ANALYTIC CONTINUATION OF THE HOLOMORPHIC DISCRETE SERIES OF A SEMI-SIMPLE LIE GROUP

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0. Introduction

In this paper, we are mainly interested in the construction of certain Hilbert spaces of holomorphic functions on an irreducible hermitian symmatric space $D=G/K$ on which G acts by a natural unitary representation. Our construction will produce some new irreducible representations of the simple group G.

In this introduction we shall indicate the nature of our methods and results, leaving the full statements to the text. We shall be looking at the unbounded realization of G/K, as a Siegel domain $D=D(\Omega, Q)$ of type II; however our methods are here more easily explained for tube domains. Thus, in this introduction, we shall consider G/K as a tube domain $D=R^n+i\Omega\subset C^n$, where Ω is a homogeneous irreducible self-dual convex cone in R^n. We reiterate that our results appear, in the text, for the general case.

Let $G(\Omega)=\{g\in GL(R^n), g(\Omega)=\Omega\}$. Let $G(D)$ be the connected component of the group of holomorphic transformations of D and $\tilde{G}(D)$ the universal covering of $G(D)$.

There is a natural unitary irreducible representation of $G(D)$ on the Hilbert space of holomorphic functions on D which are square integrable, i.e., the Bergman space

$$H_1=\{F \text{ holomorphic in } D \text{ such that } \int_{R^n+i\Omega}|F(x+iy)|^2\,dx\,dy<\infty\}. \tag{1}$$

The group $G(D)$ acts on H_1 according to the formula

$$(T_1(g)F)(z)=d(g^{-1};z)\,F(g^{-1}\cdot z) \tag{2}$$

where $d(g;z)$ is the complex Jacobian of the holomorphic map $u\to g\cdot u$ at the point z. It can be seen that this representation T_1 is a member of the discrete series of $G(D)$, i.e., contained in $L^2(G(D))$. Let $P(z-\bar{w})$ be the Bergman kernel, i.e., P is a holomorphic function

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on D such that for $P_w(z)=P(z-\bar{w})$, $F(w)=\langle F, P_w\rangle_{H_1}$ for every F in H_1. As an immediate corollary of the unitarity of T_1, we have

$$d(g;z)^{-1}P(z-\bar{w})\overline{d(g;w)}^{-1}=P(g\cdot z-\overline{g\cdot w}).$$

As Koranyì remarked in [11], the Hardy space

$$H_2=\{F \text{ holomorphic in } D; \sup_{t\in\Omega}\int_{R^n}|F(x+it)|^2dx=\|F\|^2<\infty\} \tag{3}$$

has a reproducing kernel which is a fractional power P^{α_1} $(\alpha_1<1)$ of the Bergman kernel. Then it is easily seen that the representation

$$(T(g)F)(z)=d(g^{-1};z)^{\alpha}F(g^{-1}\cdot z) \tag{4}$$

$(\alpha=\alpha_1)$ is a unitary irreducible representation of $\tilde{G}(D)$ in H_2, but this representation is no longer discrete. More precisely, it can be easily seen via boundary values, that this representation can be identified with a proper subrepresentation of a representation induced by a unitary character of a maximal parabolic subgroup of $\tilde{G}(D)$.

It is then natural to pose the following question:

Problem A. For which real numbers α is P^α the reproducing kernel for a Hilbert space of holomorphic functions on D?

We can restate this as follows: Find the set $\underset{\sim}{P}$ of α such that (P): Given $z_1, ..., z_N\in D$, and $c_1, ..., c_N\in C$,

$$\sum c_i\bar{c}_jP^\alpha(z_i-\bar{z}_j)\geqslant 0?$$

For α in $\underset{\sim}{P}$, the representation T_α given by the formula (4) is unitary and irreducible on the Hilbert space H_α of holomorphic functions on D defined as the completion of the $\sum_{i=1}^N c_iP_{z_j}^\alpha$ with the formula in (P) giving the norm of such holomorphic functions (see [14, 15]).

In this paper we shall give a complete answer to problem A. Originally we had found a half-line contained in $\underset{\sim}{P}$, and we felt that this was the entire set. However, Wallach, working with a purely algebraic formulation of this problem (as in Harish-Chandra [8]), found, in addition to our half-line, a discrete set of points which formed the entire set (to be called the Wallach set). We took up the problem again, and using a classical theorem of Nussbaum [17], we were able, independently and by completely analytic means, to find the Wallach set $\underset{\sim}{P}$, and to associate to each α in $\underset{\sim}{P}$ a concrete Hilbert space of holomorphic functions on D. The Hilbert spaces occuring in the half line are defined by a choice of norm, whereas those corresponding to the discrete set are given as solutions of certain systems of partial differential equations (similar to an example of Ehrenpreis [4]). As a corollary of special interest we produce some generalized Hardy spaces which will be naturally imbedded (in terms of appropriate boundary values) inside certain unitary principal series representations.

Let us now sketch the "plan" of this article. In chapters 1 and 2, we begin by recalling notations and results of [20a] concerning the realization of the relative discrete holomorphic series of $\tilde{G}(D)$ as spaces of holomorphic functions on D.

In chapter 3, using a theorem of Nussbaum [17], we prove that α has property (P) if and only if P^α has an integral representation:

$$P^\alpha(z-\bar{w}) = \int e^{2\pi i\langle\zeta,\, z-\bar{w}\rangle} d\mu_\alpha(\xi), \tag{5}$$

where $d\mu_\alpha(\xi)$ is a positive Borel measure supported on the closure $\bar{\Omega}$ of the cone Ω. (The corresponding representation in Siegel II domains is easily deduced from this). Using this integral representation, the Hilbert space H_α is seen to be

$$\{\check{\phi}(z) = \int e^{2i\pi\langle\xi,\, z\rangle}\phi(\xi)\, d\mu_\alpha(\xi); \quad \phi \in L^2(d\mu_\alpha) \quad \text{and} \quad \|\check{\phi}\|_{H_\alpha} = \|\phi\|_{L^2(d\mu_\alpha)}\}.$$

Many such results (as well as those of [20], a) are closely related with the paper of Koranyì-Stein [12] and the Gindikin integral representation.

The representation (5) being unique, in chapter 4 we see that $d\mu_\alpha$ has to transform under the action of $G(\Omega)$ by the character $g \to |\det_{R^n} g|^{-2\alpha}$. We determine all such measures; the description is as follows:

First of all, for *each* α, there exists a unique semi-invariant measure $d\mu_\alpha$ supported in Ω. We can calculate rather easily the integral (5) in group coordinates after identifying Ω with the "Iwasawa" solvable subgroup of $G(\Omega)$, and we show that the integral converges if and only if $\alpha > c \geqslant 0$, where c is an explicit constant associated to the cone Ω. In particular, $c = 0$ if and only if the rank r of the cone, the real rank of $G(D)$, is 1.

Secondly, the boundary $\bar{\Omega} - \Omega$ of Ω in R^n breaks into r orbits $O_1, \ldots, O_i, \ldots, O_r$ where $O_1 = \{0\}$ and $O_i \subset \bar{O}_{i+1}$. Each of these orbits carries a unique semi-invariant measure $d\mu_i$ (the character is determined); the r values of the associated character are $|\det g|^{-2x_i}$ with $x_1 = 0$, $x_r = c$, and $x_1, \ldots, x_r$ dividing $[0, c]$ into equal intervals. For each $d\mu_i$ in this discrete set, the integral (5) is convergent.

We thus obtain the following diagram of the Wallach set,

$$P = \{x_1\} \cup \ldots \cup \{x_r\} \cup \{\alpha > c\}:$$

$x_1 = 0 \quad x_2 \quad x_3 \ldots x_r = c \quad \alpha_1 \quad \alpha_2 \ldots \alpha_r = c_d$

r discrete points $\qquad$ r points $\qquad$ holomorphic discrete series

The values x_j correspond to Hilbert spaces defined by certain systems of partial differential equations, while the values α_j correspond to various "Hardy spaces"; in both cases these are determined by certain boundary orbits.

In chapter 5, we investigate the possibility of describing the norms of some of the spaces H_α for $\alpha > c$ intrinsically in terms of the holomorphic functions in H_α. Of course, if α is sufficiently large, i.e., if $\alpha > c_d$, H_α is in the holomorphic discrete series, where c_d was determined by Harish-Chandra. Recall that in [20a] we calculated c_d in terms of the convergence of some simple integral on Ω and here H_α has an intrinsic description as a space of holomorphic functions, square integrable on D, for some measure $dx\, d\mu_\alpha(y)$, ($d\mu_\alpha$ on Ω).

Now for the r points situated equidistantly between $\alpha_r = c_d$ and α_1 we show that the abstract norm in H_α is a Hardy type norm; namely

$H_{\alpha_i} = \{F$ holomorphic in $R^n + i\Omega$ such that

$$\|F\|^2 = \sup_{t \in \Omega} \int_{\substack{x \in R^n \\ y \in O_i}} |F(x + i(y+t))|^2 dx\, d\mu_i(y) < \infty\}.$$

Of course if $i = 1$, i.e., $O_i = \{0\}$ we find the usual Hardy space H^2.

We can take boundary values on $\Sigma_i = R^n + iO_i$ in the corresponding L^2-norms; and if $O_i \neq \{0\}$, we can characterize the space of boundary values as weak solutions of certain first order left invariant differential operators. These are the tangential Cauchy-Riemann equations on the real submanifold Σ_i.

In chapter 6, we produce for each α_i, $1 \leqslant i \leqslant r$, a maximal parabolic subgroup P_i of the group $\tilde{G}(D)$ and a unitary irreducible representation of P_i such that the corresponding induced representation τ_i is reducible, having H_{α_i} as proper irreducible subspace. These are new examples of reducible principal series representations. If $i \neq r$, the series are degenerate. If $i = r$, the corresponding representation was studied by Knapp and Okamoto [10].

Except when $i = 1$ in the tube case (where Σ_1 is totally real and has no "holomorphic tangent space"), these tangential Cauchy-Riemann equations on Σ_i characterize the space H_{α_i}. In this case then, we cannot find an irreducible piece of τ_i by "holomorphic induction", but we have to allow more general differential equations. Otherwise put, when, in general, we cut down a representation by introducing a complex Lie algebra $\mathfrak{h} \subset \mathfrak{g}^C$, we should not require that $\mathfrak{h}+\bar{\mathfrak{h}}$ is also a Lie algebra; our spaces H_{α_i} provide such an example.

We here acknowledge that K. Gross and R. Kunze, in their study of the decomposition of a metaplectic representation in [7], produced some of these spaces H_α, suggesting to us that there should be some analytic continuation through the limit point of Knapp-Okamoto [10] (for the universal covering of $SL(2;R)$ see [19] and [22]). Our present results concern the case of scalar valued holomorphic functions, corresponding to holomorphic sections of a line bundle on G/K. We wish to thank Nicole Conze, Ray Kunze, Mustapha Rais, Eli Stein and Nolan Wallach for much friendly help and conversation on the topics of this work.

1. An algebraic result

1.1. Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a Cartan decomposition for $\mathfrak{g}$. We shall suppose that $\mathfrak{k}$ has a non-empty center $\mathfrak{z}$; then $\mathfrak{z}=RZ$, where the eigenvalues of the adjoint action of Z on $\mathfrak{p}^C$ are $\pm i$. Let

$$\mathfrak{p}^+=\{X\in\mathfrak{p}^C;\,[Z,X]=iX\},$$
$$\mathfrak{p}^-=\{X\in\mathfrak{p}^C;\,[Z,X]=-iX\}.$$

We have $\mathfrak{k}=[\mathfrak{k},\mathfrak{k}]\oplus RZ$.

Let $\tilde{G}$ be the simply connected group with Lie algebra $\mathfrak{g}$. For $X\in\mathfrak{g}$ and ϕ a differentiable function on $\tilde{G}$, we shall let $r(X)\phi$ denote the function $(r(X)\phi)(g)=(d/dt)\phi(g\exp tX)|_{t=0}$. For $X\in\mathfrak{g}^C$, we define $r(X)$ by linearity.

Let $\tilde{K}$ be the analytic subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{k}$. Then $\tilde{G}/\tilde{K}$ is a hermitian symmetric space. The holomorphic functions on $\tilde{G}/\tilde{K}$ can be identified as the space of functions on $\tilde{G}$ which are annihilated by all the vector fields $r(X)$, with $X\in\mathfrak{k}^C+\mathfrak{p}^-$. Notice that $\tilde{K}$ is not compact: $\tilde{K}=[\tilde{K},\tilde{K}]\cdot\exp RZ$, with $[\tilde{K},\tilde{K}]$ compact.

Now let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$. We have $\mathfrak{h}=\mathfrak{h}\cap[\mathfrak{k},\mathfrak{k}]+\mathbf{R}Z$, $(\mathfrak{h}\cap[\mathfrak{k},\mathfrak{k}])^C$ is a Cartan subalgebra of $[\mathfrak{k},\mathfrak{k}]^C$ and $\mathfrak{h}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$. We shall let $\langle,\rangle$ denote the Killing form, and $x\to\bar{x}$ the conjugation in $\mathfrak{g}^C$ relative to the real form $\mathfrak{g}$ of $\mathfrak{g}^C$. Let Δ denote the system of roots of $\mathfrak{g}^C$ relative to $\mathfrak{h}^C$; these roots take purely imaginary values on $\mathfrak{h}$. We have

$$\Delta=\Delta_{\mathfrak{k}}\cup\Delta_{\mathfrak{p}}$$

where

$$\Delta_{\mathfrak{k}}=\{\gamma\in\Delta;\,(\mathfrak{g}^C)^\gamma\subset\mathfrak{k}^C\}$$
$$\Delta_{\mathfrak{p}}=\{\gamma\in\Delta;\,(\mathfrak{g}^C)^\gamma\subset\mathfrak{p}^C\}.$$

Choose an ordering on the roots so that $\mathfrak{p}^+=\sum_{\alpha\in\Delta_{\mathfrak{p}}^+}\mathfrak{g}^\alpha$, and let

$$\varrho=\tfrac{1}{2}\sum_{\alpha\in\Delta^+}\alpha.$$

If $\gamma\in\Delta$, let H_γ be the unique element of $i\mathfrak{h}\cap[(\mathfrak{g}^C)^\gamma, (\mathfrak{g}^C)^{-\gamma}]$ such that $\gamma(H_\gamma)=2$. We have

$$\frac{2}{\langle\gamma,\gamma\rangle}\langle H_\gamma, H\rangle=\gamma(H).$$

For $\gamma\in\Delta_{\mathfrak{p}}^+$, choose $E_\gamma\in(\mathfrak{g}^C)^\gamma$ so that $[E_\gamma,\bar{E}_\gamma]=H_\gamma$, and put $\bar{E}_\gamma=E_{-\gamma}$, and $X_\gamma=E_\gamma+E_{-\gamma}$. Then

$$\mathfrak{p}=\sum\oplus RX_\gamma+\sum\oplus R(iE_\gamma-iE_{-\gamma}).$$

Let r be the real rank of $\mathfrak{g}$, and γ_r the highest root; then $H_{\gamma_r}\in(\mathfrak{h}\cap[\mathfrak{k},\mathfrak{k}])^C$.

1.2. Realization of $\tilde{G}/\tilde{K}$ as a bounded domain ([8], IV, or [9])

Let G_C be the simply connected group with Lie algebra $\mathfrak{g}^C$ and G, K, K_C, P_+ and P_- the connected subgroups of G_C with Lie algebra $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}^C$, $\mathfrak{p}^+$, $\mathfrak{p}^-$, respectively. Note that $\tilde{G}/\tilde{K}$ is canonically isomorphic to G/K. Every element of $P_+K_CP_-$ can be written in a unique way:

$$g=\exp\zeta(g)\cdot k(g)\cdot\exp\zeta'(g)$$

with $\zeta(g)\in\mathfrak{p}^+$, $k(g)\in K_C$, and $\zeta'(g)\in\mathfrak{p}^-$. We have $G\subset P_+K_CP_-$, and the map $g\to k(g)$ lifts to a map also denoted $k(g)$ of $\tilde{G}$ into $\tilde{K}_C$, the universal cover of K_C. The map $g\to\zeta(g)$ induces a biholomorphism of the complex manifold G/K onto a bounded domain $\mathcal{D}$ in $\mathfrak{p}^+$. For $X\in\mathcal{D}$, we shall denote by $g\cdot X$ the unique element of $\mathcal{D}$ such that

$$g\exp X\in\exp(g\cdot X)K_CP_-.$$

We know that the action of G on $\mathcal{D}$ extends continuously to the closure $\bar{\mathcal{D}}$ of $\mathcal{D}$ in $\mathfrak{p}^+$; i.e., for any $X\in\bar{\mathcal{D}}$, $g\cdot\exp X\in P_+K_CP_-$.

1.3. The discrete holomorphic series ([8], V and VI)

Let Λ_0 be a dominant weight of $[\mathfrak{k},\mathfrak{k}]$, i.e., Λ_0 is a linear form on $(\mathfrak{h}\cap[\mathfrak{k},\mathfrak{k}])^C$ such that $\Lambda_0(H_\alpha)$ is a non-negative integer for every compact positive root α. Let U_0 be the representation of $[\tilde{K},\tilde{K}]$ with highest weight Λ_0; U_0 acts in a finite dimensional Hilbert space V_{Λ_0}. We shall let v_{Λ_0} denote the vector of highest weight Λ_0, normalized so that $\|v_{\Lambda_0}\|=1$. For λ any real number let $\Lambda=(\Lambda_0,\lambda)$ be the linear form on $\mathfrak{h}^C$ whose restriction to $(\mathfrak{h}\cap[\mathfrak{k},\mathfrak{k}])^C$ is Λ_0 and such that $\langle\Lambda, H_{\gamma_r}\rangle=\lambda$. Let U_Λ be the representation of $\tilde{K}$ of highest weight Λ; U_Λ acts on $V_\Lambda=V_{\Lambda_0}$, restricting to $[\tilde{K},\tilde{K}]$ as U_{Λ_0} and v_{Λ_0} is also the highest weight vector for U_Λ and we have

$$H\cdot v_{\Lambda_0}=\langle\Lambda, H\rangle v_{\Lambda_0}\quad\text{for all } H\in\mathfrak{h}.$$

The pair $(\Lambda_0,\lambda)=\Lambda$ parametrizes the irreducible unitary representations of $\tilde{K}$.

1.3.1. *Definition.* $O(\Lambda)=\{\phi;\ C^\infty$ function on $\tilde{G}$ with values in V_Λ such that

(i) $$\phi(gk)=U_\Lambda(k)^{-1}\phi(g), \quad g\in\tilde{G}, \quad k\in\tilde{K}.$$

(ii) $$r(X)\cdot\phi=0 \quad \text{for all } X\in\mathfrak{p}^-\}.$$

$\tilde{G}$ acts by left translations in $O(\Lambda)$, we denote this action by T_Λ: $(T_\Lambda(x)\phi)(g)=\phi(x^{-1}g)$.

Now $O(\Lambda)$ can be realized as the space $O(V_\Lambda)$ of V_Λ-valued holomorphic functions on $\tilde{G}/\tilde{K}$ as follows. Letting U_Λ also denote the (holomorphic) representation of $\tilde{K}_C$ which restricts to U_Λ on $\tilde{K}$, we put

$$\phi_\Lambda^0(g)=U_\Lambda(k(g)). \tag{1.3.2}$$

ϕ_Λ^0 is an operator-valued function on $\tilde{G}$, and we obtain

$$O(\Lambda)=\{\phi:\phi(g)=\phi_\Lambda^0(g)^{-1}F(g\tilde{K}), \quad \text{with} \quad F\in O(V_\Lambda)\}. \tag{1.3.3}$$

Thus, corresponding to any $v\in V_\Lambda$, we obtain the element ψ_Λ^v in $O(V_\Lambda)$ given by

$$\psi_\Lambda^v(g)=\phi_\Lambda^0(g)^{-1}\cdot v.$$

1.3.4. *Definition.* Given Λ and the highest weight vector v_Λ, we define the scalar function on $\tilde{G}$

$$\psi_\Lambda(g)=\langle\psi_\Lambda^{v_\Lambda}(g), v_\Lambda\rangle=\langle\phi_\Lambda^0(g)^{-1}v_\Lambda, v_\Lambda\rangle$$

(where the inner product is in V_Λ).

Let $\mathcal{L}_\Lambda$ be the subspace of $O(\Lambda)$ generated by the left translates of $\phi_\Lambda^{v_\Lambda}$.

It is easy to see that because of the invariance conditions defining $O(\Lambda)$, the correspondence

$$\phi\to\langle\phi, v_\Lambda\rangle \tag{1.3.5}$$

identifies $O(\Lambda)$ as a space of functions on $\tilde{G}$. In fact, taking $\mathfrak{g}_-=\sum_{\alpha<0}\mathfrak{g}^\alpha$, and

$$O'(\Lambda)=\{\phi;\ C^\infty \text{ functions on } \tilde{G} \text{ with scalar values such that}$$
$$r(X)\cdot\phi=0, \quad X\in\mathfrak{g}_-$$
$$r(H)\cdot\phi=-\langle\Lambda, H\rangle\phi, \quad H\in\mathfrak{h}\},$$

the correspondence (1.3.5) is an isomorphism of $O(\Lambda)$ with $O'(\Lambda)$.

Now, if $\phi\in O(\Lambda)$, ϕ transforms on the right by a unitary character of the center $Z(\tilde{G})$ of $\tilde{G}$; (as $Z(\tilde{G})\subset\tilde{K}$, and $\phi(gz)=U_\Lambda(z)^{-1}\phi(g)$). We introduce the Hilbert space

$$H(\Lambda)=\{\phi\in O(\Lambda); \int_{\tilde{G}/Z(\tilde{G})}\|\phi(g)\|_{V_\Lambda}^2\,dg=N(\Lambda;\ \phi)^2<\infty\}. \tag{1.3.6}$$

Harish-Chandra ([8], VI) has proven that $H(\Lambda) \neq \{0\}$ if and only if $\langle \Lambda + \varrho, H_{\gamma_r} \rangle < 0$. Since $H(\Lambda)$ is (viewed as a subspace of $O'(\Lambda)$) a space of functions on $\tilde{G}$, square integrable over $\tilde{G}/Z(\tilde{G})$, and (viewed as a subspace of $O(V_\Lambda)$) is a space of holomorphic vectorvalued functions on $\tilde{G}/\tilde{K}$, the representation T_Λ is seen to be an irreducible unitary representation of $\tilde{G}$, which belongs to the relative discrete series of $\tilde{G}$. Thus T_Λ is said to belong to the discrete holomorphic series of $\tilde{G}$.

Now $H(\Lambda)$ (when non-zero), is in one interpretation, a Hilbert space of holomorphic functions and admits a reproducing Kernel function. This kernel function is realized in $O'(\Lambda)$ by the function ψ_Λ. This assertion is based on the following computations ([8], V, VI):

When $H(\Lambda) \neq \{0\}$, the function $\psi_\Lambda^{v_\Lambda}$ is in $H(\Lambda)$, and for 1 the identity of $\tilde{G}$ we have, for $\phi \in H(\Lambda)$

$$\langle \phi(1), v \rangle_{V_\Lambda} = \frac{1}{\langle \psi_\Lambda^{v_\Lambda}, \psi_\Lambda^{v_\Lambda} \rangle} \langle \phi, \psi_\Lambda^{v} \rangle_{H(\Lambda)}. \tag{1.3.7}$$

Let

$$c(\Lambda) = \langle \psi_\Lambda^{v_\Lambda}, \psi_\Lambda^{v_\Lambda} \rangle. \tag{1.3.8}$$

We have

$$c(\Lambda) = \left(\prod_{\beta > 0} \frac{\langle \Lambda + \varrho, H_\beta \rangle}{\langle \varrho, H_\beta \rangle} \right)^{-1}. \tag{1.3.9}$$

Now, by (1.3.8), taking $\phi = T_\Lambda(x)\psi_\Lambda^{v_\Lambda}$, (1.3.7) becomes

$$\langle \psi_\Lambda^{v_\Lambda}, T_\Lambda(x)\psi_\Lambda^{v_\Lambda} \rangle = c(\Lambda)\psi_\Lambda(x). \tag{1.3.10}$$

This is essentially equivalent to (1.3.7), for $\mathcal{L}_\Lambda$ is dense in $H(\Lambda)$, since T_Λ is irreducible. Finally, by (1.3.10) the norm of an element of $\mathcal{L}_\Lambda$ becomes

$$\| \sum c_i T(g_i) \psi_\Lambda^{v_\Lambda} \|^2 = c(\Lambda) \left| \sum_{i,j} c_i \bar{c}_j \psi_\Lambda(g_i^{-1} g_j) \right| \tag{1.3.11}$$

1.4. Analytic continuation of the discrete series

More generally, let us fix Λ_0, a linear form on $(\mathfrak{h} \cap [\mathfrak{k}, \mathfrak{k}])^C$ such that $\Lambda_0(H_\alpha)$ is a non-negative integer, for every positive compact root α. For $\lambda \in R$, we denote by $\Lambda = (\Lambda_0, \lambda)$ the linear form on $\mathfrak{h}^C$ given by

$$\Lambda \mid (\mathfrak{h} \cap [\mathfrak{k}, \mathfrak{k}])^C = \Lambda_0, \quad \Lambda(H_{\gamma_r}) = \lambda,$$

and now in this general context, we define as above the $\mathcal{L}_\lambda$, $O(\Lambda)$, ψ_Λ.

1.4.2. *Definition.* $P_{\Lambda_0} = \{\lambda \in R;\ \psi_\Lambda$ is of positive type, i.e., for all $\{c_i \in C\}$, $\{g_i \in G\}$, $\sum_{i,j} c_i \bar{c}_j \psi_\Lambda(g_i^{-1} g_j) \geq 0\}$.

For $\lambda \in P_{\Lambda_0}$, we can define a left-invariant scalar product on $\mathcal{L}_\Lambda$ by

$$\|\sum_i c_i T_\Lambda(g_i)\psi_\Lambda^{v_\Lambda}\|^2 = \sum c_i \bar{c}_j \psi_\Lambda(g_i^{-1}g_j).$$

We shall let $H_0(\Lambda)$ denote the completion of $\mathcal{L}_\Lambda$ in this norm, and $N_0(\Lambda; \phi)$ the norm of $\phi \in \mathcal{L}_\Lambda$. Thus, by (1.4.1), if $H(\Lambda) \neq \{0\}$, i.e., if $\langle \Lambda + \varrho, H\gamma_r \rangle < 0$, ψ_Λ is of positive type, $\lambda \in P_{\Lambda_0}$, and $H_0(\Lambda) = H(\Lambda)$ with

$$N_0(\Lambda; \phi)^2 = \frac{N(\Lambda; \phi)^2}{c(\Lambda)}.$$

1.4.3. Lemma. *For $\lambda \in P_{\Lambda_0}$, $H_0(\Lambda)$ can be identified with a subspace of $O(\Lambda)$. We have $\|F\| = 0$ if and only if $F = 0$, and*

$$\langle F(1), v_\Lambda \rangle = \langle F, \psi_\Lambda^{v_\Lambda} \rangle_{H_0(\Lambda)}. \tag{1.4.4}$$

The representation by left translation, T_Λ, of $\tilde{G}$ on $H_0(\Lambda)$ is unitary and irreducible. T_Λ is a member of the relative discrete holomorphic series of $\tilde{G}$ only if $\langle \Lambda + \varrho, H\gamma_r \rangle < 0$.

