basis of Δ. If $\sigma = \sigma^+ \cup \sigma^-$ with $\sigma^+ \subset \Delta^+$ and $\sigma^- \subset \Delta^-$, then

$$f_\sigma = (-1)^{|\sigma^-|} f_{(\sigma^+ \cup \{-\sigma^-\})}.$$

Thus any function f in S_Δ can be written as a linear combination of functions f_σ, with σ a basis of Δ consisting of positive elements.

The linear form $\langle\langle \mathfrak{c}, f\rangle\rangle$ has the following properties:

A) If σ is a basis of Δ^+ (it is important to require that σ consists of positive elements), then

a) If $\mathfrak{c} \subset C(\sigma)$,

$$\langle\langle \mathfrak{c}, f_\sigma\rangle\rangle = \frac{1}{\operatorname{vol}(\sigma)},$$

b) If $\mathfrak{c} \cap C(\sigma) = \emptyset$,

$$\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle = 0.$$

B) If $\sigma = \sigma^+ \cup \sigma^-$ with $\sigma^+ \subset \Delta^+$ and $\sigma^- \subset \Delta^-$, then

$$\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle = (-1)^{|\sigma^-|} \langle\langle \mathfrak{c}, f_{(\sigma^+ \cup \{-\sigma^-\})} \rangle\rangle.$$

Remark. We can interpret the linear form $f \to \langle\langle \mathfrak{c}, f \rangle\rangle$ in terms of Laplace transform (see Section 14) . Assume σ is a basis of Δ contained in Δ^+. Then we have:

$$\operatorname{vol}(\sigma) f_\sigma(x) = \int_{C(\sigma)} e^{-\langle x, a \rangle} da$$

whenever x is in the dual cone of $C(\sigma)$. In particular, a function f in S_Δ, being a linear combination of such functions f_σ, coincide on the dual cone to $C(\Delta^+)$ with the Laplace transform of a function $\hat{f}$ locally constant on big chambers of $C(\Delta^+)$, and the linear form $\langle\langle \mathfrak{c}, f \rangle\rangle$ consists in evaluating the function $\hat{f}$ at a point of $\mathfrak{c}$.

Let $a \in V^*$. Let $f \in R_\Delta$. Consider

$$e^a f = \sum_{k=0}^{\infty} \frac{a^k}{k!} f.$$

This is an element of $\hat{R}_\Delta$. If e_k is a basis of V, and we write $a \in V^* = \sum_{i=1}^r a_i e^i$ and $x \in V$ as $x = \sum_{k=1}^r x_k e_k$ (here, and in many other local calculations, we write x_i instead of x^i, as we do not want to misinterpret an upper index as a power), then $e^a f$ is the analytic function on $V_{\mathbb{C}} - \mathcal{H}_{\mathbb{C}}$, given by $x \mapsto e^{a_1 x_1 + a_2 x_2 + \cdots + a_r x_r} f(x)$. Its total residue is defined by

$$Tres_\Delta(e^a f) = \sum_{k=0}^{\infty} Tres_\Delta(\frac{a^k}{k!} f).$$

There are only a finite number of non-zero terms in this sum. More precisely, if f is homogeneous of degree $-q$, then

$$Tres_\Delta(e^a f) = \frac{1}{(q-r)!} Tres_\Delta(a^{q-r} f).$$

Definition 6 *Define*

$$J_\Phi(a) := Tres_\Delta\left(\frac{e^a}{\prod_{\alpha\in\Phi}\alpha}\right).$$

This is a polynomial function of $a \in V^*$ with value in the vector space S_Δ. More precisely, we choose a basis $e^1, ..., e^r$ of V^*, and write $a = \sum_{i=1}^r a_i e^i$, $x = \sum_{i=1}^r x_i e_i$. Let $\mathbf{i}$ be a sequence $(i_1, i_2, ..., i_r)$ of non-negative integers. We write $|\mathbf{i}| := i_1 + i_2 + \cdots + i_r$. If $|\mathbf{i}| = N - r$, then the function

$$\frac{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}}{\prod_{\alpha\in\Phi}\alpha(x)}$$

is of homogeneous degree $-r$.

Definition 7 *Let $f_\Phi(\mathbf{i})$ be the following element of S_Δ:*

$$f_\Phi(\mathbf{i})(x) = Tres_\Delta\left(\frac{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}}{\prod_{\alpha\in\Phi}\alpha(x)}\right).$$

Lemma 8 *We have*

$$J_\Phi(a) = \sum_{|\mathbf{i}|=N-r} \frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!}\cdots\frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!} f_\Phi(\mathbf{i}).$$

Proof. We have

$$J_\Phi(a)(x) = Tres_\Delta\left(\frac{e^{a_1x_1+\ldots+a_rx_r}}{\prod_{\alpha\in\Phi}\alpha(x)}\right)$$

$$= \sum_{i_1,i_2,..,i_r} \frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!}\cdots\frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!} Tres_\Delta\left(\frac{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}}{\prod_{\alpha\in\Phi}\alpha(x)}\right).$$

The total residue is zero outside the homogeneous component of degree $-r$. This gives the formula of the lemma.

QED

Thus the function $a \mapsto J_\Phi(a)$ is an homogeneous polynomial in the variable a of degree $N - r$, with value in the finite dimensional vector space S_Δ.

Theorem 9 *(Jeffrey-Kirwan) Let $\mathfrak{c}$ be a big chamber of $C(\Delta^+)$. Then, if $a \in \overline{\mathfrak{c}}$, we have*

$$\operatorname{vol} P_\Phi(a) = \langle\langle \mathfrak{c}, J_\Phi(a)\rangle\rangle.$$

We give the proof of this theorem in the appendix (Section14).

Definition 10 *Let $\mathfrak{c}$ be a big chamber. We denote by $v(\Phi, \mathfrak{c})(a)$ the polynomial function on V^* such that*

$$v(\Phi, \mathfrak{c})(a) = \operatorname{vol} P_\Phi(a)$$

when a varies in $\overline{\mathfrak{c}}$.

More explicitly, when a varies in the closure of a big chamber $\mathfrak{c}$, we have the polynomial formula for the volume:

$$v(\Phi, \mathfrak{c})(a) = \sum_{|\mathbf{i}|=N-r} \langle\langle \mathfrak{c}, f_\Phi(\mathbf{i})\rangle\rangle \frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!} \cdots \frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!}. \tag{1}$$

Remark. The function $a \to J_\Phi(a) \in S_\Delta$ is a vector-valued polynomial function on V^* of degree $N - r$. It is thus determined by $\dim S_\Delta$ *scalar homogeneous polynomial functions of degree $(N-r)$ in r variables. As a result of THEOREM 9, the collection of polynomial functions $v(\Phi, \mathfrak{c})$ is entirely determined by the function $J_\Phi(a)$. It is very difficult to determine the number of big chambers $\mathfrak{c}$. It is usually much larger than* $\dim S_\Delta$. *For example, consider A_n^+ the positive root system of A_n. For $n = 1, 2, 3$, the number of big chambers is 1,2,7; for $n =$ 4,5,6, it is known that their number is greater or equal to than 48, 820,51133 ([S]). The dimension of S_Δ is $n!$, so respectively* $1, 2, 6, 24, 120, 720$. *So, there are many linear relations between the different linear functions $f \to \langle\langle \mathfrak{c}, f\rangle\rangle$ on S_Δ, thus many linear relations between the polynomials $v(\Phi, \mathfrak{c})$.*

The relative volume of $P_\Phi(a)$ compare the volume of $P_\Phi(a)$ to the volume of the standard simplex, so is defined by

$$\operatorname{vol}_{rel} P_\Phi(a) = (N-r)! \operatorname{vol} P_\Phi(a).$$

We have:

$$\operatorname{vol}_{rel} P_\Phi(a) = \langle\langle \mathfrak{c}, Tres_\Delta(\frac{a^{N-r}}{\prod_{\alpha\in\Phi}\alpha})\rangle\rangle.$$

Bibliographical remarks.

Results on this section are mainly due to Jeffrey-Kirwan [J-K]. The stratification on big chambers is introduced in [B-V 1]. The fact that the volume is polynomial on big chambers follows from the proof given in the Appendix (Section **??**). More details on the relations between Jeffrey-Kirwan formulae and Laplace transforms are given in [B-V 2].

3 Flow polytopes

Consider a $(r+1)$ dimensional vector space, with basis $e^1, e^2, ..., e^{r+1}$, and let A_r^+ (the positive root system of A_r) be

$$A_r^+ = \{(e^i - e^j), 1 \leq i < j \leq (r+1)\}.$$

Let E_r be the vector space spanned by the elements $(e^i - e^j)$,

$$E_r = \{a \in \mathbb{R}^{r+1}, a = a_1e^1 + ... + a_re^r + a_{r+1}e^{r+1} \text{ with } a_1 + a_2 + ... + a_r + a_{r+1} = 0\}.$$

The vector space E_r is of dimension r and is provided with the lattice generated by A_r, so has a canonical measure $d\mathbf{a}$. We have $d\mathbf{a} = da_1...da_r$. If σ is any basis of A_r, then $\text{vol}(\sigma) = 1$. The cone $C(A_r^+)$ generated by positive roots is the cone $a_1 \geq 0$, $a_1 + a_2 \geq 0$,...,$a_1 + a_2 + \cdots + a_r \geq 0$.

Definition 11 *If $a = (a_1, a_2, ..., a_r) \in \mathbb{R}^r$, we denote by $\mathbf{a} \in E_r$ the element*

$$\mathbf{a} = a_1e^1 + ... + a_re^r - (a_1 + \cdots + a_r)e^{r+1}.$$

Let Φ be a **sequence** of N elements of A_r^+ spanning E_r.

Definition 12 *We define $m_{ij} (i < j)$ to be the multiplicity of the root $(e^i - e^j)$ in Φ . Thus $m_{ij} \geq 0$.*

Associate to Φ a graph with $r+1$ vertices $1, 2, ..., r+1$, and m_{ij} edges from $i \to j$ if $(e^i - e^j)$ is in Φ (thus $i < j$). This graph is called a flow graph. The graph associated to A_r^+ is the complete flow graph (with 1 edge $i \mapsto j$ for any $i < j$).

Definition 13

- *A flow polytope is a polytope isomorphic to a polytope $P_\Phi(\mathbf{a})$ where Φ is a sequence of elements of A_r^+ and $a = (a_1, a_2, ..., a_r) \in \mathbb{R}^r$.*

- *The polytope $P_\Phi(e^1 - e^{r+1})$ will be called the fountain polytope of shape Φ.*
- *A flow polytope $P_\Phi(\mathbf{a})$ with $a_1 \geq 0, a_2 \geq 0, ..., a_r \geq 0$ will be called a cascade polytope.*

The flow polytope $P_\Phi(\mathbf{a})$ is of dimension $N-r$ when $a_1 > 0$. The polytope $P_{A_r^+}(e^1 - e^{r+1})$ associated to the complete flow graph is the Chan-Robbins-Yuen polytope. It will be described in Section 7.

Remark 1. If $a_1, ..., a_r$ are greater or equal to 0 , the polytope $P_\Phi(\mathbf{a})$ is the Minkowski sum of the polytopes $a_k P_\Phi(e^k - e^{r+1})$. If $k > 1$, the vector $e^k - e^{r+1}$ is on the boundary of $C(\Delta^+)$ and the polytope $P_\Phi(e^k - e^{r+1})$ is of smaller dimension than $N - r$. It is the fountain polytope of shape Φ' where Φ' is the system where we have deleted all $(e^i - e^j)$ with $i < k$.

Remark 2. Let d_a be the dimension of $P_\Phi(\mathbf{a})$. If s is a vertex of $P_\Phi(\mathbf{a})$ and $s + \mathbb{R}^+ v_1$, $s + \mathbb{R}^+ v_2$,..., $s + \mathbb{R}^+ v_{d_a}$ are d_a edges passing through s, directed by integral vectors v_i with minimal length, then the volume of the parallelepiped spanned by $v_k, 1 \leq k \leq d_a$ is equal to 1. Such polytope is associated to an ample line bundle on a smooth toric variety: a limit of Delzant polytopes. Reciprocally, any polytope associated to a smooth toric variety can be realized as a flow polytope $P_\Phi(\mathbf{a})$ when Φ is a sequence of elements of the positive root system A_r^+ (Szenes: personal communication).

In the rest of this article, we study volumes and Ehrhart polynomials of flow polytopes. Our method is based on the study of the vector space R_{A_r} and of the vector space S_{A_r}.

Consider the system $A_{r-1} = \{(e^i - e^j), 1 \leq i < j \leq r\}$.

Proposition 14 *The map from*

$$\sum_{i=1}^{r} S_{A_{r-1}} \otimes \frac{1}{(e^i - e^{r+1})} \to S_{A_r}$$

given by

$$\sum_{i=1}^{r} f_i \otimes \frac{1}{(e^i - e^{r+1})} \mapsto \sum_{i=1}^{r} f_i \frac{1}{(e^i - e^{r+1})}$$

is a bijection.

Proof. If f is in $S_{A_{r-1}}$, then for any $1 \leq i \leq r$, the element $f\frac{1}{(e^i-e^{r+1})}$ is in S_{A_r}, and it is easy to see that the map above is injective.

To prove that it is surjective, observe first that for any set of elements $A \subset \{1, ..., r\}$:

$$\prod_{i\in A}\frac{1}{(e^{r+1}-e^i)} = \sum_{i\in A}\frac{1}{\prod_{j\in A;i\neq j}(e^i-e^j)}\frac{1}{(e^{r+1}-e^i)}.$$

If ν is a subset of A_{r-1}, we denote by $f_\nu = \frac{1}{\prod_{\alpha\in\nu}\alpha}$. Then

$$\prod_{i\in A}\frac{1}{(e^{r+1}-e^i)}f_\nu = \sum_{i\in A}\frac{1}{\prod_{j\in A;i\neq j}(e^i-e^j)}f_\nu\frac{1}{(e^{r+1}-e^i)}.$$

If $\nu \cup \{(e^i - e^{r+1}), i \in A\}$ is a basis of A_r, then for every $i \in A$,

$$\nu \cup \{(e^j - e^{r+1}), j \in A, j \neq i\} \cup \{(e^i - e^{r+1})\}$$

is a basis of A_r. The lemma follows.

QED

It follows from the lemma above that the dimension of S_{A_r} is $r!$. We denote by Σ_r the set of permutations of $\{1, 2, ...r\}$. As seen from PROPOSITION 14 above, a particularly nice basis of S_{A_r} is given by the elements

$$f_w = w \cdot \frac{1}{(e^1-e^2)(e^2-e^3)\cdots(e^{r-1}-e^r)(e^r-e^{r+1})}$$

where $w \in \Sigma_r$ acts by permutation on the set $\{1, 2, ...r\}$.

We define as usual the character $\epsilon(w) = \pm 1$ of Σ_r with value -1 on symmetries.

Bibliographical remarks. Polytopes associated to subsets of A_r^+ are related to graphs and called flow polytopes by Stanley [S]. The representation of the Weyl group in the space S_Δ, for Δ any root system, has been studied extensively by G. Lehrer [L]. In particular, for A_r, the space S_Δ carries the regular representation of the subgroup Σ_r of the Weyl group Σ_{r+1}. We employ the corresponding basis f_w ($w \in \Sigma_r$) in these notes.

4 Chambers and Iterated residues for A_r.

Let

$$V = \{\sum_{i=1}^{r+1} x_i e_i, x_{r+1} = 0\}.$$

(as before, we write x_i instead of x^i). The vector space V has basis $e_1, ..., e_r$. An element $a = \sum_{i=1}^{r+1} a_i e^i$ of E_r gives the linear form $\sum_{i=1}^{r} a_i x_i$ on V. This identifies E_r with V^*. In this identification

$$A_r^+ = \{x_i, 1 \leq i \leq r, (x_i - x_j), 1 \leq i < j \leq r\}.$$

A function in R_{A_r} is thus a rational function $f(x_1, x_2, ..., x_r)$ on $V_{\mathbb{C}}$, with poles on the hyperplanes $x_i = x_j$ or $x_i = 0$. The basis f_w considered in the preceding section is given by the elements

$$f_w(x_1, ..., x_r) = w \cdot \frac{1}{(x_1 - x_2)(x_2 - x_3) \cdots (x_{r-1} - x_r) x_r}$$

where $w \in \Sigma_r$ acts by permutation on the set $\{1, 2, ...r\}$.

The following linear form $f \mapsto Ires_{x=0} f$ defined by

$$Ires_{x=0} f$$

$$= Res_{x_1=0} Res_{x_2=0} \cdots Res_{x_r=0} f(x_1, x_2, ..., x_r)$$

is a linear form on R_{A_r} which vanishes on the vector space of derivatives $\sum_{i=1}^{r} \partial_i R_{A_r}$. It will be called the iterated residue. It provides a linear form on S_{A_r}. If we compute it on the basis f_w of S_{A_r} indexed by the symmetric group, we have $Ires_{x=0}(f_w) = \delta_w^1$. The iterated residue depends on the order on variables. Permuting the variables by the group Σ_r, we obtain $r!$ linear forms on S_{A_r}, dual to the basis f_w. Precisely set

$$Ires_{x=0}^{\sigma} f = Res_{x_{\sigma(1)}=0} Res_{x_{\sigma(2)}=0} \cdots Res_{x_{\sigma(r)}=0} f(x_1, x_2, ..., x_r) =$$

$$Res_{x_1=0} Res_{x_2=0} \cdots Res_{x_r=0} f(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, ..., x_{\sigma^{-1}(r)}),$$

it is not difficult to check that $\sigma \cdot Ires_{x=0} f_w = Ires_{x=0} \sigma^{-1} f_w = Ires_{x=0}^{\sigma} f_w = \delta_w^{\sigma}$ while $Ires_{x=0}^1 = Ires_{x=0}$.

Iterated residues are particularly easy to calculate, thus it is important to express the linear forms associated to big chambers in terms of iterated residues.

Consider the set $\mathcal{H}^*$ of hyperplanes in E_r spanned by $(r-1)$ linearly independent vectors of A_r.

Lemma 15 *An element H of $\mathcal{H}^*$ is the kernel of a linear form $\sum_{i\in A} a_i$ where A is a subset of $1, 2, ..., r$.*

Proof. Let ν be a subset of A_r^+ spanning a $(r-1)$-dimensional vector space. If ν is contained in the set $\pm(e^i - e^j), 1 \leq i < j \leq r$, the hyperplane spanned by ν is the hyperplane $\sum_{i=1}^r a_i = 0$. If not, then ν contains a vector $(e^i - e^{r+1})$, and we conclude by induction.

QED

Definition 16 *We denote by $\mathfrak{c}^+$ the open set*

$$\mathfrak{c}^+ = \{a \in C({A_r}^+) \text{ such that } a_i > 0,\ i = 1, .., r\}.$$

Lemma 17 *The set $\mathfrak{c}^+$ is a small and a big chamber for A_r^+.* **It will be called the nice chamber.**

Proof. The set $\mathfrak{c}^+$ is a small chamber since it doesn't meet any hyperplane. But $\overline{\mathfrak{c}^+}$ is the simplicial cone generated by the elements $(e^i - e^{r+1}),\ 1 \leq i \leq r$, so that $\mathfrak{c}^+$ is also a big chamber.

QED

Let $w \in \Sigma_r$ and $n(w)$ be the number of elements such that $w(i) > w(i+1)$. We denote by $C_w^+ \subset C({A_r}^+)$ the simplicial cone generated by the vectors

$$\epsilon(1)(e^{w(1)} - e^{w(2)}), \epsilon(2)(e^{w(2)} - e^{w(3)}), ..., \epsilon(r-1)(e^{w(r-1)} - e^{w(r)}), (e^{w(r)} - e^{r+1}),$$

where $\epsilon(i)$ is 1 or -1 depending whether $w(i) < w(i+1)$ or not. When $w = 1$, then $C_1 = C(A_r^+)$. The dual basis for the vectors generating C_w^+ is given by

$$\epsilon(1)e_{w(1)}, \epsilon(2)(e_{w(1)} + e_{w(2)}), ..., \epsilon(r-1)(e_{w(1)} + .. + e_{w(r-1)}), (e_{w(1)} + .. + e_{w(r)}).$$

If we write $a = \sum_{j=1}^{r+1} a_j e^j$ in E_r, then the cone C_w^+ is given by the following system of inequalities $\sum_{j=1}^{i} a_w(j) \geq 0$, for all i such that $w(i) < w(i+1)$, but $\sum_{j=1}^{i} a_w(j) \leq 0$ if $w(i) > w(i+1)$.

