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GROUP REPRESENTATIONS ON HILBERT SPACES DEFINED IN TERMS OF $\bar{\partial}_b$-COHOMOLOGY ON THE SILOV BOUNDARY OF A SIEGEL DOMAIN

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Let Q be a C^n-valued quadratic form on C^m. Let $N(Q)$ be the 2-step nilpotent group defined on $R^n \times C^m$ by the group law

$$(x, u)\cdot(x', u') = (x + x' + 2\,\mathrm{Im}\, Q(u, u'), u + u') .$$

Then $N(Q)$ has a faithful representation as a group of complex affine transformations of C^{n+m} as follows:

$$g\cdot(z, u) = (z + x_0) + i(2Q(u, u_0) + Q(u, u_0), u_0 + u_0) ,$$

where $g = (x_0, u_0)$. The orbit of the origin is the surface

$$\Sigma = \{(z, u) \in C^{n+m};\ \mathrm{Im}\, z = Q(u, u)\} .$$

This surface is of the type introduced in [11], and has an induced $\bar{\partial}_b$-complex (as described in that paper) which is, roughly speaking, the residual part (along Σ) of the $\bar{\partial}$-complex on C^{n+m}. Since the action of $N(Q)$ is complex analytic, it lifts to an action on the spaces E^q of this complex which commutes with $\bar{\partial}_b$. Since the action of $N(Q)$ is by translations, the ordinary Euclidean inner product on C^{n+m} is $N(Q)$-invariant, and thus $N(Q)$ acts unitarily in the L^2-metrics on $C_0^\infty(E^q)$ defined by

$$\|\Sigma a_I d\bar{u}_I\|^2 = \int_\Sigma \Sigma |a_I|^2 dV$$

where dV is ordinary Lebesgue surface measure. In this way we obtain unitary representations ρ_q of $N(Q)$ on the square-integrable cohomology spaces $H^q(E)$ of the induced $\bar{\partial}_b$-complex.

These are generalizations of the so-called Fock or Segal-Bargmann representations [2, 4, 10, 13], and the representations studied by Carmona [3]. In this paper, we explicitly determine these representations and exhibit operators which intertwine the ρ_q with certain direct integrals of the Fock representations.

This is accomplished by means of a generalized Paley-Wiener theorem arising out of Fourier-Laplace transformation in the x ($\mathrm{Re}\, z$) variable. Let us describe this result. For $\xi \in R^{n*}$, let $Q_\xi(u, v) = \langle \xi, Q(u, v)\rangle$. Let $H^q(\xi)$ be the square-integrable cohomology of the $\bar{\partial}$-complex on C^m relative to the norm

$$\left\| \sum_I a_I d\bar{u}_I \right\|_\xi^2 = \sum_I \int |a_I|^2 e^{-2Q\xi(u,u)} du \, .$$

Let $U_q = \{\xi \in R^{n*};$ the quadratic form Q_ξ has q negative and $n - q$ positive eigenvalues$\}$. Let $U = \bigcup U_q$.

THEOREM. *For $\xi \in U$, $H^q(\xi) \neq \{0\}$ if and only if $\xi \in U_q$. In particular the fibration $H^q(\xi) \rightarrow \xi$ is a* (locally trivial) *Hilbert fibration on U_q, and the following result holds!*

THEOREM. *Let $H^q(F)$ be the space of square-integrable sections of the fibration $H^q(\xi) \rightarrow \xi$ over U_q. Then the Fourier-Laplace transform, defined for functions by*

$$\hat{a}(\xi, u) = \int a_I(x + iQ(u, u), u) e^{-i\langle \xi, x + iQ(u,u)\rangle} dx$$

induces an isometry of $H^q(E)$ with $H^q(F)$.

Furthermore, this transform followed by a suitable variable change (in C^m, dependent on ξ) is the sought-for intertwining operator.