Proof. By (1.3.4), the definition of ψ_Λ, we have

$$\langle T_\Lambda(g)\psi_\Lambda^{v_\Lambda}, \psi_\Lambda^{v_\Lambda} \rangle_{H_0(\Lambda)} = \psi_\Lambda(g^{-1}) = \langle T_\Lambda(g)\psi_\Lambda^{v_\Lambda}(1), v_\Lambda \rangle, \tag{1.4.5}$$

which is just (1.4.4) for $F = T_\Lambda(g)\psi_\Lambda^{v_\Lambda}$. Taking linear combinations, the equation persists, so (1.4.4) is proven. More generally, for $F \in \mathcal{L}_\Lambda$, we have

$$\langle F(g_0), k \cdot v_\Lambda \rangle_{V_\Lambda} = \langle F, T_\Lambda(g_0)\psi_\Lambda^{k \cdot v_\Lambda} \rangle_{H_0(\Lambda)},$$

so that point evaluation of functions in $\mathcal{L}_\Lambda$ is $H_0(\Lambda)$-norm continuous. Thus the identification of $\mathcal{L}_\Lambda$ as a subspace of $O(V_\Lambda)$ realizes $\mathcal{L}_\Lambda$ as a normed space of V_Λ-valued holomorphic functions on $\tilde{G}/\tilde{K}$ for which point evaluations are norm continuous. From this one easily proves in the standard way that the norm completion $H_0(\Lambda)$ of $\mathcal{L}_\Lambda$ is a Hilbert subspace of the space $O(V_\Lambda)$ of V_Λ-valued holomorphic functions on $\tilde{G}/\tilde{K}$. That T_Λ is irreducible and unitary follows as in [14].

Finally, by (1.4.5), the coefficient corresponding to $\psi_\Lambda^{v_\Lambda}$ is ψ_Λ; and ψ_Λ is square-integrable on $\tilde{G}/Z(\tilde{G})$ only if $\langle \Lambda + \varrho, H\gamma_r \rangle < 0$.

Thus, for $\lambda \in P_{\Lambda_0}$, one constructs an irreducible unitary representation T_Λ of $\tilde{G}$ in a Hilbert subspace of $O(\Lambda)$. Our problem is to determine explicitly the set P_{Λ_0}, and to identify in some reasonable way, the corresponding representations T_Λ.

1.5. Associated infinitesimal modules

We shall denote the enveloping algebra of $\mathfrak{g}^C$ by $\mathcal{U}$. Let $u \to {}^t u$ be the linear antiautomorphism of $\mathcal{U}$ which extends the map $x \to -x$ on $\mathfrak{g}^C$ and $u \to u^*$ the antilinear antiautomorphism which extends the map $x \to -\bar{x}$ on $\mathfrak{g}^C$. Let M be a $\mathcal{U}$-module and B a hermitian form on M. We shall say that B is $\mathfrak{g}$-invariant if $B(u\cdot m, m') = B(m, u^* m')$. Let $\mathfrak{g}^+ = \sum_{\alpha\in\Delta^+}\mathfrak{g}^\alpha$ and (as before) $\mathfrak{g}^- = \sum_{\alpha\in\Delta^-}\mathfrak{g}^\alpha$. We have $\mathcal{U} = (\mathcal{U}\mathfrak{g}^+ \oplus \mathfrak{g}^-\mathcal{U}) \oplus \mathcal{U}(\mathfrak{h})^C$. We shall let $\gamma(u)$ represent the unique element of $\mathcal{U}(\mathfrak{h})^C$ such that $u-\gamma(u)$ is in $\mathcal{U}\mathfrak{g}^+ \oplus \mathfrak{g}^-\mathcal{U}$. Since the algebra $\mathcal{U}(\mathfrak{h})^C$ is canonically identified with the algebra $S(\mathfrak{h})^C$ of polynomials on the dual of $\mathfrak{h}^C$, we can calculate $\langle \Lambda, u\rangle$ for all $u \in \mathcal{U}(\mathfrak{h})^C$. The form $B_\Lambda(u,v) = \langle \Lambda, \gamma(v^*u)\rangle$ is clearly a $\mathfrak{g}$-invariant hermitian form. Let ξ_Λ be the form on $\mathcal{U}$ defined by

$$\xi_\Lambda(u) = \langle \Lambda, \gamma({}^t u)\rangle.$$

Now the enveloping algebra of $\mathfrak{g}^C$ operates on the left in $O(\Lambda)$ by extending the action of $\mathfrak{g}$ given by

$$(X\cdot\phi)(g) = \frac{d}{dt}\phi(\exp(-tX)g)|_{t=0}.$$

Let $W_\Lambda = \mathcal{U}\cdot\psi_\Lambda^{v_\Lambda}$; $W_\Lambda \subset O_\Lambda$, and we have the following fact:

1.5.1. Lemma. (i) *$\psi_\Lambda^{v_\Lambda}$ is an extreme vector of weight Λ.* (ii) *W_Λ is an irreducible $\mathcal{U}$-module. The annihilator M_Λ of $\psi_\Lambda^{v_\Lambda}$ is the kernel of B_Λ:*

$$M_\Lambda = \{u;\ \langle \Lambda, \gamma(v^*u)\rangle = 0 \quad \textit{for all } v\in\mathcal{U}\}.$$

B_Λ thus defines a nondegenerate form on W_Λ, and every other $\mathfrak{g}$-invariant hermitian form is proportional to B_Λ.

Proof. Notice that the map $\phi \to \tilde{\phi}$ given by

$$\tilde{\phi}(u) = \langle ({}^t u\cdot\phi)(1), v_\Lambda\rangle$$

sends W_Λ onto $\mathcal{U}\cdot\xi_\Lambda \subset \mathcal{U}^*$. This map is injective. It is known [24a] that the module $\mathcal{U}\cdot\xi_\Lambda$ is irreducible. The rest of the lemma is in ([8], IV).

1.5.2. Proposition. *$\lambda \in P_{\Lambda_0}$ if and only if $\langle \Lambda, \gamma(u^*u)\rangle \geq 0$ for all $u\in\mathcal{U}$. In this case W_Λ can be identified as the space of K-finite vectors in $H_0(\Lambda)$ and*

$$N_0(\Lambda, u\cdot\psi_\Lambda^{v_\Lambda})^2 = \langle \Lambda, \gamma(u^*u)\rangle.$$

Proof. Suppose $\lambda \in P_{\Lambda_0}$. Then $\psi_\Lambda^{v_\Lambda}$ is in $\mathcal{H}_0(\Lambda)$ and is K-finite. Since the representation T_Λ on $\mathcal{H}_0(\Lambda)$ is irreducible, W_Λ is the space of K-finite vectors for this representation. Thus we can define

$$\{u, v\} = \langle u\cdot\psi_\Lambda^{v_\Lambda}, v\cdot\psi_\Lambda^{v_\Lambda}\rangle_{\mathcal{H}_0(\Lambda)},$$

a $\mathfrak{g}$-invariant hermitian form. Then, by 1.5.1, $\{u, v\} = c\langle\Lambda, \gamma(v^*u)\rangle$ for some c. Since $\langle\psi_\Lambda^{v_\Lambda}, \psi_\Lambda^{v_\Lambda}\rangle_{\mathcal{H}_0(\Lambda)} = 1$, $c=1$ so that

$$\langle\Lambda, \gamma(u^*u)\rangle = N_0(\Lambda, u\cdot\psi_\Lambda^{v_\Lambda})^2 \geqslant 0.$$

Conversely, suppose $\langle\Lambda, \gamma(u^*u)\rangle \geqslant 0$ for all u. Then

$$\langle u\cdot\psi_\Lambda^{v_\Lambda}, v\cdot\psi_\Lambda^{v_\Lambda}\rangle, = \langle\Lambda, \gamma(v^*u)\rangle$$

defines an invariant scalar product on W_Λ. In particular,

$$(1.5.3) \qquad \langle(u\cdot\psi_\Lambda^{v_\Lambda})(1), v_\Lambda\rangle_{V_\Lambda} = \langle\Lambda, \gamma(u)\rangle = \langle u\cdot\psi_\Lambda^{v_\Lambda}, \psi_\Lambda^{v_\Lambda}\rangle.$$

Let $\mathcal{H}_0'(\Lambda)$ be the completion of W_Λ in this inner product. Because of (1.5.3); we prove (as in Lemma 1.4.3), that $\mathcal{H}_0'(\Lambda)$ is a Hilbert space of functions in $O(\Lambda)$ on which $\tilde{G}$ acts unitarily by left-translations. By continuity, (1.5.3) is true for $u\cdot\psi_\Lambda^{v_\Lambda}$ replaced by any function ϕ in $\mathcal{H}_0'(\Lambda)$; in particular, $\phi = T(g^{-1})\psi_\Lambda^{v_\Lambda}$. We obtain

$$\psi_\Lambda(g) = \langle\psi_\Lambda^{v_\Lambda}(g), v_\Lambda\rangle_{V_\Lambda} = \langle T(g^{-1})\psi_\Lambda^{v_\Lambda}, v_\Lambda\rangle_{V_\Lambda} = \langle T(g^{-1})\psi_\Lambda^{v_\Lambda}, \psi_\Lambda^{v_\Lambda}\rangle,$$

from which it immediately follows that ψ_Λ is of positive type. Thus $\lambda \in P_{\Lambda_0}$ (and $\mathcal{H}_0(\Lambda) = \mathcal{H}_0'(\Lambda)$).

1.5.4. COROLLARY. *If $\lambda \in P_{\Lambda_0}$, then $\lambda \leqslant 0$.*

Proof. For $\lambda \in P_{\Lambda_0}$,

$$\langle\Lambda, \gamma(E_{-\gamma_r}^* E_{-\gamma_r})\rangle \geqslant 0.$$

But $E_{-\gamma_r}^* = -E_{\gamma_r}$ and $E_{\gamma_r}E_{-\gamma_r} = E_{-\gamma_r}E_{\gamma_r} + H_{\gamma_r}$, so that we must have $\lambda = \langle\Lambda, H_{\gamma_r}\rangle \leqslant 0$.

Let us now consider the Verma module $M(\Lambda+\varrho)$ which is the universal module of highest weight Λ; (see [3]), that is for I_Λ the left ideal generated by $\mathfrak{g}^+$ and $H - \Lambda(H)$ $(H \in \mathfrak{h}^C)$ and $M(\Lambda+\varrho) = U/I_\Lambda$; let 1_Λ denote the image of 1 in $M(\Lambda+\varrho)$; the module W_Λ is there the unique simple quotient of $M(\Lambda+\varrho)$.

Let a_Λ be the annihilator of v_Λ in the enveloping algebra $\mathcal{U}(\mathfrak{k}^C)$ of $\mathfrak{k}^C$. Then M_Λ, the annihilator of $\psi_\Lambda^{v_\Lambda}$ contains $J_\Lambda = I_\Lambda + \mathcal{U}a_\Lambda$ and is the unique maximal left ideal in $\mathcal{U}$ containing J_Λ. As Nicole Conze has shown to us, we have

1.5.5. PROPOSITION (N. Conze). *If $\langle\Lambda+\varrho, H_\gamma\rangle \notin \{1, 2, ..., n, ...\}$ for every noncompact positive root γ, then J_Λ is a maximal left ideal in $\mathcal{U}$.*

Proof. We have to show that $M(\Lambda+\varrho)/J_\Lambda\cdot 1_\Lambda$ is a simple module. Let N be a submodule of $M(\Lambda+\varrho)$ containing $J_\Lambda\cdot 1_\Lambda$, then there exists a form μ on $\mathfrak{h}^C$ and a vector e_μ in N such that

$$X\cdot e_\mu = 0, \quad X \in \mathfrak{g}^+$$

and $H\cdot e_\mu = \mu(H)e_\mu$, $H \in \mathfrak{h}^C$ and so $M(\mu+\varrho) \subset M(\Lambda+\varrho)$.

Let us show that $\Lambda-\mu$ is a linear combination of compact positive roots. If $\mu_1-\mu_2$ is a linear combination with non-negative integral coefficients of positive roots, we will write $\mu_1\geqslant\mu_2$. If α is a root, we will denote by S_α the symmetry with respect to the root α; i.e., $S_\alpha(\beta)=\beta-\beta(H_\alpha)\alpha$. By Bernstein-Gelfand-Gelfand (Th. 7.6.23, [3]), there exists $\alpha_1, \alpha_2, ..., \alpha_n$ positive roots such that

$$\Lambda+\varrho>S_{\alpha_1}(\Lambda+\varrho)>S_{\alpha_2}\cdot S_{\alpha_1}(\Lambda+\varrho)> \dots >S_{\alpha_n}\dots S_{\alpha_1}(\Lambda+\varrho)=\mu+\varrho$$

and so it is sufficient to prove that the α_i are compact roots; α_1 is compact, as $\Lambda+\varrho> S_{\alpha_1}(\Lambda+\varrho)$ means that $\langle\Lambda+\varrho, H_{\alpha_1}\rangle\in\{1, 2, ...,\}$.

Let us suppose that we have proved that $\alpha_1, \alpha_2, ..., \alpha_i$ are compact positive roots. We then have

$$\langle S_{\alpha_i}\dots S_{\alpha_1}(\Lambda+\varrho), H_{\alpha_{i+1}}\rangle\in\{1,2,3,\dots\}=\langle\Lambda+\varrho, H_{S_{\alpha_1}\cdots S_{\alpha_i}(\alpha_{i+1})}\rangle.$$

But if α_{i+1} was a non-compact positive root, as $\mathfrak{p}^+$ is stable by $\mathfrak{k}^C$, so will $S_{\alpha_1}\dots S_{\alpha_i}(\alpha_{i+1})$, and we will reach a contradiction, and so α_{i+1} is a compact positive root.

We conclude as in Harish-Chandra ([8], IV): i.e., as $M(\Lambda+\varrho)=\mathcal{U}(\mathfrak{p}^-)\cdot\mathcal{U}(\mathfrak{k}^C)\cdot 1_\Lambda$, we see that it is necessary that e_μ belongs to $\mathcal{U}(\mathfrak{k}^C)\cdot 1_\Lambda$. After factorizing through a_Λ, we see that e_μ, being an extreme vector of the simple module V_Λ, is equal to 0 or 1_Λ mod a_Λ.

Remark. We can see in some particular examples that the condition of Proposition 1.5.5 is not necessary for the irreducibility of J_Λ.

1.5.6. Corollary. *Let $\Lambda=(\Lambda_0;\lambda)$. If $\langle\Lambda+\varrho, H_{\gamma_r}\rangle\leqslant 1$, then $\lambda\in P_{\Lambda_0}$.*

Proof. If $\langle\Lambda+\varrho, H_{\gamma_r}\rangle<1$, then if γ is another positive noncompact root we have $\gamma=\gamma_r-\sum m_i\alpha_i$, with $\alpha_i\in\Delta_K^+$ and $m_i\geqslant 0$, so

$$H_\gamma=\frac{\langle\gamma,\gamma\rangle}{\langle\gamma_r,\gamma_r\rangle}H_{\gamma_r}-\sum m_i\frac{\langle\gamma,\gamma\rangle}{\langle\alpha_i,\alpha_i\rangle}H_{\alpha_i},$$

and as $\langle\varrho, H_{\alpha_i}\rangle>0$ for all positive roots and $\langle\gamma,\gamma\rangle\leqslant\langle\gamma_r,\gamma_r\rangle$ (γ_r is a large root), we see that

$$\langle\Lambda+\varrho, H_\gamma\rangle\leqslant\langle\Lambda+\varrho, H_{\gamma_r}\rangle<1.$$

So the module W_Λ can be identified with $\mathcal{U}/J_\Lambda=\mathcal{U}(\mathfrak{p}^-)\otimes V_\Lambda=\mathcal{U}(\mathfrak{p}^-)\otimes V_{\Lambda_0}$ which is fixed when Λ_0 is fixed. The hermitian form B_Λ is non-degenerate on this module, wherever $\lambda+\langle\varrho, H_{\gamma_r}\rangle<1$. So by continuity argument, it remains positive definite, at least where $\lambda+\langle\varrho, H_{\gamma_r}\rangle<1$, and positive semi-definite if $\lambda+\langle\varrho, H_{\gamma_r}\rangle\leqslant 1$.

By this corollary, we see already the possibility of passing through the limit point $\lambda+\langle\varrho, H_{\gamma_r}\rangle=0$ for the construction of the representation T_Λ. But in this paper, we will mainly be concerned with the case where $\Lambda_0=0$.

2. The relation of ψ_Λ with the reproducing kernel function

2.1. Realization as a Siegel domain $D(\Omega, Q)$

2.1.1. *Definition.* [18]. Let Ω be an open convex cone in a real vector space V. The *dual cone* Ω^* is the cone

$$\Omega^* = \{\xi \in V^*; \langle \xi, y\rangle > 0 \quad \text{for all } y \in \bar{\Omega} - \{0\}\}.$$

We shall say that Ω is a *proper cone* when $\Omega^* \neq \emptyset$. We shall let $D(\Omega) \subset V^C$ be the *tube over* Ω: $D(\Omega) = V + i\Omega$. Such a tube is called a *Siegel domain of type* I (SI).

Let W be a complex vector space, and Q a hermitian form on W taking values in V^C such that

$$Q(u, u) \in \bar{\Omega} - \{0\} \quad \text{for all } u \in W, \quad u \neq 0.$$

We let $D(\Omega, Q)$ be the open subset of $V^C \times W$ defined by

$$D(\Omega, Q) = \{p = (x+iy, u); y - Q(u, u) \in \Omega\}.$$

Such a domain is called a *Siegel domain of type* II (SII).

2.1.2. We know that for G, K as in the preceding section, the hermitian symmetric space G/K may be realized as an SII domain [13]. In order to fix the notation, let us recall this construction (see [13]; also [11] and [20a]).

Let Ψ be a maximal set of orthogonal non-compact positive roots, chosen as follows. Put the largest root in Ψ and successively choose the largest root orthogonal to those already chosen. This process ends when we have obtained r roots: $\Psi = \{\gamma_1, \gamma_2, \ldots, \gamma_r\}$, where, for each j, γ_j is the largest non-compact positive root orthogonal to $\gamma_{j+1}, \ldots, \gamma_r$. (Notice that γ_r is the largest root; thus our convention differs from that of C. C. Moore [16]).

For $X_\gamma = E_\gamma + E_{-\gamma}$, let $\mathfrak{a} = \sum_{\gamma\in\Psi} RX_\gamma$. $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$, and r is the real rank of $\mathfrak{g}$. Let $\mathfrak{h}_r = \sum_{\gamma\in\Psi} RH_\gamma$. Identifying γ_i with its restriction to $\mathfrak{h}_r$, we have $\gamma_i(H_{\gamma_j}) = 2\delta_i^j$ and the theorem of C. C. Moore:

2.1.3. Theorem ([16]). *The non-zero restrictions of Δ to $\mathfrak{h}_r$ form one of these two sets:*
(*Case* 1): $\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j; 1 \leqslant i \leqslant j \leqslant r\}$,
(*Case* 2): $\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j; 1 \leqslant i \leqslant j \leqslant r\} \cup \{\pm\frac{1}{2}\gamma_i; 1 \leqslant i \leqslant r\}$.

In case 1, the non-zero restrictions of the positive compact roots form the set $\{\frac{1}{2}(\gamma_j - \gamma_i); j > i\}$. The restrictions of the positive non-compact roots are non-zero and form the set $\{\frac{1}{2}(\gamma_i + \gamma_j); j \geqslant i\}$. The only root of restriction γ_i to $\mathfrak{h}_r$ is γ_i. In case 2, the non-zero restrictions of the positive compact roots form the set $\{\frac{1}{2}(\gamma_j - \gamma_i); j > i\} \cup \{\frac{1}{2}\gamma_i; 1 \leqslant i \leqslant r\}$. The restrictions of the non-compact positive roots are non-zero and form the set $\{\frac{1}{2}(\gamma_i+\gamma_j);\, j \geqslant i\} \cup \{\frac{1}{2}\gamma_i;\, 1 \leqslant i \leqslant r\}$. The only root of restriction γ_i to $\mathfrak{h}_r$ is γ_i. In both cases, the γ_i have the same length.

2.1.4. LEMMA. $H_{\gamma_i} - H_{\gamma_j} \in (\mathfrak{h} \cap [\mathfrak{k}, \mathfrak{k}])^C$.

Proof. Since γ_i and γ_j have the same length, it suffices to see that $\gamma_i - \gamma_j$ is zero on $\mathfrak{z}$. But this is clear since γ_i and $\gamma_j \in \Delta_{\mathfrak{p}}^+$.

2.1.5. *The Cayley transform.* Let G_C be the simply connected Lie group with Lie algebra $\mathfrak{g}_C$. If $\gamma \in \Delta_{\mathfrak{p}}^+$, we define $c_\gamma \in G_C$ by

$$c_\gamma = \exp\left(-\frac{\pi}{4}(E_\gamma - E_{-\gamma})\right).$$

Adc_γ is an automorphism of $\mathfrak{g}^C$ taking H_γ to X_γ. The *Cayley transfor* ined as

$$c = c_{\Psi} = \prod_{i=1}^{r} c_{\gamma_i} \quad (\Psi) = (\gamma_1, \ldots \gamma_r)).$$

(The c_{γ_i} commute since γ_i and γ_j are strongly orthogonal). We introduce the *partial Cayley transforms*

$$c_j = \prod_{i=j}^{r} c_{\gamma_i}.$$

Thus $c = c_1$, $c_r = c_{\gamma_r}$ and the following formulae are valid (as is easy to see by looking at SL(2, C) [9]).

2.1.6. For $\gamma \in \Delta_{\mathfrak{p}}^+$, $t \in R$,

$$\begin{aligned}\exp(\pi/4)(E_{-\gamma} - E_\gamma) &= \exp(-E_\gamma)\cdot\exp(\log\sqrt{2}H_\gamma)\cdot\exp(E_{-\gamma})\\ &= \exp(E_{-\gamma})\cdot\exp(-\log\sqrt{2}H_\gamma)\cdot\exp(-E_\gamma).\end{aligned}$$

$$\exp t(E_{-\gamma} + E_\gamma) = \exp(\tanh tE_\gamma)\cdot\exp(-\log(\cosh t)H_\gamma)\cdot\exp(\tanh tE_{-\gamma}).$$

$$c_\gamma(E_\gamma) = \tfrac{1}{2}(E_\gamma - E_{-\gamma} - H_\gamma). \tag{2.1.7}$$

$$\begin{aligned}c &= \exp(-\textstyle\sum E_{\gamma_i})\cdot\exp(\log\sqrt{2}\textstyle\sum H_{\gamma_i})\cdot\exp(\textstyle\sum E_{-\gamma_i})\\ &= \exp(\textstyle\sum E_{-\gamma_i})\cdot\exp(-\log\sqrt{2}\textstyle\sum H_{\gamma_i})\cdot\exp(\textstyle\sum E_{\gamma_i}).\end{aligned} \tag{2.1.8}$$

2.2. The Iwasawa decomposition

For α a linear form on $\mathfrak{a}$, we shall write

$$\eta^\alpha = \{X \in \mathfrak{g};\, [A, X] = \alpha(A)X,\, A \in \mathfrak{a}\}.$$

Let $\alpha_1, \ldots, \alpha_r$ be the restrictions to $\mathfrak{a}$ of the Cayley transforms of the roots $\gamma_1, \ldots, \gamma_r$. Then $\alpha_j(X_{\gamma_i}) = 2\delta_i^j$. We shall fix an order on the dual vector space $\mathfrak{a}^*$ so that $\alpha_1 < \alpha_2 < \ldots < \alpha_r$.

2.2.1. THEOREM. (C. C. Moore [16]). (a) *The system of positive roots of* $\mathfrak{g}$ *with respect to* $\mathfrak{a}$ *are (corresponding to the two cases of Theorem* 2.1.3):

(*Case* 1) $\{\frac{1}{2}(\alpha_i+\alpha_j);\ 1 \leqslant i \leqslant j \leqslant r\} \cup \{\frac{1}{2}(\alpha_i-\alpha_j);\ 1 \leqslant j < i \leqslant r\}$,
(*Case* 2) $\{\frac{1}{2}(\alpha_i+\alpha_j);\ 1 \leqslant i \leqslant j \leqslant r\} \cup \{\frac{1}{2}(\alpha_i-\alpha_j);\ 1 \leqslant j < i \leqslant r\} \cup \{\frac{1}{2}\alpha_i;\ 1 \leqslant i \leqslant r\}$.

The set S *of simple roots is*

(*Case* 1) $S = \{\alpha_1, \frac{1}{2}(\alpha_2-\alpha_1), \ldots, \frac{1}{2}(\alpha_r-\alpha_{r-1})\}$,
(*Case* 2) $S = \{\frac{1}{2}\alpha_1, \frac{1}{2}(\alpha_2-\alpha_1), \ldots, \frac{1}{2}(\alpha_r-\alpha_{r-1})\}$.

The vectors space η^{α_i} *is one dimensional.*

(b) *The action of the Weyl group on* $\mathfrak{a}^*$ *consists of all transformations* $\alpha_i \to \pm\alpha_{\sigma(i)}$ *for all permutations* σ *of* $\{1, \ldots, r\}$.

Part (a) of this theorem follows immediately from Theorem 2.1.3. From part (b) it follows that the integers

$$p = \dim \eta^{1/2(\alpha_i+\alpha_j)}, \quad \mu = \dim \eta^{1/2\,\alpha_i}, \tag{2.2.2}$$

are independent of i and j, and $p = \dim \eta^{1/2(\alpha_i-\alpha_j)}$. Note that for $r>1$, we have $p>0$ and $\mu \geqslant 0$. $\mu = 0$ only in case 1.