Lemma 18 *Let $\mathfrak{c}$ be a big chamber. Consider the set of elements $w \in \Sigma_r$ such that $\mathfrak{c} \subset C_w^+$. Then, for $f \in S_{A_r}$,*

$$\langle\langle \mathfrak{c}, f\rangle\rangle = \sum_{w\in\Sigma_r, \mathfrak{c}\subset C_w^+} (-1)^{n(w)} Ires_{x=0} w^{-1} f.$$

Proof. Let $f \in S_{A_r}$ and write $f = \sum_{w\in\Sigma_r} a_w^f f_w$. Using the dual base, we compute $a_s^f = Ires_{x=0}{}^s(f)$ and we find that the linear form $f \mapsto \langle\langle \mathfrak{c}, f\rangle\rangle$ is the sum over all elements $w \in \Sigma_r$ such that $\mathfrak{c} \subset C_w^+$ of the linear form $(-1)^{n(w)} Ires_{x=0}^w = (-1)^{n(w)} Ires_{x=0}^w$.

QED

Theorem 19 *The linear form* $f \mapsto \langle\langle \mathfrak{c}^+, f\rangle\rangle$ *coincide with the iterated residue* $Ires_{x=0}$.

Proof. As we just observed, the linear form $f \mapsto \langle\langle \mathfrak{c}^+, f\rangle\rangle$ is the sum over all elements w such that $\mathfrak{c}^+ \subset C_w^+$ of the linear form $(-1)^{n(w)} Ires_{x=0}^w$. So, to prove the lemma, it is enough to prove that if $\mathfrak{c}^+ \subset C_w^+$, then $w = 1$. Assume that $w \neq 1$. We let k be the smallest integer such that $w(k) > w(k+1)$. In particular if $a \in C_w^+$ then

$$a_{w(1)} \geq 0, a_{w(1)} + a_{w(2)} \geq 0, \ldots, a_{w(1)} + a_{w(2)} + \ldots + a_{w(k-1)} \geq 0,$$

$$a_{w(1)} + a_{w(2)} + \ldots + a_{w(k)} \leq 0$$

which forces $a_w(k) \leq 0$, and therefore, if $w \neq 1$, then $\mathfrak{c}^+$ is not contained in C_w^+.

QED

Bibliographical remarks.

Bases of S_Δ^* given by iterated residues are constructed for any system Δ by Szenes [Sz]. The linear forms $f \mapsto \langle\langle \mathfrak{c}, f\rangle\rangle$ are called Jeffrey-Kirwan residues. It is usually difficult to express them in functions of iterated residues.

5 Volumes of flow polytopes

Let Φ be a sequence of N vectors in A_r^+ spanning E_r. The set of regular elements for the system A_r^+ is clearly contained in the set of regular elements for the smaller system $\{\Phi\}$. Recall the map $a = (a_1, a_2, ..., a_r) \to \mathbf{a}$ given in DEFINITION 11. We will sometimes identify implicitly E_r with $\mathbb{R}^r$ using this map. Thus if $\mathfrak{c}$ is a big chamber for A_r^+, the function $a \mapsto \mathrm{vol}\, P_\Phi(\mathbf{a})$ is polynomial on $\mathfrak{c}$. We denote it by $v(\Phi, \mathfrak{c})$. We have (see Section 2, Formula 1):

$$v(\Phi, \mathfrak{c})(a) = \langle\langle \mathfrak{c}, J_\Phi(a)\rangle\rangle.$$

The function $v(\Phi, \mathfrak{c}^+)$, attached to the nice chamber $\mathfrak{c}^+$ is particularly important, due to the following lemma.

Lemma 20 *Let $\mathfrak{c}$ be a big chamber. Let $\Sigma_r(\mathfrak{c})$ be the set of elements $w \in \Sigma_r$ such that $\mathfrak{c} \subset C_w^+$. Then*

$$v(\Phi, \mathfrak{c})(a) = \sum_{w \in \Sigma_r(\mathfrak{c})} (-1)^{n(w)} v(\Phi, \mathfrak{c}^+)(w^{-1}a).$$

Proof. Due to THEOREM 19, the linear form $f \mapsto \langle\langle \mathfrak{c}^+, f \rangle\rangle$ is the iterated residue $Ires_{x=0}$. Thus $v(\phi, \mathfrak{c}^+)(a) = Ires_{x=0} J_\Phi(a)$. By LEMMA 18, the form $\langle\langle \mathfrak{c}, f \rangle\rangle$ is a signed sum of iterated residues $Ires^w_{x=0}$ over the elements w in $\Sigma_r(\mathfrak{c})$ and we obtain the lemma.

QED

We write

$$v(\Phi, \mathfrak{c})(a_1, ..., a_r) = \sum_{i_1 + i_2 + \cdots + i_r = N - r} v(\Phi, \mathfrak{c}, \mathbf{i}) \frac{a_1^{i_1}}{i_1!} \frac{a_2^{i_2}}{i_2!} \cdots \frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!} \frac{a_r^{i_r}}{i_r!}$$

where

$$v(\Phi, \mathfrak{c}, \mathbf{i}) = \langle\langle \mathfrak{c}, f_\Phi(\mathbf{i}) \rangle\rangle.$$

Lemma 21 *The coefficient $v(\Phi, \mathfrak{c}^+, \mathbf{i})$ is the non negative integer defined by:*

$$v(\Phi, \mathfrak{c}^+, \mathbf{i}) = Ires_{x=0} \frac{x_1^{i_1} ... x_r^{i_r}}{\prod_{1 \le i < j \le r} (x_i - x_j)^{m_{ij}} \prod_{1 \le i \le r} x_i^{m_{i,r+1}}}.$$

Proof. It is immediate to check that

$$v(\Phi, \mathfrak{c}^+, \mathbf{i}) = Ires_{x=0} \frac{x_1^{i_1} ... x_r^{i_r}}{\prod_{1 \le i < j \le r} (x_i - x_j)^{m_{ij}} \prod_{1 \le i \le r} x_i^{m_{i,r+1}}}$$

is a non negative integer. We will give a combinatorial interpretation of this integer in LEMMA 43 in Section 8.

QED

Proposition 22 *The polynomial $v(\Phi, \mathfrak{c}^+)(a_1, a_2, ..., a_r)$ is homogeneous of degree $|\Phi| - r$. It is divisible by $a_1^{(\sum_{k=2}^{r+1} m_{1k} - 1)}$. It is of degree less or equal to $(m_{r,r+1} - 1)$ in a_r.*

Proof. We have

$$J_\Phi(a)(x) = \frac{e^{a_1x_1+\ldots+a_{r-1}x_{r-1}+a_rx_r}}{\prod_{1\leq i<j\leq r}(x_i-x_j)^{m_{ij}}\prod_{1\leq i\leq r-1}x_i^{m_{i,r+1}}x_r^{m_{r,r+1}}}$$

$$= \frac{e^{a_1x_1+\ldots+a_{r-1}x_{r-1}+a_rx_r}}{x_1^{m_{1,r+1}}\prod_{1<j\leq r}(x_1-x_j)^{m_{1j}}\prod_{2\leq i<j\leq r}(x_i-x_j)^{m_{ij}}\prod_{2\leq i\leq r}x_i^{m_{i,r+1}}}.$$

The function $v(\Phi,\mathfrak{c}^+)(a_1,a_2,...,a_r)$ is the iterated residue of $J_\Phi(a)(x)$. If we start by taking the residue in $x_r=0$, (use the first expression), we have to develop the term $e^{a_rx_r}$ up to order $m_{r,r+1}-1$, so we obtain the second property.

On the other hand, x_1 is considered as generic up to the last step. Consider the function

$$F(x_1,a_2,...,a_r) =$$

$$Ires_{x_2=0,\ldots,x_r=0}\frac{e^{a_2x_2+\ldots+a_{r-1}x_{r-1}+a_rx_r}}{\prod_{2\leq j\leq r}(1-x_j/x_1)^{m_{1j}}\prod_{2\leq i<j\leq r}(x_i-x_j)^{m_{ij}}\prod_{2\leq i\leq r}x_i^{m_{i,r+1}}}.$$

This function is of the form $\sum_{k\geq 0}C_k(a_2,...,a_r)x_1^{-k}$, and

$$Ires_{x=0}J_\Phi(a)(x) = Res_{x_1=0}\frac{e^{a_1x_1}}{x_1^{N_1}}F(x_1,a_2,...,a_r),$$

with $N_1=\sum_{k=2}^{r+1}m_{1k}$.

Thus, expanding the exponential, we obtain

$$v(\Phi,\mathfrak{c}^+)(a_1,a_2,....,a_r) = \sum_{k\geq 0}C_k(a_2,...,a_r)\frac{a_1^{N_1+k-1}}{(N_1+k-1)!}.$$

This establishes the first property .

QED

Proposition 23 *(Schmidt-Bincer)*

Let $\mathfrak{c}^+$ be the nice chamber. Let $v_r^+=v(A_r^+,\mathfrak{c}^+)$. The function $v_r^+(a_1,a_2,...,a_r)$ is independent of a_r. It is of homogeneous degree $r(r-1)/2$, of degree less than 1 in the variable a_{r-1} and is divisible by $a_1^{r-1}(a_1+a_2+a_3+\ldots+a_{r-2}+3a_{r-1})$.

More precisely, we have:

$$3v_r^+(a_1,a_2,...,a_r)$$

$$=(a_1+a_2+\ldots+a_{r-2}+3a_{r-1})v(A_r^+ \text{ minus } (e^{r-1}-e^r),\mathfrak{c}^+)(a_1,a_2,...,a_{r-2},0,0).$$

Proof. Let

$$J(a,x)=\frac{e^{a_1x_1+\ldots+a_{r-1}x_{r-1}+a_rx_r}}{\prod_{1\le i<j\le r-1}(x_i-x_j)\prod_{1\le i\le r-1}(x_i-x_r)\prod_{1\le i\le r}x_i}.$$

We have

$$v_r^+(a_1,a_2,...,a_r)=Ires_{x=0}J(a,x).$$

We first take the residue in $x_r=0$. We obtain

$$Res_{x_r=0}J(a,x)=\frac{e^{a_1x_1+\ldots+a_{r-1}x_{r-1}}}{\prod_{1\le i\le r-1}x_i^2\prod_{1\le i<j\le r-1}(x_i-x_j)}$$

$$=\frac{e^{a_1x_1+\ldots+a_{r-1}x_{r-1}}}{\prod_{1\le i\le r-2}x_i^3\prod_{1\le i<j\le r-2}(x_i-x_j)\prod_{1\le i\le r-2}(1-x_{r-1}/x_i)x_{r-1}^2}.$$

This shows already that $v_r^+(a_1,a_2,...,a_r)$ is independent of a_r.We proceed to take the residue in $x_{r-1}=0$. There is a double pole in x_{r-1}, so that the dependence in a_{r-1} is of degree at most 1. More precisely,

$$Res_{x_{r-1}=0}[\frac{e^{a_1x_1+\ldots+a_{r-1}x_{r-1}}}{\prod_{1\le i\le r-2}x_i^3\prod_{1\le i<j\le r-2}(x_i-x_j)\prod_{1\le i\le r-2}(1-x_{r-1}/x_i)x_{r-1}^2}]$$

$$=\frac{e^{a_1x_1+\ldots+a_{r-2}x_{r-2}}}{\prod_{1\le i\le r-2}x_i^3\prod_{1\le i<j\le r-2}(x_i-x_j)}(a_{r-1}+\sum_{i=1}^{r-2}\frac{1}{x_i})$$

$$=(a_{r-1}+\frac{1}{3}(a_1+a_2+...+a_{r-2}))\frac{e^{a_1x_1+\ldots+a_{r-2}x_{r-2}}}{\prod_{1\le i\le r-2}x_i^3\prod_{1\le i<j\le r-2}(x_i-x_j)}$$

$$-\frac{1}{3}(\partial_1+\partial_2+\cdots+\partial_{r-2})\frac{e^{a_1x_1+\ldots+a_{r-2}x_{r-2}}}{\prod_{1\le i\le r-2}x_i^3\prod_{1\le i<j\le r-2}(x_i-x_j)},$$

as $(\partial_1+\partial_2+\cdots+\partial_{r-2})$ annihilates functions x_i-x_j.
Residues vanishes on derivatives, so that we obtain

$$3v_r^+(a_1,a_2,...,a_r)=(3a_{r-1}+a_1+a_2+a_3+...+a_{r-2})\times$$

$$Res_{x_1=0}...Res_{x_{r-2}=0}\frac{e^{a_1x_1+...+a_{r-2}x_{r-2}}}{\prod_{1\leq i\leq r-2}x_i^3\prod_{1\leq i<j\leq r-2}(x_i-x_j)}.$$

On the other hand, the residue computation of

$$v(A_r^+ \,\text{minus}\,(e^{r-1}-e^r),\mathfrak{c}^+)(a_1,a_2,...,a_{r-2},0,0)$$

gives

$$v(A_r^+ \,\text{minus}\,(e^{r-1}-e^r),\mathfrak{c}^+)(a_1,a_2,...,a_{r-2},0,0)$$
$$=Res_{x_1=0}...Res_{x_{r-2}=0}\frac{e^{a_1x_1+...+a_{r-2}x_{r-2}}}{\prod_{1\leq i\leq r-2}x_i^3\prod_{1\leq i<j\leq r-2}(x_i-x_j)}$$

as the step $x_r=0$ as well as the step $x_{r-1}=0$ involves only simple poles, and we obtain the divisibility property announced.

QED

Example.

The dimension of the polytope $P_{A_2^+}(a)$ is 1, of $P_{A_3^+}(a)$ is 3 and of $P_{A_4^+}(a)$ is 6. Thus the corresponding polynomials $v(A_2^+,\mathfrak{c}^+),v(A_3,\mathfrak{c}^+),v(A_4,\mathfrak{c}^+)$ are homogeneous of degrees $1,3,6$.

We have :

$$v(A_2^+,\mathfrak{c}^+)(a_1,a_2)=a_1,$$

$$v(A_3^+,\mathfrak{c}^+)(a_1,a_2,a_3)=\frac{1}{3!}a_1^3+\frac{1}{2}a_1^2a_2=\frac{1}{6}a_1^2(a_1+3a_2),$$

$$v(A_4^+,\mathfrak{c}^+)(a_1,a_2,a_3,a_4)=\frac{1}{120}a_1^3(a_1+a_2+3a_3)(a_1^2+5a_1a_2+10a_2^2).$$

More generally, we have the following proposition, with same proof.

Proposition 24 *Let Φ be a sequence of N vectors in A_r^+, generating E_r. Assume $m_{r,r+1}=1$ and $m_{r-1,r+1}+m_{r-1,r}=2$. Furthermore, assume that*

$$\frac{m_{j,r+1}+m_{j,r}+m_{j,r-1}}{m_{j,r-1}}=c$$

is independent of j for $1\leq j\leq(r-2)$, then

$$v(\Phi,\mathfrak{c}^+)(a_1,...,a_{r-1},a_r)=v(\Phi,\mathfrak{c}^+)(a_1,...,a_{r-1},0)$$
$$=(\frac{a_1+\cdots+a_{r-2}}{c}+a_{r-1})v(\Phi\,\text{minus}\,(e^{r-1}-e^r),\mathfrak{c}^+)(a_1,a_2,...,a_{r-2},0,0).$$

Consider the permutation $w_0 : [1, 2, ..., r, r+1] \mapsto [r+1, r, ..., 2, 1]$. Then $-w_0$ preserves A_r^+ and the vector $e^1 - e^{r+1}$. It transforms $\Phi = A_r^+ \text{ minus } (e^{r-1} - e^r)$ in $A_r^+ \text{ minus } (e^2 - e^3)$. Thus we obtain from LEMMA 5

Corollary 25 *We have*

$$3v(A_r^+, \mathfrak{c}^+)(e^1 - e^{r+1}) = v(A_r^+ \text{ minus } (e^{r-1} - e^r), \mathfrak{c}^+)(e^1 - e^{r+1})$$

$$= v(A_r^+ \text{ minus } (e^2 - e^3), \mathfrak{c}^+)(e^1 - e^{r+1}).$$

Bibliographical remarks.
Proposition 23 is due to [S-B].

6 Calculation of some total residues for the system A_{r+1}

Consider a $(r+2)$ dimensional real vector space, with basis $e^0, e^1, e^2, ..., e^r, e^{r+1}$. In this Section, we consider A_{r+1} as the collection of elements $(e^i - e^j)$ with $0 \leq i \leq r+1, 0 \leq j \leq r+1$, and $i \neq j$.

Let

$$\Pi = \{(e^0 - e^1), (e^1 - e^2), ..., (e^r - e^{r+1})\}$$

be the set of simple roots. Let

$$f_\Pi = \frac{1}{(e^0 - e^1)(e^1 - e^2) \cdots (e^r - e^{r+1})}.$$

This is an element of $S_{A_{r+1}}$ (in particular is homogeneous of degree $-(r+1)$).

As explained in the introduction, we are particularly interested in the Σ_r anti-invariant function of homogeneous degree $-(r+1)$ given by

$$W_{r+1} = \frac{(e^0 - e^{r+1})^{r(r+1)/2}}{\prod_{0 \leq i < j \leq r+1} (e^i - e^j)}.$$

As seen by the Example 2 of Section 1, this function is not in the space S_Δ. However, its projection on S_Δ is particularly nice.

We prove in this section the following Theorem

Theorem 26 *We have*

$$Tres_{A^+_{r+1}}\left(\frac{(e^0-e^{r+1})^{r(r+1)/2}}{\prod_{0\le i<j\le r+1}(e^i-e^j)}\right)$$

$$=\prod_{i=1}^{r-1}\frac{(2i)!}{i!(i+1)!}\sum_{w\in\Sigma_r}\epsilon(w)w\cdot\frac{1}{(e^0-e^1)(e^1-e^2)\cdots(e^{r-1}-e^r)(e^r-e^{r+1})}.$$

In fact, we will prove more general identities, which are reformulation of Morris identity.

Consider the group Σ_r of permutations of $\{1,...,r\}$. Let $0\le \ell\le r$ and denote by $P_{\ell,r}$ the Σ_r-invariant polynomial

$$P_{\ell,r}=\sum_{w\in\Sigma_r} w\cdot[(e^1-e^{r+1})(e^2-e^{r+1})...(e^\ell-e^{r+1})].$$

In particular

$$P_{0,r}=r! \qquad\qquad P_{r,r}=r!\prod_{j=1}^{r}(e^j-e^{r+1}).$$

When r is fixed, we will write P_ℓ for $P_{\ell,r}$, when $\ell>0$.

We consider the rational function given by

$$\phi_{r+1}(\ell,k_1,k_2,k_3)$$
$$=\frac{P_\ell}{(\prod_{j=1}^r(e^j-e^{r+1}))^{k_1}(\prod_{j=1}^r(e^0-e^j))^{k_2}(\prod_{1\le i<j\le r}(e^i-e^j))^{k_3}}.$$

In particular

$$\phi_{r+1}(0,k_1,k_2,k_3)$$
$$=r!\frac{1}{(\prod_{j=1}^r(e^j-e^{r+1}))^{k_1}(\prod_{j=1}^r(e^0-e^j))^{k_2}(\prod_{1\le i<j\le r}(e^i-e^j))^{k_3}}.$$

Here, k_1, k_2 and k_3 are non negative integers, so that $\phi_{r+1}(\ell,k_1,k_2,k_3)$ is an element of $R_{A_{r+1}}$ of homogeneity degree $\ell-(k_1+k_2)r-k_3\frac{r(r-1)}{2}$. If k_3 is odd, this function is anti-invariant under the group Σ_r of permutations of $\{1,...,r\}$. If k_3 is even, this function is invariant.

Remark that if $k_1 \geq 1$,

$$\phi_{r+1}(r, k_1, k_2, k_3) = \phi_{r+1}(0, k_1 - 1, k_2, k_3).$$

Let $k_1, k_2, k_3 \geq 0$, $0 \leq \ell \leq r$. Let $D = (k_1 + k_2)r + k_3\frac{r(r-1)}{2} - \ell$.

Then the function $(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3)$ is of homogeneity degree equal to $-(r+1)$.