2. A Paley-Wiener theorem for $\bar{\partial}_b$-cohomology on certain homogeneous surfaces. Let Q be a nondegenerate C^n-valued hermitian form defined on C^m. That Q is nondegenerate means that the only solution of

$$Q(u, v) = 0 \quad \text{for all} \quad u \in C^m$$

is $v = 0$. Equivalently, there is a $\xi \in R^{n*}$ sudh that the C-valued form

$$Q_\xi(u, v) = \langle \xi, Q(u, v) \rangle \tag{2.1}$$

is nondegenerate. Given such a Q we introduce the real submanifold of C^{n+m}:

$$\Sigma = \Sigma(Q) = \{(z, u) \in C^{n+m}; \operatorname{Im} z = Q(u, u)\} \, . \tag{2.2}$$

Let $N(Q)$ be the 2-step nilpotent group defined on $R^n \times C^m$ by the group law

$$(x, u) \cdot (x', u') = (x + x' + 2 \operatorname{Im} Q(u, u'), u + u') \, . \tag{2.3}$$

Then $N(Q)$ has a faithful realization in the group of complex affine transformations of C^{n+m} as follows

$$(z, u) \xrightarrow{(x_0, u_0)} (z + x_0 + i(2Q(u, u_0) + Q(u_0, u_0)), u + u_0) \, , \tag{2.4}$$

so that Σ is the orbit of 0. The correspondence $N(Q) \rightarrow \Sigma$ given by

$g \to g\cdot 0$, $(x, u) \to (x + iQ(u, u), u)$, is a diffeomorphism, and in certain contexts we may identify $N(Q)$ with Σ under this correspondence. If we let dx, du represent Lebesgue measure in R^n, C^m, then $dxdu$ is the Haar measure of $N(Q)$. We shall return, in §4, to the study of representations of $N(Q)$ connected with its realization as Σ; in this and the next section we shall carry out the relevant analysis.

Σ is a surface of the type studied in [11], Chapter I, (with $V = \{0\}$). Here we shall summarize the relevant results in that paper.

Let $A \to \Sigma$ be the complex vector bundle of antiholomorphic tangent vectors along Σ, and $E^q = \Lambda^q A^*$ the bundle of q-forms on A. For $V \to \Sigma$ any vector bundle we shall let $C^\infty(V)$ represent the sheaf of C^∞ sections of V. Let $\bar\partial_b\colon C^\infty(E^q) \to C^\infty(E^{q+1})$ be the differential operator induced (as in [10]) by exterior differentiation. The complex $(E^q, \bar\partial_b)$ is referred to as *the $\bar\partial_b$-complex on Σ.*

We can make this complex explicit as follows. Let $z_1, \cdots, z_k, \cdots, z_n, u_1, \cdots, u_\alpha, \cdots, u_m$ be coordinates for $C^n \times C^m$. Then, the (restrictions of the) forms $d\bar u_{\alpha'}$, $1 \leqq \alpha \leqq$ m form a basis for E^1. The dual vectors U_α, $1 \leqq \alpha \leqq m$ giving a basis for A are as follows:

$$U_\alpha = \frac{\partial}{\partial \bar u_\alpha} + i \sum_k Q_k(u, E_\alpha) \frac{\partial}{\partial x_k} \tag{2.5}$$

where $Q_k = z_k \circ Q$ and $\{E_\alpha\}$ is the basis of C^m dual to the coordinates u_α.

Then E^q has as basis the forms $\{d\bar u_I;\ I = (i_1, \cdots, i_q)$, with $i_1 < \cdots < i_q\}$. Any q-form is written

$$\omega = \sum_{|I|=q}{}' a_I d\bar u_I\ , \tag{2.6}$$

where Σ' refers to summation only over those q-tuples in increasing order. If J is an arbitrary q-tuple, $[J]$ will refer to the same q-tuple written in increasing order, and ε_J is the sign of the permutation $J \to [J]$. We define the coefficients a_J of ω for unordered q-tuples by $a_J = \varepsilon_J a_{[J]}$. Now, in this notation we have

$$\begin{aligned} \bar\partial_b \omega &= \sum_{|I|=q}{}' \sum_{\alpha=1}^{m} U_\alpha(a_I) d\bar u_\alpha \wedge d\bar u_I \\ &= \sum_{|J|=q+1} \Big(\sum_{\alpha=1}^{m} \varepsilon_J^{\alpha I} U_\alpha(a_I) \Big) d\bar u_I\ , \end{aligned} \tag{2.7}$$

where $\varepsilon_J^{\alpha I} = 0$ if $\alpha I \neq J$ set theoretically, and $\varepsilon_J^{\alpha I} = \varepsilon_{\alpha I}$ otherwise.