Let $\eta = \sum_{\alpha>0} \eta^\alpha$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \eta$. Let $\mathfrak{b} = \mathfrak{a} \oplus \eta$, $\mathfrak{b}^+ = \mathfrak{b}^C \cap (\mathfrak{k}^C \oplus \mathfrak{p}^+)$, $\mathfrak{b}^- = \mathfrak{b}^C \cap (\mathfrak{k}^C \oplus \mathfrak{p}^-)$, so that $\mathfrak{b}^C = \mathfrak{b}^+ \oplus \mathfrak{b}^-$. Since $\mathfrak{b}^+ \cap \mathfrak{b} = \{0\}$, $\mathfrak{b}^+$ is complementary (in $\mathfrak{b}^C$ as real spaces) to $\mathfrak{b}$, as well as $i\mathfrak{b}$. Thus $\mathfrak{b}^+$ is the graph of a transformation of $\mathfrak{b}$ to $i\mathfrak{b}$. Let $J: \mathfrak{b} \to \mathfrak{b}$ be defined so that $\mathfrak{b}^+ = \{X - iJX;\ X \in \mathfrak{b}\}$. It follows that $\mathfrak{b}^- = \{X + iJX;\ X \in \mathfrak{b}\}$. Note that

$$U_i = \tfrac{1}{2}\left(iH_{\gamma_i} - i(E_{\gamma_i} - E_{-\gamma_i})\right)$$

is real. By (2.1.7), $c_\Psi(E_{\gamma_i}) = iU_i$, so $U_i \in \eta^{\alpha_i}$. Since

$$\tfrac{1}{2} X_{\gamma_i} - iU_i = \tfrac{1}{2} H_{\gamma_i} + E_{-\gamma_i} \in \mathfrak{b},$$

$JU_i = \frac{1}{2} X_{\gamma_i}$. Now let $s = \sum_{i=1}^r U_i$, so that $Js = \frac{1}{2}\sum_{i=1}^r X_{\gamma_i}$. Js is semi-simple and has eigenvalues $0, \pm\frac{1}{2}, \pm 1$ on $\mathfrak{g}$. We consider the decomposition of $\mathfrak{g}$ into Js-eigenspaces

$$\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(-\tfrac{1}{2}) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(\tfrac{1}{2}) \oplus \mathfrak{g}(1).$$

Then $\mathfrak{b} = \mathcal{H}_0 \oplus \mathcal{H}_{1/2} \oplus \mathcal{H}_1$, with $\mathcal{H}_0 \subset \mathfrak{g}(0)$, $\mathcal{H}_{1/2} = \mathfrak{g}(\frac{1}{2})$, $\mathcal{H}_1 = \mathfrak{g}(1)$. More explicitly

$$\mathcal{H}_0 = \mathfrak{a} + \sum_{i>j} \eta^{1/2(\alpha_i-\alpha_j)}, \quad \mathcal{H}_{1/2} = \sum_i \eta^{1/2\,\alpha_i}, \quad \mathcal{H}_1 = \sum_{i\leqslant j} \eta^{1/2(\alpha_i+\alpha_j)}. \tag{2.2.3}$$

2.2.4. LEMMA. (a) $c(E_{\gamma_i}) = iU_i$.
(b) $J\mathcal{H}_0 = \mathcal{H}_1$, $J(\mathcal{H}_{1/2}) = \mathcal{H}_{1/2}$ *and* $JX = [s, X]$ *for* $X \in \mathcal{H}_0$. *In particular for* $i > j$ *fixed,*

$$J(\eta^{1/2(\alpha_i - \alpha_j)}) = \eta^{1/2(\alpha_i + \alpha_j)}, \quad \text{and} \quad JX = [U_j, X], \quad J(\eta^{1/2\alpha_i}) = \eta^{1/2\alpha_i}.$$

(c) *Let* $\mathcal{H}^+_{1/2} = \mathfrak{b}^+ \cap \mathcal{H}^C_{1/2}$, $\mathcal{H}^-_{1/2} = \mathfrak{b}^- \cap \mathcal{H}^C_{1/2}$. *Then* $\mathcal{H}^C_{1/2} = \mathcal{H}^+_{1/2} \oplus \mathcal{H}^-_{1/2}$, *and* $\mathcal{H}^+_{1/2} = c(\mathfrak{p}^+) \cap \mathcal{H}^C_{1/2}$, $\mathcal{H}^-_{1/2} = c(\mathfrak{k}^C) \cap \mathcal{H}^C_{1/2}$, $c(\mathfrak{p}^+) = \mathcal{H}^+_{1/2} \oplus \mathcal{H}^C_1$.

This is easily proven. For example, for (c): let γ be a root of Δ with restriction $\frac{1}{2}\gamma_i$ on $\mathfrak{h}_r$. Using Theorem 2.1.3, it follows from (2.1.8) that

$$c(E_\gamma) = \frac{1}{\sqrt{2}}(E_\gamma + [E_{-\gamma_i}, E_\gamma]).$$

Thus, if γ is compact, $c(E_\gamma) \in \mathcal{H}^-_{1/2}$, and if γ is noncompact and thus positive, $c(E_\gamma) \in \mathcal{H}^+_{1/2}$.

2.3. Let B, A, N, H_0 be the connected subgroups of G with Lie algebra $\mathfrak{b}, \mathfrak{a}, \eta, \mathcal{H}_0$ respectively. Since these are simply connected groups we can denote by the same symbols the corresponding subgroups of $\tilde{G}$. Then $\tilde{G} = \tilde{K} \cdot B$ and $G = K \cdot B$. B is a solvable group, and every element of B can be written uniquely in the form $b = h_0 \cdot \exp U \cdot \exp X$ with $h_0 \in H_0$, $U \in \mathcal{H}_{1/2}$, $X \in \mathcal{H}_1$. Let $\tilde{M} = \{g \in \tilde{K};\ ga = ag \text{ for all } a \in A\}$. Then $\tilde{M}$ has Lie algebra $m = \{X \in \mathfrak{k};\ [X, A] = 0 \text{ for all } A \in \mathfrak{a}\}$. Let $\tilde{G}(0) = \{g \in \tilde{G};\ g \cdot Js = Js\}$. $\tilde{G}(0)$ has $\mathfrak{g}(0)$ as its Lie algebra, and leaves $\mathcal{H}_{1/2}$, $\mathcal{H}_1$ invariant under the adjoint action. Similarly, let $G(0) = \{g \in G;\ g \cdot Js = Js\}$.

2.3.1. *Definition.* Let $\Omega = G(0) \cdot s$, the orbit of s under the adjoint action of $G(0)$ on $\mathcal{H}_1$. Let

$$\Omega^* = \{\xi \in \mathcal{H}_1^*;\ \langle \xi, X \rangle > 0 \quad \text{for all } X \in \bar{\Omega} - \{0\}\}.$$

Let θ be the Cartan involution ($\theta(X) = X$, $X \in \mathfrak{k}$, $\theta(X) = -X$, $X \in \mathfrak{p}$). The form $S(X, Y) = -\langle X, \theta Y \rangle$ is a symmetric positive definite form on $\mathfrak{g}$ and thus its restriction to $\mathcal{H}_1$ defines an isomorphism ξ: $\mathcal{H}_1 \to \mathcal{H}_1^*$. Let $\xi_0 = \sum U_i^*$; ξ_0 is determined by the equations

$$\xi_0\Big(\sum_{i \neq j} \eta^{1/2(\alpha_i + \alpha_j)}\Big) = 0, \quad \xi_0(U_i) = 1.$$

ξ_0 is a positive multiple of $\xi(s)$.

Let $g \to g^*$ be the involutive antiautomorphism of G defined by $(\exp X)^* = (\exp \theta(X))^{-1}$. Then $K = \{g;\ gg^* = id\}$, and $S(g \cdot X, Y) = S(X, g^* Y)$ for all $g \in G$. The involution $g \to g^*$ preserves $G(0)$, and K, and therefore $K(0) = G(0) \cap K$. It follows that

$$\xi(g \cdot X) = (g^*)^{-1} \cdot \xi(X), \quad g \in G(0),\ X \in \mathcal{H}_1. \tag{2.3.2}$$

2.3.3. PROPOSITION. [13] Ω *is an open convex cone in* $\mathcal{H}_1$. *The correspondence* ξ *is an isomorphism of* Ω *and* Ω^*. $K(0)$ *is the isotropy group of* $s \in \Omega$ *and* $\xi_0 \in \Omega^*$. *The map* $h \to h \cdot s$ *is a diffeomorphism of* H_0 *onto* Ω. *Similarly* $h \to h \cdot \xi_0$ *is a diffeomorphism of* H_0 *onto* Ω^*.

For $t \in \Omega$, we shall let $t \to h(t)$ denote the inverse to the above diffeomorphism of H_0 onto Ω; i.e., $h(t)$ is the unique element of H_0 such that $t = h(t) \cdot s$. Similarly, for $\xi \in \Omega^*$, $h(\xi) \in H_0$ is defined by $\xi = h(\xi) \cdot \xi_0$.

If ϕ is a continuous, compactly supported function defined on Ω, resp. Ω^*, we have

$$\int_\Omega \phi(t) \det_{\mathcal{H}_1}(h(t))^{-1} dt = \int_{H_0} \phi(h \cdot s)\, dh = \int_{G(0)} \phi(g \cdot s)\, dg,$$

$$\int_{\Omega^*} \phi(\xi) \det_{\mathcal{H}_1}(h(\xi))\, d\xi = \int_{H_0} \phi(h \cdot \xi_0)\, dh = \int_{G(0)} \phi(g \cdot \xi_0)\, dg$$

(here dh is the left-invariant Haar measure of H_0 and dg is the Haar measure of $G(0)$ (which is unimodular)).

Now, $\mathcal{H}_{1/2}$ is J-stable and, since $[\mathcal{H}_{1/2}^+, \mathcal{H}_{1/2}^+] \subset \mathfrak{b}^+ \cap \mathcal{H}_1 = \{0\}$, we have $[Ju, Jv] = [u, v]$ for all $u, v \in \mathcal{H}_{1/2}^+$. Since $\mathcal{H}_{1/2}^+$ is the $+i$ eigenspace of J on $\mathcal{H}_{1/2}^C$, the map τ: $\mathcal{H}_{1/2} \to \mathcal{H}_{1/2}^+$ defined by $\tau(u) = \frac{1}{2}(u - iJu)$ is a complex isomorphism of $\mathcal{H}_{1/2}$ (furnished with the complex structure J) and $\mathcal{H}_{1/2}^+$. $G(0)$ leaves $\mathcal{H}_{1/2}^+$ invariant. We introduce the hermitian $\mathcal{H}_1^C$-valued form Q on $\mathcal{H}_{1/2}^+$:

$$Q(u, v) = \frac{i}{2}[u, \bar{v}].$$

2.3.4. Lemma. *Q is an Ω-hermitian form. We have*

$$Q(g_0 \cdot u, g_0 \cdot v) = g_0 \cdot Q(u, v) \quad \textit{for all } u, v \in \mathcal{H}_{1/2},\ g_0 \in G(0).$$

We shall see that G/K is isomorphic to the Siegel domain $D(\Omega, Q) \subset \mathcal{H}_1^C \oplus \mathcal{H}_{1/2}^+$.

Let us recall the map (section 1.2) ζ of G into $\mathfrak{p}^+$ which determines an isomorphism of G/K onto a bounded domain $\mathcal{D}$ in $\mathfrak{p}^+$. We know that $c^{-1}G \subset P_+ K_C P_-$. Thus we can define α: $G \to \mathcal{H}_1^C \oplus \mathcal{H}_{1/2}^+$ by $\alpha(g) = c \cdot \zeta(c^{-1}g)$.

2.3.5. Proposition. *α determines a G-invariant biholomorphism of G/K onto $D(\Omega; Q)$. In this realization B acts as a group of affine transformations of $\mathcal{H}_1^C \oplus \mathcal{H}_{1/2}^+$.*

More precisely, let $b = h_0 \cdot \exp X_0 \cdot \exp U_0 \in B$. Then

$$b \cdot (z, u) = (h_0 \cdot X_0 + h_0 \cdot z + Q(h_0 \cdot u, h_0 \cdot \tau(U_0)),\ h_0 \cdot u + h_0 \cdot \tau(U_0)). \tag{2.3.6}$$

Let $\theta_0(t) = \det_{\mathcal{H}_{1/2}} h(t)^{-1} \det_{\mathcal{H}_1} h(t)^{-2}$ (these determinants are taken in the sense of real vector spaces). If F is a continuous compactly supported function on $D(\Omega, Q)$,

$$\int F(\alpha g)\, dg = \int_B F(\alpha b)\, db = \int_{D(\Omega, Q)} F(x + iy, u)\, \theta_0(y - Q(u, u))\, dx\, dy\, du,$$

where dg is Haar measure of G, db is left-invariant Haar measure for B and $dx\,dy\,du$ is Lebesgue measure on $\mathcal{H}_1^C \oplus \mathcal{H}_{1/2}^+$.

2.3.7. *Definition*. Let $P=G(0)\exp(\mathcal{H}_{1/2}\oplus\mathcal{H}_1)$.

P is a maximal parabolic subgroup of G. It can be shown that P acts on $D(\Omega, Q)$ by affine transformations given by the formula (2.3.6) (with g_0 replacing h_0). Then P is the group of all affine transformations leaving $D(\Omega, Q)$ invariant. We have $c^{-1}Pc\subset K^C P_+$ and $c^{-1}G(0)c\subset K^C$. If $g\in\tilde{G}(0)$, we can thus define the element $c^{-1}gc$ in K^C. We shall let U_Λ denote the representation of $\tilde{K}^C P_+$ which trivially prolongs U_Λ on $\tilde{K}^C$ along P_+.

2.4. Identification of $O(\Lambda)$ as a space of holomorphic functions on $D(\Omega, Q)$

We now recall, and amplify, the discussion of $O(\Lambda)$ given in chapter 1. Since $c^{-1}G\subset P_+K^CP_-$, we can consider for $g\in G$, $k(c^{-1}g)\in K^C$. Extend the mapping to a map of $\tilde{G}$ into $\tilde{K}^C$. Let

$$\Phi_\Lambda(g) = U_\Lambda(k(c^{-1}))^{-1}\cdot U_\Lambda(k(c^{-1}g)), \tag{2.4.1}$$

so that $\Phi_\Lambda(e)=id$. Clearly

$$\Phi_\Lambda(g\cdot k) = \Phi_\Lambda(g)\circ U_\Lambda(k), \quad g\in\tilde{G} \quad k\in\tilde{K},$$

$$r(X)\cdot\Phi_\Lambda = 0, \quad X\in\mathfrak{p}^-.$$

Consequently, for any $\phi\in O(V_\Lambda)$,

$$(P_\Lambda\phi)(g) = \Phi_\Lambda(g)^{-1}\phi(g)\in O(\Lambda). \tag{2.4.2}$$

2.4.3. PROPOSITION. *Φ_Λ is a representation of P in $GL(V_\Lambda)$:*

(i) $$\text{for } g\in G(0), \quad \Phi_\Lambda(g) = U_\Lambda(c^{-1}gc),$$

(ii) $$\text{for } U\in\mathcal{H}_{1/2}, \quad \Phi_\Lambda(\exp U) = U\left(\exp\frac{1}{\sqrt{2}}c^{-1}U\right),$$

(iii) $$\text{for } X\in\mathcal{H}_1, \quad \phi_\Lambda(\exp X) = id.$$

Proof. For $g\in P$, $c^{-1}gc\in\tilde{K}_C P_+$ and

$$\phi_\Lambda(g) = (U_\Lambda(k(c^{-1}))^{-1}\cdot U_\Lambda(c^{-1}gc)\cdot U_\Lambda(k(c^{-1})).$$

Now from (2.1.8), $k(c^{-1})=\exp(\log\sqrt{2}\sum H_{\gamma_i})$. Since $c^{-1}(\mathcal{H}_0)$ commutes with $\sum H_{\gamma_i}$ and $\sum H_{\gamma_i}$ acts as the identity on $c^{-1}(\mathcal{H}_{1/2})$, the proposition follows.

2.4.4. *Definition*. Let $\Theta_\Lambda\colon D(\Omega, Q)\to GL(V_\Lambda)$ be defined by $\Theta_\Lambda(\alpha(b))=\phi_\Lambda(b)$.

Then

$$\Theta_\Lambda(x+iy, u) = \Theta''_\Lambda(u)\circ\Theta'_\Lambda(y-Q(u,u)) = \Theta_\Lambda(y,u),$$

with

$$\Theta''_\Lambda(u) = U_\Lambda(\exp\sqrt{2}c^{-1}(\tau^{-1}(u))),$$

$$\Theta'_\Lambda(y-Q(u,u)) = \phi_\Lambda(h(y-Q(u,u)),$$

and Θ''_Λ is a real polynomial in u.

Now P_Λ, as defined by (2.4.2) is an isomorphism of $O(V_\Lambda)$ with $O(\Lambda)$. Left translation $T_\Lambda(g_0)$, for $g_0 \in \tilde{G}$ is written in $O(V_\Lambda)$ by (2.4.5) $T_\Lambda(g_0)(F)(g) = \phi_\Lambda(g) \circ \Phi_\Lambda(g_0^{-1}g)^{-1} F(g_0^{-1}g)$. We shall let $J_\Lambda(g_0, g) = \phi_\Lambda(g_0 g) \circ \phi_\Lambda(g)^{-1}$, so that (2.4.5) becomes

$$T_\Lambda(g_0)(F)(g) = J_\Lambda(g_0^{-1}, g)^{-1} F(g_0^{-1}g). \tag{2.4.6}$$

J_Λ is a function on $G \times G/K$, holomorphic in $p \in G/K(=D(\Omega, Q))$, and satisfying

$$J_\Lambda(g_0 g_1, p) = J_\Lambda(g_0, g_1 p) \cdot J_\Lambda(g_1, p).$$

If $p_0 = \alpha(1) = (is, 0)$, $J_\Lambda(k_0; p_0) = U_\Lambda(k_0)$. For $g \in P$, and $p \in D(\Omega, Q)$, $J_\Lambda(g; p) = \Phi_\Lambda(g)$, and so is independent of p. Thus P_Λ intertwines the restriction of T_Λ to P and the representation $\tilde{T}$ of P in $O(V_\Lambda)$ given by

$$(\tilde{T}(g_0) F)(p) = \Phi_\Lambda(g_0) F(g_0^{-1}p).$$

Furthermore,

$$(P_\Lambda^{-1} \psi_\Lambda^{v_\Lambda})(g) = \Phi_\Lambda(g) \cdot (\Phi_\Lambda^0(g)^{-1} \cdot v_\Lambda).$$

Thus $P_\Lambda^{-1}\psi_\Lambda^{v_\Lambda}$ is a holomorphic function on G/K with values in $V_\Lambda = V_{\Lambda_0}$, and for $\Lambda = (\Lambda_0; \lambda)$, clearly $(P_\Lambda^{-1}\psi_\Lambda^{v_\Lambda})(p)$ varies analytically in λ.

2.4.7. *Definition. Let*

$$\mathcal{H}(\Lambda) = \{F, \text{ holomorphic on } D(\Omega, Q) \text{ with values in } V_{\Lambda_0};$$
$$N(\Lambda : F)^2 = \int_{D(\Omega,\, Q)} \| \Theta_\Lambda(y, u)^{-1} F(x+iy, u) \|^2_{V_{\Lambda_0}} \theta_0(y - Q(u, u))\, dx\, dy\, du < \infty \}.$$

The following proposition follows easily from the above remarks.

2.4.8. PROPOSITION ([20a]). *P_Λ is a unitary isomorphism between $\mathcal{H}(\Lambda)$ and $H(\Lambda)$.*

2.5. The kernel function

Suppose that $\langle \Lambda + \varrho, H_{\gamma_r} \rangle < 0$, i.e., $\mathcal{H}(\Lambda) \neq \{0\}$. For $p \in D(\Omega, Q)$ and $v \in V_\Lambda$, the functional $F \to \langle F(p), v \rangle_{V_\Lambda}$ is a continuous linear functional on the Hilbert space $\mathcal{H}(\Lambda)$. It follows that there is a function $R_\Lambda(p, p')$ with values in $GL(V_\Lambda)$ such that

$$R_\Lambda(p, p')^* = R_\Lambda(p', p), \tag{a}$$

$$R_\Lambda(\cdot, p') \text{ is holomorphic for all } p', \tag{b}$$

$$\langle F(p), v \rangle_{V_\Lambda} = \langle F, R_\Lambda(\cdot, p) v \rangle_{\mathcal{H}(\Lambda)}. \tag{c}$$

R_Λ will be called the *reproducing kernel function of* $\mathcal{H}(\Lambda)$. Since T_Λ is unitary on $\mathcal{H}(\Lambda)$ we easily obtain

$$R_\Lambda(g\cdot p, g\cdot p') = J_\Lambda(g;p)\,R_\Lambda(p,p')\,J_\Lambda(g;p')^*. \tag{2.5.1}$$

2.5.2. PROPOSITION. *If* $\langle \Lambda+\varrho, H_{\gamma_r}\rangle < 0$,

$$P_\Lambda^{-1}\psi_\Lambda^v = \frac{c(\Lambda)}{\langle v,v\rangle}(R_\Lambda(\cdot\,,p_0)\cdot v).$$

In fact, since $(P_\Lambda^{-1}\phi)(p_0) = \phi(1)$, we have

$$\langle P_\Lambda^{-1}\phi, P_\Lambda^{-1}\psi_\Lambda^v\rangle = \langle \phi, \psi_\Lambda^v\rangle = c(\Lambda)\frac{\langle \phi(1), v\rangle}{\langle v,v\rangle} = \frac{c(\Lambda)}{\langle v,v\rangle}\langle (P_\Lambda^{-1}\phi)(p_0), v\rangle$$

$$= \frac{c(\Lambda)}{\langle v,v\rangle}\langle P_\Lambda^{-1}\phi, R_\Lambda(\cdot\,,p_0)\,v\rangle.$$

2.5.3. *Definition.* If $\langle \Lambda+\varrho, H_{\gamma_r}\rangle < 0$, let

$$R_\Lambda^0(p,p') = c(\Lambda)\,R_\Lambda^0(p,p').$$

The formulas (2.5.1, 2.5.2) show that R_Λ^0 is defined for all $\Lambda = (\Lambda_0, \lambda)$, and is analytic in λ. We shall thus consider R_Λ^0 so defined for all $\Lambda = (\Lambda_0, \lambda)$. R_Λ^0 has these properties:

(2.5.4) (a) $R_\Lambda^0(\cdot\,,p)$ is holomorphic on $D(\Omega, Q)$ for all p,

(b) $R_\Lambda^0(p,p')^* = R_\Lambda^0(p',p)$

(c) $R_\Lambda^0(p_0,p_0)\cdot v_\Lambda = v_\Lambda$

(d) $R_\Lambda^0(g\cdot p_1, g\cdot p_2) = J_\Lambda(g,p_1)\,R_\Lambda^0(p_1,p_2)\,J_\Lambda(g,p_2)^*$.

In fact, at p_0, for $v \in V_\Lambda$,

$$R_\Lambda^0(\cdot\,,p_0)\cdot v = (\langle v,v\rangle)\,P_\Lambda^{-1}\psi_\Lambda^v,$$

and (d) serves to continue R_Λ^0 throughout $D(\Omega,Q)\times D(\Omega,Q)$.

2.5.5. *Definition.* R^0 is a *kernel function* if the following property (P) is satisfied:

(P) For all $N \in Z^+$, $p_1, \ldots, p_N \in D(\Omega,Q)$, $v_1, \ldots, v_N \in V_\Lambda$,

$$\sum_{i,j}\langle R_\Lambda^0(p_i,p_j)\,v_i, v_j\rangle \geqslant 0.$$

Clearly if $\mathcal{H}(\Lambda) \neq \{0\}$, $R_\Lambda^0 = c(\Lambda)\,R_\Lambda$, and since R_Λ is a reproducing kernel for $\mathcal{H}(\Lambda)$, it satisfies (P), and so also does R_Λ^0. Now, in any case, when property (P) holds, one can (see

[15]) determine uniquely a Hilbert space $\mathcal{H}_0(\Lambda)$ of V_Λ-valued functions holomorphic on $D(\Omega, Q)$ whose reproducing kernel is R^0_Λ. More precisely, define the norm

$$\|\cdot\| \quad \text{on} \quad \{F = \sum_{i=1}^{N} R^0_\Lambda(\,\cdot\,, p_i)\, v_i\}$$

by

$$\|F\|^2 = \sum_{i,j} \langle R^0_\Lambda(p_i, p_j)\, v_i, v_j\rangle,$$

and let $\mathcal{H}_0(\Lambda)$ be the completion. The relation (2.5.4 d) shows that the representation

$$(T_\Lambda(g_0)\,F)\,(p) = J_\Lambda(g_0^{-1}; p)^{-1} \cdot F(g_0^{-1} \cdot p)$$

is a unitary irreducible representation of $\tilde{G}$ in $\mathcal{H}_0(\Lambda)$. Recalling definition 1.4.2 and the relationship of R^0_Λ with $\psi^{v_\Lambda}_\Lambda$, we easily obtain the result:

2.5.6. THEOREM. *For $\Lambda = (\Lambda_0, \lambda)$, $\lambda \in P_{\Lambda_0}$ if and only if R_{Λ_0} verifies* (P). *For such λ, P_Λ is a unitary transformation of $\mathcal{H}_0(\Lambda)$ onto $H_0(\Lambda)$ which intertwines the representations of $\tilde{G}$ and satisfies*

$$P_\Lambda(R^0_\Lambda(\,\cdot\,, p_0) \cdot v_\Lambda) = \psi^{v_\Lambda}_\Lambda.$$

Now, since B acts transitively on $D(\Omega, Q)$ and $J_\Lambda(b, p_0) = \phi_\Lambda(b)$, the formulas (2.5.4c and d) allow us (in principle) to compute the value of $R^0_\Lambda(p, p)$ on the diagonal of $D(\Omega, Q) \times D(\Omega, Q)$. Furthermore, since $R^0_\Lambda(p, p')$ is holomorphic in $(p, \bar{p}')$, it is determined by its restriction to the diagonal, so R^0_Λ is completely determined by ϕ_Λ.

2.6. The case of a character

In the case $\Lambda_0 = 0$, where U_Λ is a character of λ of $\tilde{K}$, these expressions become particularly simple. We shall make explicit the expression for R^0_Λ (to be denoted R^0_λ).

First of all, if $\Lambda_0 = 0$, $\langle \Lambda, H_{\gamma_i}\rangle = \langle \Lambda, H_{\gamma_r}\rangle = \lambda$ (from Lemma 2.1.4), and thus ϕ_Λ is a character of $G(0)$, given by $\phi_\lambda(\exp tX_{\gamma_i}) = e^{t\lambda}$, $\phi_\lambda = 1$ on $\exp(\mathcal{H}_{1/2} \oplus \mathcal{H}_1)$.

2.6.1. *Definition. The Koecher function K* for the domain $D(\Omega)$ is defined by

$$K(z) = \int_{\Omega^*} e^{2\pi i\langle \xi, z\rangle}\, d\xi.$$

($d\xi$ is normalized so that $K(is) = \int e^{-2\pi\langle \xi, s\rangle} d\xi = 1$).

This integral converges absolutely for all $z \in D(\Omega)$ to a non-vanishing holomorphic function [21]. Since $D(\Omega)$ is simply connected, we can uniquely define all powers K^α of K with $K^\alpha(is)=1$. Let us notice two important facts:

$$K(i(h\cdot s)) = (\det_{\mathcal{H}_1} h)^{-1} \quad \text{for } h \in H_0 \tag{2.6.2}$$

(because Ω^* is H_0-invariant).