We have in particular

$$W_{r+1} = \frac{1}{r!}(e^0 - e^{r+1})^{r(r+1)/2}\phi_{r+1}(0, 1, 1, 1).$$

Theorem 27 *Let $k_1, k_2, k_3 \geq 0$, $0 \leq \ell \leq r$. Let $D = (k_1+k_2)r+k_3\frac{r(r-1)}{2}-\ell$.*

Then the function $(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3)$ is of homogeneity degree equal to $-(r+1)$, and we have

- *If k_3 is odd,*

$$Tres_{A^+_{r+1}}((e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3))$$

$$= C_{r+1}(\ell, k_1, k_2, k_3)[\sum_{w\in\Sigma_r} \epsilon(w)w.f_\Pi].$$

- *If k_3 is even,*

$$Tres_{A^+_{r+1}}((e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3))$$

$$= C_{r+1}(\ell, k_1, k_2, k_3)[\sum_{w\in\Sigma_r} w.f_\Pi].$$

The constants $C_{r+1}(\ell, k_1, k_2, k_3)$ are determined uniquely by the relations:

- *for $1 \leq \ell \leq r$,*

$$(k_1 + k_2 - 2 + \frac{k_3}{2}(2r - \ell - 1))C_{r+1}(\ell, k_1, k_2, k_3)$$

$$= (k_1 - 1 + \frac{k_3}{2}(r - \ell))C_{r+1}(\ell - 1, k_1, k_2, k_3),$$

- $$C_{r+1}(r,k_1,k_2,k_3)=C_{r+1}(0,k_1-1,k_2,k_3),$$
- $$C_{r+1}(r-1,1,k_2,k_3)=C_r(0,k_3,k_2,k_3).$$
- $$C_{r+1}(0,k_1,k_2,k_3)=C_{r+1}(0,k_2,k_1,k_3).$$
- $$C_{r+1}(0,1,1,0)=r!.$$
- *If k_1 or $k_2=0$,*
$$C_{r+1}(\ell,k_1,k_2,k_3)=0.$$

Remark 1. The function $(e^0-e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell,k_1,k_2,k_3)$ is invariant or anti-invariant under the group Σ_r depending on the parity of k_3, then its total residue must be an element of $S_{A_{r+1}}$ which is invariant or anti-invariant by Σ_r. There are $(r+1)$ linearly independent such functions. Let us consider the basis $w\cdot f_\Pi$ of $S_{A_{r+1}}$ with w a permutation of $\{0,1,2,...,r\}$. For homogeneity reasons, it is easy to see that the iterated residue $Ires^\sigma_{x=0}$ of the function $x_0^{D-(r+1)}\phi_{r+1}(\ell,k_1,k_2,k_3)(x_0,x_1,....,x_r,0)$ is equal to 0, if the permutation σ of $\{0,1,....r\}$ does not leave 0 fixed. Thus the total residue of $(e^0-e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell,k_1,k_2,k_3)$ belongs to the vector space spanned by elements f_w with $w\in\Sigma_r$, and the total residue of the function

$$(e^0-e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell,k_1,k_2,k_3))$$

is proportional to either $[\sum_{w\in\Sigma_r}\epsilon(w)w.f_\Pi]$ or $[\sum_{w\in\Sigma_r}w.f_\Pi]$. The calculation of the constant of proportionality is thus equivalent to the Morris iterated constant term identity. However, we will give here a direct proof.

The recurrence formula above determines entirely the constants

$$C_{r+1}(\ell,k_1,k_2,k_3).$$

Indeed, assume first $k_3>0$. Then, in the first relation, when $k_3>0$ and $r>1$, all constants $(k_1-1)+\frac{k_3}{2}(r-\ell)$ are strictly positive, so if $k_1>1$, we can increase ℓ to $\ell=r$, then using the second relation, we decrease k_1 to k_1-1. Now, if $k_1-1>1$, then using one we can again increase $\ell=0$

to $\ell = r$, and using the second decrease to $k_1 - 2$. In conclusion, we may determine using alternatively one and two, the constant $C_{r+1}(\ell, k_1, k_2, k_3)$ from $C_{r+1}(\ell, 1, k_2, k_3)$. If $k_1 = 1$, $k_3 > 0$ and $r > 1$, we can increase ℓ up to $(r-1)$, using the first relation. Then using the third, we decrease $r+1$ to r. In conclusion we determine $C_{r+1}(\ell, k_1, k_2, k_3)$, if $k_3 > 0$ from the value of $C_2(\ell, k_1, k_2, k_3)$. But if $r = 1$, there is no factor corresponding to k_3, so that $C_2(\ell, k_1, k_2, k_3) = C_2(\ell, k_1, k_2, 0)$. Now, check that constants $C_{r+1}(\ell, k_1, k_2, 0)$ are uniquely determined by the recurrence relations above. The first one reads, for $1 \leq \ell \leq r$,

$$(k_1 + k_2 - 2)C_{r+1}(\ell, k_1, k_2, 0) = (k_1 - 1)C_{r+1}(\ell - 1, k_1, k_2, 0).$$

In the same way, using alternatively one and two, we compute $C_{r+1}(\ell, k_1, k_2, 0)$ from the value of $C_{r+1}(\ell, 1, k_2, 0)$. Relation above shows that $C_{r+1}(\ell, 1, k_2, 0) = 0$, if ℓ is not equal to 0. It remains to compute $C_{r+1}(0, 1, k, 0)$. By the symmetry relation, we can also assume $k = 1$. We are finally reduced to $C_{r+1}(0, 1, 1, 0)$.

Corollary 28 *Assume $r > 1$.*

- *If $k_3 > 0$, or if $k_1 + k_2 > 2$, then for $1 \leq \ell \leq r$,*

$$C_{r+1}(\ell, k_1, k_2, k_3) = \prod_{j=1}^{\ell} \frac{k_1 - 1 + (r-j)\frac{k_3}{2}}{k_1 + k_2 - 2 + (2r - j - 1)\frac{k_3}{2}} C_{r+1}(0, k_1, k_2, k_3).$$

- *If $k_1 + k_2 \geq 2$,*

$$C_{r+1}(0, k_1, k_2, k_3) = r! \prod_{j=0}^{r-1} \frac{\Gamma(1 + \frac{k_3}{2})\Gamma(k_1 + k_2 - 1 + (r + j - 1)\frac{k_3}{2})}{\Gamma(1 + (j+1)\frac{k_3}{2})\Gamma(k_1 + j\frac{k_3}{2})\Gamma(k_2 + j\frac{k_3}{2})}.$$

Corollary 29 *We have*

- $$C_{r+1}(\ell, 1, 1, 1) = \prod_{j=1}^{\ell} \frac{(r-j)}{(2r - j - 1)} C_{r+1}(0, 1, 1, 1).$$

- $$C_{r+1}(0,1,1,1) = r! \prod_{i=1}^{r-1} C_i,$$

  where $C_i = \frac{(2i)!}{i!(i+1)!}$ is the i-th Catalan number.

- $$C_{r+1}(0,k,1,1) = r! \prod_{i=k-1}^{r+k-3} \frac{1}{2i+1} \binom{r+k+i-1}{2i}.$$

- $$C_{r+1}(\ell,k,1,1) = \prod_{j=1}^{\ell} \frac{2(k-1)+(r-j)}{2(k-1)+(2r-j-1)} C_{r+1}(0,k,1,1).$$

The second corollary is of course a consequence of the first, using several times the duplication formula for the Gamma function, but it is somewhat easier to use directly the recurrence formulas in k_1, k_2, k_3, with $k_3 = 1$ as the value of k_3 remains constant and equal to 1, through the recurrence.

THEOREM 26 is then a corollary of THEOREM 27 and COROLLARY 29.

Remark that

$$\frac{1}{r!} C_{r+1}(0,1,1,1) = \frac{1}{(r-1)!} C_r(0,2,1,1) = \prod_{i=1}^{r-1} C_i = \prod_{i=1}^{r-2} \frac{1}{2i+1} \binom{r+i}{2i}.$$

Let us first verify the corollaries , assuming THEOREM 27.

To verify the first corollary, we verify the recurrence relations, the first being obvious, we check the second:

$$C_{r+1}(r, k_1, k_2, k_3) = C_{r+1}(0, k_1 - 1, k_2, k_3).$$

We write

$$\begin{aligned} &C_{r+1}(r, k_1, k_2, k_3) \\ &= r! \prod_{j=1}^{r} \frac{k_1 - 1 + (r-j)\frac{k_3}{2}}{k_1 + k_2 - 2 + (2r-j-1)\frac{k_3}{2}} \\ &\times \prod_{j=0}^{r-1} \frac{\Gamma(1+\frac{k_3}{2})\Gamma(k_1+k_2-1+(r+j-1)\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_1+j\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}. \end{aligned}$$

In the first product, we change j in $(r-j)$, in the second we use $\Gamma(z+1) = z\Gamma(z)$ and we obtain:

$$r!\prod_{j=0}^{r-1}\frac{k_1-1+j\frac{k_3}{2}}{k_1+k_2-2+(r+j-1)\frac{k_3}{2}}\frac{\Gamma(1+\frac{k_3}{2})(k_1+k_2-2+(r+j-1)\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})(k_1-1+j\frac{k_3}{2})}$$

$$\times\prod_{j=0}^{r-1}\frac{\Gamma(k_1-1+k_2-1+(r+j-1)\frac{k_3}{2})}{\Gamma(k_1-1+j\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}$$

$$= C_{r+1}(0,k_1-1,k_2,k_3).$$

We verify the third condition.

We write

$$C_{r+1}(r-1,1,k_2,k_3)$$

$$= r!\prod_{j=1}^{r-1}\frac{(r-j)\frac{k_3}{2}}{k_2-1+(2r-j-1)\frac{k_3}{2}}\prod_{j=0}^{r-1}\frac{\Gamma(1+\frac{k_3}{2})\Gamma(k_2+(r+j-1)\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(1+j\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}$$

$$= r!\prod_{j=1}^{r-1}\frac{(r-j)\frac{k_3}{2}}{k_2-1+(2r-j-1)\frac{k_3}{2}}\prod_{j=0}^{r-1}\frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}$$

$$\times\Gamma(k_2+(r-1)\frac{k_3}{2})\prod_{j=1}^{r-1}\frac{\Gamma(k_2+(r+j-1)\frac{k_3}{2})}{\Gamma(1+j\frac{k_3}{2})}$$

In the first product, we change j in $(r-j)$, in the last we use $\Gamma(z+1) = z\Gamma(z)$, and we obtain after simplification that $C_{r+1}(r-1,1,k_2,k_3)$ is equal to

$$r!\prod_{j=0}^{r-1}\frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}\Gamma(k_2+(r-1)\frac{k_3}{2})$$

$$\times\prod_{j=1}^{r-1}\frac{\Gamma(k_2-1+(r+j-1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})}$$

$$= r!\prod_{j=0}^{r-2}\frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}\frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+r\frac{k_3}{2})}\prod_{j=1}^{r-1}\frac{\Gamma(k_2-1+(r+j-1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})}$$

$$= (r-1)! \prod_{j=0}^{r-2} \frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})} \frac{\Gamma(\frac{k_3}{2})}{\Gamma(r\frac{k_3}{2})} \prod_{j=1}^{r-1} \frac{\Gamma(k_2-1+(r+j-1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})}$$

while

$$C_r(0,k_3,k_2,k_3) = (r-1)! \prod_{j=0}^{r-2} \frac{\Gamma(1+\frac{k_3}{2})\Gamma(k_3+k_2-1+(r-1+j-1)\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_3+j\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}.$$

$$= (r-1)! \prod_{j=0}^{r-2} \frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})} \prod_{j=0}^{r-2} \frac{\Gamma(k_2-1+(r+j)\frac{k_3}{2})}{\Gamma((j+2)\frac{k_3}{2})}$$

It remains to verify

$$\frac{\Gamma(\frac{k_3}{2})}{\Gamma(r\frac{k_3}{2})} \prod_{j=1}^{r-1} \frac{\Gamma(k_2-1+(r+j-1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})} = \prod_{j=0}^{r-2} \frac{\Gamma(k_2-1+(r+j)\frac{k_3}{2})}{\Gamma((j+2)\frac{k_3}{2})}.$$

which is true.

The remaining properties are obvious.

We now prove THEOREM 27 by induction on r.

Proof. If $k_1 = 0$, the remaining roots $(e^i - e^j)$ occurring in the denominator of $\phi_{r+1}(\ell, 0, k_2, k_3)$ are contained in the hyperplane $\sum_{i=0}^r e^i = 0$. So the total residue of $\phi_{r+1}(\ell, 0, k_2, k_3)$ is 0. The same argument shows that $\phi_{r+1}(\ell, k_1, 0, k_3)$ is 0 .

We thus may assume that $k_1, k_2 > 0$. We first show that the function $(e^0 - e^{r+1})\phi_{r+1}(\ell-1, k_1, k_2, k_3)$ is proportional to the function $\phi_{r+1}(\ell, k_1, k_2, k_3)$ modulo $\sum_{i=1}^r \partial_i R_{A_{r+1}^+}$. Thus the functions $(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell-1, k_1, k_2, k_3)$ and $(e^0 - e^{r+1})^{D-r}\phi_{r+1}(\ell-1, k_1, k_2, k_3)$ will be proportional too, modulo the vector space $\sum_{i=1}^r \partial_i R_{A_{r+1}^+}$, their total residues will be proportional and we will get the first recursive relations for the constant C_{r+1}.

In the following, we use x_i instead of e^i, etc... as it is a more familiar notation for computing derivatives.

We write $U = (\prod_{j=1}^r (x_j - x_{r+1}))^{-k_1} (\prod_{j=1}^r (x_0 - x_j))^{-k_2} (\prod_{1 \le i < j \le r} (x_i - x_j))^{-k_3}$ so that $P_\ell U = \phi_{r+1}(\ell, k_1, k_2, k_3)$.

We compute

$$\partial_1[(x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U]$$

This is equal to

$$\begin{gathered}
-(1-k_2)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1})U \\
+(1-k_1)(x_0 - x_1)(x_2 - x_{r+1})...(x_\ell - x_{r+1})U \\
-k_3(x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1}) \sum_{j=2}^{r} \frac{1}{x_1 - x_j} U.
\end{gathered}$$

Using $(x_0 - x_1) = (x_0 - x_{r+1}) + (x_{r+1} - x_1)$, this is also equal to

$$\begin{gathered}
= (k_1 + k_2 - 2)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1})U \\
+(1-k_1)(x_0 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1})U \\
-k_3(x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1}) \sum_{j=2}^{r} \frac{1}{x_1 - x_j} U.
\end{gathered}$$

Assume first that k_3 is odd, so that U is anti-invariant by the group Σ_r.

Let us antisymmetrize over permutations. We obtain

$$\sum_{w \in \Sigma_r} \epsilon(w) w \cdot (\partial_1((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1})U)) =$$

$$\begin{gathered}
(k_1 + k_2 - 2)P_\ell U + (1-k_1)(x_0 - x_{r+1})P_{\ell-1} U \\
-k_3 \sum_{w \in \Sigma_r} w \cdot \left((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1}) \sum_{j=2}^{r} \frac{1}{x_1 - x_j} \right) U.
\end{gathered}$$

To compute $\sum_{w \in \Sigma_r} w \cdot ((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1}) \frac{1}{x_1 - x_j})$, we first sum over the transposition $(j, 1)$.

If $2 \leq j \leq \ell$, we use

$$\frac{(x_0 - x_1)(x_1 - x_{r+1})(x_j - x_{r+1})}{x_1 - x_j} + \frac{(x_0 - x_j)(x_j - x_{r+1})(x_1 - x_{r+1})}{x_j - x_1}$$

$$= -(x_1 - x_{r+1})(x_j - x_{r+1}).$$

If $j > \ell$, we use

$$\frac{(x_1 - x_{r+1})(x_0 - x_1)}{x_1 - x_j} + \frac{(x_j - x_{r+1})(x_0 - x_j)}{x_j - x_1}$$

$$= (x_0 - x_1) + (x_{r+1} - x_j) = (x_0 - x_{r+1}) + (x_{r+1} - x_1) + (x_{r+1} - x_j).$$

We obtain that

$$2 \sum_{w \in \Sigma_r} w \cdot ((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1}) \sum_{j=2}^{r} \frac{1}{x_1 - x_j})$$

is equal to

$$(-(\ell - 1) - 2(r - \ell))P_\ell + (x_0 - x_{r+1})(r - \ell)P_{\ell-1}.$$

Thus finally, we obtain

$$\sum_{w \in \Sigma_r} \epsilon(w) w \cdot (\partial_1((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1})\phi_{r+1}(0, k_1, k_2, k_3)) =$$

$$(k_1 + k_2 - 2 + \frac{k_3}{2}(2r - \ell - 1))\phi_{r+1}(\ell, k_1, k_2, k_3)$$

$$+(x_0 - x_{r+1})(-k_1 + 1 - \frac{k_3}{2}(r - \ell))\phi_{r+1}(\ell - 1, k_1, k_2, k_3).$$

If k_3 is even, we also obtain

$$\sum_{w \in \Sigma_r} w \cdot (\partial_1((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1}))\phi_{r+1}(0, k_1, k_2, k_3)) =$$

$$(k_1 + k_2 - 2 + \frac{k_3}{2}(2r - \ell - 1))\phi_{r+1}(\ell, k_1, k_2, k_3)$$

$$+(x_0 - x_{r+1})(-k_1 + 1 - \frac{k_3}{2}(r - \ell))\phi_{r+1}(\ell - 1, k_1, k_2, k_3).$$

Thus we see that

$$(x_0 - x_{r+1})\phi_{r+1}(\ell - 1, k_1, k_2, k_3) \text{ is proportional to } \phi_{r+1}(\ell, k_1, k_2, k_3),$$

modulo derivatives with respect to x_1, x_2,..., x_r. In particular the total residue of the function $(x_0 - x_{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3)$ is proportional to the total residue of the function $(x_0 - x_{r+1})^{D-(r+1)+1}\phi_{r+1}(\ell - 1, k_1, k_2, k_3)$. This proves the first property.

We proceed to the proof of the second property. We return to the notation $x_i = e^i$.

To avoid confusion in the following argument we will write explicitly the dependence by the parameters of D, that is we will write, whenever necessary,

$$D = D_{A_{r+1}}(\ell, k_1, k_2, k_3) = (k_1 + k_2)r + k_3\frac{r(r-1)}{2} - \ell.$$

We now compute the total residue of

$$(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(r-1, 1, k_2, k_3)$$

with $D = D_{A_{r+1}}(r-1, 1, k_2, k_3) = (1 + k_2)r + k_3\frac{r(r-1)}{2} - (r-1)$.

We have:

$$r!(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(r-1, 1, k_2, k_3)$$

$$= (e^0 - e^{r+1})^{D-(r+1)}\frac{P_{r-1}}{(e^1 - e^{r+1})(e^2 - e^{r+1})...(e^r - e^{r+1})}\phi_{r+1}(0, 0, k_2, k_3).$$

with

$$\frac{P_{r-1}}{(e^1 - e^{r+1})(e^2 - e^{r+1})...(e^r - e^{r+1})}$$

$$\frac{\sum_{w\in\Sigma_r} w \cdot [(e^1 - e^{r+1})(e^2 - e^{r+1})...(e^{r-1} - e^{r+1})]}{(e^1 - e^{r+1})(e^2 - e^{r+1})...(e^r - e^{r+1})}$$

$$= (r-1)!\sum_{j=1}^{r}\frac{1}{(e^j - e^{r+1})}.$$

Consider the subgroup Cyc_r generated by the circular permutation of $1, ..., r$. Then

$$\sum_{j=1}^{r}\frac{1}{(e^j - e^{r+1})} = \sum_{w\in Cyc_r} w \cdot \frac{1}{(e^r - e^{r+1})}.$$

Assume k_3 odd. Thus

$$\frac{r!}{(r-1)!}\phi_{r+1}(r-1, 1, k_2, k_3) = [\sum_{w\in Cyc_r} w \cdot \frac{1}{(e^r - e^{r+1})}]\phi_{r+1}(0, 0, k_2, k_3)$$

$$= \sum_{w \in Cyc_r} \epsilon(w) w \cdot [\frac{1}{(e^r - e^{r+1})} \phi_{r+1}(0,0,k_2,k_3)]$$

as $\phi_{r+1}(0,0,k_2,k_3)$ is anti-invariant under Σ_r.

Remark that:

$$\phi_{r+1}(0,0,k_2,k_3) = r \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0,k_3,k_2,k_3)$$

so that

$$(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \phi_{r+1}(0,0,k_2,k_3)$$

$$= r(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0,k_3,k_2,k_3).$$

It follows that we have

$$(e^0 - e^{r+1})^{D-(r+1)} \phi_{r+1}(r-1,1,k_2,k_3)$$

$$= \sum_{w \in Cyc_r} \epsilon(w) w \cdot [(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0,k_3,k_2,k_3)].$$

We now use LEMMA 3 to compute the total residue of the last term in the equality.