Now, we turn to $R^{n*} \times C^m$. We shall refer to the coordinate of R^{n*} by ξ. Let A_u be the vector bundle on $R^{n*} \times C^m$ of antiholomorphic vector fields along the C^m-leaves: the leaves $\xi =$ constant. Let F^q be the vector bundle of q-forms on A_u, and $\bar\partial_u\colon C^\infty(F^q) \to C^\infty(F^{q+1})$ the differential operator induced by exterior differentiation.

We make this complex explicit as follows. Let $\xi_1, \cdots, \xi_n, u_1, \cdots, u_m$ be coordinates in $R^{n*} \times C^m$. Then, with the same conventions as above, F^q has the basis $\{d\bar{u}_I; I = (i_1, \cdots, i_q), i_1 < \cdots < i_q\}$ and any $\omega \in C^\infty(F^q)$ has the form

$$\omega = \sum_{|I|=q}' \phi_I d\bar{u}_I . \tag{2.8}$$

We have

$$\bar{\partial}_u \omega = \sum_{|I|=q} \sum_{\alpha=1}^{m} \frac{\partial \phi_I}{\partial \bar{u}_\alpha} d\bar{u}_\alpha \wedge d\bar{u}_I . \tag{2.9}$$

We now bring in Lemma I. 3. 2 of [11] which relates these two complexes.

2.10. DEFINITION. Let $\pi: R^n \times C^m \to R^n (\pi: R^{n*} \times C^m \to R^{n*})$ be the projection on the first factor. Let $C_0^\infty(E^q)(C_0^\infty(F^q))$ be the set of $\omega \in C^\infty(E^q)(C^\infty(F^q))$ such that π(support of ω) is relatively compact. For $\omega = \Sigma' a_I d\bar{u}_I \in C_0^\infty(E^q)$, define $\hat{\omega} \in C^\infty(F^q)$ by $\Sigma' \hat{a}_I d\bar{u}_I$, where, for functions

$$\begin{aligned} \hat{a}(\xi, u) &= \int_{R^n} a(x + iQ(u, u), u) e^{-i\langle \xi, x+iQ(u,u)\rangle} dx \\ &= (\mathscr{F}_x a)(\xi, u) e^{Q_\xi(u,u)} \end{aligned} \tag{2.11}$$

where $\mathscr{F}_x$ is the partial (in the x-variables) Fourier transform.

2.12. LEMMA (See I.3.2 of [11].) $(\bar{\partial}_b \omega)^\wedge = \bar{\partial}_u \hat{\omega}$.

Here we shall introduce inner products of the spaces $C^\infty(E^q)$, $C^\infty(F^q)$. (Although the expressions we use to define norms could be infinite, by *completion* we shall mean in the following, the completion of the space of norm-finite forms.) First, we consider C^{m*} as endowed with the standard hermitian inner product in which the set of vectors $\{(0, \cdots, 1, \cdots, 0)\}$ is orthonormal. Let $u_1, \cdots, u_m$ be an orthonormal basis of C^{m*}; we shall call $\{u_1, \cdots, u_m\}$ an orthonormal coordinate set. The following definitions are independent of such a choice of orthonormal coordinate set.

2.13. DEFINITION. For $\omega = \Sigma' a_I d\bar{u}_I$ in $C^\infty(E^q)$, define

$$||\omega||_b^2 = \sum_I{}' \int_\Sigma |a_I|^2 dx du .$$

For $\omega = \Sigma' \phi_I d\bar{u}_I$ in $C^\infty(F^q)$, define

$$||\omega||_u^2 = \sum_I \int_{R^{n*} \times C^m} |\phi_I|^2 e^{-2Q_\xi(u,u)} d\xi du .$$

2.14. LEMMA. *If* $\omega \in C_0^\infty(E^q)$, *we have* $\hat\omega \in C^\infty(F^q)$ *and* $\|\hat\omega\|_u^2 = \|\omega\|_b^2$.

Proof. This is an immediate consequence of the Plancherel formula.