If (z_1, u_1), $(z_2, u_2) \in D(\Omega, Q)$, then $z_1 - \bar{z}_2 - 2iQ(u_1, u_2) \in D(\Omega)$. For, if we let $z_i = x_i + i(t_i + Q(u_i, u_i))$, with $t_i \in \Omega$, then

$$\begin{aligned} \operatorname{Im}(z_1 - \bar{z}_2 - 2iQ(u_1, u_2)) &= t_1 + t_2 + Q(u_1, u_1) + Q(u_2, u_2) - Q(u_1, u_2) - Q(u_2, u_1) \\ &= t_1 + t_2 + Q(u_1 - u_2, u_1 - u_2) \in \Omega, \end{aligned} \tag{2.6.3}$$

since $Q(u, u) \in \bar{\Omega}$ for all $u \in \mathcal{H}_{1/2}$, and $t_i \in \Omega$, and Ω is a convex open cone.

Let $n = \dim \mathcal{H}_1 = r + \frac{r(r-1)p}{2}$, $(p = \dim \eta^{1/2(\alpha_i + \alpha_j)})$ by (2.2.1,2)

2.6.4. Proposition.

$$R_\lambda^0((z_1, u_1), (z_2, u_2)) = K\left(\frac{z_1 - \bar{z}_2 - 2iQ(u_1, u_2)}{2}\right)^{-((\lambda r)/n)}.$$

Proof. It suffices to show that both functions coincide on the diagonal of $D(\Omega, Q) \times D(\Omega, Q)$. Let $p = (x + i(t + Q(u, u)), u)$, with $t = h(t)\cdot s$. Then

$$p = \exp X \cdot \exp \tau^{-1}(U) \cdot h(t) \cdot (is, 0),$$

so that

$$R_\lambda^0(p, p) = \phi_\lambda(h(t))^2.$$

The right hand side is

$$K(ih(t)\cdot s)^{-((\lambda r)/n)} = (\det_{\mathcal{H}_1} h(t))^{(\lambda r)/n}.$$

Now for $h = \exp tX_{\gamma_i}$, we can compute that these two expressions are the same:

$$\phi_\lambda(\exp tX_{\gamma_i})^2 = e^{2\lambda t}$$

$$(\det_{\mathcal{H}_1} \exp tX_\gamma)^{(\lambda r)/n} = \exp t\left(\frac{(2 + (r-1)p)\lambda r}{n}\right) = e^{2\lambda t}.$$

We thus conclude that $\lambda \in P$, $((0, \lambda) \in P_0)$ if and only if $\sum \lambda_i \bar{\lambda}_j R_\lambda^0((z_i, u_i), (z_j, u_j)) \geqslant 0$ for all $(z_i, u_i) \in D(\Omega, Q)$.

On the other hand, we shall keep in mind that $P \subset \{\lambda; \lambda \leqslant 0\}$.

3. Hilbert spaces of holomorphic functions on a Siegel domain $D(\Omega, Q)$ and reproducing kernel functions

3.1 An abstract theorem of Nussbaum

Let Ω be a proper convex cone in the real vector space V, and let Q be an Ω-hermitian form on the complex vector space W. We form the Siegel domain

$$D(\Omega, Q) = \{(z, u) \in V^C \oplus W;\ \operatorname{Im} z - Q(u, u) \in \Omega\},$$

without assuming homogeneity nor symmetry. For F a holomorphic function on $D(\Omega, Q)$, we introduce the function R_F holomorphic on $D(\Omega, Q) \times \overline{D(\Omega, Q)}$ by

$$R_F((z_1, u_1), (z_2, u_2)) = F(z_1 - \bar{z}_2 - 2iQ(u_1, u_2)).$$

We need conditions on F in order that R_F be a reproducing kernel, i.e., satisfies the property

$$(P): \sum_{i,j=1}^{N} \lambda_i \bar{\lambda}_j R_F(p_j, p_i) \geqslant 0, \quad \{p_1, \ldots, p_N\} \subset D(\Omega, Q), \quad \{\lambda_1, \ldots, \lambda_N\} \subset C.$$

Furthermore, when (P) is satisfied, we wish to describe, in a concrete way, the Hilbert space $\mathcal{H}(R_F)$ for which R_F is the reproducing kernel.

Letting $\phi(y) = F(iy)$; $y \in \Omega$, the property (P) implies, in particular, that

$$\sum_{i,j} \lambda_i \bar{\lambda}_j \phi(y_i + y_j) \geqslant 0. \tag{3.1.1}$$

Now, such functions on semi-groups determine Hilbert spaces, on which translation acts unitarily, thus giving a representation of the semi-group Ω. Such representations of semi-groups have been widely studied; we shall need the following representation theorem of A. E. Nussbaum [17]. The condition (3.1.3) below is a condition on the uniform continuity of the representation, which we shall be able to verify in our case:

3.1.2. THEOREM [17]. *Let ϕ be a continuous function on Ω satisfying* (3.1.1) *and*

$$\phi(y_0 + y) \leqslant \phi(y) \quad \text{for } y_0, y \in \Omega. \tag{3.1.3}$$

Then there is a positive measure μ supported in

$$\bar{\Omega}^* = \{\xi \in V^*;\ \langle \xi, y \rangle \geqslant 0 \quad \textit{for all } y \in \Omega\},$$

such that

$$\phi(y) = \int e^{-2\pi\langle \xi, y \rangle} d\mu(\xi). \tag{3.1.4}$$

Conversely, given any such a measure μ, clearly the function ϕ defined by (3.1.4) *satisfies* (3.1.1) *and* (3.1.3).

It should be pointed out that Nussbaum allows 0 as a value of y in (3.1.1) and asks that ϕ be continuous at 0, and concludes that μ is a finite measure. However the connection between this added hypothesis and conclusion is direct, and its deletion leaves the above result. In our case Nussbaum's result produces the following.

3.1.5. PROPOSITION. *Let F be holomorphic on $D(\Omega)$. Suppose that*

$$F(iy) = \int e^{-2\pi\langle \xi, y\rangle} d\mu(\xi) \tag{3.1.6}$$

for some positive measure μ supported on $\bar{\Omega}^*$. Then

$$F(z) = \int e^{2\pi i\langle \xi, z\rangle} d\mu(\xi), \tag{3.1.7}$$

and R_F satisfies property (P). *Conversely, if R_F satisfies property* (P) *and, in addition*

$$F(i(y_0+y)) \leqslant F(iy) \quad \textit{for } y, y_0 \in \Omega, \tag{3.1.8}$$

then there is a positive measure μ supported on $\bar{\Omega}^*$ such that (3.1.7) *holds.*

Proof. The convergence of the integral in (3.1.6) means that the integral in (3.1.7) converges absolutely. That integral is holomorphic, as we can see by Morera's criterion: for Γ a closed curve in a complex plane, we have

$$\int_\Gamma \left[\int e^{2\pi i\langle \xi, z\rangle} d\mu(\xi)\right] dz = \int \left[\int_\Gamma e^{2\pi i\langle \zeta, z\rangle} dz\right] d\mu(\xi) = 0,$$

by Cauchy's theorem. Thus, if F, holomorphic on $D(\Omega)$ satisfies (3.1.6), F must be given by (3.1.7), for a holomorphic function on $D(\Omega)$ is determined by its values on $\{\operatorname{Re} z = 0\}$.

Now, we verify that R_F satisfies property (P). Let $p_i = (z_i, u_i)$, $1 \leqslant i \leqslant N$, be in $D(\Omega)$, and $\lambda_i \in C$, $1 \leqslant i \leqslant N$.

$$\sum_{i,j} \lambda_i \bar{\lambda}_j R_F(p_i, p_j) = \int_{\bar{\Omega}^*} \sum_{i,j} \lambda_i \bar{\lambda}_j e^{2\pi i\langle \zeta, z_i - \bar{z}_j - 2iQ(u_i, u_j)\rangle} d\mu(\xi).$$

It suffices to show that the integrand is nonnegative on $\bar{\Omega}^*$. But that integrand is $\sum \mu_i \bar{\mu}_j \exp(m_{ij})$, where $\mu_i = \lambda_i \exp(2\pi i\langle \xi, z_i\rangle)$, $m_{ij} = \langle \xi, Q(u_i, u_j)\rangle$. Now, for $\xi \in \bar{\Omega}^*$, $u \in \Omega$, $\langle \xi, Q(u,u)\rangle \geqslant 0$, so the matrix (m_{ij}) is positive semidefinite. It follows that the matrices $(1/n!)(m_{ij}^n)$ are also positive semidefinite, and thus also $(\exp(m_{ij})) = \sum_{n=0}^{\infty}(m_{ij}^n/n!)$. Thus R_F verifies (P).

Conversely given F so that R_F satisfies (P) and (3.1.8) holds, by the theorem of Nussbaum,

$$F(iy) = \int_{\bar{\Omega}^*} e^{-2\pi\langle \zeta, y\rangle} d\mu(\xi)$$

for some positive measure μ. Thus, as observed above, F is given by (3.1.7).

3.2. Description of $H(R_F)$

Let μ be a positive measure supported on $\overline{\Omega}^*$. We shall assume that the form

$$Q_\xi(u, u) = \langle \xi, Q(u, u) \rangle$$

has constant rank for ξ in a set $O_\mu \subset \overline{\Omega}^*$ whose complement is of μ-measure zero. For $\xi \in \overline{\Omega}^*$, define

$$W_\xi = \{u_0 \in W;\ Q_\xi(u_0, u_0) = 0\} = \{u_0 \in W;\ Q_\xi(u, u_0) = 0 \quad \text{for all } u \in W\}.$$

Then W_ξ is of constant dimension on O_μ.

3.2.2. *Definition*. For $\xi \in \overline{\Omega}^*$, let $d_\xi \dot{u}$ refer to Lebesgue measure on W/W_ξ so normalized that

$$\int_{W/W_\xi} e^{-4\pi Q_\xi(u, u)} d_\xi \dot{u} = 1.$$

(Since Q_ξ induces a positive definite form on W/W_ξ, clearly $\exp(-4\pi Q_\xi(u, u))$ is integrable).

Let $H(\xi)$ be the space of holomorphic functions F on W such that

(a) $$F(u + u_0) = F(u) \quad \text{for } u_0 \in W_\xi,$$

(b) $$\|F\|_\xi^2 = \int_{W/W_\xi} |F(u)|^2 \exp(-4\pi Q_\xi(u, u))\, d_\xi \dot{u} < \infty.$$

$H(\xi)$ is nonempty: it includes all polynomials independent of $W(\xi)$; this space of functions is dense.

Let us note that for $\xi \in \Omega^*$, $W_\xi = \{0\}$, and $W/W_\xi = W$. If we let du represent Lebesque measure (relative to a basis of W fixed once and for all) on W, then

$$\int_W \exp(-4\pi Q_\xi(u, u))\, du = (\det 4Q_\xi)^{-1},$$

where $\det Q_\xi$ is the determinant of the hermitian form Q_ξ relative to this basis. Thus $d_\xi \dot{u} = (\det 4Q_\xi)\, du$ for $\xi \in \Omega^*$.

3.2.3. LEMMA. *For $\xi \in \overline{\Omega}^*$, $H(\xi)$ has the reproducing kernel*

$$k_\xi(u, v) = e^{4\pi \langle \xi, Q(u, v) \rangle}.$$

Proof. For $f \in H(\xi)$, we have

$$f(0) = \int_{W/W_\xi} f(u)\, e^{-4\pi Q_\xi(u, u)} d_\xi \dot{u} = \langle f, 1 \rangle,$$

as is easily seen by the mean-value theorem (integrating in polar coordinates on W/W_ξ relative to the unit sphere in the Q_ξ-norm). Thus $k_{\xi,0}=1$. Now, for $v\in W$, we define

$$(T_\xi(v)F)(u)=e^{-2\pi Q_\xi(v,v)}e^{4\pi Q_\xi(u,v)}F(u-v).$$

$T_\xi(v)$ acts unitarily on $H(\xi)$. Thus, for $F\in H(\xi)$,

$$\langle F, k_{\xi,v}\rangle = F(v) = e^{2\pi Q_\xi(v,v)}T_\xi(v)^{-1}F(0) = e^{2\pi Q_\xi(v,v)}\langle T_\xi(v)^{-1}F, 1\rangle = e^{2\pi Q_\xi(v,v)}\langle F, T_\xi(v)1\rangle.$$

Thus

$$k_\xi(u,v)=k_{\xi,v}(u)=e^{2\pi Q_\xi(v,v)}T_\xi(v)(1)(u)=e^{4\pi Q_\xi(u,v)}.$$

It follows from this, that $\{\sum c_i e^{4\pi Q_\xi(v,u_i)};\ c_i\in C,\ v_i\in W\}$ is dense in $H(\xi)$, and that the norm of such a function is given by

$$\|\sum c_i e^{4\pi Q_\xi(u,v_i)}\|^2=\sum c_i\bar{c}_j e^{4\pi Q_\xi(v_j,v_i)}.$$

Also we have for $f\in H(\xi)$,

$$e^{-4\pi Q_\xi(u,u)}|f(u)|^2\leq \|f\|^2_{H(\xi)}, \tag{3.2.4}$$

since $|f(u)|^2=\langle f, k_{\xi,u}\rangle^2$, applying the Schwarz inequality.

3.2.5. *Definition.* Let $\mathfrak{L}(\mu;Q)$ be the space of functions on $O_\mu\times W$ of the form

$$F(\xi,u)=\sum_{i=1}^N l_i(\xi)\,e^{4\pi\langle\xi,\,Q(u,v_i)\rangle},\quad l_i\in C_0(\overline{\Omega^*}),\quad v_i\in W. \tag{3.2.6}$$

Define

$$\|F\|^2_\mu=\sum_{i,j}\int l_i(\xi)\,\bar{l}_j(\xi)\,e^{4\pi\langle\xi,\,Q(v_j,v_i)\rangle}\,d\mu(\xi).$$

and let $H(\mu,Q)$ be the Hilbert space completion of this space.

Otherwise put, $H(\mu,Q)$ is the space of square integrable sections of the fibration $H(\xi)\to\xi$ over O_μ. The following characterization is more valuable:

3.2.7. LEMMA. *Let $\mathcal{H}(\mu,Q)$ be the space of Borel measurable functions F defined on $\overline{\Omega}^*\times W$ such that*

(a) $$F(\xi,\cdot)\in H(\xi)\quad \textit{for almost all } \xi(d\mu),$$

(b) $$\|F\|_0^2=\int\|F(\xi,\cdot)\|^2 d\mu(\xi)<\infty.$$

Then $H(\mu,Q)$ is naturally identified with the equivalence classes of $\mathcal{H}(\mu,Q)$, modulo $d\mu\times du$-null functions.

Proof. Let $H_0(\mu, Q)$ be this space of equivalence classes, and $\Phi: \mathcal{H}(\mu, Q) \to H_0(\mu, Q)$ the equivalence relation. First of all, $H_0(\mu, Q)$ is a Hilbert space in the norm $\|\cdot\|_0$ given above.

For $\xi \in O_\mu$, $G \in H(\xi)$ and B a fixed ball in W, we have clearly

$$\int_B |G(u)|^2 du \leqslant CM(\xi) \|G\|_\xi^2,$$

where $M(\xi) = \max \{\exp(+4\pi Q_\xi(u, u)); u \in B\}$, and C depends only on B. Then if U is a small open set in $\bar{\Omega}^*$, and $M = \max\{M(\xi); \xi \in U\}$, for any $G \in H(\mu, Q)$,

$$\int_{B \times U} |G(\xi, u)|^2 du\, d\mu \leqslant K \|G\|_0^2,$$

where K depends only on $B \times U$.

Now, let $\{F_n\} \subset \mathcal{H}(\mu, Q)$ be Cauchy in $\|\cdot\|_0$. Then, for such B, U, $\{F_n\}$ is Cauchy in $L^2(B \times U, du \times d\mu)$. Thus $\{F_n\}$ has a limit in $L^2_{\text{loc}}(du \times d\mu)$ on $\bar{\Omega}^* \times W$; let F be this limit. We then have

$$\int_U \left[\int_B |F_n(\xi, u) - F(\xi, u)|^2 du\right] d\mu \to 0,$$

so, as functions of ξ, the inner integrals converge to 0 in $L^1(U, d\mu)$. We can choose a subsequence F_{n_k} such that the inner integrals converge pointwise to zero a.e. $(d\mu)$; or, what is the same $F_n(\xi, \cdot) \to F(\xi, \cdot)$ in $L^2(B, du)$ for almost all ξ. Thus $F(\xi, \cdot)$ is holomorphic on B for almost all $\xi(d\mu)$. Covering W by a countable set of such B, we see that for almost all ξ, $F(\xi, \cdot)$ is holomorphic on W. Similarly, since $\{F_n\}$ is $\|\cdot\|_0$-Cauchy, we can conclude that for almost all ξ, $\{F_n(\xi, \cdot)\}$ is Cauchy in $H(\xi)$, and $F_n(\xi, \cdot) \to F(\xi, \cdot)$ on W. Thus $F(\xi, \cdot) \in H(\xi)$ also and $\|F_n(\xi, \cdot) - F(\xi, \cdot)\|^2 \to 0$ in $L^1(d\mu)$. We conclude that a $\|\cdot\|_0$-Cauchy sequence in $H_0(\mu, Q)$ actually converges to an element of $H_0(\mu, Q)$, so $H_0(\mu, Q)$ is a Hilbert space.

Now, for F of the form (3.2.6), clearly $F \in \mathcal{H}(\mu, Q)$. Since $\exp(4\pi Q_\xi(u, v))$ is the kernel function of $H(\xi)$, we have

$$\|F\|_\mu^2 = \int \|F(\xi, \cdot)\|_\xi^2 d\mu(\xi) = \|F\|_0^2,$$

so the correspondence $F \to \Phi F$ is an isometry of $\mathcal{L}(\mu, Q)$ into $H_0(\mu, Q)$. It remains to show that the image is dense.

Let $\phi \in H_0(\mu, Q)$, and suppose that ϕ is orthogonal to the image of $\mathcal{L}(\mu, Q)$. Then for all $l \in C_0(\bar{\Omega}^*)$, $v \in W$, $\phi(\xi, u)[l(\xi) \exp(4\pi Q_\xi(u, v))]^-$ is in $L^1(\exp(-4\pi Q_\xi(u, u) du_\xi du)$, so by Fubini's theorem,

$$0 = \int \overline{l(\xi)} \left[\int_{W/W_\xi} \phi(\xi, u)\, e^{\overline{4\pi Q_\xi(u, v)}} e^{-4\pi Q_\xi(u, u)} d\dot{u}_\xi\right] d\mu(\xi).$$

Now, since $e^{4\pi Q_\xi(u,v)}$ is the kernel function for $H(\xi)$, the inner integral is $\phi(\xi, v)$, a.e. (ξ). Thus, for all $v \in W$, all $l \in C_0(\overline{\Omega}^*)$, $\int \phi(\xi, v)\overline{l(\xi)}\,d\mu(\xi) = 0$. This implies that $\phi(\xi, v) = 0$ for almost all ξ, i.e., $\phi = 0$ in $H_0(\mu, Q)$.

3.2.8. THEOREM. *Let μ be a positive measure on $\overline{\Omega}^*$ such that*

$$F(iy) = \int e^{-2\pi\langle \xi, y\rangle}\,d\mu(\xi)$$

converges for all $y \in \Omega$. Let

$$F(z) = \int e^{2\pi i\langle \xi, z\rangle}\,d\mu(\xi); \quad z \in D(\Omega).$$

For $\phi \in \mathcal{H}(\mu, Q)$ (as defined in Lemma 3.2.7), the integral

$$\check{\phi}(z, u) = \int e^{2\pi i\langle \xi, z\rangle}\phi(\xi, u)\,d\mu(\xi) \tag{3.2.9}$$

converges absolutely for all $(z, u) \in D(\Omega, Q)$. Let

$$\check{H}(\mu, Q) = \{\check{\phi} : \|\check{\phi}\|_\mu^2 = \|\phi\|_\mu^2\}.$$

$\check{H}$ is a Hilbert space of functions holomorphic on $D(\Omega, Q)$ with $R_F((z_1, u_1), (z_2, u_2)) = F(z_1 - \bar{z}_2 - 2iQ(u_1, u_2))$ as reproducing kernel.

Proof. Let $\phi \in \mathcal{H}(\mu, Q)$. Let $(z, u) \in D(\Omega, Q)$, with $z = x + i(t + Q(u, u))$, $t \in \Omega$. We have

$$\int \left| e^{2\pi i\langle \xi, z\rangle}\phi(\xi, u)\right| d\mu(\xi) = \int e^{-2\pi\langle \xi, t\rangle} e^{-2\pi\langle \xi, Q(u,u)\rangle}|\phi(\xi, u)|\,d\mu(\xi)$$
$$\leq \left[\int e^{-4\pi\langle \xi, t\rangle}\,d\mu(\xi)\right]^{1/2} \left[\int e^{-4\pi\langle \xi, Q(u,u)\rangle}|\phi(\xi, u)|^2\,d\mu(\xi)\right]^{1/2},$$

and by (3.2.4), since $e^{-4\pi\langle \xi, Q(u,u)\rangle}|\phi(\xi, u)|^2 \leq \|\phi(\xi, \cdot)\|_{H(\xi)}^2$, this last is dominated by $F(2it)^{1/2}\|\phi\|_\mu$, so the integral is absolutely convergent. In that case, applying Morera's criterion, we can use Fubini's theorem to prove that since $e^{2\pi i\langle \xi, z\rangle}\phi(\xi, u)$ is holomorphic in $D(\Omega, Q)$ for almost all ξ, so is (3.2.9). It remains to verify that R_F is the kernel function. Let $\phi \in \mathcal{H}(\mu, Q)$.

$$\check{\phi}(z, u) = \int_{\overline{\Omega}^*} \phi(\xi, u)\,e^{2\pi i\langle \xi, z\rangle}\,d\mu(\xi)$$
$$= \int_{\overline{\Omega}^* \times W} \left(\phi(\xi, v)\,e^{2\pi i\langle \xi, z\rangle}\,\overline{e^{4\pi\Omega_\xi(v,u)}}\,e^{-4\pi Q_\xi(v,v)}\right) d_\xi \dot{v}\,d\mu(\xi)$$
$$= \int \phi(\xi, v)\,\overline{\exp(-2\pi i\langle \xi, \bar{z} + 2iQ(v, u)\rangle)}\,e^{-4Q_\xi(v,v)}\,d_\xi \dot{v}\,d\mu(\xi) = \langle \phi, \Phi_{(z,u)}\rangle$$

by Lemma 3.2.7, where $\Phi_{(z,u)}$ is the expression under the bar. But, by definition, this is $\langle \check{\phi}, \check{\Phi}_{(z,u)}\rangle$, so $\check{\Phi}_{(z,u)}$ is the kernel function for $H(\mu, Q)$. But

$$\check{\Phi}_{(z,u)}(z', u') = \int e^{2\pi i\langle \xi, z'\rangle} \exp(-2\pi i\langle \xi, \bar{z} + 2iQ(u', u)\rangle)\, d\mu(\xi) = R_F((z', u'), (z, u)).$$

3.3. The case where $d\mu$ is absolutely continous

In this section we shall describe the inverse isomorphism of $\check{H}(\xi, Q)$ with $\mathcal{H}(\xi, Q)$ when $d\mu(\xi) = f(\xi)\,d\xi$ on Ω^*. This includes, in particular, the spaces $\mathcal{H}(\lambda)$ corresponding to the discrete holomorphic series. We shall make explicit their reproducing kernels in the following section.

For J a continuous positive function on the cone Ω^*, we consider the measure $d\mu(\xi) = J(\xi)^{-1}\det Q_\xi\, d\xi$. Here $O_\mu = \Omega^*$, and for $\xi \in \Omega^*$, $H(\xi)$ is the space of functions F holomorphic on W such that

$$\|F\|_\xi^2 = \int |F(u)|^2 e^{-4\pi Q_\xi(u,u)} \det Q_\xi\, du < \infty.$$

The correspondence $\alpha \to \beta$, defined by

$$\beta(\xi, u) = J(\xi)^{-1} \det Q_\xi\, \alpha(\xi, u),$$

determines an isomorphism of $\mathcal{H}(\mu, Q)$ with the space $\mathcal{H}(\Omega^*, Q; J)$ of functions ϕ measurable on $\Omega^* \times W$ such that

(3.3.1)
(a) $\phi(\xi, \cdot)$ is holomorphic for almost all ξ,

(b) $\int e^{-4\pi Q_\xi(u,u)} |\phi(\xi, u)|^2 J(\xi)\, d\xi\, du < \infty.$

As in [20a], p. 345, we introduce the space $P(\Omega^*, Q)$ of functions of the form $\sum l_i(\xi) P_i(u)$, $l_i \in C_0(\Omega^*)$, P_i a complex polynomial on W, and its completion $H(\Omega^*, Q; J)$ in the norm

$$\|\phi\|^2 = \int_{\Omega^* \times W} |\phi(\xi, u)|^2 e^{-4\pi\langle \xi, Q(u,u)\rangle} J(\xi)\, d\xi\, du.$$

3.3.2. Lemma. *$H(\Omega^*, Q; J)$ is the space of equivalence classes (modulo null functions relative to Lebesgue measure $d\xi\, du$) of functions in $\mathcal{H}(\Omega^*, Q; J)$.*

Proof. Let $H_0(\Omega^*, Q; J)$ be the space of such equivalence classes; by Lemma 3.2.7, this is a Hilbert space in the norm (3.3.1). Clearly, if $\phi \in P(\Omega^*, Q)$, it is in $\mathcal{H}(\Omega^*, Q; J)$, and the correspondence is isometric. It remains only to show that $P(\Omega^*, Q)$ is dense. But this (replacing I_ψ by J), is just the argument on the top of p. 346 of [20a].

Remark. It is here appropriate for us to disclaim Theorem 2.30 of [20a], which, if it made sense, would still be incorrect.

3.3.3. *Definition.* We shall say that J satisfies the condition (C) if

$$F(iy) = \int_{\Omega^*} e^{-2\pi\langle \xi, y\rangle} J(\xi)^{-1} \det Q_\xi \, d\xi$$

converges for all $y \in \Omega$.

Notice that whenever the integral converges for some $y_0 \in \Omega$, it converges for $y \in y_0 + \Omega$ (since $\exp(-2\pi\langle \xi, t\rangle) \leqslant 1$ for $t \in \Omega$). In particular, if J is homogeneous with respect to homotheties ($J(ty) = t^\alpha J(y)$ for some real α, all $t \in R^+$), then J satisfies condition (C) if and only if the integral converges for some $y_0 \in \Omega$. When condition (C) holds,

$$F(z) = \int_{\Omega^*} e^{2\pi i\langle \xi, z\rangle} J(\xi)^{-1} \det Q_\xi \, d\xi$$

is holomorphic in $D(\Omega)$. The following proposition paraphrases Theorem 3.2.8 for the present case.