We write the vector space E_{r+1} as $E_r \oplus \mathbb{R}(e^r - e^{r+1})$, and we consider $\Delta' = A_r \cup \{(e^r - e^{r+1})\}$

Using the decomposition $(e^0 - e^{r+1}) = (e^r - e^{r+1}) + (e^0 - e^r)$, we write

$$(e^0 - e^{r+1})^{D-(r+1)} = \sum_{i \geq 0, j \geq 0, i+j = D-(r+1)} c_{ij} (e^r - e^{r+1})^i (e^0 - e^r)^j.$$

Thus

$$(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0,k_3,k_2,k_3)$$

$$= \sum_{i \geq 0, j \geq 0, i+j = D-(r+1)} c_{ij} \frac{(e^r - e^{r+1})^i}{(e^r - e^{r+1})} \frac{(e^0 - e^r)^j}{(e^0 - e^r)^{k_2}} \phi_r(0,k_3,k_2,k_3)$$

belongs to the vector space $R_{\{(e^r-e^{r+1})\}}\otimes R_{A_r}$ and we can easily compute the total residue using LEMMA 3 of Section 1, as well as the obvious calculation for a one dimensional space. Precisely

$$Tres_{A_{r+1}}[(e^0-e^{r+1})^{D-(r+1)}\frac{1}{(e^r-e^{r+1})}\frac{1}{(e^0-e^r)^{k_2}}\phi_r(0,k_3,k_2,k_3)]$$

$$=\sum_{i\geq 0,j\geq 0,i+j=D-(r+1)} c_{ij}Tres_{(e^r-e^{r+1})}(\frac{(e^r-e^{r+1})^i}{(e^r-e^{r+1})})$$

$$\times Tres_{A_r}[\frac{(e^0-e^r)^j}{(e^0-e^r)^{k_2}}\phi_r(0,k_3,k_2,k_3)].$$

Only the term $i=0$ gives a non zero residue, so we obtain

$$Tres_{A_{r+1}}[(e^0-e^{r+1})^{D-(r+1)}\frac{1}{(e^r-e^{r+1})}\frac{1}{(e^0-e^r)^{k_2}}\phi_r(0,k_3,k_2,k_3)]$$

$$=\frac{1}{(e^r-e^{r+1})}Tres_{A_r}[(e^0-e^r)^{D-(r+1)-k_2}\phi_r(0,k_3,k_2,k_3)].$$

Now

$$D-(r+1)-k_2=D_{A_{r+1}}(r-1,1,k_2,k_3)-(r+1)-k_2=D_{A_r}(0,k_3,k_2,k_3)-r.$$

So we obtain

$$Tres_{A_{r+1}}[(e^0-e^{r+1})^{D-(r+1)}\frac{1}{(e^r-e^{r+1})}(e^0-e^r)^{k_2}\phi_r(0,k_3,k_2,k_3)]$$

$$=\frac{1}{(e^r-e^{r+1})}Tres_{A_r}[(e^0-e^r)^{D_{A_r}(0,k_3,k_2,k_3)-r}\phi_r(0,k_3,k_2,k_3)].$$

We apply induction hypothesis on r. We have

$$Tres_{A_r}[(e^0-e^r)^{D_{A_r}(0,k_3,k_2,k_3)-r}\phi_r(0,k_3,k_2,k_3)]$$

$$=C_r(0,k_3,k_2,k_3)\sum_{w'\in\Sigma_{r-1}}\epsilon(w')w'\cdot[\frac{1}{(e^0-e^1)(e^1-e^2)\cdots(e^{r-1}-e^r)}].$$

By FORMULA 6, as the total residue commutes with the action of W, we obtain:

$$Tres_{A_{r+1}}(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(r-1,1,k_2,k_3)$$
$$= C_r(0,k_3,k_2,k_3) \sum_{w\in Cyc_r} \epsilon(w)w\cdot[\frac{1}{(e^r - e^{r+1})}]$$
$$\times \sum_{w'\in\Sigma_{r-1}} \epsilon(w')w'\cdot[\frac{1}{(e^0-e^1)\cdots(e^{r-1}-e^r)}].$$

But

$$\sum_{w\in Cyc_r} \epsilon(w)w\cdot[\frac{1}{(e^r - e^{r+1})}] \sum_{w'\in\Sigma_{r-1}} \epsilon(w')w'\cdot[\frac{1}{(e^0-e^1)(e^1-e^2)\cdots(e^{r-1}-e^r)}]$$

$$= \sum_{ww'\in Cyc_r\Sigma_{r-1}=\Sigma_r} \epsilon(ww')ww'\cdot[\frac{1}{(e^r - e^{r+1})}\frac{1}{(e^0-e^1)(e^1-e^2)\cdots(e^{r-1}-e^r)}]$$

$$= \sum_{w\in\Sigma_r} \epsilon(w)w.f_\Pi.$$

Thus we obtain the second relation. The case k_3 even is completely analogous, so the proof of the first and second relation is complete. The symmetry property in k_1, k_2 is obvious.

Let us check

$$C_{r+1}(0,1,1,0) = r!.$$

More precisely, we have the following exact formula (without projection on S_Δ).

Lemma 30

$$\frac{(e^0-e^{r+1})^{(r-1)}}{\prod_{i=1}^r (e^0-e^i)(e^i-e^r)} = \sum_{w\in\Sigma_r} w\cdot f_\Pi.$$

Indeed, by reduction to the same denominator, the right hand side can be written as

$$\frac{P}{\prod_{i=1}^r (e^0-e^i)(e^i-e^r)\prod_{1\le i<j\le r}(e^i-e^j)}$$

¿From invariance consideration, P has to be anti-invariant under Σ_r , so is divisible by $\prod_{1\leq i<j\leq r}(e^i - e^j)$. From degree consideration, we obtain the desired equality, and $C_{r+1}(0,1,1,0) = r!$.

QED

Bibliographical remarks.

For the proofs of Theorem 26 and 27, we followed the indications of Zeilberger [Z]. Indeed, as explained in the remark following Theorem 27, these formulae are closely related to calculations of Selberg integral [Se], and to Morris identity [M]. We follow here closely Aomoto's [A0] proof of the Selberg formula .

7 Chan-Robbins-Yuen polytope

Let B_n be the polytope of $n \times n$ doubly stochastic matrices; that is the set of $n \times n$ matrices with non negative entries, and such that the sum of entries in any row or column is equal to 1. The vector space of $n \times n$ matrices is equipped with the lattice of matrices with integral coefficients. The intersection of this lattice with the affine space spanned by B_n is an affine lattice, thus determines an unity of volume. The volume of B_n is defined with respect to this unity. It has been computed only up to $n = 8$ ([CY]). In [C-R-Y], some conjectures on the volume of some faces of B_n were stated. We show that they follow from Jeffrey-Kirwan formula for volumes, together with the explicit calculation of the preceding paragraph.

Definition 31 *Define the vector space Y_n as the vector space of $(n \times n)$ matrices, with $x_{ij} = 0$, if $j \geq (i+2)$. The Chan-Robbins-Yuen polytope is the polytope $CRY_n = B_n \cap Y_n$.*

Consider the vector space of $(n+1) \times (n+1)$ lower triangular matrices such that first and last coefficients in the diagonal are equal to 0. Clearly this vector space is isomorphic to Y_n, by adding to a matrix $X \in Y_n$ a row above identically equal to 0, and a last column identically equal to 0.

Example $n = 5$.

The vector space Y_5 consists of 5×5 matrices $X = (x_{ij})$ such that

$$X = \begin{pmatrix} x_{11} & x_{12} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 \\ x_{31} & x_{32} & x_{33} & x_{34} & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix}$$

The augmented matrix $aug(X)$ is the 6×6 lower triangular matrix

$$aug(X) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ x_{11} & x_{12} & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & x_{34} & 0 & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & 0 \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & 0 \end{pmatrix}$$

It is clear that the transformation $X \mapsto aug(X)$ transforms the polytope CRY_n to the polytope C'_n of $(n+1) \times (n+1)$ lower triangular matrices with non negative entries, first row and last column identically equal to 0, and other rows and columns summing to 1.

Example $n = 5$**.** The polytope C'_5 consists of matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & 0 \end{pmatrix}$$

such that $a_{ij} \geq 0$, and such that

$$a_{21} + a_{31} + a_{41} + a_{51} + a_{61} = 1,$$

$$a_{21} + a_{22} = 1 = a_{22} + a_{32} + a_{42} + a_{52} + a_{62},$$

....

$$a_{51} + a_{52} + a_{53} + a_{54} + a_{55} = 1 = a_{55} + a_{65},$$

$$a_{61} + a_{62} + a_{63} + a_{64} + a_{65} = 1.$$

Let $\mathfrak{n}$ be the vector space of strictly lower triangular $(n+1)\times(n+1)$-matrices, with basis $E_{ij}, (j<i)$ with unique non zero entry at the i-th row and j column. Consider a $(n+1)$ dimensional vector space with basis $e_i, 1\leq i\leq (n+1)$ and dual basis e^i. Let $A_n=\{(e^k-e^\ell); k\neq \ell\}$ with positive root system $\{(e^k-e^\ell); 1\leq k<\ell\leq (n+1)\}$. Let p be the map sending the basis E_{ji} of the vector space $\mathfrak{n}$ to (e^j-e^i), for $1\leq j<i\leq (n+1)$. The span of A_n is the n-dimensional vector space E_n consisting of elements $a=\sum_{i=1}^{n+1} a_i e^i$, with $\sum_{i=1}^{n+1} a_i=0$

Lemma 32 *The polytope CRY_n is isomorphic to $P_{A_n^+}(e^1-e^{n+1})$.*

We give the proof for $n=5$, as it is most transparent on an example. By definition the polytope $P_{A_5^+}(e^1-e^6)$ consists of all (6×6) lower triangular matrices

$$B=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ b_{21} & 0 & 0 & 0 & 0 & 0\\ b_{31} & b_{32} & 0 & 0 & 0 & 0\\ b_{41} & b_{42} & b_{43} & 0 & 0 & 0\\ b_{51} & b_{52} & b_{53} & b_{54} & 0 & 0\\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & 0\end{pmatrix}$$

with non negative coefficients b_{ij} and such that

$$\sum b_{ij}(e^j-e^i)=e^1-e^6.$$

This gives the 6 equations:

$$\begin{gathered} b_{21}+b_{31}+b_{41}+b_{51}+b_{61}=1,\\ -b_{21}+b_{32}+b_{42}+b_{52}+b_{62}=0,\\ -b_{31}-b_{32}+b_{43}+b_{53}+b_{63}=0,\\ \dots\dots\dots\\ -b_{61}-b_{62}-b_{63}-b_{64}-b_{65}=-1.\end{gathered}$$

Define

$$\begin{gathered} b'_{22}=1-b_{21},\quad b'_{33}=1-(b_{31}+b_{32})\\ b'_{44}=1-(b_{41}+b_{42}+b_{43})\\ b'_{55}=1-(b_{51}+b_{52}+b_{53}+b_{54}).\end{gathered}$$

These coefficients are positive: indeed the total sum of coefficients of the entries of the matrix B in the rectangle R_{ij} consisting on elements in a row of index greater than i and on a column of index less than j is identically equal to 1, so that any partial sum is less then 1. For example, adding first 3 inequalities, we obtain that

$$b_{41}+b_{42}+b_{43}+$$
$$b_{51}+b_{52}+b_{53}+$$
$$b_{61}+b_{62}+b_{63}$$

is equal to 1, so that $b_{41}+b_{42}+b_{43}$ is less than 1.

Then the matrix

$$B' = \begin{pmatrix} b_{21} & b'_{22} & 0 & 0 & 0 \\ b_{31} & b_{32} & b'_{33} & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b'_{44} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b'_{55} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} \end{pmatrix}$$

is in the polytope CRY_5.

We now show that the conjectures of Chan-Robbins-Yuen on their polytope CRY_n and related polytopes are consequences of THEOREM 26 and PROPOSITION 23

Theorem 33 *(Zeilberger) The relative volume of CRY_n is equal to*

$$\prod_{i=1}^{n-2} \frac{1}{i+1}\binom{2i}{i}.$$

Proof. The polytope CRY_n is isomorphic to the polytope $P_{A_n^+}(e^1-e^{n+1})$. The linear isomorphism described above using the map aug preserves the volume, as it preserves the corresponding lattices. We thus compute the relative volume of $P_{A_n^+}(e^1-e^{n+1})$. Consider

$$J_n = \frac{(e^1-e^{n+1})^{n(n-1)/2}}{\prod_{1\leq i<j\leq (n+1)}(e^i-e^j)}.$$

It is an element of R_{A_n} of homogeneous degree $-n$. We choose the big chamber $\mathfrak{c}^+$ in E_n consisting of elements $a=\sum_{i=1}^{n+1} a_i e^i$, with $a_i>0$, for

$i = 1, 2, ..., n$. The element $a = e^1 - e^{n+1}$ is in the closure of this big chamber. Jeffrey-Kirwan formula is:

$$\mathrm{vol}_{rel}\, P_{A_n^+}(e^1 - e^{n+1}) = \langle\langle \mathfrak{c}^+, Tres_{A_n}(J_n)\rangle\rangle.$$

The function J_n up to renumbering has been introduced in Section 5. This is the function W_n. Recall the formula for its total residue given in THEOREM 26.

Let

$$f_\Pi = \frac{1}{(e^1 - e^2)(e^2 - e^3)\cdots(e^n - e^{n+1})}$$

be the particular element of S_Δ obtained by taking the inverse of the product of simple roots. Then

$$Tres_{A_n}(J_n) = \prod_{i=1}^{n-2} \frac{1}{i+1}\binom{2i}{i}[\sum_{w\in\Sigma[2,\ldots,n]} \epsilon(w) w.f_\Pi.]$$

The linear form $\langle\langle \mathfrak{c}^+, f\rangle\rangle$ has been computed in Section 4. We write

$$j_n(x) = \frac{x_1^{n(n-1)/2}}{x_1 \ldots x_n \prod_{1\leq i<j\leq n}(x_i - x_j)}.$$

It is the function J_n restricted at elements $\sum_{i=1}^n x_i e_i$. Then

$$\langle\langle \mathfrak{c}^+, J_n\rangle\rangle = Ires_{x=0} j_n(x).$$

As iterated residues form a dual basis to the basis $w.f_\Pi$ of S_Δ (with $w \in \Sigma[1, 2, ..., n]$), we obtain $\langle\langle \mathfrak{c}^+, J_n\rangle\rangle = \prod_{i=1}^{n-2} \frac{1}{i+1}\binom{2i}{i}$.

QED

Consider the polytope $CRY_n(\widehat{(2,2)})$, consisting on elements of CRY_n with the entry $(2, 2)$ equal to 0. It is a face of codimension 1 of CRY_n, isomorphic to the polytope $P_{A_n^+ \text{ minus } (e^2-e^3)}(e^1 - e^{n+1})$. Taking in account the fact that the dimension of CRY_n is $\binom{n}{2}$, we obtain from PROPOSITION 23 the following corollary.

Lemma 34 *(Conjecture 4 of [C-R-Y])*

We have

$$\binom{n}{2} \mathrm{vol}_{rel}\, CRY_n(\widehat{(2,2)}) = 3\, \mathrm{vol}_{rel}\, CRY_n.$$

Bibliographical remarks.

The Chan-Robbins-Yuen polytope is described in [C-R-Y]. Its volume is computed by Zeilberger [Z], using a formula of Postnikov- Stanley [S] or of Chan-Robbins-Yuen [C-R-Y] for the volume of CRY_n , as a particular value of Kostant partition function (see Lemma 44 in next Section 8). Our proof follows directly from the Jeffrey-Kirwan residue formulae for the volume, but we use also the crucial identity of Theorem 27. The conjectured relations between volumes of flow polytopes and the volume of a particular face is deduced from Proposition 23. However, the volume of the entire boundary of the polytope CRY_n is unknown .

8 Ehrhart polynomial of a flow polytope.

We consider the positive root system of A_r realized in $E_r \subset \mathbb{R}^{r+1}$. We will use sometimes implicitly the map

$$a = (a_1, a_2, ..., a_r) \to \mathbf{a} = a_1 e^1 + ... + a_r e^r - (a_1 + \cdots + a_r) e^{r+1}$$

to identify E_r and $\mathbb{R}^r$.

Let $M = r(r+1)/2$. Let $\mathbf{a} \in \mathbb{Z}^{r+1} \cap C(A_r^+)$. Then $a_1 \geq 0$, $a_1 + a_2 \geq 0$, ..., $a_1 + a_2 + \cdots + a_r \geq 0, a_1 + a_2 + \cdots + a_{r+1} = 0$. Let $k_{A_r^+}(a)$ be the number of solutions $(u_1, u_2, ..., u_M)$, in non negative integers u_m, of the equation

$$\sum_{m=1}^{M} u_m \alpha^m = \mathbf{a}.$$

Here α^m runs though the $r(r+1)/2$ positive roots $(e^i - e^j)$ of A_r. The function $k_{A_r^+}(a)$ is the number of integral points in the polytope $P_{A_r^+}(\mathbf{a})$ and is called the **Kostant partition function** (for A_r).

Consider the permutation $w_0 : [1, 2, ..., r, r+1] \mapsto [r+1, r, ..., 2, 1]$. Then $-w_0$ preserves A_r^+. A solution $(u_1, u_2, ..., u_M)$, in non negative integers u_m, of the equation

$$\sum_{m=1}^{M} u_m \alpha^m = \mathbf{a}$$

gives a solution of the equation

$$\sum_{m=1}^{M} u_m (-w_0 \cdot \alpha^m) = -w_0 \cdot \mathbf{a}.$$

So for $a_1 + a_2 + ... + a_{r+1} = 0$, we have

$$k_{A_r^+}(a_1 e^1 + a_2 e^2 + \cdots + a_r e^r + a_{r+1} e^{r+1})$$

$$= k_{A_r^+}(-a_{r+1} e^1 - a_r e^2 - \cdots - a_2 e^r - a_1 e^{r+1}).$$

Almost by definition, the function $k_{A_r^+}$ is given by an iterated constant term.

Lemma 35 *Let* $\mathbf{a} = a_1 e^1 + a_2 e^2 + ... + a_r e^r + a_{r+1} e^{r+1}$, *with* $a_1 + a_2 + ... + a_r + a_{r+1} = 0$. *The value* $k_{A_r^+}(\mathbf{a})$ *is given by the iterated constant term formula:*

$$k_{A_r^+}(\mathbf{a}) = CT_{z_1=0} CT_{z_2=0} ... CT_{z_{r+1}=0} \left(\frac{z_1^{a_1} z_2^{a_2} ... z_{r+1}^{a_{r+1}}}{\prod_{1 \leq i < j \leq r+1} (1 - z_j/z_i)} \right).$$

Here $\frac{1}{(1-z_{r+1}/z_i)}$ is expanded as a power series in z_{r+1}. The constant term at $z_{r+1} = 0$ of $\frac{z_{r+1}^{a_{r+1}}}{\prod_{1 \leq i \leq r+1}(1-z_{r+1}/z_i)}$ is in the ring $R[z_1^{-1}, z_2^{-1}, ..., z_r^{-1}]$ and we reiterate.

Thus, some explicit values for Kostant partition function can be obtained from the total residue formula of Section 6, as iterated constant term of a function f coincide with the iterated residue of this function divided by $z_1 z_2 ... z_r$.

For example

Lemma 36 *For* $r \geq 2$,

$$k_{A_r^+}((1+d), (2+d), ..., (r+d)) = \prod_{i=d+1}^{r+d-1} \frac{1}{2i+1} \binom{r+d+i+1}{2i},$$

$$k_{A_r^+}(1, 2, ..., r) = \prod_{i=1}^{r} C_i.$$

Let $\Phi = \{\alpha^1, \alpha^2, ..., \alpha^N\}$ be a sequence of elements of A_r^+ generating E_r.

Consider the surjective linear map $p : \mathbb{R}^N \to E_r$ such that $p(w^i) = \alpha^i$. Then, for $a \in \mathbb{Z}^r$, the polytope $P_\Phi(\mathbf{a}) \subset \mathbb{R}_+^N \cap p^{-1}(\mathbf{a})$ has vertices with integral coordinates.

Definition 37 *For $a \in \mathbb{Z}^r$, define*

$$k_\Phi(a) = |P_\Phi(\mathbf{a}) \cap \mathbb{Z}^N|.$$

Thus $k_\Phi(a)$ is the number of solutions $(u_1, u_2, ..., u_N)$, in non negative integers u_m, of the equation

$$\sum_{m=1}^{N} u_m \alpha^m = \mathbf{a}.$$

Define $k'_\Phi(a)$ as the number of solutions $(u_1, u_2, ..., u_N)$, in strictly positive integers u_m, of the equation

$$\sum_{m=1}^{N} u_m \alpha^m = \mathbf{a}.$$

Remark. Let $a = \sum_{i=1}^{r+1} a_i e^i$. If $k'_\Phi(a)$ is non zero, then necessarily $a_1 \geq \sum_{j=2}^{r+1} m_{1j}$.

We consider the system $-w_0(\Phi)$. Then clearly $k_\Phi(-w_0(a)) = k_{-w_0(\Phi)}(a)$.