3.3.4. PROPOSITION. *Let J be a positive continuous function on Ω^* satisfying condition* (C). *For ϕ a measurable function on $\Omega^* \times W$, holomorphic on W for almost all $\xi \in \Omega^*$ such that*

$$\|\phi\|_J^2 = \int |\phi(\xi, u)|^2 \exp(-4\pi Q_\xi(u, u)) J(\xi) \, d\xi \, du < \infty,$$

(*we shall say $\phi \in H(D, J)$*), *then the integral*

$$\check{\phi}(z, u) = \int_{\Omega^*} e^{2\pi i\langle \xi, z\rangle} \phi(\xi, u) \, d\xi \tag{3.3.5}$$

converges absolutely for all (z, u) in $D(\Omega, Q)$.

The space $\check{H}(D, J)$ of all such $\check{\phi}$, endowed with the norm $\|\check{\phi}\|_J = \|\phi\|_J$, is a Hilbert space of holomorphic functions on $D(\Omega, Q)$, with kernel function

$$k_J(z_1, u_1), (z_0, u_0) = \int_{\Omega^*} \exp(2\pi i\langle \xi, z_1 - \bar{z}_0 - 2iQ(u_1, u_0)\rangle) J(\xi)^{-1} \det Q_\xi \, d\xi.$$

We note that, in case $J=1$, the space $\check{H}(D; J)$ is already studied by Koranyi-Stein (see [12]). In this case $\check{H}(D; 1)$ is the Hardy space H^2 of holomorphic functions on $D(\Omega, Q)$ such that

$$\|F\|_J^2 = \sup_{t\in\Omega} \int |F(x+i(t+Q(u,u)), u)|^2 dx\, du < \infty.$$

3.4. Condition (*HC*)

We begin by recalling the Plaey-Wiener theorem of [20a] (see also [20b]). For F a C^∞ function on $D(\Omega, Q)$ such that $F(x+iy, u)$ is a Schwartz test function of x for all fixed (y, u), define

$$\hat{F}(\xi, y, u) = e^{2\pi\langle \xi, y\rangle} \int F(x+iy, u)\, e^{-2\pi i\langle \xi, x\rangle}\, dx. \tag{3.4.1}$$

Now, let ψ be a positive continuous function on Ω which is homogeneous relative to homotheties. We introduce the following:

$H^2(D, \psi) = \{F$ holomorphic on $D(\Omega, Q)$, such that

$$\|F\|_\psi^2 = \int |F(x+iy, u)|^2 \psi(y - Q(u, u)\, dx\, dy\, du < \infty\}. \tag{3.4.2}$$

$$I_\psi(\xi) = \int e^{-4\pi\langle \xi, y\rangle} \psi(y)\, dy. \tag{3.4.3}$$

3.4.4. LEMMA. (a) *$I_\psi(\xi)$ is a convex function of ξ which is identically infinite off $\bar{\Omega}^*$.*

(b) *$I_\psi(\xi_0) < \infty$ for some $\xi_0 \in \Omega^*$ if and only if $I_\psi(\xi) < \infty$ for all $\xi \in \Omega^*$.*

(c) *$H^2(D, \psi) \neq \{0\}$ if and only if $I_\psi(\xi) < \infty$ in Ω^*.*

3.4.5. *Definition.* For such a ψ, we shall say that ψ satisfies *condition* (HC) if I_ψ is finite somewhere (and thus finite throughout Ω^*). Given J defined on Ω^*, we shall say that J satisfies *condition* (HC) if $J = I_\psi$ for such a ψ.

3.4.6. THEOREM. *Suppose ψ satisfies condition* (HC). *Then the correspondence* (3.4.1) *induces an isometry of $H^2(D, \psi)$ with $H(D; I_\psi)$ defined as in Proposition* 3.3.4.

We shall denote this isometry by the formula (3.4.1) where the integral is interpreted as the Fourier transform. Notice that for $F \in H^2(D, \psi)$, $\hat{F}(\xi, y, u)$ is independent of y. We shall denote this as $\hat{F}(\xi, u)$.

3.4.7. THEOREM. *If ψ satisfies condition* (HC), *then $\check{H}(D, I_\psi) = H^2(D, \psi)$; in particular I_ψ satisfies condition* (C).

Proof. The first assertion is just the affirmation of Fourier inversion, whether or not I_ψ satisfies condition (C). That it does so follows easily, as we now show:

Let $\phi(\xi, u) = l(\xi)P(u)$ a function in $P(\Omega^*; Q)$. By Fourier inversion, for $y \in \Omega$ we have

$$\check{\phi}\left(\frac{iy}{2}, 0\right) = \int e^{-\pi\langle\xi, y\rangle} l(\xi) P(0) \, d\xi. \tag{3.4.8}$$

On the other hand, we know that $H^2(D; \psi)$ being a Hilbert space of holomorphic functions square integrable on the domain D has a reproducing kernel and so by 3.4.6 we have

$$\check{\phi}\left(\frac{iy}{2}, 0\right) = \int l(\xi) P(u) \overline{k(\xi, u)} e^{-4\pi\langle\xi, Q(u, u)\rangle} J(\xi) \, d\xi \, du \tag{3.4.9}$$

with $k(\xi, u) \in H(D; I_\psi)$. Since (3.4.8), (3.4.9) hold for all $l \in C_0(\Omega^*)$, the integrating factors are identical, so that

$$\int P(u) \overline{k(\xi, u)} e^{-4\pi\langle\xi, Q(u, u)\rangle} du = J(\xi)^{-1} e^{-\pi\langle\xi, y\rangle} P(0)$$

for every polynomial P.

Since the polynomials are dense, in $H(\xi)$ and $k(\xi, u) \in H(\xi)$ for almost all ξ, $k(\xi, u) J(\xi) e^{\pi\langle\xi, y\rangle}$ is the reproducing kernel of $H(\xi)$ at 0, so

$$k(\xi, u) = e^{-\pi\langle\xi, y\rangle} J(\xi)^{-1} \det Q_\xi,$$

for all y. Since $k(\xi, u) \in H(D, I_\psi)$,

$$\int e^{-2\pi\langle\xi, y\rangle} J(\xi)^{-1} \det Q_\xi \, d\xi < \infty,$$

as desired.

Thus condition (HC) implies condition (C), but not conversely, since $J = 1$ satisfies condition (C), but is not of the form I_ψ for some ψ. Finally, from the last assertion of Proposition 3.3.4 we obtain

3.4.10. Proposition. *The kernel function of $H^2(D, \psi)$ is*

$$k_\psi((z_1, u_1), (z_0, u_0)) = \int_{\Omega^*} \exp\left(2\pi i\langle\xi, z_1 - \bar{z}_0 - 2iQ(u_1, u_0)\rangle\right) I_\psi(\xi)^{-1} \det Q_\xi \, d\xi.$$

3.5. Description of the spaces $\mathcal{H}(\lambda)$

Let us now suppose that $D(\Omega; Q)$ is a homogeneous symmetric Siegel domain; then we have $\alpha: G/K \xrightarrow{\sim} D(\Omega, Q)$, $W = \mathcal{H}^+_{1/2}$, $\Omega \subset \mathcal{H}^1$, in the notation of chapter 2. Let λ be a character of $\tilde{K}$, and $\Lambda=(0, \lambda)$. Let $\phi_\lambda=\Phi_\Lambda$. Then ϕ_λ is a character of B such that

$$\phi_\lambda(\exp tX_{\gamma_j})=e^{t\lambda}.$$

Let μ_λ be the function defined on Ω by

$$\mu_\lambda(h\cdot s)=\phi_\lambda(h)^{-2}(\det_{\mathcal{H}_{1/2}} h)^{-1}(det_{\mathcal{H}_1} h)^{-2}.$$

Then, if we let $\mathcal{H}(\lambda)$ designate the space of the holomorphic discrete series associated to the character λ of $\tilde{K}$, we have $\mathcal{H}(\lambda)=\{F$ holomorphic on $D(\Omega, Q)$ such that

$$N(\lambda, F)^2=\int |F(x+iy, u)|^2\mu_\lambda(y-Q(u, u))\,dx\,dy\,du<+\infty\}. \tag{3.5.1}$$

Thus we are led to consider

$$I_\lambda(\xi)=\int_\Omega e^{-4\pi\langle\xi, t\rangle}\mu_\lambda(t)\,dt=\int_{H_0} e^{-4\pi\langle\xi, h\cdot s\rangle}\phi_\lambda(h)^{-2}(\det_{\mathcal{H}_{1/2}} h)^{-1}(\det_{\mathcal{H}_1} h)^{-1}dh, \tag{3.5.2}$$

so that

$$I_\lambda(h\cdot\xi)=\phi_\lambda(h)^{-2}(\det_{\mathcal{H}_1} h)^{-1}(\det_{\mathcal{H}_{1/2}} h)^{-1}I_\lambda(\xi_0).$$

We shall calculate $I_\lambda(\xi_0)$ explicitly later (in section 4.4) and show that $I_\lambda(\xi_0)<\infty$ if and only if $\lambda+\langle\varrho, H_{\gamma_r}\rangle<0$, which is fortunate, for that is Harish-Chandra's condition for $H(\lambda)\neq\{0\}$, and we have affirmed that

$$H(\lambda)\neq\{0\}\text{ if and only if }\mathcal{H}(\lambda)\neq\{0\}.$$

Define J_λ on Ω^* by

$$J_\lambda(h\cdot\xi_0)=\phi_\lambda(h)^{-2}(\det_{\mathcal{H}_1} h)^{-1}(\det_{\mathcal{H}_{1/2}} h)^{-1} \tag{2.5.3}$$

J_λ is defined for all $\lambda\in R$, but if $\lambda+\langle\varrho, H_{\gamma_r}\rangle<0$, then $I_\lambda(\xi)=I_\lambda(\xi_0)J_\lambda(\xi)$, thus J_λ is of the form I_ψ; i.e., J_λ satisfies condition (HC) if and only if λ satisfies the condition of Harish-Chandra for $(0, \lambda)$ to be in the discrete holomorphic series. This gives us the following description of $\mathcal{H}(\lambda)$.

3.5.4. PROPOSITION. *Let $\lambda+\langle\varrho, H_{\gamma_r}\rangle<0$. Then the correspondence $F\to\hat{F}$ is an isometry of $\mathcal{H}(\lambda)$ with $H(\Omega^*, Q; I_\lambda)$. For every $\phi\in H(\Omega^*, Q; J_\lambda)$, the integral*

$$\check{\phi}(z, u)=\int e^{2\pi i\langle\xi, z\rangle}\phi(\xi, u)\,d\xi \tag{3.5.5}$$

converges absolutely to a function of $H(\lambda)$, and $\hat{\check{\phi}}=\phi$, $N(\lambda; \check{\phi})^2=I_\lambda(\xi_0)\|\phi\|^2_{J_\lambda}$. More explicitly,

$$\int |F(x+iy, u)|^2\mu_\lambda(y-Q(u, u))\,dx\,dy\,du=I_\lambda(\xi_0)\int_{\Omega^*\times\mathcal{H}^+_{1/2}}|\hat{F}(\xi, u)|^2e^{-4\pi Q_\xi(u, u)}J_\lambda(\xi)\,d\xi\,du$$

for all $F \in \mathcal{H}(\lambda)$. *The reproducing kernel* K_λ *of* $\mathcal{H}(\lambda)$ *is*

$$I_\lambda(\xi_0)^{-1}\int e^{2\pi i\langle \xi,\, z_1-\bar{z}_0-2iQ(u_1,\, u_0)\rangle} J_\lambda(\xi)^{-1} \det Q_\xi \, d\xi, \tag{3.5.6}$$

and the integral converges absolutely.

We shall see that in the next chapter J_λ satisfies condition (C) for values of $\lambda > -\langle \varrho, H_{\gamma_r}\rangle$. In this case, the spaces $H(\Omega^*, Q, J_\lambda)$ shall be isometric to the spaces $\mathcal{H}_0(\lambda)$ and as such, define irreducible unitary representations of $\tilde{G}$. In particular, if $\lambda = -((r-1)p+1+\mu)$, then $J_\lambda = 1$ (by 3.5.3) and $\check{H}$ is the Hardy space H^2. This value is larger than $-\langle \varrho, H_{\gamma_r}\rangle = (r-1)p+1+\mu/2$.

4. Hilbert spaces of holomorphic functions on the symmetric domains $D(\Omega, Q)$

We shall consider the realization of G/K as a Siegel domain $D(\Omega, Q)$ as described in chapter 2. We shall consider specifically the function F_λ on $D(\Omega)$ defined by

$$F_\lambda(z) = K(z)^{-\lambda r/n},$$

(see section 2.6). We shall describe the set P of those $\lambda \leqslant 0$ such that R_{F_λ} (see 2.6.4) has the property (P). This set was described completely first by Wallach ([24b], II) using completely different techniques.

Note that, since $K(z) = \int_{\Omega^*} \exp 2\pi i\langle \xi, z\rangle d\xi$, we have, automatically $K(i(y+y_0)) \leqslant K(iy)$, and thus since $\lambda \leqslant 0$, $F_\lambda(i(y+y_0)) \leqslant F_\lambda(iy)$, so we may apply the theorem of Nussbaum (3.1.2). Thus R_{F_λ} satisfies property (P) if and only if

$$F_\lambda(iy) = \int e^{-2\pi\langle \xi,\, y\rangle} d\mu_\lambda(\xi)$$

with $d\mu_\lambda$ a positive measure supported on $\bar{\Omega}^*$. Such a representation is unique [17]. We shall now use the homogeneity of our situation to discover all permissible μ_λ.

Since F_λ transforms (under $G(0)$) by the character χ_λ,

$$\chi_\lambda(\exp tX_{\gamma_i}) = e^{2t\lambda},$$

it follows that $d\mu_\lambda$ is semi-invariant under the $G(0)$ action:

$$\langle d\mu_\lambda, l(x_0)\phi\rangle = \chi_\lambda(x_0)\langle d\mu_\lambda, \phi\rangle$$

for all $\phi \in C_0(\bar{\Omega}^*)$, where $l(x_0)$ is left translation by x_0^{-1}. We thus need to describe the semi-invariant measures (under $G(0)$ action) on $\bar{\Omega}^*$, i.e., the measure transforming by a character of $G(0)$.

4.1. Orbits of $G(0)$ on $\bar{\Omega}$

The map $\xi: \mathcal{H}_1 \to \mathcal{H}_1^*$ given in 2.3.1 has the property $\xi(g\cdot X) = (g^*)^{-1}\cdot\xi(X)$, so the G-orbits in $\bar{\Omega}$ and $\bar{\Omega}^*$ are in one-one correspondence

4.1.1. PROPOSITION. *Let $s_1 = 0$, $s_e = \sum_{i=1}^{e-1} U_i$. Then $\{s_1, ..., s_r, s\}$ is a complete system of representatives of the $G(0)$-orbits in $\bar{\Omega}$. We have $G(0)\cdot s_1 = \{0\}$, and $s_i \in \overline{G(0)\cdot s_{i+1}}$.*

Proof. Since $s_i = \lim_{t\to-\infty} (\exp tX_{\gamma_i}) s_{i+1}$, we have $s_i \in \overline{G(0)\cdot s_{i+1}}$, and thus $s_i \in \bar{\Omega}$, $1 \leqslant i \leqslant r$. Now, let $\alpha \in \bar{\Omega}$, $\alpha = \lim g_n\cdot s$. Since $G(0) = K_0 A K_0$ with K_0 compact, we can write $g_n = k_n\cdot a_n\cdot k_n'$ and $\alpha = \lim (k_n\cdot a_n\cdot s)$. Extracting a convergent subsequence, we see that α is conjugate (under K_0) to an element of the form $s_I = \sum_{i\in I} U_i$. Thus, we can find a permutation ω in the Weyl group, fixing Js and thus in $G(0)$, such that $\omega\cdot s_I = s_e$ for some e (unless already $\alpha \in \Omega$ and $s_I = s$).

Let O_e denote the $G(0)$-orbit of s_e. Thus $\bar{\Omega} = \Omega \cup \bigcup_{e=1}^{r} O_e$. We introduce the following notation: fix e, $1 \leqslant e \leqslant r$, and let $C_e = \{1, 2, ..., e-1\}$, $C_e' = \{e, ..., r\}$. Let $\mathfrak{g}(C_e)$ be the derived algebra of

$$\mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{i,j\in C_e} \eta^{\pm((\alpha_i\pm\alpha_j)/2)} \oplus \sum_{i\in C_e} \eta^{\pm(\alpha_i/2)}.$$

Let

$$\begin{aligned} &\mathfrak{g}_0(C_e) = \mathfrak{g}(C_e) \cap \mathfrak{g}(0),\\ &\mathcal{H}_0(C_e) = \mathfrak{g}(C_e) \cap \mathcal{H}_0 = \sum_{i\in C_e} RX_{\gamma_i} \oplus \sum_{i,j\in C_e} \eta^{(\alpha_i-\alpha_j)/2},\\ &\mathfrak{k}(C_e) = \mathfrak{g}(C_e) \cap \mathfrak{k}, \quad \mathfrak{p}(C_e) = \mathfrak{g}(C_e) \cap \mathfrak{p},\\ &\mathfrak{k}_0(C_e) = \mathfrak{g}_0(C_e) \cap \mathfrak{k}, \end{aligned} \tag{4.1.2}$$

and $G(C_e)$, $G_0(C_e)$, $H_0(C_e)$, $K(C_e)$, $K_0(C_e)$ the corresponding connected subgroups of G. Let $\tilde{G}(C_e)$, $\tilde{G}_0(C_e)$, $H_0(C_e)$, $\tilde{K}(C_e)$, $\tilde{K}_0(C_e)$ be the corresponding subgroups of $\tilde{G}$. Finally, let

$$\mathcal{H}_{0,e}' = \sum_{\substack{i\in C_e'\\ j\in C_e}} \eta^{(\alpha_i-\alpha_j)/2},$$

$$\mathcal{V}_{0,e} = \sum_{\substack{i\in C_e'\\ j\in C_e}} \eta^{-((\alpha_i-\alpha_j)/2)}.$$

We shall calculate the stabilizer S_e^0 of s_e in $G(0)$, and its Lie algebra $\mathcal{S}_e^0$.

4.1.3. LEMMA. $\mathcal{S}_e^0 = \mathfrak{k}_0(C_e) \oplus \mathfrak{g}_0(C_e') \oplus \mathfrak{m} \oplus \mathcal{V}_{0,e}$.

Proof. It is easy to see that the space on the right is contained in $\mathcal{S}_e^0$; on the other hand the map $X \to [X, s_e]$ coincides with J and is bijective on the complement $\mathcal{H}_0(C_e) \oplus \mathcal{H}_{0,e}'$.

Let $\tau_e = \mathcal{H}_0(C_e) \oplus \mathcal{H}_{0,e}'$. τ_e is a subalgebra of $\mathcal{H}_0$; let T_e be the corresponding connected subgroup of H_0.

4.1.4. LEMMA. $S_e^0 = G_0(C'_e)K_0(C_e)M \exp \mathfrak{V}_{0,e}$.

Clearly S_e^0 contains the group on the right; we shall prove equality in chapter 6 (Corollary 6.2.5)).

4.1.5. LEMMA. $T_e \cdot S_e^0$ has a complement of measure zero for the Haar measure on $G(0)$.

Proof. $T_e \cdot S_e^0$ contains the fat cell of $G(0)$.

4.1.6. LEMMA. *The orbit $T_e \cdot s_e$ has a complement of measure zero for the (unique) class of quasi-invariant measures on O_e.*

Proof. Using Lemma 4.1.4, we have $G(0) = K_0 H_0(C_e) S_e^0$. The functional $\phi \to \int \phi(kh \cdot s_e)\,dk\,dh$ ($\phi \in C_0(O_e)$) defines a quasi-invariant measure on O_e. If $k \cdot h \cdot s_e \notin T_e \cdot s_e$, then $k \notin T_e \cdot S_e^0$. For, otherwise $k \cdot h$ would belong to $T_e S_e^0 H_0(C_e) = T_e H_0(C_e) S_e^0 = T_e S_e^0$. But, by Lemma 4.1.4, the complement of $T_e S_e^0$ is of dk-measure zero.

4.2. Semi-invariant measures on the orbit O_e

Let us consider the character χ_e on S_e^0 defined by

$$\chi_e(h) = \det_{\mathfrak{g}(0)/\mathfrak{s}_e^0} h.$$

χ_e is trivial on $[G_0(C'_e), G_0(C'_e)] \cdot K_0(C_e) M \exp \mathfrak{V}_{0,e}$. On the other hand, if $X_{\gamma_i} \in \mathfrak{g}_0(C'_e)$ (i.e., if $i \geqslant e$) then

$$Tr_{\mathfrak{g}_0/\mathfrak{s}_e^0} X_{\gamma_i} = Tr_{\mathcal{H}'_{0,e}} X_{\gamma_i} = (e-1)\,p.$$

Thus χ_e admits a unique extension (since $e \neq r+1$) to a character $\tilde{\chi}_e$ of $G(0)$. $\tilde{\chi}_e$ takes positive real values and

$$\tilde{\chi}_e(\exp tX_{\gamma_i}) = e^{t(e-1)p}, \quad 1 \leqslant i \leqslant r.$$

Let $K_e = \{\phi \in C(G(0));\ \phi$ has compact support mod S_e and $\phi(gh) = \chi_e(h)\phi(g)$, $g \in G(0)$, $h \in S_e^0\}$, $\mathcal{C}(O_e) = \{\phi \in C(G(0));\ \phi$ has compact support mod S_e and ϕ is invariant under right multiplication by $S_e^0\}$. $\mathcal{C}(O_e)$ can be identified with the set of continuous functions on O_e of compact support. Let I_e designate the unique positive linear functional on K_e which is invariant under left translation by G_0. Since $\phi \to \tilde{\chi}_e \cdot \phi$ is an isomorphism of $\mathcal{C}(O_e)$ with K_e, we obtain from I_e (by composition) a positive measure $d\mu_e$ on $\mathcal{C}(O_e)$ which satisfies

$$\langle d\mu_e, l(x_0) \cdot \phi \rangle = \tilde{\chi}_e(x_0) \langle d\mu_e, \phi \rangle.$$

Furthermore, $d\mu_e$ is the *unique* semi-invariant measure on O_e; i.e., if $d\mu'$ is semi-invariant, transforming by the character χ', then $\chi' = \chi_e$ (see [27], chapter 7, § 2.6, Corollary 1), and $d\mu'$ is proportional to $d\mu_e$.

Since $T_e \cdot s_e$ has a complement of $d\mu_e$-measure zero, it follows that for $\phi \in C(O_e)$,

$$\langle d\mu_e, \phi\rangle = \int_{T_e} \phi(t \cdot s_e)\,\tilde{\chi}_e(t)\,dt.$$

Notice that Ω has a $G(0)$-invariant measure $d\mu = \det_{\mathcal{H}_1} h(t)^{-1} dt$, and thus all the measures $(\det_{\mathcal{H}_1} h(t))^{\alpha} dt$ are $G(0)$-semi-invariant.

Now, we move these results to $\overline{\Omega}^*$ using the map ξ. Let $\xi_e = \sum_{i=1}^{e-1} U_i^*$, $\xi_1 = 0$, $\xi_{r+1} = \xi_0$. These $\{\xi_e\}$ form a complete system of representatives of the $G(0)$-orbits in $\overline{\Omega}^*$. Let O_e^* be the orbit of ξ_e, and its isotopy group is $(S_e^0)^*$. The T_e-orbit of ξ_e, under the action $\xi \to (t^*)^{-1} \cdot \xi$, has a complement of measure zero in O_e^* (relative to $d\mu_e^*$, the transport of $d\mu_e$ to O_e^*). We have

$$\langle d\mu_e^*, l(x_0)\phi\rangle = \tilde{\chi}_e(x_0)^{-1}\langle d\mu_e^*, \phi\rangle,$$

where

$$\tilde{\chi}_e(\exp tX_{\gamma_i}) = \exp t(e-1)\,p, \quad e \leqslant r, \quad \tilde{\chi}_{r+1} = 1.$$

$$\langle d\mu_e^*, \phi\rangle = \int_{T_e} \phi((t^*)^{-1}\xi_e)\,\tilde{\chi}_e(t)\,dt, \quad (\phi \in C(O_e)).$$

If $e \leqslant r$, every semi-invariant measure on O_e^* is proportional to $d\mu_e^*$. On Ω^* $(e = r+1)$, every measure, semi-invariant by the character χ_λ:

$$\chi_\lambda(\exp\ tX_{\gamma_i}) = \exp 2t\lambda,$$

is proportional to

$$\alpha_\lambda(\xi)\,(\det_{\mathcal{H}_1} h(\xi))\,d\xi, \tag{4.2.1}$$

where $\alpha_\lambda(g \cdot \xi_0) = \chi_\lambda(g)$. For $\phi \in C_0(\Omega^*)$,

$$\int \phi(\xi)\,\alpha_\lambda(\xi)\,\det_{\mathcal{H}_1} h(\xi)\,d\xi = \int_{H_0} \phi(h_0 \cdot \xi_0)\,\chi_\lambda(h_0)\,dh_0 = \int_{G_0} \phi(g \cdot \xi_0)\,\chi_\lambda(g)\,dg$$

$$= \int_{G_0} \phi((g^*)^{-1} \cdot \xi_0)\,\chi_\lambda(g)^{-1} dg = \int_{H_0} \phi((h^*)^{-1} \cdot \xi_0)\,\chi_\lambda(h)^{-1} dh.$$

4.3

We now ask: for what λ does there exist a positive measure $d\mu_\lambda$ on $\overline{\Omega}^*$ with $F_\lambda(iy) = \int e^{-2\pi\langle \xi, y\rangle} d\mu_\lambda(\xi)$.

Such a $d\mu_\lambda$ is semi-invariant by the character χ_λ. Letting μ_λ^e by the restriction of μ_λ to O_e^* $(\mu_\lambda^e(E) = \mu_\lambda(E \cap O_e))$, we have $\mu_\lambda = \sum_e \mu_\lambda^e$, and each μ_λ^e is also semi-invariant. Thus

4.3.1. PROPOSITION. *If $-\lambda \notin \{0, p/2, ..., \frac{1}{2}(r-1)p\}$, the only possibility is that $d\mu_\lambda$ is proportional to $\alpha_\lambda(\xi)\, \det_{\mathcal{H}_1} h(\xi)\, d\xi$. If $\lambda = -\frac{1}{2}(e-1)p$ $(1 \leq e \leq r)$, then $d\mu_\lambda$ is in the linear span of $d\mu_e^*$ and $\alpha_\lambda(\xi)\, \det_{\mathcal{H}_1} h(\xi)\, d\xi$.*

Finally, we must find out, for which λ and the possible $d\mu_\lambda$ is the integral $\int e^{-\langle \xi, y\rangle} d\mu_\lambda$ convergent for $y \in \Omega$. It suffices to consider only $y = s$. Since

$$\langle (t^*)^{-1}\cdot \xi_e, s\rangle = \langle \xi_e, t^*\cdot s\rangle = S(s_e, t^* s) = S(ts_e, s) = \langle \xi_0, t\cdot s_e\rangle,$$

we must thus calculate

$$\tilde{F}_\lambda(is) = \int_{H_0} e^{-2\pi\langle \xi_0, t\cdot s\rangle} \chi_\lambda(t)^{-1} dt, \quad \text{all} \quad \lambda \in R, \tag{4.3.2}$$

and

$$\tilde{F}_e(is) = \int_{T_e} e^{-2\pi\langle \xi_0, t\cdot s_e\rangle} \tilde{\chi}_e(t)\, dt, \quad 1 \leq e \leq r. \tag{4.3.3}$$

4.4. Certain integrals on the cone Ω (see [5])

In this paragraph we shall calculate the above integral (4.3.2) as well as the integral $I_\lambda(\xi_0)$ as given by (3.5.2). The condition (HC) for $H(\lambda) \neq \{0\}$ comes out of this calulation. In addition, these calculations shall determine, for $J_\lambda(\xi) = I_\lambda(\xi)/I_\lambda(\xi_0)$ the condition for J_λ to satisfy condition (C) (recall 3.3.3). Since (recall 3.5.3)

$$J_\lambda(\xi)^{-1} \det Q_\xi d\xi = \alpha_\lambda(\xi)\, \det_{\mathcal{H}_1} h(\xi)\, d\xi,$$

and $\det Q_{h\cdot\xi} = (\det_{\mathcal{H}_{1/2}} h)^{-1} \det Q_\xi$, it follows that J_λ satisfies condition (C) if and only if $F_\lambda(is) < +\infty$.

4.4.1. PROPOSITION. *There is a basis $\{E^\alpha_{i,j}\}$ of $\eta^{\frac{1}{2}(\alpha_j - \alpha_i)}$, $1 \leq \alpha \leq p$, $(j > i)$ such that for*

$$h_0 = \prod_{i=1}^{r} \exp a_i X_{\gamma_i} \cdot \exp L_1 \dots \exp L_{r-1},$$

and

$$L_i \in \sum_{j, j>i} \eta^{\frac{1}{2}(\alpha_j - \alpha_i)}, \qquad L_i = \sum_{\alpha, j} x^\alpha_{i,j} E^\alpha_{i,j},$$

and $\xi = \sum u_k U_k^*$, we have

$$\langle \xi, h_0\cdot s_e\rangle = \sum_{i=1}^{e-1} (e^{2a_i} u_i + \sum_{j>1} u_j e^{2a_j} (\sum_\alpha (x^\alpha_{i,j})^2)), \tag{4.4.2}$$

Proof. This is just Theorem 4.10 of [20a], where $s = \sum_{i=1}^r U_i$ is replaced by $s_e = \sum_{i=1}^{e-1} U_i$.

This proposition gives the formula on which our computations are based. We consider a generic integral, which includes both $\tilde{F}_\lambda(is)$ and $I_\lambda(\xi_0)$ as cases. Let Λ be an arbitrary purely imaginary linear form on $\mathfrak{h}$. Let ϕ_Λ be the character on H_0 defined by

$$\phi_\Lambda(\exp tH_i) = e^{t\Lambda_i}, \quad \text{where} \quad \Lambda_i = \langle \Lambda, H_{\gamma_i} \rangle.$$

Let

$$I(\xi_0; \Lambda) = \int_{H_0} e^{-2\pi \langle \xi_0, h \cdot s \rangle} \phi_\Lambda(h)^{-2} (\det_{\mathcal{H}_1} h)^{-1} (\det_{\mathcal{H}_{1/2}} h)^{-1} dh. \tag{4.4.3}$$

Then $\tilde{F}_\lambda(is) = I(\xi_0; \Lambda)$ for

$$\Lambda_i = \lambda - \tfrac{1}{2}(2 + \mu + p(r-1)).$$

In the coordinates given by Proposition 4.1, we have

$$dh = da_1 \dots da_r \cdot \Pi dx^\alpha_{i,j},$$

$$Tr_{\mathcal{H}_1} ad X_{\gamma_i} = (r-1)p + 2, \quad Tr_{\mathcal{H}_{1/2}} ad X_{\gamma_i} = \mu,$$

2.2.2, and thus

$$I(\xi_0, \Lambda) = \int \exp\left[-2\pi\left(\sum_{i=1}^r e^{2a_i} + \sum_{j>1} e^{2a_j}\left(\sum_\alpha (x^\alpha_{i,j})^2\right)\right)\right]$$

$$\times \exp\left[-2\sum_{j=1}^r \Lambda_j a_j - \sum_{j=1}^r ((r-1)p + 2 + \mu) a_j\right] da_1 \dots da_r \prod dx^\alpha_{i,j}.$$

Make the change of variable $x^\alpha_{i,j} \to e^{a_j} x^\alpha_{i,j}$ (j fixed). There are $(j-1)p$ such variables, thus

$$I(\xi_0, \Lambda) = \int \exp\left[-2\pi\left(\sum_{i=1}^r e^{2a_i} + \sum_{j>i} (x^\alpha_{i,j})^2\right)\right]$$

$$\times \exp\left[-\sum_{j=1}^r a_j(2\Lambda_j + (j-1)p + (r-1)p + 2 + \mu\right] da_1 \dots da_r \prod dx^\alpha_{i,j}.$$

4.4.4. *Definition.* For $1 \leqslant i \leqslant r$, let $l_i = \frac{1}{2}[(r+i-2)p + 2 + \mu]$.

4.4.5. THEOREM. *$I(\xi_0, \Lambda) < +\infty$ if and only if $\Lambda_i + l_i < 0$, $1 \leqslant i \leqslant r$. In this case*

$$I(\xi_0, \Lambda) = \prod_{i=1}^r \tfrac{1}{2}(2\pi)^{\Lambda_i + l_i} \Gamma(-\Lambda_i - l_i).$$

Proof. Normalize the measure $\Pi dx^\alpha_{i,j}$ so that $\exp(-2\pi \Sigma (x^\alpha_{i,j})^2)$ has integral 1. Then

$$I(\xi_0, \Lambda) = \prod_{i=1}^r \int \exp[-2\pi e^{2a_i} - a_i(2\Lambda_i + (i-1)p + (r-1)p + 2 + \mu)] \, da_i$$

Letting $t = 2\pi e^{2a_i}$, the theorem falls out.

Note that $l_r=\langle\varrho, H_{\gamma_r}\rangle=(r-1)p+1+\mu/2$ and that $l_1<\ldots<l_r$. If Λ is a dominant weight, we also have $\Lambda_i\leqslant\Lambda_r$. In fact, since γ_r is the largest root, we have $\gamma_i=\gamma_r-\sum m_i\alpha_i$ with $m_i\geqslant 0$ and the α_i are compact positive roots. Since γ_i, γ_r have the same length, it follows that $H_{\gamma_i}=H_{\gamma_r}-\sum m_i' H_{\alpha_i}$ with $m_i'\geqslant 0$, and thus $\langle\Lambda, H_{\gamma_i}\rangle\leqslant\langle\Lambda, H_{\gamma_r}\rangle$. The condition of Harish-Chandra easily follows ([20a]).

4.4.6. COROLLARY. *If $\Lambda_i\leqslant\Lambda_r$ for all i, then $I(\Lambda;\,\xi_0)$ converges if and only if $\langle\Lambda+\varrho, H_{\gamma_r}\rangle<0$.*

4.4.7. COROLLARY. *Let λ be a character of $\tilde{K}$. Then $I_\lambda(\xi_0)<\infty$ if and only if $\lambda+\langle\varrho, H_{\gamma_r}\rangle<0$ and in this case*

$$I_\lambda(\xi_0)=\prod_{i=1}^{r}\tfrac{1}{2}(2\pi)^{\lambda+l_i}\Gamma(-(\lambda+l_i)).$$

4.4.8. COROLLARY. *$F_\lambda(is)$ converges if and only if $\lambda+\frac{1}{2}(r-1)p<0$, and in this case*

$$\tilde{F}_\lambda(is)=\prod_{i=1}^{r}\tfrac{1}{2}(2\pi)^{\lambda+\frac{1}{2}(i-1)p}\Gamma(-(\lambda+\tfrac{1}{2}(i-1)p)).$$

4.4.9. COROLLARY. *J_λ satisfies condition* (C) *if and only if $\lambda+\frac{1}{2}(r-1)p<0$. In this case F_λ satisfies condition* (P) *and $\lambda\in P$.*

4.5. Certain integrals on the orbit O_e

It remains to consider the integrals (4.3.3): the case $\lambda=-\frac{1}{2}(e-1)p$ with $d\mu_\lambda=d\mu_e^*$. As above, we consider

$$I_e(\xi_0;\Lambda)=\int_{T_e} e^{-2\pi\langle\xi_0,\,h\cdot s_e\rangle}\phi_\Lambda(h)^{-2}(\det\nolimits_{\mathcal{H}_1}h)^{-1}(\det\nolimits_{\mathcal{H}_{1/2}}h)^{-1}dh,$$

so that $\tilde{F}_e(is)=I_e(\xi_0;\Lambda)$ with

$$\Lambda_i=\Lambda_{e-1}=-\frac{(e-1)p}{2}-\frac{2+\mu+p(r-1)}{2},\quad 1\leqslant i\leqslant r. \tag{4.5.1}$$

4.5.2. THEOREM. *$I_e(\xi_0,\Lambda)<+\infty$ if and only if $\Lambda_i+l_i<0$ for $1\leqslant i\leqslant e-1$. In this case*

$$I_e(\xi_0,\Lambda)=\prod_{i=1}^{e-1}(\tfrac{1}{2})(2\pi)^{\Lambda_i+l_i}\Gamma(-\Lambda_i-l_i).$$

Proof. We follow the proof of Theorem 4.4.5, noting that if $h_0\in T_e$,

$$h_0=\prod_{i=1}^{e-1}\exp a_i X_{\gamma_i}\cdot\exp L_1\ldots\exp L_{e-1}$$

(as in 4.4.1), so all the variables corresponding to X_{γ_i}, L_i, $i \geq e$ are missing. Then, simply

$$(4.5.3)\quad I_e(\xi_0, \Lambda) = \prod_{i=1}^{e-1} \int \exp[-2\pi e^{2a_i}] \exp[-a_i(2\Lambda_i + (i-1)p + (r-1)p + 2 + \mu)]\, da_i.$$

4.5.4. COROLLARY. $\tilde{F}_e(is) < +\infty$, $1 \leq e \leq r$.

Proof. Since the l_i are increasing, it suffices to note (using (4.4.4) and (4.5.1), $\Lambda_{e-1} + l_{e-1} = -(p/2) < 0$.

4.6. Description of the set P and the corresponding spaces $\mathcal{H}_0(\lambda)$

It follows from the preceding sections that the set P of λ such that R_{F_λ} is of positive type (see 2.6.4) is as determined by Wallach [24b]:

$$P = \left\{\lambda; \lambda < -\frac{(r-1)p}{2}\right\} \cup \{\lambda_1, \ldots, \lambda_r\},$$

$$\lambda_e = -\frac{(e-1)p}{2}, \quad 1 \leq e \leq r.$$

First, let us consider $\lambda < -\frac{1}{2}(r-1)p$. As in section 4.4, consider

$$J_\lambda(h \cdot \xi_0) = \phi_\lambda(h)^{-2} (\det_{\mathcal{H}_1} h)^{-1} (\det_{\mathcal{H}_{1/2}} h)^{-1}.$$

J_λ satisfies condition (C) (recall (3.3.3)) and

$$F_\lambda(z) = \tilde{F}_\lambda(is)^{-1} \int e^{2\pi i \langle \xi, z \rangle} J_\lambda(\xi)^{-1} \det Q_\xi \, d\xi.$$

Let $\mathcal{H}(\Omega^*, Q; J_\lambda)$ be the space of measurable functions ϕ of ξ, u such that

(a) $\phi(\xi, u)$ is holomorphic in u, for almost all $\xi \in \Omega^*$,

(b) $$\|\phi\|^2_{J_\lambda} = \int |\phi(\xi, u)|^2 e^{-4\pi Q_\xi(u, u)} J_\lambda(\xi)\, d\xi < +\infty.$$

For $\phi \in \mathcal{H}(\Omega^*, Q; J_\lambda)$, let

$$(4.6.1)\quad \check{\phi}(z, u) = \int_{\Omega^*} e^{2\pi i \langle \xi, z \rangle} \phi(\xi, u)\, d\xi.$$

This integral is absolutely convergent for all $(z, u) \in D(\Omega, Q)$ and the correspondence $\phi \to \check{\phi}$ is an isomorphism between $\mathcal{H}(\Omega^*, Q; J_\lambda)$ and $\mathcal{H}_0(\lambda)$ with

$$N_0(\lambda : \check{\phi})^2 = \tilde{F}_\lambda(is) \|\phi\|^2_{J_\lambda}$$

$$(4.6.2)\quad \tilde{F}_\lambda(is) = \prod_{i=1}^{r} \tfrac{1}{2}(2\pi)^{\frac{1}{2}(\lambda + (i-1)p)} \Gamma(-\tfrac{1}{2}(\lambda + (i-1)p)).$$

In particular, if $\lambda=-l_1=-\frac{1}{2}((r-1)p+\mu+2)$, then $J_{-l_1}=1$ and $\phi\to\check{\phi}$ is an isometry of $\mathcal{H}(\Omega^*, Q; J_\lambda)$ onto the Hardy space H^2 of functions holomorphic on $D(\Omega, Q)$ such that

$$\sup_{t\in\Omega}\int|F(x+i(t+Q(u,u)),u)|^2\,dx\,du=\|F\|^2_{H^2}<+\infty.$$

Now, if the additional condition (HC) is satisfied: $\lambda+\langle\varrho, H_{\gamma_r}\rangle<0$, then $\phi\to\check{\phi}$ is an isomorphism of $\mathcal{H}(\Omega^*, Q; J_\lambda)$ with

$$\mathcal{H}(\lambda)=\{F\in O(D);\ N(\lambda, F)^2=\int|F(x+iy,u)|^2\mu_\lambda(y-Q(u,u))\,dx\,dy\,du<\infty\}.$$

and
$$N(\lambda;\check{\phi})^2=I_\lambda(\xi_0)\|\phi\|^2_{J_\lambda},$$

$$I_\lambda(\xi_0)=\prod_{i=1}^{r}\tfrac{1}{2}(2\pi)^{\lambda+\langle\varrho, H_{\gamma_r}\rangle+\frac{1}{2}(i-1)p}\,\Gamma(-(\lambda+\langle\varrho, H_{\gamma_r}\rangle+\tfrac{1}{2}(i-1)p)).$$

Now let us turn to $\lambda_e=-\frac{1}{2}(e-1)p$, $1\leqslant e\leqslant r$, and to describe the space $\mathcal{H}_0(\lambda_e)$ Here

$$F_\lambda(z)=\tilde{F}_e(is)^{-1}\int_{O_e^*}e^{2\pi i\langle\xi, z\rangle}\,d\mu_e^*.$$

For $\xi_e=\sum_{i=1}^{e-1}U_i^*$, the form Q_{ξ_e} has as kernel $W_{\xi_e}=\sum_{i=e}^{r}\eta^{(\alpha_i/2)}$. Let $\mathcal{H}_{1/2}(C_e)=\sum_{i=1}^{e-1}\eta^{(\alpha_i/2)}$, $\mathcal{H}_{1/2}(C_e)^+=\mathcal{H}_{1/2}(C_e)^C\cap\mathcal{H}_{1/2}^+$, and Q_e the restriction of Q to $\mathcal{H}_{1/2}(C_e)^+$. Then $\mathcal{H}(\xi_e)$ (recall definition 3.2.2) is easily identified with the space of holomorphic functions on $\mathcal{H}_{1/2}(C_e)^+$ such that

$$\|F\|^2_{H(\xi_e)}=\int_{\mathcal{H}_{1/2}(C_e)^+}e^{-\langle\xi_e, Q_e(u,u)\rangle}|F(u)|^2\,du<+\infty.$$

Now, $t\to(t^*)^{-1}\cdot\xi_e$ is an isomorphism of the group $T_e\subset H_0$ on the set $T_e\cdot\xi_e$ whose complement in O_e^* is of $d\mu_e^*$-measure zero. Consequently, the correspondence

$$(A\phi)(t,u)=\tilde{\chi}_e(t)^{1/2}\phi((t^*)^{-1}\cdot\xi_e, (t^*)^{-1}\cdot u),\quad t\in T_e,\quad u\in\mathcal{H}_{1/2}(C_e)^+$$

gives an isomorphism of $H(d\mu_e^*, Q)$ (recall 3.2.7) and $L^2(T_e, H(\xi_e))$. For $\phi\in L^2(T_e, H(\xi_e))$, the integral

$$\check{\phi}(z,u)=\int_{T_e}e^{2\pi i\langle\xi_e, t^*z\rangle}\phi(t, t^*u)\,\tilde{\chi}_e(t)^{1/2}\,dt$$

converges absolutely and the correspondence $\phi\to\check{\phi}$ is an isomorphism of $L^2(T_e, \mathcal{H}(\lambda_e))$ on $H_0(\lambda_e)$.

Note that for $\lambda_1=0$, we have $\mathcal{H}_0(\lambda_e)=C$.

5. The space $\mathcal{H}_0(R_\lambda)$ and boundary values of holomorphic functions

5.1. Boundary orbits (work of Korányi and Wolf)

In this section we state some of the results of Korányi and Wolf on boundary orbits which are useful in this paper.

Let us temporarily return to the realization, via the map ζ, of G/K as a bounded domain $\mathcal{D}$ in $\mathfrak{p}^+$ as in section 1.2. Then the action of G on $\mathcal{D}$ extends smoothly to $\bar{\mathcal{D}}$ and the topological boundary $\bar{\mathcal{D}}-\mathcal{D}$ is a union of r G-orbits. There is the following description of these orbits (see [13], [25]). As in 2.1.5, let $c_e = c_{\gamma_e} \cdot c_{\gamma_{e+1}} \dots c_{\gamma_r}$ be the partial Cayley transforms (for $1 \leqslant e \leqslant r$). The formulae 2.1.6 show that $c_e \in P^+ K_C P^-$ so that $\zeta(c_e) \in \mathfrak{p}^+$. Letting $\exp tX_{\gamma_e} = \exp t(E_{\gamma_e} + E_{-\gamma_e})$ act, we see that $\zeta(c_e)$ is in $G \cdot \overline{\zeta(c_{e+1})}$ and the points $\{\zeta(c_e);\ 1 \leqslant e \leqslant r\}$ form a complete system of representatives of the orbits of G on $\bar{\mathcal{D}}-\mathcal{D}$. We have $c_1 = c$ and $G \cdot \zeta(c)$ is the Silov boundary of $\mathcal{D}$.

We now turn to the unbounded realization $D = D(\Omega, Q)$ of G/K given by the Cayley transform, and relate these G-orbits with the orbits described in 4.1. Let $\alpha\colon \mathcal{D} \to D$ be given by $\alpha(x) = c \cdot \zeta(c^{-1} \exp x)$. Since D is unbounded, some points of $\bar{\mathcal{D}}-\mathcal{D}$ are sent to infinity, so the action of G cannot extend to $\bar{\mathcal{D}}$. However, if we recall the group P (2.3.7) of affine automorphisms of D, we can compare an orbit $G \cdot \zeta(c_e)$ with the orbit $P \cdot \zeta(c_e)$. We shall see in the next chapter (section 6.2) that $G \cdot \zeta(c_e) - P \cdot \zeta(c_e)$ is of measure zero, with respect to the (unique) class of quasiinvariant measures on $G \cdot \zeta(c_e)$. Furthermore, by 2.1.6,

$$\alpha(\zeta(c_e)) = c \cdot \zeta(c^{-1} \cdot c_e) = c \cdot \zeta(c_{\gamma_1}^{-1} \dots c_{\gamma_{e-1}}^{-1}) = c\left(\sum_{i=1}^{e-1} E_{\gamma_i}\right) = (is_e, 0).$$

Thus α extends smoothly to $P \cdot \zeta(c_e)$ and sends it onto $P(is_e, 0) = \Sigma_e$. Thus except for an ambiguity on a set of measure zero, the $\{\Sigma_e\}$ are the "orbits" of G in $\bar{\mathcal{D}}-\mathcal{D}$.

Now (recalling 4.1), let $O_e = G(0) \cdot s_e$. Then

$$\begin{aligned} \Sigma_e &= \{(x + i(t + Q(u,u)), u);\quad t \in O_e,\quad x \in \mathcal{H}_1,\quad u \in \mathcal{H}_{1/2}^+\} \\ &= \{(x + iy, u);\ y - Q(u,u) \in O_e\}. \end{aligned}$$

Let $l_1, l_2, \dots, l_r$ be the number defined by (4.4.4). We shall show that for $\lambda = -l_e$, $1 \leqslant e \leqslant r$, we can describe the norm of a function $F \in \mathcal{H}_0(\lambda)$ as the integral of its boundary value on Σ_e. For example: the case $e = 1$ of the Hardy class on the Silov boundary. Here

$$\mathcal{H}_0(-l_1) = H_2 = \left\{F \in O(D);\quad \sup_{t \in \Omega} \int |F(x + i(t + Q(u,u); u))|^2 dx\, du < \infty\right\}.$$

For $t \in \Omega$, $(x, u) \in \Sigma_1$, the Silov boundary of D, and for $F \in H_2$, letting $F_t(x, u) = F(x + i(t + Q(u, u)), u)$, it is known that F_t converges in L^2, as $t \to 0$, to a function $\tilde{F}$ in $L^2(\Sigma_1, dx\,du)$, and $F \to \tilde{F}$ is an isometry of H_2 into $L^2(\Sigma_1, dx\,du)$ [12]. Our result exactly generalizes this theorem, replacing Σ_1 by Σ_e.

5.2. Geometric description of Σ_e and the space $\mathcal{H}_0(-l_e)$ $(e \neq 1)$

Once again, O_e is the $G(0)$-orbit of s_e in $\bar{\Omega} - \Omega$, and

$$\Sigma_e = \{(x + iy, u);\ y - Q(u, u) \in O_e\}.$$

5.2.1. LEMMA. *For $e \neq 1$, the closed convex hull of O_e is $\bar{\Omega}$.*

Proof. Clearly the closed convex hull of an orbit is also $G(0)$-invariant, so it suffices to show that there is a point in Ω which is a convex combination of points in O_e. Given a permutation α of $\{1, ..., r\}$, there is a $\omega_\alpha \in G$ transforming X_{γ_l} to $X_{\sigma(\gamma_l)}$. Thus $\omega_\sigma \in G(0)$, and $\omega_\sigma(s_e) \in O_e$. Now $\sum_\sigma \omega_\sigma(s_e)$ is clearly proportional to $s \in \Omega$, so is also in Ω.

Now let J_e be the function defined by 3.5.3 with $\lambda = -l_e$:

$$J_e(h \cdot \xi_0) = \phi_{-l_e}(h)^{-2}(\det_{\mathcal{H}_1} h)^{-1}(\det_{\mathcal{H}_{1/2}} h)^{-1}. \tag{5.2.2}$$

Since $Tr_{\mathcal{H}_1 + \mathcal{H}_{1/2}} X_{\gamma_i} = 2 + \mu + (r-1)p$ and $2l_e = 2 + \mu + (r-1)p + (e-1)p$, we see that J_e transforms by the character $\chi_e^{-1}(\chi_e(\exp tX_{\gamma_i}) = \exp t(e-1)p)$. Let us dualize the results of the preceding chapter. For any λ, let $f_\lambda^* \in \Omega^*$ be defined by

$$f_\lambda^*(\xi) = \left(\int e^{-2\pi\langle \xi, y\rangle} dy\right)^{-\lambda r/n} \quad (n = \dim \mathcal{H}_1). \tag{5.2.3}$$

Then f_λ^* transforms by the character χ_λ and we have

$$J_e(\xi) = f_\lambda^*(\xi) \quad \text{for } \lambda = -(e-1)p/2. \tag{5.2.4}$$

(We shall simply write $f_{(e)}^*$ for this value of λ. Now in the preceding chapter (i.e., Proposition 4.3.1) we obtained an integral representation for $f_{(e)}$ over the orbit O_e^*. Interchanging the roles of $\mathcal{H}_1$ and $\mathcal{H}_1^*$, ξ and y, we obtain the representation for $f_{(e)}^* = J_e$:

$$J_e(\xi) = c \int_{O_e} e^{-4\pi\langle \xi, t\rangle} d\mu_e(t), \tag{5.2.5}$$

where $d\mu_e$ is the unique $G(0)$-semi-invariant measure on O_e. By Corollary 4.5.4, c is a positive and finite constant.

We return to the space $\mathcal{H}_0(-l_e)$ of holomorphic functions of the form

$$\check{\phi}(z, u) = \int_\Omega e^{2\pi i\langle \xi, z\rangle} \phi(\xi, u)\, d\xi,$$

where $\phi(\xi, u)$ is holomorphic in u for almost all ξ, and

$$\int |\phi(\xi, u)|^2 e^{-4\pi\langle \xi, Q(u,u)\rangle} J_e(\xi)\, d\xi\, du < +\infty.$$

Now Σ_e is naturally parametrized by $\mathcal{H}_1 \times \mathcal{H}^+_{1/2} \times O_e$: we consider the measure $d\sigma_e = dx\, du\, d\mu_e$ on Σ_e. For $t \in \Omega$, $\sum_{e,t} = \{\sigma + (it, 0);\ \sigma \in \Sigma_e\} \subset D$.

5.2.6. THEOREM. *Let* $\mathcal{H}_e = \{F$ *holomorphic in* D,

$$\|F\|_e^2 = \sup_{t\in\Omega}\left\{\int_{\Sigma_{e,t}} |F(z,u)|^2 d\sigma_e < +\infty\right\}.$$

Then $\mathcal{H}_e = \mathcal{H}_0(-l_e)$ *and the norms are proportional.*

Proof. This theorem and the following are direct applications of the results in [20 b]. We need only show that, for $e \neq 1$, $J_e(\xi) = +\infty$ for $\xi \notin \bar{\Omega}^*$. Now, since O_e is a cone, $O_e = R^+ \times S_e$ where R^+ is the group of positive real numbers and S_e is the intersection of O_e with the unit sphere in $\mathcal{H}_1$. Let (r, t) be the coordinates of this product representation. Since $d\sigma_e$ is $G(0)$-semi-invariant and $G(0)$ includes the dilation in R^+, we have for any $s > 0$, $d\mu_e(sr, t) = s^\lambda d\mu_e(r, t)$. It follows easily from the uniqueness of the dilation invariant measure on $\mathbf{R}^+$ that $d\mu_e(r, t) = r^{-(\lambda+1)} dr \times d\nu(t)$, where $d\nu$ is a measure on S_e, thus

$$J_e(\xi = c\int_{O_e} e^{-4\pi\langle \xi, t\rangle} d\mu_e(t) = c\int_{S_e}\left(\int_0^{+\infty} e^{-4\pi r\langle \xi, t\rangle} r^{-(\lambda+1)} dr\right) d\nu(t). \tag{5.2.7}$$

Now if $\xi \notin \bar{\Omega}^*$ we have $\langle \xi, t\rangle < 0$ for some $t \in \bar{\Omega} - \{0\}$, and thus since the closed hull of O_e is $\bar{\Omega}$, we must have $\langle \xi, t\rangle < 0$ for some $t \in O_e$. By linearity and continuity this implies that $\langle \xi, t\rangle < 0$ for t in an open subset U of S_e. Since $d\mu_e$ is semi-invariant, U has positive $d\nu$-measure, and clearly for $t \in U$, the inner integrand in (5.2.7) is infinite. Thus $J_e(\xi) = +\infty$.