Lemma 38 *Let $\mathbf{a} = a_1 e^1 + a_2 e^2 + ... + a_r e^r + a_{r+1} e^{r+1}$, with $a_1 + a_2 + ... + a_r + a_{r+1} = 0$. The value $k_\Phi(\mathbf{a})$ is given by the iterated constant term formula:*

$$k_\Phi(\mathbf{a}) = CT_{z_1=0} CT_{z_2=0} ... CT_{z_{r+1}=0} \left(\frac{z_1^{a_1} z_2^{a_2} ... z_{r+1}^{a_{r+1}}}{\prod_{1 \leq i < j \leq r+1} (1 - z_j/z_i)^{m_{ij}}} \right).$$

We also consider the number $k'(\Phi, \mathfrak{c})(a)$ of solutions in strictly positive integers u_m of the equation

$$\sum_{m=1}^{N} u_m \alpha^m = \mathbf{a}.$$

Let $\mathfrak{c}$ be a big chamber for the system $\Delta^+ = \{\Phi\}$. It follows from Ehrhart results on polytopes with integral vertices that the function $k_\Phi(a)$ is given by a polynomial formula when a varies in $\bar{\mathfrak{c}} \cap \mathbb{Z}^r$. This polynomial is called **the Ehrhart polynomial** of the family of polytopes $P_\Phi(\mathbf{a})$, **on the big chamber $\mathfrak{c}$**.

For $\alpha \in \Phi$, the rational function on E_r^*

$$\frac{\alpha}{1-e^{-\alpha}} = 1 + \frac{1}{2}\alpha + \frac{1}{12}\alpha^2 + ...$$

is analytic at the origin, and can be expanded as a Taylor series. Thus

$$\frac{1}{1-e^{-\alpha}} = \frac{1}{\alpha}(\frac{\alpha}{1-e^{-\alpha}})$$

as an element of $\hat{R}_\Delta$, and for any $\mathbf{a} \in E_r$, the function $\frac{e^{\mathbf{a}}}{\prod_{i=1}^N (1-e^{-\alpha^i})}$ is an element of $\hat{R}_\Delta$.

Define

$$K_\Phi(\mathbf{a}) = Tres_\Delta \left(\frac{e^{\mathbf{a}}}{\prod_{\alpha\in\Phi}(1-e^{-\alpha})} \right).$$

As $\frac{1}{\prod_{i=1}^N (1-e^{-\alpha^i})} = \frac{1}{\prod_{i=1}^N \alpha^i} + \sum_{k>-N} f_k$, we have

$$Tres_\Delta(\frac{e^{\mathbf{a}}}{\prod_{i=1}^N (1-e^{-\alpha_i})})$$

$$= \frac{1}{(N-r)!} Tres_\Delta(\frac{\mathbf{a}^{N-r}}{\prod_{i=1}^N \alpha^i}) + \sum_{q<(N-r)} \frac{1}{q!} Tres_\Delta(\mathbf{a}^q f_{-r-q}).$$

Thus the function

$$a \mapsto Tres_\Delta(\frac{e^{\mathbf{a}}}{\prod_{i=1}^N 1-e^{-\alpha^i}})$$

is a polynomial function of degree $(N-r)$ with value in S_Δ. Recall DEFINITION 6 of the function J_Φ. We see that the homogeneous component of degree $(N-r)$ of $K_\Phi(a)$ is

$$J_\Phi(a) = Tres_\Delta(\frac{e^{\mathbf{a}}}{\prod_{i=1}^N \alpha^i})$$

$$= \sum_{|\mathbf{i}|=N-r} f_\Phi(\mathbf{i}) \frac{a_1^{i_1}}{i_1!}\frac{a^{i_2}}{i_2!} \cdots \frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!}.$$

Khovanski-Pukhlikhov formula [KP] says that the locally polynomial function $k_\Phi(a)$ is deduced (as for any partition family of polytopes, see [B-V 1, Section 3.5]) from the locally polynomial function vol $P_\Phi(a)$ via a Todd operator. Using Jeffrey-Kirwan residue formula for the volume, we obtain (see [B-V 1 , Section 3.5]) the following residue formula for the Ehrhart polynomials $k_\Phi(a)$ and $k'_\Phi(a)$.

Theorem 39 *(Multidimensional residue theorem)*
Let $\mathbf{a} \in C(\Delta^+)$. Let $\mathfrak{c} \subset C(\Delta^+)$ be a big chamber such that $\mathbf{a} \in \overline{\mathfrak{c}}$, then

$$k_\Phi(a) = \langle\langle \mathfrak{c}, K_\Phi(a)\rangle\rangle.$$

If $a \in \mathfrak{c}$, then

$$k'_\Phi(a) = (-1)^{|\Phi|}\langle\langle \mathfrak{c}, K_\Phi(-a)\rangle\rangle.$$

For example, in case $\Delta = A_1$ and Φ with multiplicity m, the preceding theorem asserts that for $a \geq 0$

$$k_\Phi(ae^1 - ae^2) = CT_{z=0}\frac{z^{-a}}{(1-z)^m} = Res_{x=0}\frac{e^{ax}}{(1-e^{-x})^m}.$$

The first expression is the residue of the 1-form $\frac{z^{-a}}{(1-z)^m}\frac{dz}{z}$ at $z = 0$. The second is the residue at $z = 1$ of this 1-form (we choose $z = e^{-x}$ as local coordinate, near $x = 0$). As $a + m > 0$, the rational function $\frac{z^{-a}}{(1-z)^m}\frac{dz}{z}$ has no poles at $z = \infty$ and we obtain the result from the one dimensional residue theorem.

Definition 40 *We denote by $k(\Phi, \mathfrak{c})(a)$ the polynomial such that $k(\Phi, \mathfrak{c})(a) = k_\Phi(a)$ when $a \in \overline{\mathfrak{c}}$*

Thus

$$k(\Phi, \mathfrak{c})(a) = \langle\langle \mathfrak{c}, K_\Phi(a)\rangle\rangle.$$

This is a polynomial of degree $N - r$ and the homogeneous component of degree $N - r$ of $k(\Phi, \mathfrak{c})$ is the polynomial

$$v(\Phi, \mathfrak{c})(a) = \langle\langle \mathfrak{c}, J_\Phi(a)\rangle\rangle.$$

As in LEMMA 20, the knowledge of $k(\Phi, \mathfrak{c}^+)(a)$ determines (in principle) the other functions $k(\Phi, \mathfrak{c})$.

Lemma 41 *Let $\mathfrak{c}$ be a big chamber. Let $\Sigma_r(\mathfrak{c})$ be the set of elements $w \in \Sigma_r$ such that $\mathfrak{c} \subset C_w^+$. Then*

$$k(\Phi, \mathfrak{c})(a) = \sum_{w \in \Sigma_r(\mathfrak{c})} (-1)^{n(w)} k(\Phi, \mathfrak{c}^+)(w^{-1}a).$$

We write

$$v(\Phi, \mathfrak{c})(a_1, ..., a_r) = \sum_{i_1+i_2+\cdots+i_r=N-r} v(\Phi, \mathfrak{c}, \mathbf{i}) \frac{a_1^{i_1}}{i_1!} \frac{a_2^{i_2}}{i_2!} \cdots \frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!} \frac{a_r^{i_r}}{i_r!}.$$

Definition 42 *Define*

$$t_j^\Phi = \sum_{k=j+1}^{r+1} m_{jk} - 1,$$

$$s_j^\Phi = 1 - \sum_{k=1}^{j-1} m_{kj}.$$

We consider the nice chamber $\mathfrak{c}^+$. Then we can express the coefficients of the function $v(\Phi, \mathfrak{c})(a)$ as values at very particular points depending on $\mathbf{i}$ of the function $k_{\Phi'}$, where Φ' is the system where we have deleted all roots $e^i - e^{r+1}$.

The system Φ' is a sequence of vectors in the positive root system of A_{r-1}, which spans the vector space $E_{r-1} = \{a = \sum_{i=1}^r a_i e^i \text{, with } \sum_{i=1}^r a_i = 0\}$. Remark that if $i_1 + i_2 + ... + i_r = N - r$, then the vector $(i_1 - t_1)e^1 + (i_2 - t_2)e^2 + ... + (i_r - t_r)e^r$ is in E_{r-1}.

Proposition 43 *(Postnikov-Stanley)*

For the big chamber $\mathfrak{c}^+$ of $C(A_r^+)$ we have

$$v(\Phi, \mathfrak{c}^+, \mathbf{i}) = k_{\Phi'}((i_1 - t_1)e^1 + (i_2 - t_2)e^2 + ... + (i_r - t_r)e^r).$$

Proof. We use the iterated residue formula for $f \mapsto \langle\langle \mathfrak{c}^+, f \rangle\rangle$ and the iterated constant term formula (LEMMA 38) for $k_{\Phi'}$. Indeed

$$v(\Phi, \mathfrak{c}^+, \mathbf{i})$$

$$= Res_{x_1=0} Res_{x_2=0} \cdots Res_{x_r=0} \left(\frac{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}}{x_1^{m_{1,r+1}} ... x_r^{m_{r,r+1}} \prod_{1 \le i < j \le r} (x_i - x_j)^{m_{ij}}} \right)$$

$$= CT_{x_1=0}CT_{x_2=0}\cdots CT_{x_r=0}\left(\frac{x_1^{i_1+1-m_{1,r+1}}x_2^{i_2+1-m_{2,r+1}}\cdots x_r^{i_r+1-m_{r,r+1}}}{\prod_{1\leq i<j\leq r}(x_i-x_j)^{m_{ij}}}\right)$$

$$= CT_{x_1=0}CT_{x_2=0}\cdots CT_{x_r=0}\left(\frac{x_1^{i_1-t_1}x_2^{i_2-t_2}\cdots x_r^{i_r-t_r}}{\prod_{1\leq i<j\leq}(1-x_j/x_i)^{m_{ij}}}\right)$$

$$= k_{\Phi'}((i_1-t_1)e^1+(i_2-t_2)e^2+...+(i_r-t_r)e^r).$$

QED

Consider now the system $\Phi = A_r^+$. We have $t_i^\Phi = (r-i)$. The system Φ' is A_{r-1}^+. We have thus

$$v(A_r^+,\mathfrak{c}^+,\mathbf{i}) = k_{A_{r-1}^+}((i_1-(r-1))e^1+(i_2+(r-2))e^2+\cdots++i_{r-1}e^{r-1}+i_re^r)$$

In the right hand side, if $i_r>0$, the element $(i_1-(r-1))e^1+\cdots+i_{r-1}e^{r-1}+i_re^r$ cannot be in the cone generated by (e^i-e^j) with $1\leq i<j\leq r$, as seen by looking at the component on e^r which would be negative. Furthermore, if $i_r=0$, we see that the coefficients of roots e^i-e^r in a solution of the equation

$$\sum_{\alpha^m\in A_{r-1}^+} x_m\alpha^m = (i_1-(r-1))e^1+(i_2+(r-2))e^2+\cdots+i_{r-1}e^{r-1}$$

(with $x_m\geq 0$) are necessarily 0. So, consider the system A_{r-2}^+ realized as $(e^i-e^j), 1\leq i<j\leq(r-1)$. Then we see that

$$v(A_r^+,\mathfrak{c}^+;(i_1,i_2,...,i_{r-1},i_r))=0$$

if $i_r>0$ and

$$v(A_r^+,\mathfrak{c}^+;(i_1,i_2,...,i_{r-1},0))$$
$$= k_{A_{r-2}^+}((i_1-(r-1))e^1+(i_2-(r-2))e^2+\cdots+(i_{r-1}-1)e^{r-1}).$$

In particular, we obtain for monomials, the relation between the volume of the CRY polytope $P_{A_r}(e^1-e^{r+1})$ and Kostant Partition function. Indeed, the only $\mathbf{i}$ to consider is $i_1=r(r-1)/2, i_2=0,...,i_r=0$. The point $(i_1-(r-1))e^1-(r-2)e^2+...-2e^{r-2}-e^{r-1}$ is flipped to the point $e^1+2e^2+...+(r-2)e^{r-2}+(r-1)-i_1)e^{r-1}$ under the transformation $-w_0$ of the system A_{r-2}^+. We then obtain:

Corollary 44 *(Postnikov-Stanley)*

$$\mathrm{vol}\, P_{A_r^+}(e^1 - e^{r+1}) = k_{A_{r-2}^+}(1, 2, 3, 4, ..., (r-2)).$$

Similarly, for any sequence Φ, the relative volume of the flow polytope $P_\Phi(e^1 - e^{r+1})$ is an integer, given by the Kostant partition function $k_{\Phi'}$ at a particular point. Combinatorists are happy of this result, only if they can explain this by giving an explicit simplicial decomposition of the corresponding flow polytope.

Bibliographical remarks.

Lemma 36, as well as many other results on Kostant Partition function is in [Ki]. Theorem 39 follows from [K-P] and [J-K], at least for generic values. A generalization to any rational polytope is given in [B-V 1]. Proposition 43 and corollary 44 are due to Postnikov- Stanley [P-S].

9 Change of variables for the total residue

We return to the notations of Section 2.

Let $F : V_\mathbb{C} \to V_\mathbb{C}$ be an analytic map, such that $F(0) = 0$ and preserving the open set $U_\Delta = V_\mathbb{C} - \mathcal{H}_\mathbb{C}$ of $V_\mathbb{C}$. If $f \in \hat{R}_\Delta$, the function $(F^*f)(x) = f(F(x))$ is again in $\hat{R}_\Delta$. Let $Jac(F)$ be the Jacobian of the map F. We assume $Jac(F)(0) \neq 0$. We write $F(x) = L(F)(x) + r(x)$ where $L(F)$ is a linear invertible map, and $r(x)$ vanishes at 0 at order 2. Thus $L(F)$ permutes the hyperplanes $\{\alpha = 0\}$. If f is in S_Δ, the function $L(F)^*(f)$ is again in S_Δ.

Theorem 45 *For any f in $\hat{R}_\Delta$, we have the equality in S_Δ:*

$$Jac(F)(0)L(F)^*(Tres_\Delta(f)) = Tres_\Delta(Jac(F)(F^*f)).$$

Proof. Let $f \in \hat{R}_\Delta$. Then $(f - Tres_\Delta(f))dx$ is the differential of some $(r-1)$-form $\sum_{k=1}^r f_k dx_1 \wedge dx_2 \wedge \widehat{dx_k} \wedge \cdots \wedge dx_r$, with $f_k \in \hat{R}_\Delta$. The vector space $\sum_{k=1}^r f_k dx_1 \wedge dx_2 \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n$ with $f_k \in \hat{R}_\Delta$ is stable by the action of F^* on differential forms. As F^* commutes with d, $(F^*f)F^*dx - F^*(Tres_\Delta(f))F^*dx$ is the differential of some $(r-1)$-form $\sum_{k=1}^r \Phi_k dx_1 \wedge dx_2 \wedge \widehat{dx_k} \wedge \cdots \wedge dx_r$, with $\Phi_k \in \hat{R}_\Delta$. We have $(F^*f)F^*dx = Jac(F)(F^*f)dx$.

We now analyze the differential form $(F^*f)(F^*dx) = (F^*f)Jac(F)dx$ with $f \in S_\Delta$. The function f is the inverse of the product of r linear forms α^k. As F preserves U_Δ, we have $\alpha^k(F(x)) = \beta^k(x)g^k(x)$ where $\beta^k(x)$ is another linear form in the system Δ and $g^k(x)$ is holomorphic at 0. Thus $(F^*f)(x) = \frac{1}{\prod_k \beta^k(x)} \prod_k (g^k(x))^{-1}$ is again in $\hat{R}_\Delta$.

Furthermore, we see that $(F^*f) - (L(F)^*f)$ is an element of R_Δ of degree $\rangle - r$. Thus $(F^*f)J(F) - (L(F)^*f)J(F)(0)$ is an element of $\hat{R}_\Delta$ of degree $> -n$. It follows that $(F^*f)F^*dx - L(F)^*(f)J(F)(0)dx$ is the differential of some $(r-1)$-form $\sum_{k=1}^r h_k(x)dx_1 \wedge dx_2 \wedge \hat{dx_k} \wedge \cdots \wedge dx_r$, with $h_k \in \hat{R}_\Delta$.

Adding these two informations, we see that for any $f \in \hat{R}_\Delta$,

$$Jac(F)(F^*f)dx - J(F)(0)(L(F)^*Tres_\Delta(f))dx$$

is the differential of some $r-1$-form $\sum_{k=1}^r m_k(x)dx_1 \wedge dx_2 \wedge \hat{dx_k} \wedge \cdots \wedge dx_r$, with $m_k \in \hat{R}_\Delta$. This implies the formula of the lemma.

QED

10 A nice formula for Kostant restricted partition function

Let Φ be a sequence of N vectors in $A_r^+ \subset E_r$ generating E_r. In the same spirit that Lidskii formula for Kostant partition function ([L]), we will give a closed formula for $k_\Phi(a)$ in function of $\operatorname{vol} P_\Phi(a)$. In fact, we will express the S_Δ-valued polynomial $K_\Phi(a)$ in function of the S_Δ-valued polynomial $J_\Phi(a)$ of Section 12.

Let
$$\left(\binom{u}{k}\right) = \frac{u(u+1)(u+2)\cdots(u+(k-1))}{k!}.$$

Theorem 46 *Let*

$$J_\Phi(a) = \sum_{i_1+i_2+\cdots+i_r=N-r} f_\Phi(\mathbf{i}) \frac{a_1^{i_1}}{i_1!}\frac{a_2^{i_2}}{i_2!}\cdots\frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!}\frac{a_r^{i_r}}{i_r!}.$$

Then

$$K_\Phi(a) = \sum_{|\mathbf{i}|=N-r} f_\Phi(\mathbf{i}) \binom{a_1+t_1^\Phi}{i_1}\binom{a_2+t_2^\Phi}{i_2}\cdots\binom{a_{r-1}+t_{r-1}^\Phi}{i_{r-1}}\binom{a_r+t_r^\Phi}{i_r}.$$

We have as well

$$K_\Phi(a) = \sum_{|\mathbf{i}|=N-r} f_\Phi(\mathbf{i}) \left(\binom{a_1+s_1^\Phi}{i_1}\right) \cdots \left(\binom{a_{r-1}+s_{r-1}^\Phi}{i_{r-1}}\right) \left(\binom{a_r+s_r^\Phi}{i_r}\right).$$

Proof. We realize A_r in $\mathbb{R}^r$ as the system $(e^i - e^j), 1 \le i < j \le r, e^i, 1 \le i \le r$.

Then

$$K_{\Phi(a)}$$

$$= Tres_\Delta\left[\frac{e^{a_1x_1}e^{a_2x_2}\cdots e^{a_rx_r}}{\prod_{i=1}^r (1-e^{-x_i})^{m_{i,r+1}} \prod_{1\le i<j\le r}(1-e^{-(x_i-x_j)})^{m_{ij}}}\right].$$

We use LEMMA 45 for changing variables in residues. Let $x = \sum_{i=1}^r x_ie_i$ in V. Define $F(x) = \sum_{i=1}^r (e^{x_i}-1)e_i$. This change of variables preserves the vector space R_Δ. Indeed $(e^{x_i} - e^{x_j})$ is divisible by $(x_i - x_j)$. The differential of F at the origin is the identity. We have $Jac(F) = e^{x_1}e^{x_2}\cdots e^{x_r}$.

We write t_k, s_k instead of t_k^Φ, s_k^Φ. Let $a_1, a_2, ..., a_r$ be integers and let

$$f(a_1, a_2, ..., a_r)(x) = \frac{(1+x_1)^{a_1+t_1}(1+x_2)^{a_2+t_2}\cdots(1+x_r)^{a_r+t_r}}{x_1^{m_{1,r+1}}x_2^{m_{2,r+2}}\cdots x_r^{m_{r,r+1}}\prod_{1\le i<j\le r}(x_i-x_j)^{m_{ij}}}.$$

Then the function $(F^*f)Jac(F)$ is equal to

$$\frac{e^{a_1x_1}e^{a_2x_2}\cdots e^{a_rx_r}}{(1-e^{-x_1})^{m_{1,r+1}}(1-e^{-x_2})^{m_{2,r+1}}\cdots(1-e^{-x_r})^{m_{r,r+1}}}$$

$$\times\frac{1}{\prod_{1\le i<j\le r}(1-e^{-(x_i-x_j)})^{m_{ij}}}.$$

Thus, we obtain from LEMMA 45:

$$Tres_\Delta\left(\frac{e^{\langle a,x\rangle}}{\prod_{\alpha\in\Phi}(1-e^{-\langle\alpha,x\rangle})}\right) = Tres_\Delta\left(\frac{(1+x_1)^{a_1+t_1}\cdots(1+x_r)^{a_r+t_r}}{\prod_{\alpha\in\Phi}\alpha(x)}\right).$$

To compute the total residue of the last expression, as the denominator $\prod_{\alpha\in\Phi}\alpha(x)$ is homogeneous of degree N, we have to seek for the term in the numerator which is homogeneous of degree $N-r$, thus we seek the coefficient of each term of the form $x_1^{i_1}\cdots x_r^{i_r}$ with $i_1+i_2+\cdots+i_n=N-r$. We thus obtain the first part of Proposition 46.