Now, as in [20b] the space $\mathcal{H}_e$ can be described in terms of its boundary values on Σ_e. As in that article, we introduce the tangential Cauchy-Riemann operator, to be denoted $\bar{\partial}_e$. This can be defined as follows: for each point $p \in \Sigma_e$, let A_p be the largest subspace of the tangent space $T_p(\Sigma_e)$ which is invariant under multiplication by i. Then we say, for $F \in C^\infty(\Sigma_e)$, *F satisfies the induced Cauchy-Riemann equations*, or $\bar{\partial}_e F = 0$ if the differential of F restricted to A_p is linear over the complexes, for all p.

5.2.8. *Definition.* $\mathrm{H}_e^2=\{F$, measurable on Σ_e; $\int_{\Sigma_e}|F(\sigma)|^2 d\sigma_e<+\infty$, satisfying $\bar{\partial}_e F=0$ in the sense of distributions$\}$.

5.2.9. THEOREM. *For $F\in\mathcal{H}_e$, and $t\in\Omega$, $F|\Sigma_{e,t}=F_t$ converges as $t\to 0$ to a function $\nu_e(F)$ in H_e^2. The correspondence $F\to\nu_e(F)$ is isometric. If $e\neq 1$, or if $e=1$, but G/K is not a tube domain, ν_e is an isometry onto H_e^2.*

This theorem is a corollary of the main theorem in [20 b]; for if $e\neq 1$, we know by Lemma 5.2.1 that O_e generates $\bar{\Omega}$. If $e=1$, and G/K is not a tube domain, then the values of $Q(u,u)$ generate $\bar{\Omega}$ as a convex hull. In 5.2.9 the case $e=1$ was originally proven by Vagi [23]. (When G/K is a tube domain, there are no tangential Cauchy-Riemann equations and therefore no intrinsic defining conditions for the space of boundary values $\nu_e(F)$).

5.3. Realization of D as a Siegel domain of type III

We shall now give a description of Σ_e which is more appropriate to the study of sub-representations of the principal series.

Let $\tilde{O}_e=T_e\cdot s_e=H_0\cdot s_e$. $\tilde{O}_e$ is "almost" equal to O_e (in the sense of the unique quasi-invariant class of measures). Similarly, we consider

$$\tilde{\Sigma}_e=\{(x+iy,u);\ y-Q(u,u)\in\tilde{O}_e\}.$$

Since O_e and O_e generate the same convex cone, it follows from the theorem of [20b] that the restriction from Σ_e to $\tilde{\Sigma}_e$ is a unitary isomorphism which sends H_e^2 onto $\{f\in L^2(\Sigma_e, d\sigma_e)$; $\bar{\partial}_e f=0$ in the sense of distributions$\}$. We shall thus refer to this latter space as H_e^2 also. We now describe the representation of the situation which we are after.

First, we shall find a fibration π: $D\to D_e$, where D_e is a Siegel domain in fewer dimensions. Associated to this fibration is a representation of D as a Siegel domain of type III ([26]; see also [18], 1969 edition):

$$D=\{(p_1,z',u');\ p_1\in D_e,\ \operatorname{Im} z'-L_{p_1}(u',u')\in\Omega'\},$$

where L_{p_1} is the real part of a semi-hermitian form, depending differentiably on p_1 and Ω' is a proper cone in a subspace $V'\subset\mathcal{H}_1$. The fiber map is the projection $\pi_e(p_1,z',u')=p_1$. Letting

$$\Sigma_{p_1}=\{(p_1,z',u')'\quad \operatorname{Im} z'=L_{p_1}(u',u')\}$$

be the Silov boundary of the fiber D_{p_1} above p_1, it is the case that $\tilde{\Sigma}_e=\bigcup_{p_1\in D_e}\Sigma_{p_1}$; i.e., π_e: $\tilde{\Sigma}_e\to D_e$, and the fibers of this projection are the surfaces Σ_{p_1}. In this setup, for $F\in H_e^2$, $F|\Sigma_{p_1}$ is in the Hardy space $H_{p_1}^2$ associated to D_{p_1}, for almost all $p_1\in D_e$.

In order to obtain this description of H_e^2, we will need results of Korányi and Wolf [26], which we shall now expose in our present context. First, we recall the terminology of section 4.1, where $C_e=\{1, ..., e-1\}$, $C_e'=\{e, ..., r\}$. Here it is appropriate for us to minimize the notational complications, thus we shall let $\mathcal{H}_0(e)=\mathcal{H}_0(C_e)$, $\mathfrak{g}_0(e)=\mathfrak{g}_0(C_e)$, $\mathcal{H}_0'=\mathcal{H}_{0,e}'$, where the right hand sides are as defined in section 4.1. In addition, let

$$\mathcal{H}_1(e)=\sum_{i,j\in C_e}\eta^{\frac{1}{2}(\alpha_i+\alpha_j)},\quad \mathcal{H}_{1/2}(e)=\sum_{i\in C_e}\eta^{\frac{1}{2}\alpha_i},\quad \mathcal{H}_{1/2}^+(e)=\mathcal{H}_{1/2}(e)^C\cap\mathcal{H}_{1/2}^+, \tag{5.3.1}$$

and let $\mathcal{H}_0(e')$, $\mathcal{H}_{1/2}(e')$, $H_{1/2}^+(e')$, $\mathcal{H}_1(e')$ be similarly defined with C_e' replacing C_e. Let

$$\mathcal{H}_1'=\sum_{\substack{i\in C_e'\\ j\in C_e}}\eta^{\frac{1}{2}(\alpha_i+\alpha_j)}.$$

Then, we have the decompositions:

$$\mathcal{H}_1=\mathcal{H}_1(e)\oplus\mathcal{H}_1(e')\oplus\mathcal{H}_1',\quad \mathcal{H}_{1/2}^+=\mathcal{H}_{1/2}^+(e)\oplus\mathcal{H}_{1/2}^+(e').$$

Let $\mathcal{H}_{1/2}'=\mathcal{H}_{1/2}(e')\oplus H_0'\oplus\mathcal{H}_1'$; this is the subspace of $\mathfrak{g}$ with eigenvalue $\frac{1}{2}$ for Js_e'. It follows, from computation of the eigenvalues of Js_e' that

$$\begin{aligned}[]
[\mathcal{H}_0(e), \mathcal{H}_0']&\subset\mathcal{H}_0',\\
[\mathcal{H}_0(e), \mathcal{H}_1']&\subset\mathcal{H}_1',\\
[\mathcal{H}_0(e'), \mathcal{H}_0']&\subset\mathcal{H}_0',\\
[\mathcal{H}_0(e'), \mathcal{H}_1']&\subset\mathcal{H}_1'.
\end{aligned}\tag{5.3.2}$$

Let $\mathfrak{b}_e=\mathcal{H}_0(e)\oplus\mathcal{H}_{1/2}(e)\oplus\mathcal{H}_1(e)$, $\Omega_e=G_0(e)\cdot s_e=H_0(e)\cdot s_e$, (recall section 4.1), which is an open proper convex cone in $\mathcal{H}_1(e)$. Let Q_e be the restriction of Q to $\mathcal{H}_{1/2}^+(e)$, and $\xi_e\in\Omega_c^*$ given by $\xi_e=\sum_{i=1}^{e-1}U_i^*$. Finally

$$D_e=D_e(\Omega_e; Q_e)=\{p_1=(z_1, u_1)\in\mathcal{H}_1(e)^C\oplus\mathcal{H}_{1/2}^+(e); \operatorname{Im} z_1-Q(u_1, u_1)\in\Omega_e\}.$$

Let α_e: $B_e\to D(\Omega_e, Q_e)$ be given by $\alpha_e(b_1)=b_1\cdot(is_e, 0)$.

For $p=(z, u)\in D(\Omega, Q)$ write $p=(z_1+z_2+z', u_1+u_2)$ with $z_1\in\mathcal{H}_1(e)^C$, $z_2\in\mathcal{H}_1(e')^C$, $z'\in\mathcal{H}_1'^C$, $u_1\in H_{1/2}^+(e)$, $u_2\in H_{1/2}^+(e')$. Then

$$\pi_e(z, u)=(z_1, u_1). \tag{5.3.4}$$

5.3.5. Lemma. *π_e is a surjective map of D on D_e.*

Proof. Let $\Omega_\pi=\{x\in\mathcal{H}_1(e);$ there is a $t\in\mathcal{H}_1(e')\oplus\mathcal{H}_1'$ such that $x+t\in\Omega\}$. Clearly π_e maps D onto $D(\Omega_\pi, Q_e)$; so we must show $\Omega_\pi=\Omega_e$. Let $s'=\sum_{i=e}^{r}U_i$. Then $\Omega_e+s'=H_0(e)\cdot s_e+s'=H_0(e)\cdot s\subset\Omega$, so $\Omega_e\subset\Omega_\pi$.

Now let $t_1 \in \Omega_\pi$. There are $t_2 \in \mathcal{H}_1(e')$, $t' \in \mathcal{H}_1'$ such that $t_1 + t_2 + t' \in \Omega$. Let $h_1 \in H_0(e)$, $h_2 \in H_0(e')$; then $h_1 h_2(t_1 + t_2 + t') \in \Omega$ also. For $\xi_0 = \sum_{i=1}^{e-1} U_i^* + \sum_{i=e}^{r} U_i^* = \xi_e + \xi_e'$,

$$0 < \langle \xi_0, h_1 h_2(t_1 + t_2 + t') \rangle = \langle \xi_e, h_1 \cdot t_1 \rangle + \langle \xi_e', h_2 \cdot t_2 \rangle,$$

so

$$\langle \xi_e, h_1 \cdot t_1 \rangle + \langle h_2^{-1} \cdot \xi_e', t_2 \rangle > 0.$$

Since we can choose h_2 so that $h_2^{-1} \cdot \xi_e'$ is as close to the origin as we please, we have $\langle \xi_e, h_1 \cdot t_1 \rangle \geqslant 0$ for all $h_1 \in H_0(e)$. Thus $t_1 \in \overline{\Omega}$, since $\overline{\Omega}$ can be described as $\{t \in \mathcal{H}_1(e); \langle \xi_e, H_0(e) \cdot t \rangle \geqslant 0\}$. Thus $\Omega_e \subset \Omega_\pi \subset \overline{\Omega}_e$. Since Ω_π is obviously open, we must have $\Omega_\pi = \Omega_e$.

Now let

$$\mathfrak{b}' = \mathcal{H}_0(e') + \mathcal{H}_{1/2}' + \mathcal{H}_1(e'). \tag{5.3.6}$$

This is the decomposition of $\mathfrak{b}'$ into eigenspaces according to Js_e' with eigenvalues $0, \frac{1}{2}, 1$ respectively. $\mathfrak{b}'$ is invariant under J and is an ideal in $\mathfrak{b}$. Since $\mathcal{H}_{1/2}'$ is J-stable we can define $\mathcal{H}_{1/2}'^{+}$ the eigenspace according to J of eigenvalue $+i$ in $\mathcal{H}_{1/2}'^{C}$. Let Q' be defined by

$$Q'(u', u') = \frac{i}{2}[u', \bar{u}'].$$

On $\mathcal{H}_{1/2}'$ we have $[Ju, Jv] = [u, v]$. Let $\tau'\colon \mathcal{H}_{1/2}' \to \mathcal{H}_{1/2}'^{+}$ be defined by

$$\tau'(u') = \tfrac{1}{2}(u' - iJu').$$

Since $\mathcal{H}_{1/2}' = \mathcal{H}_{1/2}(e') \oplus \mathcal{H}_0' \oplus H_1'$ and $\mathcal{H}_0' = J\mathcal{H}_1'$, we have defined an isomorphism of complex vector spaces: $\mathcal{H}_1'^{C} + \mathcal{H}_{1/2}^{+}(e') \to \mathcal{H}_{1/2}'^{+}$ by $(x' + iy', u_2) \to (\tau'(x' + Jy'), u_2)$. Transporting Q' via this isomorphism we find

$$Q'(x' + iy' + u_2, x' + iy' + u_2) = \frac{1}{4}([Jx', x'] + [Jy', y']) + Q_e'(u_2, u_2).$$

Let $D' = D(\Omega_e'; Q') = \{(z_2, z' + u_2); z_2 \in \mathcal{H}_1(e')^C, z' \in \mathcal{H}_1'^{C}, u_2 \in \mathcal{H}_{1/2}^{+}(e'), \operatorname{Im} z_2 - Q'(z' + u_2, z' + u_2) \in \Omega_e'\}$. The map $\alpha'\colon B' \to D(\Omega_e', Q)$ given by $\alpha'(b') = b' \cdot (is_e', 0)$ defines an isomorphism of B' onto D'.

Since $\mathfrak{b} = \mathfrak{b}_e \oplus \mathfrak{b}'$, and $\mathfrak{b}'$ is an ideal in $\mathfrak{b}$, we can write $B = B(e) \cdot B'$. Identifying B with D via α, $B(e)$ with D_e via α_e, B' with D' via α', this defines a diffeomorphism i of $D_e \times D'$ onto D.

5.3.7. LEMMA. *Let $p_1 = (z_1, u_1) \in D_e$ with $y_1 - Q_e(u_1, u_1) = h_1 \cdot s_e$, $h_1 \in H_0(e)$. Let $p' = (z_2, z' + u_2) \in D'$. Then*

$$i(p_1, p') = (z_1 + z_2 - \frac{1}{4}[Jz', z'] + 2iQ(u_2, u_1) + h_1 \cdot z', u_1 + u_2).$$

Proof. This is a direct calculation, which we omit.

5.3.8. COROLLARY. *$\pi_e(u(p_1, p')) = p_1$. The map $i_{p_1}(p') = i(p_1, p')$ is a diffeomorphism of D' onto the fiber $\pi_e^{-1}(p_1)$.*

Now, let, as in ([20 b], (3.2)), $z'' = h_1^{-1}(z' - 2iQ(u_2, u_1))$ and $S_0(z' + u_2, z' + u_2)$ be the real quadratic form on $\mathcal{H}_1'^C \oplus \mathcal{H}_{1/2}(e')^C$ given by

$$S_0(z' + u_2, z' + u_2) = \tfrac{1}{2}[Jy', y'] + Q(u_2, u_2).$$

For $p_1 = (z_1, u_1)$, and $h_1 \cdot s_e = y_1 - Q(u_1, u_1)$,

$$S_{p_1}(z' + u_2, z' + u_2) = S_0(z'' + u_2, z'' + u_2).$$

Then we have

$$D = \{(z_1 + z_2 + z', u_1 + u_2);\ (z_1, u_1) \in D_e,\quad \operatorname{Im} z_2 - S_{p_1}(z' + u_2, z' + u_2)) \in \Omega_e'\}.$$

This is the realization of D as a Siegel domain of type III.

5.4. The fibering of $\tilde{\Sigma}_e$

The map i extends continuously to a map of $D_e \times \bar{D}$. Let Σ' be the Silov boundary of D', and for $t' \in \Omega'$, $\Sigma'_{t'}$ is the surface of level t' in D':

$$\Sigma'_{t'} = \{(z', u');\ \operatorname{Im} z' - Q'(u', u') = t'\} = \Sigma' + (it', 0).$$

5.4.1. PROPOSITION. *$\tilde{\Sigma}_e = \{(z_1 + z_2 + z',\ u_1 + u_2);\ \operatorname{Im}\ z_2 - S_{p_1}(z' + u_2, z' + u_2) = 0\} = \bigcup_{p_1 \in D_e} i_{p_1}(\Sigma')$. Further, $\tilde{\Sigma}_e + (it', 0) = \bigcup_{p_1 \in D_e} i_{p_1}(\Sigma'_{t'}) \subset D$.*

Proof. Let $N_e = N' = \exp(\mathcal{H}'_{1/2} \oplus \mathcal{H}_1(e')) = \exp(\mathcal{H}'_0 \oplus \mathcal{H}_{1/2}(e') \oplus \mathcal{H}'_1 \oplus \mathcal{H}_1(e'))$. N' acts simply transitively on Σ' and on Σ'_t. The formulae of Lemma 5.3.7 show that

$$i_{p_1}(\Sigma'_{t'}) = \{(z_1 + z_2 + z', u_1 + u_2);\quad \operatorname{Im} z_2 - S_{p_1}(z' + u_2, z' + u_2) = t'\},$$

$$i_{p_1}(\Sigma') = \{(z_1 + z_2 + z', u_1 + u_2);\quad \operatorname{Im} z_2 - S_{p_1}(z' + u_2, z' + u_2) = 0\}.$$

Let $p_1 = b_1 \cdot (is_e, 0)$ and $p' = n' \cdot (it', 0) = n' \cdot h_2(i') \cdot (is'_e, 0)$. By definition, we have

$$i_{p_1}(p') = b_1 \cdot n' \cdot h_2(t') \cdot (i(s_e + s'_e), 0) = b_1 \cdot n' \cdot (i(s_e + t'), 0),$$

which is in $D(\Omega, Q)$. Thus

$$\bigcup_{p_1} i_{p_1}(\Sigma') = B(e) \cdot N' \cdot (is_e, 0).$$

On the other hand, $B = B(e) \cdot N' \cdot H_0(e')$. Since $H_0(e')$ stabilizes $(is_e, 0)$, $\bigcup_{p_1} i_{p_1}(\Sigma') = B \cdot (is_e, 0) = \tilde{\Sigma}_e$, and thus $\tilde{\Sigma}_e$ is the union of the Silov boundaries of the fibers of the fibration π_e. The rest of the proposition is easy.

We return now to the map $\alpha'_e: b \to b\cdot(is_e, 0)$ of B onto $\tilde{\Sigma}_e$. As noted in Proposition 3.1.6 of [20b] $\phi \in C^\infty(\tilde{\Sigma}_e)$ satisfies the tangential Cauchy-Riemann equations if and only if $\phi(b\cdot(is_e, 0))$ is annihilated by all vector fields $r(X)$, $X \in \mathfrak{h}_e^- = \mathcal{H}_0(e') \oplus \mathcal{H}'^-_{1/2} \oplus \mathfrak{b}(e)^-$. We can identify $\tilde{\Sigma}_e$ with $B(e)\cdot N'$ under the map considered above: $i(b_1, n') = b_1\cdot n'\cdot(is_e, 0)$. We also may consider the identification $\tilde{\Sigma}_e \to N'\cdot B(e)$ by $j(n', b_1) = n'\cdot b_1\cdot(is_e, 0)$.

5.4.2. LEMMA. *In the representation j: $N'\cdot B(e) \to \tilde{\Sigma}_e$, the measure $d\sigma_e$ is $\phi_\lambda(b_1)^{-2} db_1 dn'$ ($\lambda = -l_e$).*

Proof. By definition $d\sigma_e = dx\,du\,d\mu_e(t)$ (under the parametrization $\tilde{\Sigma}_e \to N(Q)\cdot T_e\cdot(is_e, 0)$). The lemma follows from a computation based on a variable change.

Let j: $D' \times D_e \to D$ be given by

$$j(b'\cdot(is', 0), b_1\cdot(is_e, 0)) = b'\cdot b_1\cdot(is, 0) = b'\cdot(z_1 + is_e, u_1) = i(b_1\cdot(is, 0), b_1^{-1}b'b_1\cdot(is', 0)).$$

j thus extends continuously to $\bar{D}' \times D_e$ and clearly $\tilde{\Sigma}_e = j(\Sigma' \times D_e)$. Clearly if $p \in \bar{D}'$ is fixed, the map $j_{p'}$: $p_1 \to j(p', p_1)$ is holomorphic in p_1. This j defines a foliation of $\tilde{\Sigma}_e$ whose leaves are complex analytic manifolds isomorphic to D_e (these are the holomorphic arc-components of [26]) and transversal to the fibration π_e. Clearly the tangential Cauchy-Riemann operator includes the Cauchy-Riemann operator along the leaves (however, these are not the only condition). Then

5.4.3. LEMMA. *For $F \in H^2_e$, $F(j(n', p_1))$ is holomorphic in p_1, for almost all $p_1 \in D_e$.*

Since j is *not* biholomorphic, the holomorphic structure on the fibers of π_e pulled back via the map j is not that of D' and varies with p_1. Let D'_{p_1} be the space D' furnished with this structure. (These differing structures are obtained by conjugation by $B(e)$ acting on the subgroup B').

For F a holomorphic function on D, let

$$\|F\|^2_{H_2, p_1} = \sup_{t' \in \Omega'} \int_{\Sigma'} |F(j(\sigma' + it', p_1)|^2 d\sigma',$$

i.e., this is the Hardy norm of F on the space D'_{p_1}. Let $H^2_{p_1}$ be the space of boundary values on Σ_{p_1}. Note that $j(\Sigma'_{t'} \times D_e) = \Sigma_{e,t'}$. Let dm^e be the measure on D_e which corresponds to $\phi_\lambda(b_1)^{-2} db_1$ ($\lambda = -l_e$) under the identification $b_1 \to b_1\cdot(is_e, 0)$, i.e.,

$$dm^e = \phi_\lambda(h_1(y_1 - Q(u_1, u_1)))^{-2} \det_{\mathcal{H}_1(e)} h_1^{-2} \det_{\mathcal{H}_{1/2}(e)} h_1^{-1}\, dx_1\, dy_1\, du_1.$$

Let $\mathcal{H}(D_e, -l_e) = \{F$ holomorphic on D_e such that

$$\int |F(x_1 + iy_1, u_1)|^2 dm^e(p_1) < \infty\}.$$

Thus, for $F \in \mathcal{H}_0(-l_e)$ we have that $F(j(\cdot, p_1))$ is in the Hardy space of D'_{p_1} for all p_1, and $F(j(p', \cdot))$ is in $\mathcal{H}(D_e, -l_e)$ for all $p' \in D'$, and

$$\|F\|^2_{\mathcal{H}_0(-l_e)} = \int \|F(j(\cdot, p_1))\|^2_{H_2, p_1} dm^e(p_1).$$

In conclusion

5.4.4. Theorem. *If $F \in \mathcal{H}_e$, the map $F \to (\nu_e F) j(n', p_1)$ is an isomoetry of $\mathcal{H}_e$ onto a proper subspace of $L^2(N'; \mathcal{H}(D_e, -l_e))$. If we identify $L^2(N', \mathcal{H}(D_e, -l_e))$ as a space of holomorphic functions on D_e with values in $L^2(N')$, this subspace is defined by the property $F(p_1, \cdot) \in H^2_{p_1}$ for all $p_1 \in D_e$.*

6. The space $\mathcal{H}_0(-l_e)$ and invariant subspaces of principal series modules

6.1

First we describe, in a rough way, the purpose of this section. Let λ be a character of K (*not* $\tilde{K}$) and let $O(\lambda)$ be the space of C^∞ functions ϕ defined on G satisfying

$$\phi(gk) = \lambda(k)^{-1}\phi(g),$$
$$r(X)\phi = 0, \quad X \in \mathfrak{p}^-.$$

Such a function can be extended as a holomorphic function on $GK_C P_- = \exp \mathcal{D} K_C P_- \subset G_C$, which transforms on the right under $K_C P_-$ according to the character $\tilde{\lambda}$ which extends λ trivially to P_-. Since the left action of G extends to $\bar{\mathcal{D}}$, left translation by G preserves the subspace of such functions which extend to $\exp \bar{\mathcal{D}} K_C P_-$. Clearly ψ_λ is such a function, and thus every function ϕ in $\mathcal{L}_\lambda$ (recall definition 1.3.4) extends continuously to $\tilde{\phi}$ on $\exp \bar{\mathcal{D}} K_C P_-$. Since $G \cdot c_e \subset \exp \bar{\mathcal{D}} K_C P_-$, we can define, for $\phi \in \mathcal{L}_\lambda$,

$$(A_e\phi)(g) = \tilde{\phi}(g \cdot c_e). \tag{6.1.1}$$

Now let $\mathfrak{w}_e = c_e(\mathfrak{k}^C + \mathfrak{p}^-)$ and let $c_e(\lambda)$ be the character on the algebra $\mathfrak{w}_e$ defined by $\langle c_e(\lambda), c_e(X)\rangle = \langle \tilde{\lambda}, X\rangle$. Clearly, for $\phi \in \mathcal{L}_\lambda$, A_e satisfies

$$r(X) \cdot A_e\phi = -\langle c_e(\lambda), X\rangle \phi, \quad X \in \mathfrak{w}_e, \tag{6.1}$$

$$(A_e\phi)(g \cdot m) = \lambda(m)^{-1}(A_e\phi)(g), \quad m \in M. \tag{6.1.3}$$

Since A_e is defined by right multiplication, it commutes with the left action of G. Thus, "formally" A_e is an interwining operator between the representation "holomorphically induced" from the character λ on $\mathfrak{k}^C + \mathfrak{p}^-$ and the representation "CR-induced" from the character $c_e(\lambda)$ on $\mathfrak{w}_e$. (Note that $\mathfrak{w}_e+\bar{\mathfrak{w}}_e$ is *not* a subalgebra of $\mathfrak{g}^C$, and in fact, generates $\mathfrak{g}^C$ as a Lie algebra).

Now, when $\lambda=-l_e$, the isomorphism P_λ (defined in 2.4) identifies $\mathcal{L}_\lambda$ with a subspace of $\mathcal{H}_0(-l_e)=\mathcal{H}_e$. Similarly, we shall define an isomorphism $\tilde{P}_\lambda$ between the space of functions satisfying (6.1.2,3) and the space of CR boundary values H^2_e. We shall show that these isomorphisms transport A_e into the isometry ν_e of Theorem 5.4.4, while, at the same time, we will see that $L^2(N'; \mathcal{H}(D_e; -l_e))$ can be identified as the space of a representation τ_e belonging to a unitary principal series. We exhibit a proper invariant subspace for τ_e. Furthermore, as l_e is either an integer or a half-integer, all the representations which occur are representations of a group G_1 (between $\tilde{G}$ and G) with a finite center, we can then embed G_1 in a complexified group G_1^C and calculate the preceeding formulae for $A_e\phi$, $\phi\in\mathcal{L}(\lambda)$ in G_1^C. In order to minimize the notation, we will continue as if $G=G_1$.

6.2. Study of the G-orbits on $\bar{\mathcal{D}}$

Let us now write the orbit $G\cdot\zeta(c_e)$ as $\mathcal{D}_e$, and otherwise continue the notations of chapters 4 and 5. $G(e)/K(e)$ (recall 4.1.2) is identified with a bounded domain $\mathcal{D}_e$ in $\mathfrak{p}_e^+\subset\mathfrak{p}^+$ via the map ζ_e. Clearly

$$G(e)\cdot\zeta(c_e)=B(e)\cdot\zeta(c_e)=\mathcal{D}_e=\zeta(c_e)\subset\mathcal{D}\subset\mathfrak{p}^+.$$

We know ([26], see also [25]) that $\mathcal{D}_e+\zeta(c_e)$ is the holomorphic arc-component passing through $\zeta(c_e)$. Let P_e be the maximal parabolic corresponding to the subset S_e of simple roots defined by

$$S_e=S-\{\tfrac{1}{2}(\alpha_e-\alpha_{e-1})\},\quad (e>1),$$

$$S_e=S-\{\alpha_1\},\quad e=1 \text{ in the tube case},$$

$$S_e=S-\left\{\frac{\alpha_1}{2}\right\},\quad \text{otherwise}.$$

Let $\mathfrak{a}_e=\bigcap_{\alpha\in S_e}\operatorname{Ker}\alpha$. Then $\mathfrak{a}_e=RJs'_e$ with $Js'_e=\frac{1}{2}\sum_e^r X_{\gamma_i}$. Let

$$\mathfrak{g}=\mathfrak{g}_e(-1)\oplus\mathfrak{g}_e(-\tfrac{1}{2})\oplus\mathfrak{g}_e(0)\oplus\mathfrak{g}_e(\tfrac{1}{2})\oplus\mathfrak{g}_e(1)$$

be the eigenspace decomposition of $\mathfrak{g}$ under $AdJs'_e$. In particular

$$\mathfrak{g}_e(\tfrac{1}{2})=\mathcal{H}'_{1/2}=\mathcal{H}'_0\oplus\mathcal{H}_{1/2}(e)\oplus\mathcal{H}'_1,$$

$$\mathfrak{g}_e(1)=\mathcal{H}_1(e'),\quad \mathfrak{g}_e(0)=\mathfrak{g}(e)\oplus\mathfrak{g}_0(e').$$

Define

$$\mathfrak{n}_e=\mathfrak{g}_e(\tfrac{1}{2})\oplus\mathfrak{g}_e(1)=\mathfrak{n}',$$

$$\mathfrak{v}_e=\mathfrak{g}_e(-\tfrac{1}{2})\oplus\mathfrak{g}_e(-1)=\mathfrak{v}',$$

$$\mathfrak{p}_e=\mathfrak{g}_e(0)\oplus\mathfrak{v}'=\mathfrak{g}(e)\oplus\mathfrak{g}_0(e')\oplus\mathfrak{v}',$$

and let N', V' be the connected subgroups of G corresponding to $\mathfrak{n}'$, $\mathfrak{v}'$. Let $M_e=\{g\in G; g\cdot Js'_e=Js'_e\}$. M_e has a a Lie algebra $\mathfrak{g}_e(0)$, and (by [1, 26]), $M=M\cdot G(e)\cdot G_0(e')$. M normalizes $G(e)$ and $G_0(e')$. We define $P_e=M_e\cdot V'$. P_e is a maximal parabolic subgroup of G having p_e as its Lie algebra. Every maximal parabolic is conjugate to one of these P_e $(1\leqslant e\leqslant r)$ [26]. In the complexification, we have

$$\mathfrak{g}(e)^C=\mathfrak{k}(e)^C\oplus\mathfrak{p}(e)^+\oplus\mathfrak{p}(e)^-.$$

Let

$$\mathfrak{p}_e^-=\mathfrak{k}(e)^C\oplus\mathfrak{p}(e)^-\oplus\mathfrak{g}_0(e')^C\oplus\mathfrak{v}'^C,$$

$$\mathfrak{w}_e=c_e(\mathfrak{k}^C\oplus\mathfrak{p}^-).$$

6.2.1. PROPOSITION. $\mathfrak{w}_e=\mathfrak{p}_e^-\oplus\mathcal{H}'^-_{1/2}$, *so that* $c_e^{-1}(\mathfrak{p}_e^-)\subset\mathfrak{k}^C\oplus\mathfrak{p}^-$,

and

$$\mathfrak{w}_e\cap\overline{\mathfrak{w}}_e=\mathfrak{p}_e^-\cap\overline{\mathfrak{p}}_e^-=\mathfrak{k}(e)^C\oplus\mathfrak{g}_0(e')\oplus(\mathfrak{v}')^C.$$

Proof. We have $c_e^{-1}(Js'_e)=\frac{1}{2}(\sum_e^r H_{\gamma_i})$. The decompositions of $\mathfrak{k}^C$, $\mathfrak{p}^+$, $\mathfrak{p}^-$ into eigenspaces for $c_e^{-1}(Js'_e)$ are

$$\mathfrak{k}^C=\mathfrak{k}_e(0)\oplus\mathfrak{k}_e(-\tfrac{1}{2})\oplus\mathfrak{k}_e(\tfrac{1}{2}),$$

$$\mathfrak{p}^+=\mathfrak{p}_e^+(0)\oplus\mathfrak{p}_e(\tfrac{1}{2})\oplus\mathfrak{p}_e(1),$$

$$\mathfrak{p}^-=\mathfrak{p}_e^-(0)\oplus\mathfrak{p}_e(-\tfrac{1}{2})\oplus\mathfrak{p}_e(-1).$$

We first establish the

6.2.2. LEMMA. *Writing* $\mathcal{H}'^C_{1/2}=\mathcal{H}'^+_{1/2}\oplus\mathcal{H}'^-_{1/2}$, *we have* $\mathcal{H}'^+_{1/2}=c_e(\mathfrak{p}_e(\frac{1}{2}))$, $\mathcal{H}'^-_{1/2}=c_e(\mathfrak{k}_e(\frac{1}{2}))$.

Proof. $\mathcal{H}'^C_{1/2}=c_e(\mathfrak{k}_e(\frac{1}{2})\oplus\mathfrak{p}_e(\frac{1}{2}))$. Now if α is a root with restriction to $\mathfrak{h}_r$ given by one of

$$\tfrac{1}{2}(\gamma_i-\gamma_j),\ i\in C'_e,\ j\in C_e;\quad \tfrac{1}{2}(\gamma_i+\gamma_j),\ i\in C'_e,\ j\in C_e;\quad \tfrac{1}{2}\gamma_i,\ i\in C'_e,$$

the formula for c_{e_r}, analogous to (2.1.8) shows that $c_e(X)$ is proportional to $X+[\sum_e^r E_{-\gamma_i}, X]$. Thus, if $X\in\mathfrak{p}_e(\frac{1}{2})$,

$$c_e(X)\in(\mathfrak{k}^C\oplus\mathfrak{p}^+)\cap\mathcal{H}'^C_{1/2}=\mathcal{H}'_{1/2},$$

and if $\quad X\in\mathfrak{k}_e(\frac{1}{2}),\quad c_e(X)\in(\mathfrak{k}^C\oplus\mathfrak{p}^-)\cap\mathcal{H}'^C_{1/2}=\mathcal{H}'^-_{1/2}.$

We return to the proposition. Since c_e is the identity on $\mathfrak{g}(e)$, and coincides with c on $\mathfrak{g}_0(e')$, we see that $c_e^{-1}(\mathfrak{k}(e)^C\oplus\mathfrak{p}(e)^-\oplus\mathfrak{g}_0(e')^C)\subset\mathfrak{k}^C+\mathfrak{p}^-$. On the other hand, since $\mathfrak{v}'$ is the sum of the eigenspaces with negative eigenvalues for $\mathrm{Ad}Js'_e$, it follows that $c_e^{-1}(\mathfrak{v}')\subset\mathfrak{k}^C\oplus\mathfrak{p}^-$, Thus $c_e^{-1}(\mathfrak{p}_e^-\oplus\mathcal{H}'^-_{1/2})\subset\mathfrak{k}^C\oplus\mathfrak{p}^-$. But $\mathfrak{g}=\mathfrak{p}(e)^+\oplus\mathfrak{g}_e(1)\oplus\mathcal{H}^+_{1/2}\oplus\mathfrak{p}_e^-\oplus\mathcal{H}'^-_{1/2}$ and $c_e^{-1}(\mathfrak{p}(e)^+\oplus\mathfrak{g}_e(1)\oplus\mathcal{H}'^+_{1/2})\subset\mathfrak{p}^+$.

The following proposition is known.

6.2.3. PROPOSITION ([26], see also [25]). *The subgroup of G which leaves $\mathcal{D}_e+\zeta(c_e)$ invariant is the subgroup P.*

We deduce

6.2.4. LEMMA. *The stabilizer S_e of $\zeta(c_e)$ in G is the group $M\cdot K(e)\cdot G_0(e')\cdot V'$.*

Proof. It is easy to verify that S_e contains the above group. Now, if g fixes $\zeta(c_e)$, it leaves invariant its holomorphic arc-component $\mathcal{D}_e+\zeta(c_e)$, and thus $g\in P_e=B(e)\cdot M\cdot K(e)\cdot G_0(e')\cdot V'$. Since $B(e)$ acts simply (without fixed points) on $\mathcal{D}_e+\zeta(c_e)$, the lemma follows.

We may now give the proof promised in chapters 4 and 5.

6.2.5. COROLLARY. (See Lemma 4.1.4). *The stabilizer S_e^0 of s_e in $G(0)$ is $S_e^0=G_0(e')\cdot K_0(e)\cdot M\cdot\exp V_{0,e}$.*

Proof. If $g\in G(0)$ leaves s_e fixed, then it also leaves $\zeta(c_e)$ fixed, so it is in S_e. But $S_e\cap G(0)$ is clearly the group on the right.

And now the density asserted in section 5.1:

6.2.6. PROPOSITION. *$B\cdot\zeta(c_e)$ has a complement of zero measure with respect to the unique class of quasi-invariant measures on $G\cdot\zeta(c_e)=O_e$.*

Proof. $G=K\cdot P_e=K\cdot B(e)\cdot S_e=K\cdot G(e)\cdot S_e$, and thus the measure

$$d\mu\colon \phi\to\int\phi(k\cdot g\cdot\zeta(c_e))\,dk\,dg$$

is a quasi-invariant measure on O_e. Now if $k\in N'\cdot P_e$ then $k\cdot G(e)\cdot\zeta(c_e)\in B\cdot\zeta(c_e)$ since

$$N'(\mathcal{D}_e+\zeta(c_e))=N'\cdot B(e)\cdot\zeta(c_e)=B\cdot\zeta(c_e).$$

But $N'\cdot P_e$ has a complement of zero measure for dg and $(N'\cdot P_e)\cap K$ has a complement of zero measure in K. It follows that $((N'\cdot P_e)\cap K)\cdot G(e)\cdot\zeta(c_e)\subset B\cdot\varrho(c_e)$ has a complement of zero measure with respect to $d\mu$.

We now turn to the map α which transforms the bounded realization of G/K into the realization as $D(\Omega;Q)$: α extends continuously to $P\cdot\zeta(c_e)$ and sends $P\cdot\zeta(c_e)$ onto $P\cdot(is_s,0)=\sum_e$. Thus $\sum_e$ is "almost" the transform (under α) of the orbit $G\cdot\zeta(c_e)$ on $\bar{\mathcal{D}}-\mathcal{D}$.

6.3. An irreducible unitary representation of P_e

For $\lambda=-l_e$, extended trivially to $\mathfrak{p}^-$, let $c_e(\lambda)$ represent the character of $\mathfrak{p}_e^-$ defined by

$$\langle c_e(\lambda),X\rangle=\langle\lambda,c_e^{-1}(X)\rangle.$$

We consider the space

$\mathcal{C}(\lambda, \mathfrak{p}_e^-; P_e) = \{\phi$, C^∞ functions on P_e; $r(X)\phi = -\langle c_e(\lambda), X\rangle\phi$, for all $X \in \mathfrak{p}_e^-$, $\phi(gm) = \lambda(m)^{-1}\phi(g)$, $g \in P_e$, $m \in M\}$. Such a function is completely determined by its restriction to $B(e)$. Introduce the norm

$$\|\phi\|^2 = \int_{B(e)} |\phi(b)|^2 db = \int_{G(e)} |\phi(g)|^2 dg,$$

and let $H(\lambda; P_e)$ be the Hilbert space of norm-finite functions. The correspondence

$$(P_\lambda^e F)(b_1) = \phi_{\lambda_e}(b_1)^{-1} \cdot F(\alpha_e(b_1))$$

defines a unitary isomorphism between $\mathcal{H}(D_e, -l_e)$ and $H(\lambda; P_e)$. Let w_e be the representation of P_e on this space given by left translation. Let $\varrho_e(X) = \frac{1}{2}\mathrm{Tr}_{\mathfrak{p}_e}\mathrm{ad}X$ for $X \in \mathfrak{p}_e$, and let ϱ_e represent also the corresponding character on P_e.

6.3.1. Proposition. *$\varrho_e^{-1} \otimes w_e$ is an irreducible unitary representation of P_e which coincides on $G(e)$ with the holomorphic discrete series of $G(e)$ corresponding to the character $-l_e$ of $\mathfrak{k}(e)$ and is trivial on $G_0(e')\,V'$.*

Proof. For $g \in G(e)$, $\phi \in \mathcal{C}(\lambda; p_e^-, P_e)$, we have

$$(w_e(g_0)\phi)(g) = \phi(g_0^{-1}g), \quad g_0 \in G(e),$$
$$(w_e(m)\phi)(g) = \lambda(m)\phi(m^{-1}g_0 m), \quad m \in M,$$

and w_e is trivial on $[G_0(e'), G_0(e')]\,V'$. On the other hand, if $i \geqslant e$, $X_{\gamma_i} \in \mathfrak{g}_0(e')$ and $(w_e(\exp tX_{\gamma_i})\phi)(g) = e^{-tl_e}\phi(g)$. However

$$-\varrho_e(X_{\gamma_i}) = -\tfrac{1}{2}\mathrm{Tr}_{\mathfrak{p}_e} X_{\gamma_i} = \tfrac{1}{2}\mathrm{Tr}_{\mathfrak{n}'} X_{\gamma_i} = \tfrac{1}{2}\mathrm{Tr}_{\mathcal{H}_0'} X_{\gamma_i} \oplus \tfrac{1}{2}\mathrm{Tr}_{\mathcal{H}_{1/2}} X_{\gamma_i} + \tfrac{1}{2}\mathrm{Tr}_{\mathcal{H}_1} X_{\gamma_i}$$
$$= \tfrac{1}{2}((e-1)p + \mu + 2 + (r-1)p) = l_e.$$

The proposition easily follows from these formulae.

6.4. The corresponding principal series representation

The representation $\tau_e = \mathrm{Ind}_{P_e \uparrow G}(\varrho_e^{-1} \otimes w_e)$ is then a representation in the principal series for G. Let us make this representation explicit. Let $\mathcal{K}(w_e, G)$ be the space of continuous functions on G with compact support modulo P_e, and values in $H(\lambda; P_e)$, verifying $\phi(gp) = w_e(p)^{-1}\phi(g)$. For such a function

$$\|\phi(gp)\|^2_{H(\lambda;\, P_e)} = \chi_e(p)\,\|\phi(g)\|^2_{H(\lambda;\, P_e)},$$

with $\chi_e(p) = |\det \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_e} p|$.

As in the notation of ([2], chapter 5) we form the norm

$$\oint_{G/P_e} \|\phi(g)\|^2 dg = \|\phi\|^2,$$

and τ_e is realized by left translation in the Hilbert space completion $H(\tau_e)$ of $\mathcal{K}(w_e, G)$ in this norm. Since $G = N' \cdot P_e$ but for a set of measure zero, this norm is given by $\int_{N'} \|\phi(n')\|^2 dn'$, and thus $H(\tau_e)$ can be realized as $L^2(N'; H(\lambda; P_e)) = L^2(N'; H(D_e; -l_e))$, using yet another isometry. Thus

6.4.1. LEMMA. *Define I on $\mathcal{K}(w_e; G)$ by*

$$(I\phi)(n, b_1) = \phi_\lambda(b_1)(\phi(n))(b_1).$$

I induces a unitary transformation of $H(\tau_e)$ onto $L^2(N'; \mathcal{H}(D_e; -l_e))$.

Now, the Lie algebra of S_e is $\mathfrak{S}_e = \mathfrak{p}_e^- \cap \mathfrak{g}$. Let us consider the one-dimensional unitary representation α_λ of S_e, which is trivial on $G_0(e')V'$ and equal to λ on $M \cdot K(e)$. As α_λ is compatible with $c_e(\lambda)$ on S_e, we can form the holomorphically induced representation of P_e, corresponding to $(\alpha_\lambda, c_e(\lambda))$, i.e., it is the subspace of the induced representation $\mathrm{Ind}_{S_e \uparrow P_e} \alpha_\lambda$ formed by the the functions on P_e satisfying in addition $r(X)\phi = -\langle c_e(\lambda), X\rangle \phi$ for X in $\mathfrak{p}_e^-$ (see the exact definition in [2], chapter 5). Clearly this is the representation $\varrho_e^{-1} \otimes w_e$.

From the property of transitivity of holomorphically induced representations ([2], chapter 5), we see that the space $H(\tau_e)$ is the completion of the space $\mathcal{K}(\lambda, \mathfrak{p}_e^-, G)$ of C^∞ functions ϕ on G satisfying

$$r(X)\phi = -\langle \lambda, c_e^{-1}(X)\rangle \phi, \quad X \in \mathfrak{p}_e^-,$$

$$\phi(gm) = \lambda(m)^{-1}\phi(g), \quad g \in G, \quad m \in M,$$

in the norm

$$\|\phi\|^2 = \int |\phi(n' b_1)|^2 dn' \, db_1.$$

The correspondence I is written, in this realization, on $\mathcal{K}(\lambda, \mathfrak{p}_e^-, G)$ as $(I\phi)(n')(b_1) = \phi_\lambda(b_1)\phi(n' b_1)$.

6.4.2. THEOREM. *The map $F \to \nu_e(F)$ of Theorem 5.4.4 is an operator intertwining the representation of G on $\mathcal{H}_0(-l_e)$ with a proper subspace of the representation τ realized in $L^2(N', \mathcal{H}(D_e - l_e))$.*

Proof. Consider A_e as in the beginning of this chapter. As $\mathfrak{p}_e^- \subset \mathfrak{w}_e$, we have $A_e(\mathcal{L}(\lambda)) \subset \mathcal{K}(\lambda; \mathfrak{p}_e^-; G)$ and with all the conventions here established, it is easy to verify that the diagram

$$\begin{array}{ccc} \mathcal{L}(\lambda) & \longrightarrow & \mathcal{H}_e \\ \Big\downarrow A_e & & \Big\downarrow \nu_e \\ \mathcal{K}(\lambda; \mathfrak{p}_e^-; G) & \longrightarrow & L^2(N'; \mathcal{H}(D_e; -l_e)) = H(\tau_e) \end{array}$$

is commutative. Since A_e commutes with left translations, and the left sides are dense in the spaces on the right, the theorem follows.

6.5. Spaces of C.R. functions and subspaces of some principal series

If $e \neq 1$, or if $e = 1$ and G/K is not a tube domain, we can give a characterization by first order differential equations of the proper subspace of $H(\tau_e)$ obtained as $\nu_e(\mathcal{H}_e)$. Let $\mathcal{K}(\lambda; \mathfrak{w}_e; G)$ be the subspace of functions in $\mathcal{K}(\lambda; \mathfrak{p}_e^-; G)$ satisfying $r(X)\phi = -\langle \lambda, c_e^{-1}(X)\rangle \phi$, $X \in \mathfrak{w}_e$. Let $H(\lambda; \mathfrak{w}_e; G)$ be the closure of $\mathcal{K}(\lambda; \mathfrak{w}_e; G)$ in $H(\tau_e)$.

6.5.1. Proposition. *The map I defines an isomorphism of $H(\lambda; \mathfrak{w}_e; G)$ with H_2^e.*

Proof. Since $\mathfrak{w}_e \cap \mathfrak{b}^C = \mathfrak{h}_e^-$ (see 5.4), the image of $K(\lambda; \mathfrak{w}_e; G)$ clearly satisfies the tangential Cauchy-Riemann equations on $\tilde{\Sigma}_e$. Thus, $I: H(\lambda; \mathfrak{w}_e; G) \to H_2^e$. On the other hand, if $e \neq 1$, or if $e = 1$ and G/K is not a tube domain, the representation of G in H_2^e is irreducible ($H_2^e \cong \mathcal{H}_0(-l_e)$), so I is surjective. The remaining case is trivial.

Thus, if $e \neq 1$, or $e = 1$ and G/K is not a tube domain, we obtain an irreducible proper invariant subspace of $H(\tau_e)$ by adding to the defining equations coming from $\mathfrak{p}_e$ those coming from all of $\mathfrak{w}_e = \mathfrak{p}_e^- \oplus \mathcal{H}'^-_{1/2}$. These are the tangential Cauchy-Riemann equations of Σ_e. In the case $e = 1$, G/K a tube domain, the tangential Cauchy-Riemann equations are trivial; but the space in question is nevertheless well-understood: it is the Hardy space [12]. We shall say that $\mathfrak{w}_e$ is a CR-polarization. Here $\mathfrak{w}_e \oplus \bar{\mathfrak{w}}_e$ is *not* a Lie algebra; on the contrary, it generates all of $\mathfrak{g}^C$ as a Lie algebra. We shall say that the representation of G in $H(\lambda; \mathfrak{w}_e; G)$ is a CR-induced representation which selects an irreducible proper subspace of the holomorphically induced representation $H(\lambda; \mathfrak{p}_e^-; G)$. Thus, it is seen that it is sometimes necessary to cut down induced representations by algebras more general than polarizations.

6.6. A simple example

The representation τ_1 (corresponding to $\lambda = -l_1$) is a representation induced by a unitary character of P_1. Let $G = \mathrm{Sp}(n; R) = \{g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$; A, B, C, D $n \times n$ matrices verifying

${}^tA\circ C={}^tC\circ A,\ {}^tD\circ B={}^tB\circ D,\ {}^tD\circ A-{}^tB\circ C=Id\}$. Then the representation τ_1 is realized in the space $L^2(V)$ of real symmetric $(n\times n)$-matrices via the formula

$$(T(g^{-1})F)(X)=\det\,(CX+D)^{-(n+1)/2}F((AX+B)\cdot(CX+D)^{-1})$$

(if $n+1$ is not even, this is a representation of the metaplectic group). The representation is reducible since the Hardy space H_2 is a proper subspace.

In particular, if 4 divides $n+1$, this representation is the quasi-regular representation induced by the parabolic $P=\left\{\begin{pmatrix}A & B\\ O & D\end{pmatrix}\right\}$. Thus one finds examples of representations induced by the identity representation of a parabolic, which are reducible (see [6]).

It would be interesting to study the decomposition of this representation into irreducible ones.

More generally, it will be interesting to study the decomposition of the representation τ_1, i.e., the decomposition of the action of G in $L^2(\Sigma_1)$ where $\Sigma=\Sigma_1$ is the Silov boundary. In a subsequent article we shall consider the decomposition of the action of B in $L^2(\Sigma)$.

References

[1]. Baily, W. & Borel, A., Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math.*, (2) 84 (1966), 442–528.

[2]. Bernat, P., et al., *Représentations des groupes de Lie résolubles.* Monographies de la Société Mathématique de France, Dunod, Paris, 1972.

[3]. Dixmier, J., *Algèbres enveloppantes.* Gauthier-Villars, Paris, 1974.

[4]. Ehrenpreis, L., Group Representations and hyperbolic differential equations.

[5]. Gindikin, S. G., Analysis on homogeneous domains. *Russian Math. Surveys,* 19 (1964), 1–89.

[6]. Gross, K., Restriction to a parabolic subgroup and irreducibility of degenerate principal series of Sp(2; C). *Bull. Amer. Math. Soc.*, 76 (1970) 1281–1285.

[7]. Gross, K. & Kunze, R., Generalized Bessel transforms, and unitary representations. *Harmonic analysis on hom. spaces;* Proceedings of Sumposia in Pure Mathematics, 26, A.M.S., 1973.

[8]. Harish-Chandra, Representations of semi-simple Lie groups.
IV. *Amer. J. Math.*, 77 (1955) 743–777.
V. *Amer. J. Math.*, 78 (1956) 1–41.
VI. *Amer. J. Math.*, 78 (1956) 564–628.

[9]. Helgason, S., *Differential Geometry and Symmetric Spaces*, Academic, New York, 1962.

[10]. Knapp, A. & Okamoto, K., Limits of holomorphic discrete series, *J. Funct. Anal.*, 9 (1972) 375–409.

[11]. Korányi, A., *Holomorphic and harmonic functions on bounded symmetric domains.* Centro Internazionale Matematico Estivo (C.I.M.E.), III Ciclo, Urbino, 1962. Edizioni Cremonese, Roma, 1968.

[12]. Korányi, A. & Stein, E., H²-spaces of generalized half planes. *Studia Mathematica* 44 (1972), 379–388.

[13]. KORÁNYI, A. & WOLF, J., Realization of Hermitian symmetric spaces as generalized half planes. *Ann. of Math.*, 81 (1965) 575–596.

[14]. KUNZE, R., On the irreducibility on certain multiplier representations. *Bull. Amer. Math. Soc.*, 68 (1962) 93–94.

[15]. —— Positive definite operator valued kernels and unitary representations, *Prof. Conf. Functional Anal., Irvine,* (ed. Gelbaum).

[16]. MOORE, C. C., Compactification of symmetric spaces II. *Amer. J. Math.*, 86 (1964) 358–378.

[17]. NUSSBAUM, E. A., The Hansdorff-Bernstein-Widder theorem for semigroups in locally compact Abelian groups. *Duke Math. J.*, 22 (1955) 573–582.

[18]. PIATECKII-SAPIRO, I. I., *Geometry of Classical Domains and the Theory of Automorphic Functions*, Fizmatgiz, Moscow, 1961; Dunod, Paris, 1966; Gordon and Breach, New York, 1969.

[19]. PUKANSZKY, L., The Plancherel Formula for the Universal Covering group of SL(R; 2). *Math. Ann.* 156 (1964), 96–143.

[20]. ROSSI, H. & VERGNE, M.
(a) Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semi-simple Lie group, *J. Funct. Anal.*, 13 (4) (1973), 324–389.
(b) To appear.

[21]. ROTHAUS, O., Domains of positivity. *Abh. Math. Sem, Univ. Hamburg,* (24) (1960) 189–235.

[22]. SALLY, P., *Analytic continuation of the irreducible unitary representation of the universal covering group of* $SL(2; R)$. Memoires of the Amer. Math. Soc., 69, Providence, R.I., 1967.

[23]. VAGI, S., On the boundary values of holomorphic functions. *Rev. Un. Mat. Argentina,* 25 (1970), 123–136.

[24]. WALLACH, N.
(a) Induced Representations of Lie algebras II. *Proc. Amer. Math. Soc.*, 21 (1969), 161–166.
(b) Analytic continuation of the discrete series (I), (II). To appear.

[25]. WOLF, J., Fine structure of hermitian symmetric spaces, *Symmetric Spaces, Short Lectures.* Marcel Dekker, New York, 1972.

[26]. WOLF, J. & KORANYI, A., Generalized Cayley transformations of bounded symmetric domains. *Amer. J. Math.*, 87 (1965) 899–939.

[27]. BOURBAKI, *Integration*, Hermann, Paris, 1965, 67.

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