The second is proved the same way, using $F(x)=\sum_{i=1}^r(1-e^{-x_i})e_i$ and the function

$$\frac{(1+x_1)^{-(a_1+s_1)}\cdots(1+x_r)^{-(a_2+s_r)}}{x_1^{m_{1,r+1}}x_2^{m_{2,r+1}}\cdots x_r^{m_{r,r+1}}\prod_{1\leq i<j\leq r}(x_i-x_j)^{m_{ij}}}$$

QED

Thus we obtain the following formula for $k_\Phi(a)$.

Theorem 47 *Let $\mathfrak{c}$ be a big chamber and $a\in\overline{\mathfrak{c}}$. Then*

$$k_\Phi(a)=\sum_{|\mathbf{i}|=N-r}f_{\mathfrak{c}}(\mathbf{i})\binom{a_1+t_1^\Phi}{i_1}\binom{a_2+t_2^\Phi}{i_2}\cdots\binom{a_{r-1}+t_{r-1}^\Phi}{i_{r-1}}\binom{a_r+t_r^\Phi}{i_r}.$$

We have as well

$$k_\Phi(a)=\sum_{|\mathbf{i}|=N-r}f_{\mathfrak{c}}(\mathbf{i})\left(\binom{a_1+s_1^\Phi}{i_1}\right)\cdots\left(\binom{a_{r-1}+s_{r-1}^\Phi}{i_{r-1}}\right)\left(\binom{a_r+s_r^\Phi}{i_r}\right).$$

Comparing with formula 1 for the volume, we see that the function $k_\Phi(a)$ is immediately deduced from the polynomial function $\operatorname{vol}P_\Phi(a)$ (i.e; its highest degree component) by replacing the monomial $\frac{a_k^{i_k}}{i_k!}$ by the function $\binom{a_k+t_k^\Phi}{i_k}$ (with same leading term).

Corollary 48 *Let $q^\Phi=(\sum_{k=2}^{r+1}m_{1k})-1$. The polynomial $k(\Phi,\mathfrak{c}^+)$ is divisible by $(a_1+1)(a_1+2)...(a_1+q^\Phi)$.*

Proof. Indeed, on the big chamber $\mathfrak{c}^+$, we have seen in PROPOSITION 22 that that multiindices $\mathbf{i}$ such that $f_\Phi(\mathbf{i})$ is non zero are such that $i_1\geq q^\Phi$. We have $s_1^\Phi=1$, thus the corollary follows.
QED

Remark This follows also from the remark following definition 37, and Theorem 39.

If $\Phi=A_r^+$, we then obtain the relation for Kostant partition $k_{\mathfrak{c}}$ on a big chamber $\mathfrak{c}$. Define

$$f_{\mathfrak{c}}(\mathbf{i})=\langle\langle\mathfrak{c},Tres_\Delta\left(\frac{x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r}}{x_1x_2...x_r\prod_{i<j}(x_i-x_j)}\right)\rangle\rangle.$$

Then we have

Proposition 49 *(Lidskii)*

We have:

$$k(A_r^+, \mathfrak{c})(a)$$

$$= \sum_{|\mathbf{i}|=\binom{r}{2}} f_{\mathfrak{c}}(\mathbf{i}) \binom{a_1+r-1}{i_1}\binom{a_2+r-2}{i_2}\cdots\binom{a_{r-1}+1}{i_{r-1}}\binom{a_r}{i_r}.$$

We have as well:

$$k(A_r^+, \mathfrak{c})(a)$$

$$= \sum_{|\mathbf{i}|=\binom{r}{2}} f_{\mathfrak{c}}(\mathbf{i}) \left(\!\binom{a_1+1}{i_1}\!\right)\left(\!\binom{a_2}{i_2}\!\right)\cdots\left(\!\binom{a_{r-1}+3-r}{i_{r-1}}\!\right)\left(\!\binom{a_r+2-r}{i_r}\!\right).$$

Bibliographical remarks.

Proposition 49 is due to B.V. Lidskii [L]. Theorem 47 is due to Stanley, who suggested to look for a proof via residues, as given here.

11 Volumes and Ehrhart polynomials of the Stanley-Pitman polytope

Section 10 reduces the computation of the Ehrhart polynomial to the computation of mixed volumes.

Here is an example. Let

$$\Pi_r(a) = \{y \in \mathbb{R}^r; y_i \geq 0, y_1 + \cdots + y_i \leq a_1 + a_2 + \cdots + a_i\}$$

with $a_i \geq 0$.

Let Φ be the following sequence of $2r$ elements $\{\beta^i, \gamma^j\}, 1 \leq i \leq r, 1 \leq j \leq r\}$ of A_r^+

$$\Phi = \{\beta^1 = (e^1 - e^{r+1}), ..., \beta^r = (e^r - e^{r+1})\}$$
$$\cup\{\gamma^1 = (e^1 - e^2), \gamma^2 = (e^2 - e^3), \cdots \gamma^{r-1} = (e^{r-1} - e^r), \gamma^r = (e^r - e^{r+1})\}$$

The multiplicity of $(e^r - e^{r+1})$ in Φ is 2. We write $t_i = t_i^\Phi$. Thus $t_1 = t_2 = ... = t_{r-1} = t_r = 1$, while $s_1 = 1$, $s_2 = s_3 = ... = s_r = 0$.

Let $\mathfrak{n}_\Phi = \mathbb{R}^{2r}$ with basis $v_{\beta^k}, w_{\gamma^j}$. Let C_{2r}^+ be the standard cone

$$\oplus_{k=1}^r y_{i,r+1} v_{\beta^k} \oplus_{k=1}^r z_{k,k+1} w_{\gamma^k}$$

with $y_{i,r+1} \geq 0$, $z_{k,k+1} \geq 0$

Let $p : \mathbb{R}^{2r} \to E_r$ be the map sending v_β to β and w_γ to γ. Then for $a = a_1 e^1 + \cdots + a_r e^r$, with $a_i \geq 0$ the polytope $\Pi_r(a)$ is isomorphic to $P_\Phi(a) = p^{-1}(\mathbf{a}) \cap C_{2r}^+$. Indeed the point $\sum_{k=1}^r y_{k,r+1} v_{\beta^k} + \sum_{k=1}^r z_{k,k+1} w_{\gamma^k}$ with $y_{i,r+1} \geq 0$, $z_{i,i+1} \geq 0$ is in $P_\Phi(a)$ if and only if

$$\sum_{k=1}^r y_{k,r+1}(e^k - e^{r+1}) + \sum_{i=1}^r z_{i,i+1}(e^i - e^{i+1})$$
$$= a_1 e^1 + \cdots + a_r e^r - (a_1 + a_2 + ... + a_r) e^{r+1}.$$

This gives

$$\begin{aligned} y_{1,r+1} + z_{1,2} &= a_1, \\ y_{2,r+1} + z_{2,3} &= a_2 + z_{1,2} \\ y_{3,r+1} + z_{3,4} &= a_3 + z_{1,2} + z_{2,3} \end{aligned}$$

$$......$$

so that

$$y_{1,r+1} \leq a_1,$$
$$y_{1,r+1} + y_{2,r+1} \leq a_1 + a_2,$$
$$y_{1,r+1} + y_{2,r+1} + y_{3,r+1} \leq a_1 + a_2 + a_3$$
$$\cdots$$

and the point $(y_{1,r+1}, y_{2,r+1}, ..., y_{r,r+1}) \in \Pi_r(a)$.

We compute the volume

$$\operatorname{vol} \Pi_r(a) = \sum_{i_1+i_2+\cdots+i_r=r} f_\Phi(\mathbf{i}) \frac{a_1^{i_1}}{i_1!} \frac{a_2^{i_2}}{i_2!} \cdots \frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!} \frac{a_r^{i_r}}{i_r!}$$

As $a \in \mathfrak{c}^+$, we have, by LEMMA 43,

$$f_\Phi(\mathbf{i}) = k_{\Phi'}((i_1 - 1)e^1 + (i_2 - 1)e^2 + ... + (i_r - 1)e^r).$$

The system Φ' is the set of simple roots $(e^1 - e^2), ..., (e^{r-1} - e^r)$. They are linearly independent and generate a simplicial cone $C(\Phi')$. Thus the function $k_{\Phi'}$ is identically 1 on the cone $C(\Phi') \cap \mathbb{Z}^r$. Thus we obtain $f(i_1, i_2, i_3, ..., i_r) = 0$ or 1. It is 1 if and only $(i_1 - 1)e^1 + (i_2 - 1)e^2 + ... + (i_r - 1)e^r$ is in the cone $C(A_{r-1}^+)$. We thus need

$$(i_1, i_2, i_3, ..., i_r) \in K_r$$

where

$$K_r = \{(i_1, i_2, i_3, ..., i_r), i_1 \geq 1, i_1 + i_2 \geq 2, ..., i_1 + i_2 + \cdots + i_r = r\}.$$

We obtain the formula for the volume of $\Pi_r(a)$ and its Ehrhart polynomial, given in [Pi-S] .

Proposition 50 *(Pitman-Stanley)*

$$\operatorname{vol} \Pi_r(a) = \sum_{\mathbf{i} \in K_r} \frac{a_1^{i_1}}{i_1!} \frac{a_2^{i_2}}{i_2!} \cdots \frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!} \frac{a_r^{i_r}}{i_r!},$$

$$k_\Phi(a) = \sum_{\mathbf{i} \in K_r} \left(\binom{a_1 + 1}{i_1} \right) \left(\binom{a_2}{i_2} \right) \cdots \left(\binom{a_{r-1}}{i_{r-1}} \right) \left(\binom{a_r}{i_r} \right).$$

The same beautiful occurs for any family of polytopes associated to a smooth toric variety, giving rather nice formulae deduced immediately from the mixed volumes.

Bibliographical remarks. Results of this section are due to [Pi-S].

12 Divisibility property of the Kostant partition function

.

Here we list some properties of the polynomial $k(A_r^+, \mathfrak{c}^+)$. They are similar to the properties of its highest degree term $v(A_r^+, \mathfrak{c}^+)$ established in Section 5. For example, the following Lemma implies of course LEMMA 23.

Proposition 51 *(Schmidt-Bincer)*

The function $k(A_r^+, \mathfrak{c}^+)(a_1, a_2, ..., a_r)$ is independent of a_r. It is of degree less or equal to 1 in the variable a_{r-1} and is divisible by $(a_1 + a_2 + a_3 + ... + a_{r-2} + 3a_{r-1} + 3)$.

More precisely, we have:

$$3k(A_r^+, \mathfrak{c}^+)(a_1, a_2, ..., a_{r-1}, a_r) = 3k(A_r^+, \mathfrak{c}^+)(a_1, a_2, ..., a_{r-1}, 0)$$

$$= (a_1 + a_2 + ... + a_{r-2} + 3a_{r-1} + 3)k(A_r^+, \mathfrak{c}^+)(a_1, a_2, ..., a_{r-1}, 0)$$

$$= (a_1+a_2+...+a_{r-2}+3a_{r-1}+3)k(A_r^+ \text{ minus } (e^{r-1}-e^r), \mathfrak{c}^+)(a_1, a_2, ..., a_{r-2}, 0, 0).$$

Proof. The proof is almost identical to the proof of PROPOSITION 23. Let

$$K(a, x)$$

$$= \frac{e^{a_1 x_1 + ... + a_{r-1} x_{r-1} + a_r x_r}}{\prod_{1 \le i < j \le r-1}(1 - e^{-(x_i - x_j)}) \prod_{1 \le i \le r-1}(1 - e^{-(x_i - x_r)}) \prod_{1 \le i \le r}(1 - e^{-x_i})}.$$

We have:

$$k(A_r^+, \mathfrak{c}^+)(a_1, a_2, ..., a_r) = Ires_{x=0} K(a, x).$$

We write $k(A_r^+, \mathfrak{c}^+) = k_r^+$. We first take the residue of $K(a, x)$ in $x_r = 0$. We obtain

$$Res_{x_r=0} K(a, x) = \frac{e^{a_1 x_1 + ... + a_{r-1} x_{r-1}}}{\prod_{1 \le i < j \le r-1}(1 - e^{-(x_i - x_j)}) \prod_{1 \le i \le r-1}(1 - e^{-x_i})^2}.$$

This shows already that $k_r^+(a_1, a_2, ..., a_r)$ is independent of a_r. We proceed now to take the residue in $x_{r-1} = 0$. There is a double pole in x_{r-1}, so that the dependence in a_{r-1} is of degree at most 1. More precisely, a simple calculation shows that

$$Res_{x_{r-1}=0} \frac{e^{a_1 x_1 + ... + a_{r-1} x_{r-1}}}{\prod_{1 \le i < j \le r-1}(1 - e^{-(x_i - x_j)}) \prod_{1 \le i \le r-1}(1 - e^{-x_i})^2}$$

$$= \frac{e^{a_1 x_1 + ... + a_{r-2} x_{r-2}}}{\prod_{1 \le i < j \le r-2}(1 - e^{-(x_i - x_j)}) \prod_{1 \le i \le r-2}(1 - e^{-x_i})^3} (a_{r-1} + 1 + \sum_{i=1}^{r-2} \frac{e^{-x_i}}{1 - e^{-x_i}})$$

$$= (a_{r-1}+1+\frac{1}{3}(a_1+a_2+...+a_{r-2}))\frac{e^{a_1x_1+...+a_{r-2}x_{r-2}}}{\prod_{1\leq i<j\leq r-2}(1-e^{-(x_i-x_j)})\prod_{1\leq i\leq r-2}(1-e^{-x_i})^3}$$

$$-\frac{1}{3}(\partial_1+\partial_2+\cdots+\partial_{r-2})\frac{e^{a_1x_1+...+a_{r-2}x_{r-2}}}{\prod_{1\leq i\leq r-2}(1-e^{-x_i})^3\prod_{1\leq i<j\leq r-2}(1-e^{-(x_i-x_j)})}.$$

As residue vanishes on derivatives, we obtain

$$k_r^+(a_1,a_2,...,a_{r-1},a_r)$$

$$= (a_{r-1}+1+\frac{1}{3}(a_1+a_2+...+a_{r-2}))\times$$

$$Res_{x_1=0}...Res_{x_{r-2}=0}\frac{e^{a_1x_1+...+a_{r-2}x_{r-2}}}{\prod_{1\leq i<j\leq r-2}(1-e^{-(x_i-x_j)})\prod_{1\leq i\leq r-2}(1-e^{-x_i})^3}.$$

On the other hand, the residue computation of

$$k(A_r^+ \,\mathrm{minus}\,(e^{r-1}-e^r),\mathfrak{c}^+)(a_1,a_2,...,a_{r-2},0,0)$$

gives

$$k(A_r^+ \,\mathrm{minus}\,(e^{r-1}-e^r),\mathfrak{c}^+)(a_1,a_2,...,a_{r-2},0,0)$$

$$= Res_{x_1=0}...Res_{x_{r-2}=0}\frac{e^{a_1x_1+...+a_{r-2}x_{r-2}}}{\prod_{1\leq i<j\leq r-2}(1-e^{-(x_i-x_j)})\prod_{1\leq i\leq r-2}(1-e^{-x_i})^3}$$

as the step $x_r=0$ as well as the step $x_{r-1}=0$ involves only simple poles, and we obtain the divisibility property announced.

QED

More generally, we have the following lemma, with same proof.

Lemma 52 *Let Φ a sequence of vectors in A_r^+ generating E_r. Assume $m_{r,r+1}=1$ and $m_{r-1,r+1}+m_{r-1,r}=2$. Furthermore, assume that*

$$\frac{m_{j,r+1}+m_{j,r}+m_{j,r-1}}{m_{j,r-1}}=c$$

is independent of j for $1\leq j\leq (r-2)$, then

$$k(\Phi,\mathfrak{c}^+)(a_1,...,a_{r-1},a_r)=k(\Phi,\mathfrak{c}^+)(a_1,...,a_{r-1},0)$$

$$= (\frac{a_1+\cdots+a_{r-2}}{c}+a_{r-1}+1)v(\Phi\,\mathrm{minus}\,(e^{r-1}-e^r),\mathfrak{c}^+)(a_1,a_2,...,a_{r-2},0,0).$$

Bibliographical remarks.
Proposition 51 is due to [S-B].

13 A “not so obvious ” symmetry property of the volume and Ehrhart polynomial

The full Weyl group Σ_{r+1} of the root system A_r acts on S_{A_r} as it permutes the elements $(e^i - e^j), 1 \leq i < j \leq (r+1)$. We show that the existence of this action implies some constraints on the coefficients of the S_Δ-valued polynomial function $J_{A_r^+}(a)$ and $K_{A_r^+}(a)$. In particular, the Kostant polynomial $k(A_r^+, \mathfrak{c}^+)(a) = Ires_{x=0}(K_{A_r^+}(a))$ satisfies some symmetries.

Lemma 53 *For any $w \in \Sigma_{r+1}$, we have*

$$J_{A_r^+}(w.a) = \epsilon(w) w \cdot J_{A_r^+}(a).$$

Let q be a linear form on S_Δ. Then the map $a \mapsto \langle q, J_{A_r^+}(a)\rangle$ is a polynomial of degree $|A_{r-1}| = r(r-1)/2$ on E_r. This way, we obtain a map from the $r!$-dimensional space S_Δ^* to the space of polynomials of degree $|A_{r-1}| = r(r-1)/2$ on E_r, commuting with the representation of Σ_{r+1}.

A subset of $\{1, 2, ..., N\}$ will be called an interval, if it is of the form $\{a, a+1, a+2, ..., a+k\}$, for a an integer between 1 and N (and k an integer between 0 and $N-a$). Let us represent a permutation in Σ_N as a list of N elements . For example, if we write $w = [Id_{(N-3)}, N, N-2, N-1]$, the first $N-3$ indices are fixed by the permutation w , $N-2$ is sent to N, $N-1$ to $N-2$ and N to $N-1$.

Definition 54 *The subset B_N is the subset of elements w of Σ_N such that for any $1 \leq k \leq N$ the subset $w^{-1}\{1, 2, 3, ..., k\}$ is an interval.*

Lemma 55 *The subset B_N is of cardinal 2^{N-1}.*

Proof. We prove it by induction on N. If $v \in B_{N+1}$, the set $v^{-1}\{1, 2, 3, ..., N\}$ is an interval. So it is either equal to $\{1, 2, 3, ..., N\}$ or to $\{2, 3, ..., N, N+1\}$. Thus a transformation w of B_N gives rise to two transformations of B_{N+1} namely $[w, N+1]$ and $[N+1, w]$.
QED

Definition 56 *For $1 \leq i \leq r$, we define $W^{i,r}$ as the subset of Σ_r consisting of elements w such that:*

- $w(k) = k,\ 1 \leq k \leq i-1.$
- *For $0 \leq s \leq (r-i)$, the set $w^{-1}\{i, r, r-1, r-2, r-3, ..., r-s\}$ is an interval (of $\{i, ..., r\}$).*

Clearly the subset $W^{i,r}$ is isomophic to $B_{r-(i-1)}$ by relabelling indices. Thus the set $W^{i,r}$ is of cardinality 2^{r-i}. A list of r elements representing an element of $W^{i,r}$ is constructed as follows: we start writing i in the middle of a horizontal line, then we write r either at the immediate left of i, or at the immediate right. After, we write $(r-1)$ either at the immediate left of the unordered set $\{i, r\}$, or at the immediate right, and we go on until we have written all elements of the set $\{i, r.., i+1\} = \{i, i+1, ..., r\}$ on this line. At this step, we finally write the ordered set $\{1, 2, ..., i-1\}$ at the immediate left of the unordered set $\{i, i+1, ..., r\}$.

Example. Let us list elements of $W^{i,r} \subset \Sigma_r$ for $i = r, r-1, r-2$.

We have

$$W^{r,r} = \{[Id_r]\}$$

$$W^{(r-1),r} = \{[Id_r], [Id_{r-2}, r, r-1]\}.$$

$$W^{(r-2),r} =$$

$$\{[Id_{r-3}, r-2, r, r-1], [Id_{r-3}, r, r-2, r-1], [Id_{r-3}, r-1, r-2, r], [Id_{r-3}, r-1, r, r-2]\}.$$

Let $w \in \Sigma_r$. Consider the element $\hat{r}(w)$ of Σ_{r-1} such that the list representing $\hat{r}(w)$ is the list w where the element r is omitted. In other words, if u is such that $w(u) = r$, then $\hat{r}(w)(k) = w(k)$ if $k < u$, and $\hat{r}(w)(k) = w(k+1)$ if $k \geq u$. We have the lemma.

Lemma 57 *The element $w \in W^{i,r}$ if and only if:*

- $w^{-1}\{i, r\}$ *is an interval.*
- $\hat{r}(w) \in W^{i,r-1}$.

We denote by $T(i, r+1) \in \Sigma_{r+1}$ the transposition of i and $(r+1)$. It acts on S_Δ, thus on S^*_Δ.

Proposition 58 *Let $1 \leq i \leq r$. Then the following holds:*

$$T(i, r+1) \cdot Ires_{x=0} = \sum_{w \in W^{i,r}} (-1)^{r+1-w^{-1}(i)} w \cdot Ires_{x=0}.$$

Proof. If $g \in \Sigma_{r+1}$, we have $g \cdot Ires_{x=0} = \sum_{w \in \Sigma_r} c_w^g Ires_{x=0}^w$, as elements $Ires_{x=0}^w = w \cdot Ires_{x=0}$ form a basis of linear forms on S_Δ. Let, for $w \in \Sigma_r$,

$$f_w = \frac{1}{\prod_{1 \leq p \leq r-1} (e^{w(p)} - e^{w(p+1)})(e^{w(r)} - e^{r+1})}.$$

The coefficient c_w^g is equal to $\langle g \cdot Ires_{x=0}, f_w \rangle = Ires_{x=0}(g^{-1} f_w)$. We need to prove

- If w is not in $W^{i,r}$, then

$$\langle Ires_{x=0}, T(i, r+1) f_w \rangle = 0.$$

- If $w \in W^{i,r}$, then

$$\langle Ires_{x=0}, T(i, r+1) f_w \rangle = (-1)^{r+1-j}$$

  with $j = w^{-1}(i)$.

Let $\phi = \frac{1}{\prod_{\alpha \in \sigma} \alpha}$ be in S_{A_r}. The set σ consists on r linearly independent elements of E_r. The partial residue $Res_{x_r=0}\phi$ is non zero if and only one of the elements α in the denominator of ϕ is proportional to $(e^r - e^{r+1})$. Indeed, in computing $Res_{x_r=0}\phi$, we replace e^i by x_i, when $i \leq r$, and e^{r+1} by 0. Thus the only factor creating a pole on $x_r = 0$ is $(e^r - e^{r+1})$. It follows that $Res_{x_r=0}(T(i, r+1) f_w)$ is zero unless there exists p, $1 \leq p \leq r$, such that $(e^{w(p)} - e^{w(p+1)})$ is proportional to the linear form $T(i, r+1) \cdot (e^r - e^{r+1})$.

If $i = r$, then $T(r, r+1) \cdot (e^r - e^{r+1}) = -(e^r - e^{r+1})$ and we see that necessarily $p = r$ and $w(r) = r$. Then

$$Ires_{x=0} T(r, r+1) \cdot f_w$$

$$= Res_{x_1=0} \dots Res_{x_{r-1}=0} \frac{-1}{\prod_{k=1}^{r-2} (x_{w(k)} - x_{w(k+1)}) x_{w(r-1)}} = -\delta_w^1$$

since $w(r) = r$ and that the basis f_w, with $w \in \Sigma_{r-1}$ is dual to the basis of iterated residues on $S_{A_{r-1}}$, as seen in Section 4. On the other hand $W^{r,r} = [Id_{(r)}]$, and thus the case $i = r$ is completed.

Let $i < r$. We proceed by induction on r. Let $w \in \Sigma_r$ and j such that $w(j) = i$. Then, with $g = T(i, r+1)$,

$$g \cdot f_w = \frac{1}{(e^{w(1)} - e^{w(2)})(e^{w(2)} - e^{w(3)}) \ldots (e^{w(j-2)} - e^{w(j-1)})} \times$$

$$\times \frac{1}{(e^{w(j-1)} - e^{r+1})(e^{r+1} - e^{w(j+1)}) \ldots (e^{w(j+1)} - e^{w(j+2)}) \ldots (e^{w(r)} - e^{i})}$$

and

$$Ires_{x=0} g \cdot f_w = -Ires_{x=0} \frac{1}{(x_{w(1)} - x_{w(2)})(x_{w(2)} - x_{w(3)}) \ldots (x_{w(j-2)} - x_{w(j-1)})}$$

$$\times \frac{1}{x_{w(j-1)} x_{w(j+1)} (x_{w(j+1)} - x_{w(j+2)}) \ldots (x_{w(r)} - x_i)}.$$

The function $g \cdot f_w$ has only two simple poles in $x_{w(j-1)}$ and in $x_{w(j+1)}$, thus $Res_{x_r=0} g \cdot f_w \neq 0$ iff $w(j-1) = r$, or $w(j+1) = r$. Precisely

- if $w(j-1) = r$, then $Res_{x_r=0} g \cdot f_w$
  $= \frac{-1}{(x_{w(1)} - x_{w(2)}) \ldots (x_{w(j-3)} - x_{w(j-2)}) x_{w(j-2)} \cdot x_{w(j+1)} (x_{w(j+1)} - x_{w(j+2)}) \ldots (x_{w(r)} - x_i)}$
- and if $w(j+1) = r$, then $Res_{x_r=0} g \cdot f_w$
  $= \frac{1}{(x_{w(1)} - x_{w(2)}) \ldots (x_{w(j-2)} - x_{w(j-1)}) x_{w(j-1)} \cdot x_{w(j+2)} (x_{w(j+2)} - x_{w(j+3)}) \ldots (x_{w(r)} - x_i)}$

In particular we see that if $w^{-1}\{i, r\}$ is not an interval, then $Res_{x_r=0} g \cdot f_w$ is zero, so a fortiori $Ires_{x=0} g \cdot f_w$ is equal to 0.
If $w^{-1}\{i, r\}$ is an interval, then we check on the preeceding formula that

- if $w(j-1) = r$, then
  $$Ires_{x=0} T(i, r+1) \cdot f_w = Res_{x_1=0} \ldots Res_{x_{r-1}=0} T(i, r) \cdot f_{\hat{r}(w)}.$$
- if $w(j+1) = r$, then
  $$Ires_{x=0} T(i, r+1) \cdot f_w = -Res_{x_1=0} \ldots Res_{x_{r-1}=0} T(i, r) \cdot f_{\hat{r}(w)}.$$

We conclude by induction, using Lemma 57.
QED

We now consider the volume polynomial $v(A_r^+, \mathfrak{c}^+)(a) = Ires_{x=0}(J_{A_r^+}(a))$.
As the function $J_{A_r^+}(a)$ is anti-invariant under the full group Σ_{r+1}, we obtain from PROPOSITION 58

Proposition 59 *Let* $a = \sum_{i=1}^{r+1} a_i e^i$ *with* $\sum_{k=1}^{r+1} a_k = 0$*. Then we have*

$$v(A_r^+, \mathfrak{c}^+)(T(i, r+1) \cdot a) = \sum_{w \in W^{i,r}} \epsilon(w)(-1)^{r-w^{-1}(i)} v(A_r^+, \mathfrak{c}^+)(w^{-1} \cdot a).$$

Example: We know that the function $v(A_r^+, \mathfrak{c}^+)(a)$ is a function of $(r-1)$ variables $v^+(a_1, a_2, ..., a_{r-1})$. Then, for $i = r,\ r-1,\ r-2$, we get from Proposition 59, and the description given of the corresponding sets $W^{i,r}$, that the function $v^+(x)$ satisfies the identities:

- $i = r$: $v^+(x) = v^+(x)$.
- $i = r-1$: for any values $x_1, ..., x_r$, we have

$$v^+(x_1, ..., x_{r-2}, -(x_1 + ... + x_{r-2} + x_{r-1} + x_r)) =$$
$$-v^+(x_1, x_2, ..., x_{r-2}, x_{r-1}) - v^+(x_1, ..., x_{r-2}, x_r).$$

- $i = r-2$: for any values $x_1, ..., x_r$, we have

$$v^+(x_1, x_2, ..., x_{r-3}, -(x_1 + x_2 + ... + x_r), x_{r-1}) =$$
$$-v^+(x_1, x_2, ..., x_{r-3}, x_{r-2}, x_r) + v^+(x_1, x_2, ..., x_{r-3}, x_{r-1}, x_{r-2})$$
$$-v^+(x_1, x_2, ..., x_{r-3}, x_r, x_{r-2}) + v^+(x_1, x_2, ..., x_{r-3}, x_{r-1}, x_r).$$

Remark: A function $v(x_1, ..., x_{r-1})$ of the form $w(x_1, ..., x_{r-2})(x_1 + x_2 + ... + x_{r-2} + 3x_{r-1})$ satisfies (2). This is in agreement with the divisibility of $v(A_r^+, \mathfrak{c}^+)(a)$ by the linear factor $(a_1 + a_2 + ... + a_{r-2} + 3a_{r-1})$ proved in Section 5.

We now give a stronger result on the symmetry for the Kostant partition polynomial $k(A_r^+, \mathfrak{c}^+)$ attached to the nice chamber $\mathfrak{c}^+$.

Let $\rho = \frac{1}{2} \sum_{\alpha \in A_r^+} \alpha$. We have

$$\rho = \frac{1}{2}(re^1 + (r-2)e^2 + \cdots - (r-2)e^r - re^{r+1}).$$

As usual, we have a shifted symmetry property for $K_{A_r^+}$.

Lemma 60 *For any $w \in \Sigma_{r+1}$, the function*

$$K_{A_r^+}(\mathbf{a}) = Tres_\Delta \left(\frac{e^{\mathbf{a}}}{\prod_{\alpha\in\Phi}(1-e^{-\alpha})} \right)$$

satisfies the relation:

$$w \cdot K_{A_r^+}(\mathbf{a}) = \epsilon(w) K_{A_r^+}(w \cdot \mathbf{a} + w \cdot \rho - \rho).$$

Thus PROPOSITION 13 implies the following symmetry relation for $k(A_r^+, \mathfrak{c}^+)(a) = Ires_{x=0} K_{A_r^+}(a)$.

Proposition 61 *Let $a = \sum_{k=1}^{r+1} a_k e^k$ with $\sum_{k=1}^{r+1} a_k = 0$. Then we have for, every $1 \le i \le r$,*

$$\begin{aligned} &k(A_r^+, \mathfrak{c}^+)(T(i, r+1) \cdot a - (r-i+1)(e^i - e^{r+1})) \\ = &\sum_{w\in W^{i,r}} \epsilon(w)(-1)^{r-w^{-1}(i)} k(A_r^+, \mathfrak{c}^+)(w^{-1}a + w^{-1} \cdot \rho - \rho). \end{aligned}$$

Example: We know that the function $k(A_r^+, \mathfrak{c}^+)(a)$ is a function of $(r-1)$ variables $k^+(a_1, a_2, ..., a_{r-1})$. Then, for $i = r$, $r-1$, $r-2$, we get from Proposition 61, that the function $k^+(x)$ satisfies the identities:

- $i = r$: $k^+(x) = k^+(x)$.
- $i = r-1$: for any values $x_1, ..., x_r$, we have

$$\begin{aligned} &k^+(x_1, ..., x_{r-2}, -(x_1 + ... + x_{r-2} + x_{r-1} + x_r + 2)) = \\ &-k^+(x_1, x_2, ..., x_{r-2}, x_{r-1}) - k^+(x_1, ..., x_{r-2}, x_r - 1). \end{aligned}$$

- $i = r-2$: for any values $x_1, ..., x_r$, we have

$$\begin{aligned} &k^+(x_1, x_2, ..., x_{r-3}, -(x_1 + x_2 + ... + x_r + 3), x_{r-1}) = \\ &-k^+(x_1, x_2, ..., x_{r-3}, x_{r-2}, x_r - 1) + k^+(x_1, x_2, ..., x_{r-3}, x_{r-1} - 1, x_{r-2} + 1) \\ &-k^+(x_1, x_2, ..., x_{r-3}, x_r - 2, x_{r-2} + 1) + k^+(x_1, x_2, ..., x_{r-3}, x_{r-1} - 1, x_r - 1). \end{aligned}$$

Remark: A function $k(x_1, ..., x_{r-1})$ of the form $w(x_1, ..., x_{r-2})(x_1 + x_2 + ... + x_{r-2} + 3x_{r-1} + 3)$ satisfies (2). This is in agreement with the divisibility of $k(A_r^+, \mathfrak{c}^+)(a)$ by the linear factor $(a_1 + a_2 + ... + a_{r-2} + 3a_{r-1} + 3)$ proved in Section 12.

Bibliographical remarks.

As explained as the beginning, the relations given here are transcription of the "hidden" action of Σ_{r+1} on $S(A_r)$.

14 Appendix 1: Jeffrey-Kirwan residue formula for the volume.

We return to the notations of Section 3. Let $\Phi = \{\alpha^1, ..., \alpha^N\}$ be a sequence of elements of Δ^+. Let $p : \mathbb{R}^N \to V^*$ be the map such that

$$p(u_1 w^1 + u_2 w^2 + ... + u_N w^N) = u_1\alpha^1 + u_2\alpha^2 + \cdots + u_N\alpha^N.$$

.

Consider **the family** of polytopes $P_\Phi(a)$ where a varies in $C(\Delta^+) \subset V^*$. Let $x \in V$, such that $\langle \alpha^k, x\rangle > 0$ for all $\alpha^k \in \Phi$. We have

$$\frac{1}{\prod_{k=1}^N \langle \alpha^k, x\rangle} = \int_{\mathbb{R}_+^N} e^{-\sum_{k=1}^N u_k\langle \alpha^k, x\rangle} du_1 \cdots du_N = \int_{\mathbb{R}_+^N} e^{-\langle p(w), x\rangle} dw.$$

Let $s : V^* \to \mathbb{R}^N$ be a section from V^* to E such that

$$\mathbb{R}^N = s(V^*) \oplus \mathrm{Ker}(p).$$

We write $w = s(a) + m$, with $p(w) = a$ and $p(m) = 0$. Our choice of measure is such that $dw = dadm$. By Fubini, we obtain

$$\int_{\mathbb{R}_+^N} e^{-\langle p(w), x\rangle} dw = \int_{C(\Delta^+)} e^{-\langle a, x\rangle} (\int_{P_\Phi(a)} dm) da.$$

Thus

$$\frac{1}{\prod_{k=1}^N \langle \alpha^k, x\rangle} = \int_{C(\Delta^+)} e^{-\langle x, a\rangle} \operatorname{vol} P_\Phi(a) da,$$

i.e. the Laplace transform of the function $\operatorname{vol} P_\Phi(a) da$ is $\prod_{k=1}^N \frac{1}{\alpha^k(x)}$. We need to inverse this formula to find $\operatorname{vol} P_\Phi(a)$.

We denote by G_Δ the subspace of R_Δ spanned by the functions of the form

$$\frac{1}{\prod_{\alpha \in \sigma} \alpha^{n_\alpha}}$$

where σ is a basis of Δ and the n_α are positive integers.

Lemma 62 *Let κ be a sequence of elements of Δ^+ such that κ generates V^*. Then the function*

$$\frac{1}{\prod_{\alpha \in \kappa} \alpha}$$

belongs to the vector space G_Δ.

Proof. We argue on the cardinal on the underlying set $\{\kappa\}$ to κ. If the cardinal of $\{\kappa\}$ is minimum, then $\{\kappa\}$ is a basis of Δ. If not, there is a linear relation $\beta = \sum_j c_j \alpha^j$ between elements β, α^j belonging to κ. Then

$$\frac{1}{\beta \prod_j \alpha^j} = \sum_j \frac{c_j}{\beta^2 \prod_{i \neq j} \alpha^i}.$$

We conclude by induction.

QED

Let $\sigma = \{\alpha^1, \alpha^2, ..., \alpha^r\}$ be a basis of Δ consisting of elements of Δ^+. Let

$$f_\sigma = \frac{1}{\alpha^1 \cdots \alpha^r}$$

and let $[C(\sigma)]$ be the characteristic function of the cone $C(\sigma)$.

Consider the function

$$F(x) = \frac{1}{\langle \alpha^1, x\rangle^{k_1+1} \cdots \langle \alpha^r, x\rangle^{k_r+1}}$$

where k_j are non negative integers. From the one dimensional formula:

$$\frac{1}{x^{k+1}} = \int_{\mathbb{R}^+} e^{-ax} \frac{a^k}{k!} da$$

we see that

$$F(x) = \int_{V^*} e^{-\langle a, x\rangle} v(a)[C(\sigma)](a) da$$

where v is a polynomial.

Let $\mathfrak{c}$ be a big chamber of Δ^+. The verification of the inversion formula:

$$[C(\sigma)](a)v(a) = \langle\langle \mathfrak{c}, Tres_\Delta(e^a F)\rangle\rangle \tag{2}$$

for $a \in \mathfrak{c}$ is straightforward. Indeed, we write $a = \sum_{k=1}^r a_k \alpha^k$, then $da = \mathrm{vol}(\sigma) da_1 da_2 ... da_r$ and $v(a) = \frac{1}{\mathrm{vol}(\sigma)} \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!}$. On the other hand the function $F(x)e^{\langle a, x\rangle}$ is equal to

$$\frac{e^{a_1\langle \alpha^1, x\rangle}}{\langle \alpha^1, x\rangle^{k_1+1}} \cdots \frac{e^{a_r\langle \alpha^r, x\rangle}}{\langle \alpha^r, x\rangle^{k_r+1}}$$

and its total residue is the function

$$\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}\frac{1}{\langle\alpha^1,x\rangle}\cdots\frac{1}{\langle\alpha^r,x\rangle}.$$

So

$$Tres_\Delta(e^a F)=\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}f_\sigma.$$

Let $\mathfrak{c}$ be a big chamber. If $\mathfrak{c}\subset C(\sigma)$, then the left hand side of the equation above is $v(a)$, and the right hand side is

$$\langle\langle\mathfrak{c},f_\sigma\rangle\rangle\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}=\frac{1}{\mathrm{vol}(\sigma)}\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}=v(a).$$

If $\mathfrak{c}$ is not contained in $C(\sigma)$, then $\mathfrak{c}\cap C(\sigma)=\emptyset$ and both sides are equal to 0.

Thus the inversion formula for any function of G_Δ is established and we obtain THEOREM 9.

15 Appendix 2. Chambers and basis for A_2 and A_3

We briefly recall the setting. Following Section 4 we realize V^* as E_r with basis $\{e^1, e^2, ..e^r\}$, so that we may write:

$$A_r = \{\pm e^1, \pm e^2, .., \pm e^r, \pm(e^i - e^j), 1 \leq i < j \leq r\}.$$

Then $|A_r^+| = N = r(r+1)/2$ and the set of positive roots is

$$A_r^+ = \{e^1, e^2, ..e^r, (e^i - e^j), 1 \leq i < j \leq r\}.$$

We consider the group Σ_r of permutations of $\{1, 2, ..., r\}$. We denote a permutation in Σ_r as a list of r elements . For example [231] represents the permutation in 3 elements $\{1, 2, 3\}$ sending 1 to 2, 2 to 3 and 3 to 1. The group Σ_r acts naturally on V^*.

The knowledge of the S_Δ-valued polynomial function

$$J_{A_r^+}(a)(x) = Tres_{A_r^+}[\frac{e^{\langle a,x\rangle}}{\prod_{\alpha \in A_r^+ 0}\langle \alpha, x\rangle}] = \frac{1}{(\frac{r(r-1)}{2})!} Tres_{A_r^+}[\frac{\langle a, x\rangle^{(\frac{r(r-1)}{2})}}{\prod_{\alpha \in A_r^+}\langle \alpha, x\rangle}]$$

determines the various polynomial functions associated to big chambers.

The function $J_{A_r^+}$ is anti-invariant under the group Σ_r:

$$J_{A_r^+}(w \cdot a) = \epsilon(w) J_{A_r^+}(a)$$

thus we need only to determine the polynomial $Ires_{x=0} J_{A_r^+}(a)$. It is an homogeneous polynomial of degree $|A_{r-1}|$. (In fact, $J_{A_r^+}(a)$ is also anti-invariant under Σ_{r+1} leading to some constraints on coefficients of $J_{A_r^+}(a)$.)

Recall that $C(\sigma)$ denote the simplicial cone determined by a basis σ of A_r^+. If we write $a = \sum_{i=1}^r a_i e^i$ for $a \in V^*$, the cone generated by A_r^+ is

$$C(A_r^+) = \{a \in V^* \text{such that } a_1 \geq 0, a_1 + a_2 \geq 0, ..., a_1 + .. + a_r \geq 0\}$$

The small chambers, denoted by $\mathfrak{a}$, are defined as the open connected component of $V^* - \mathcal{H}^*$ in $C(A_r^+)$. To any $h \in C(A_r^+)$ we associate the intersection of all the simplicial cones $C(\sigma)$ which contain h. The interior of the maximal cones of this subdivision of $C(A_r^+)$ into polyhedral cones are called big chambers. The relevance of the big chambers lies in the fact that the polynomial volume is the same on all the small chambers that make up a big chamber. Recall that the nice chamber $\mathfrak{c}^+$ is given by $a_i > 0$. The nice chamber $\mathfrak{c}^+$ is the cone $\sum_{i=1}^r a_i e^i$ with $a_i > 0,\ 1 \leq i \leq r$. Using the permutation $w_0 \in \Sigma_{r+1}$ reversing order on $\{1, 2, ..., (r+1)\}$, there is the "opposite" nice chamber $\mathfrak{c}^-$, which is the cone spanned by the roots $\{e^1, (e^1 - e^2), (e^1 - e^3), ..(e^1 - e^r)\}$.

If $w \in \Sigma_r$, we denote by $C_w^+ \subset C(A_r{}^+)$ the simplicial cone generated by the vectors

$$\epsilon(1)(e^{w(1)} - e^{w(2)}), ..., \epsilon(r-1)(e^{w(r-1)} - e^{w(r)}), (e^{w(r)} - e^{r+1}),$$

where $\epsilon(i)$ is 1 or -1 depending whether $w(i) < w(i+1)$ or not.

The space S_{A_r} is of dimension $r!$. As basis of S_{A_r}, we choose elements f_w indexed by $w \in \Sigma_r$, with

$$f_w = w \cdot \frac{1}{(e^1 - e^2)(e^2 - e^3)....(e^{r-1} - e^r)e^r}.$$

To a big chamber $\mathfrak{c}$ is associated a linear form $f \to \langle\langle \mathfrak{c}, f \rangle\rangle$ on S_{A_r}. By definition, $\langle\langle \mathfrak{c}, f_w \rangle\rangle$ is equal to 0 if $\mathfrak{c}$ is not contained in C_w^+. Otherwise, $\langle\langle \mathfrak{c}, f_w \rangle\rangle = (-1)^{n(w)}$, where $n(w)$ be the number of elements such that $w(i) > w(i+1)$. Thus to compute $\langle\langle \mathfrak{c}, f \rangle\rangle = (-1)^{n(w)} Ires_{x=0}^w f$, we need to determine elements w such that $\mathfrak{c} \subset C_w^+$ and $n(w)$.

We compute the big chambers forms $\mathfrak{c}$, and the corresponding form $\langle\langle \mathfrak{c}, f \rangle\rangle$, in the A_2 and A_3 case. Only the form associated to $\mathfrak{c}^+$ has a simple expression in the basis f_w. (Of course this basis is somewhat arbitrary).

For $a \in \bar{\mathfrak{c}}$, $\mathfrak{c}$ a big chamber we recall the transmutation formula for the volume and the Kostant partition function as given in Sections 10:

$$\operatorname{vol} P_{A_r^+}(a) = v(A_r^+, \mathfrak{c}) = \sum_{|\mathbf{i}| = \binom{r}{2}} f_{\mathfrak{c}}(\mathbf{i}) \frac{a_1^{i_1}}{i_1!} \frac{a_2^{i_2}}{i_2!} \cdots \frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!} \frac{a_r^{i_r}}{i_r!}$$

and

$$k(A_r^+, \mathfrak{c})(a) = \sum_{|\mathbf{i}| = \binom{r}{2}} f_{\mathfrak{c}}(\mathbf{i}) \binom{a_1 + r - 1}{i_1} \binom{a_2 + r - 2}{i_2} \cdots \binom{a_{r-1} + 1}{i_{r-1}} \binom{a_r}{i_r}.$$

where $\langle\langle \mathfrak{c}, f_{A_r^+}(\mathbf{i}) \rangle\rangle = f_{\mathfrak{c}}(\mathbf{i})$

For A_2 we may draw the following picture:

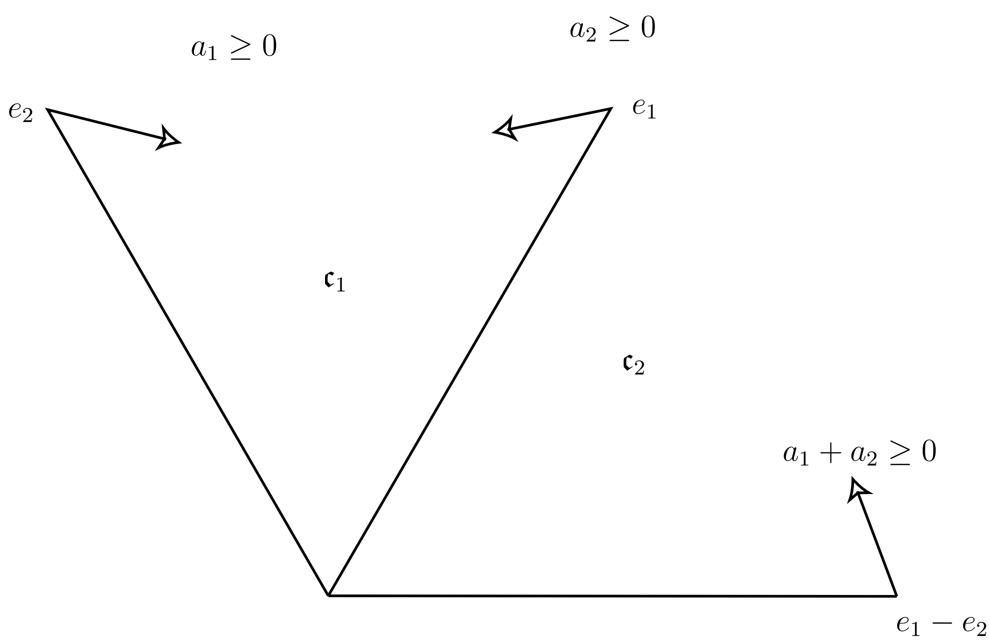


Figure 3: Chambers for A_2 with $a = a_1 e^1 + a_2 e^2$.

The equations defined by the hyperplanes are:

$$a_1 = 0, a_2 = 0, a_1 + a_2 = 0.$$

In dimension 2, small and big chambers coincide.

There are 3 bases in A_2^+,$\sigma_1 = \{e^1, e^2\}$, $\sigma_2 = \{e^1, (e^1 - e^2)\}$, and $\sigma_3 = \{e^2, (e^1 - e^2)\}$ bases of Δ. The space S_Δ is spanned by f_{σ_2},f_{σ_3}. We have the linear relation $f_{\sigma_1} = f_{\sigma_3} - f_{\sigma_2}$.

Therefore , if $a = a_1 e^1 + a_2 e^2$, then

$$J_{A_2^+}(\mathbf{a}) = a_1 f_{[12]} - a_2 f_{[21]},$$

$$v(A_2^+, \mathfrak{c}^+) = a_1,$$
$$k(A_2^+, \mathfrak{c}^+) = a_1 + 1.$$

$$\operatorname{vol} P_{A_r^+}(a) = a_1, \text{ if } a \in \mathfrak{c}_1$$

and

$$\mathrm{vol}\, P_{A_r^+}(a) = a_1 + a_2, \text{ if } a \in \mathfrak{c}_2.$$

As for the Ehrhart polynomials, we have

$$k(A_r^+, \mathfrak{c}_1)(a) = a_1 + 1 \text{ and } \; k(A_r^+, \mathfrak{c}_2)(a) = a_1 + a_2 + 1.$$

When $r = 3$ we may draw the following picture on a plane with equation $e_1 =$ constant.

We see that there are 8 small chambers $\mathfrak{a}_i$ and 7 big chambers $\mathfrak{c}_k$.

The big chambers are

$$\begin{array}{lll}
\mathfrak{c}_1 & = \mathfrak{a}_1 & = C(e_1, e_2, e_3). \\
\mathfrak{c}_2 & = \mathfrak{a}_2 & = C(e_1, e_1 - e_2, e_1 - e_3). \\
\mathfrak{c}_3 & = \mathfrak{a}_3 & = C(e_1, e_2 - e_3, e_2) \cap C(e_1, e_2, e_1 - e_3). \\
\mathfrak{c}_4 & = \mathfrak{a}_4 & = C(e_1 - e_3, e_2 - e_3, e_1) \cap C(e_1, e_2, e_1 - e_3). \\
\mathfrak{c}_5 & = \mathfrak{a}_5 & = C(e_1, e_2 - e_3, e_2) \cap C(e_2, e_1 - e_3, e_2 - e_3). \\
\mathfrak{c}_6 & = \mathfrak{a}_6 & = C(e_2, e_1 - e_3, e_2 - e_3) \cap C(e_1, e_2 - e_3, e_1 - e_3). \\
\mathfrak{c}_7 & = \mathfrak{a}_7 \cup \mathfrak{a}_8 & = C(e_1 - e_2, e_1, e_3).
\end{array}$$

For all $w \in \Sigma_3$, we have $n(w) = 1$, except if $w = [123]$, where $n(w) = 0$ and $w = [321]$ where $n(w) = 2$. We have

$$\begin{array}{ll}
C^+_{[123]} & = \mathfrak{c}_1 \cup \mathfrak{c}_2 \cup \mathfrak{c}_3 \cup \mathfrak{c}_4 \cup \mathfrak{c}_5 \cup \mathfrak{c}_6 \cup \mathfrak{c}_7 \\
C^+_{[213]} & = \mathfrak{c}_2 \cup \mathfrak{c}_7 \\
C^+_{[132]} & = \mathfrak{c}_5 \cup \mathfrak{c}_6 \\
C^+_{[231]} & = \mathfrak{c}_4 \cup \mathfrak{c}_6 \\
C^+_{[312]} & = \mathfrak{c}_2 \cup \mathfrak{c}_3 \cup \mathfrak{c}_4 \\
C^+_{[321]} & = \mathfrak{c}_2 \cup \mathfrak{c}_4 \cup \mathfrak{c}_6.
\end{array}$$

Thus, we have rather complicated formula for the linear forms $\langle\langle \mathfrak{c}, f \rangle\rangle$, except on the "nice" chamber $\mathfrak{c}^+ = \mathfrak{c}_1$.

$$\begin{array}{ll}
\langle\langle \mathfrak{c}_1, f \rangle\rangle & = Ires^{[123]}_{x=0} f. \\
\langle\langle \mathfrak{c}_2, f \rangle\rangle & = Ires^{[123]}_{x=0} f - Ires^{[213]}_{x=0} f - Ires^{[312]}_{x=0} f + Ires^{[321]}_{x=0} f. \\
\langle\langle \mathfrak{c}_3, f \rangle\rangle & = Ires^{[123]}_{x=0} f - Ires^{[312]}_{x=0} f. \\
\langle\langle \mathfrak{c}_4, f \rangle\rangle & = Ires^{[123]}_{x=0} f - Ires^{[231]}_{x=0} f - Ires^{[312]}_{x=0} f + Ires^{[321]}_{x=0} f. \\
\langle\langle \mathfrak{c}_5, f \rangle\rangle & = Ires^{[123]}_{x=0} f - Ires^{[132]}_{x=0} f. \\
\langle\langle \mathfrak{c}_6, f \rangle\rangle & = Ires^{[123]}_{x=0} f - Ires^{[132]}_{x=0} f - Ires^{[231]}_{x=0} f + Ires^{[321]}_{x=0} f. \\
\langle\langle \mathfrak{c}_7, f \rangle\rangle & = Ires^{[123]}_{x=0} f - Ires^{[213]}_{x=0} f.
\end{array}$$

We easily compute the coefficients $f_{\mathbf{i}}$ of the function

$$J_{A_3^+}(a_1 e^1 + a_2 e^2 + a_3 e^3) = \frac{1}{6} Tres_{A_3^+}\left(\frac{(a_1 x_1 + a_2 x_2 + a_3 x_3)^3}{x_1 x_2 x_3 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\right).$$

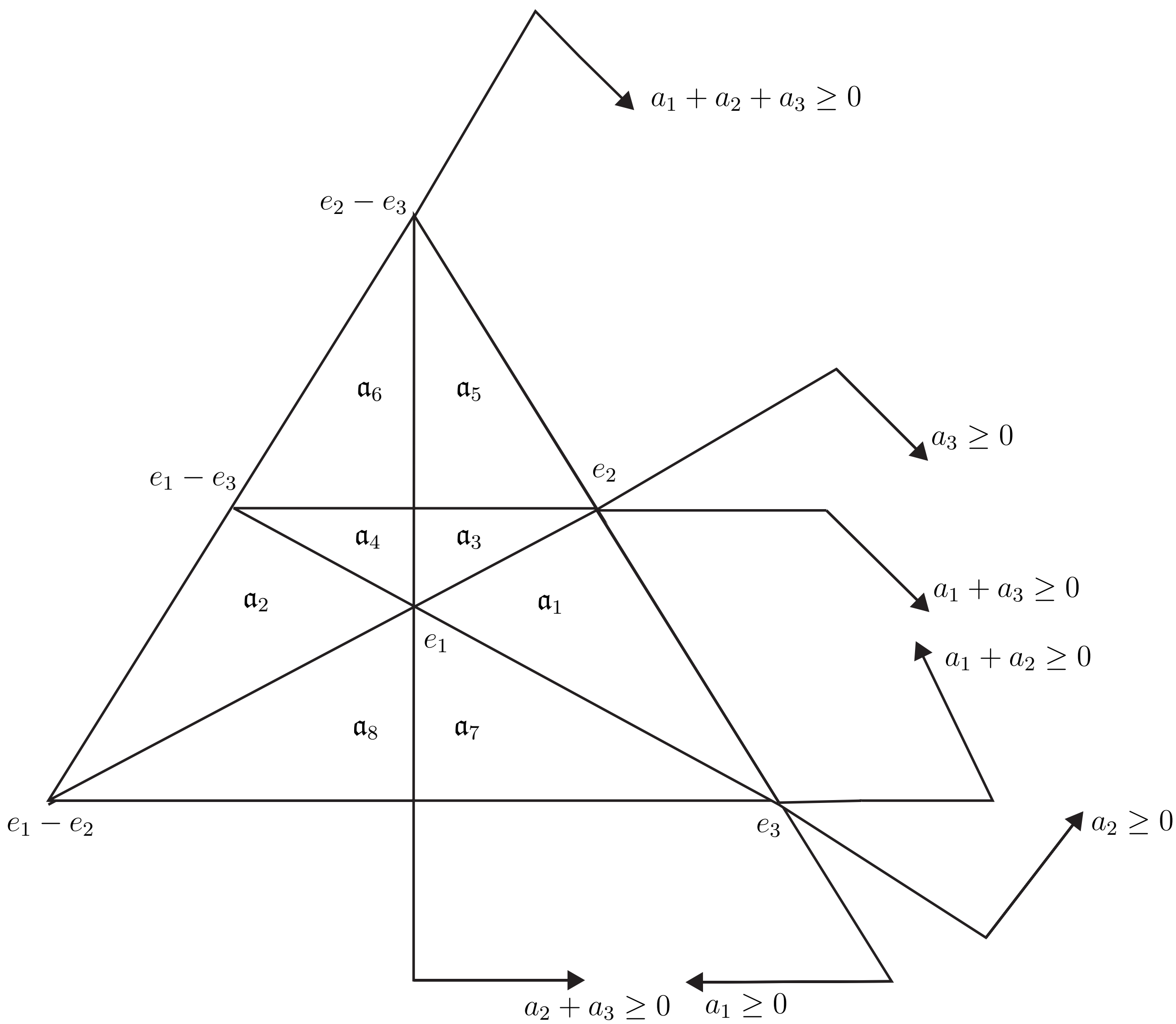


Figure 4: Chambers for A_3 with $a = a_1 e^1 + a_2 e^2 + a_3 e^3$

Let $v_+(a) = \frac{1}{6}a_1^3 + \frac{1}{2}a_1^2 a_2$.
We obtain:

$$J_{A_r^+}(a_1 e^1 + a_2 e^2 + a_3 e^3) = \sum_{w \in \Sigma_r} \epsilon(w) v_+(w^{-1} \cdot a) f_w.$$

Thus, we see that

$$v(A_3^+, \mathfrak{c}^+)(a) = \frac{1}{6} a_1^2 (a_1 + 3a_2).$$

$$k(A_3^+, \mathfrak{c}^+)(a) = \frac{1}{6}(a_1 + 1)(a_1 + 2)(a_1 + 3a_2 + 3).$$

Thus here is the list of formulas for the volume and partition functions. We see that the polynomials $v(A_3^+, \mathfrak{c}_i)$ are all differents so that the big chambers are indeed the minimal domains where the function vol is expressed by a polynomial formula.

$v(A_3^+, \mathfrak{c}_1)(a) \quad = \frac{1}{6}a_1^2(a_1 + 3a_2)$.
$k(A_3^+, \mathfrak{c}_1)(a) \quad = \frac{1}{6}(a_1 + 1)(a_1 + 2)(a_1 + 3a_2 + 3)$.
$v(A_3^+, \mathfrak{c}_2)(a) \quad = \frac{1}{6}(a_1 + a_2 + a_3)^2(a_1 + a_2 - 2a_3)$.
$k(A_3^+, \mathfrak{c}_2)(a) \quad = \frac{1}{6}(a_1 + a_2 + a_3 + 1)(a_1 + a_2 + a_3 + 2)(a_1 + a_2 - 2a_3 + 3)$.
$v(A_3^+, \mathfrak{c}_3)(a) \quad = \frac{1}{6}a_1^3 + \frac{1}{2}a_1^2 a_2 - \frac{1}{2}a_1 a_3^2 - \frac{1}{6}a_3^3$.
$k(A_3^+, \mathfrak{c}_3)(a) \quad = v(A_r^+, \mathfrak{c}_3)(a) + a_1^2 + \frac{3}{2}a_1 a_2 + \frac{1}{2}a_1 a_3 - \frac{1}{2}a_3^2$
$\qquad + \frac{11}{6}a_1 + a_2 + \frac{2}{3}a_3 + 1$.
$v(A_3^+, \mathfrak{c}_4)(a) \quad = \frac{1}{6}a_1^3 + \frac{1}{2}a_1^2 a_2 - \frac{1}{2}a_1 a_3^2 - \frac{1}{6}a_2^3 - \frac{1}{2}a_2^2 a_3 - \frac{1}{2}a_2 a_3^2 - \frac{1}{3}a_3^3$.
$k(A_3^+, \mathfrak{c}_4)(a) \quad = v(A_r^+, \mathfrak{c}_4)(a) + a_1^2 + \frac{3}{2}a_1 a_2 + \frac{1}{2}a_1 a_3 - \frac{1}{2}a_3^2$
$\qquad + \frac{11}{6}a_1 + \frac{7}{6}a_2 + \frac{5}{6}a_3 + 1$.
$v(A_3^+, \mathfrak{c}_5)(a) \quad = \frac{1}{6}a_1^2(2a_1 + 3a_2 + 3a_3)$.
$k(A_3^+, \mathfrak{c}_5)(a) \quad = \frac{1}{6}(a_1 + 2)(a_1 + 1)(2a_1 + 3a_2 + 3a_3 + 3)$.
$v(A_3^+, \mathfrak{c}_6)(a) \quad = \frac{1}{6}(a_1 + a_2 + a_3)^2(2a_1 - a_2 - a_3)$.
$k(A_3^+, \mathfrak{c}_6)(a) \quad = \frac{1}{6}(a_1 + a_2 + a_3 + 1)(a_1 + a_2 + a_3 + 2)(2a_1 - a_2 - a_3 + 3)$
$v(A_3^+, \mathfrak{c}_7)(a) \quad = \frac{1}{6}(a_1 + a_2)^3$.
$k(A_3^+, \mathfrak{c}_7)(a) \quad = \frac{1}{6}(a_1 + a_2 + 1)(a_1 + a_2 + 2)(a_1 + a_2 + 3)$.

Volume functions vanishes on the boundary of the cone $C(\Delta^+)$, thus all functions $v(A_3^+, \mathfrak{c}_k)$ for $k \neq 3, 4$ must have a linear factor. The order of this factor is computed as $L - 1$, where L is the number of roots not on the face boundaring the chamber (see for example jump formula in [B-V 2]), thus is 2 for $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_5, \mathfrak{c}_6$ and 3 for $\mathfrak{c}_7$.

It is also reassuring to check that functions $k(A_3^+, \mathfrak{c}_k)$ define a continuous function on $C(\Delta^+)$ (polynomials $k(A_3^+, \mathfrak{c}_{k_1})$ and $k(A_3^+, \mathfrak{c}_{k_2})$ agree on $\mathfrak{c}_{k_1} \cap$

$\mathfrak{c}_{k_2}$. For example, if $a_2 = a_3 = 0$, the five polynomials $k(A_3^+, \mathfrak{c}_k)(xe_1)$ $k = 1, 2, 3, 4, 7$ restricts to $\frac{1}{6}(x+1)(x+2)(x+3)$.

Bibliographical remarks. Tables for $k(A_3^+, \mathfrak{c})$ are given for example in [S-B].

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