The following formalism (which is fairly standard; see [5, 8]) developing the L^2-cohomology associated to the complex applies equally well to either complex. We shall make our definitions for a complex $(G^q, \bar\partial)$ which refers to either one of the given complexes. In the sequel we shall distinguish between them by a subscript (b or u).

2.15. DEFINITION. The *formal adjoint* $\vartheta: C^\infty(G^q) \to C^\infty(G^{q-1})$ is that differential operator defined by the equation

$$(\bar\partial\alpha, \omega) = (\alpha, \vartheta\omega) \quad \text{(for all } \alpha \text{ of compact support)} .$$

We can find the expression for ϑ by integrating by parts. For example, on E^q it is given by

$$\vartheta_b(\Sigma' a_I d\bar u_I) = \sum_{|J|=q-1}{}' \Big(\sum_{j=1}^m \bar U_\alpha(a_{\alpha J})\Big) d\bar u_J . \tag{2.16}$$

2.17. DEFINITION. Let L^q be the Hilbert space completion of (the norm finite ω in) $C_0^\infty(G^q)$. Define the *W-norm on* $C_0^\infty(G^q)$ by

$$W^2(\omega) = W(\omega, \omega) = \|\omega\|^2 + \|\bar\partial\omega\|^2 + \|\vartheta\omega\|^2 .$$

Let W^q be the Hilbert space completion of $C_0^\infty(G^q)$ in the W-norm.

Notice that $\bar\partial: C_0^\infty(G^q) \to L^{q+1}$, $\vartheta: C_0^\infty(G^q) \to L^{q-1}$ extend continuously to W^q. We shall denote their extensions by the same symbols.

2.18. LEMMA. *If* $\omega \in C^\infty(G^q)$ *and* $W^2(\omega) < \infty$, *then* $\omega \in W^q$.

Proof. We must show that ω is approximable in the W-norm by elements in $C_0^\infty(G^q)$. Let $h \in C^\infty(R)$ be such that

(i) $0 \leq h(t) \leq 1$ for all t
(ii) $h(t) = 1$ if $t \leq 1/2$
(iii) $h(t) = 0$ if $t \geq 1$.

Define h_ν on $R^n(R^{n*})$ by

$$h_\nu(t) = h(|t|/2^\nu),\ t \in R^n(R^{n*}) .$$

For $\omega \in C^\infty(G^q)$, let $\omega_\nu = h_\nu \cdot \omega$. Since $h_\nu \to 1$ boundedly, so long as $\omega \in L^q$, $\omega_\nu \to \omega$ in L^q, by dominated convergence. Since $\bar\partial$, ϑ involve no differentiations in ξ, $\bar\partial\omega_\nu = h_\nu\bar\partial\omega$, $\vartheta\omega_\nu = h_\nu\vartheta\omega$. Thus $\omega_\nu \to \omega$, $\bar\partial\omega_\nu \to \bar\partial\omega$, $\vartheta\omega_\nu \to \vartheta\omega$ in L^q or, what is the same $\omega_\nu \to \omega$ in W^q.

2.19. DEFINITION. The qth *L^2-cohomology space* of the complex $(G^q, \bar\partial)$ is

$$H^q(G) = \{\omega \in W^q;\ \bar\partial\omega = \vartheta\omega = 0\} .$$

2.20 THEOREM. *The correspondence $\omega \to \hat\omega$ induces an isometry $H^q(E) \cong H^q(F)$.*

Proof. (i) We first observe that, by Fourier inversion, the Lemma 2.12 can be worked from F to E. More precisely, let $\phi = \Sigma'\phi_I d\bar u_I \in C_0^\infty(F^q)$. Define

$$\check\phi = \Sigma'\check\phi_I d\bar u_I$$

where, for a function ϕ,

$$\check\phi(z, u) = \frac{1}{(2\pi)^n}\int_{R^{n*}}\phi(\xi, u)e^{i\langle\xi, z\rangle}d\xi . \tag{2.21}$$

Then, just as in the proof of Lemma 2.12 (see [11]) we can verify

$$(\bar\partial_u\phi)^\vee = \bar\partial_b\check\phi . \tag{2.22}$$

(ii) Using the above, we can verify that

$$(\vartheta_b\omega)^\wedge = \vartheta_u\hat\omega,\ \omega \in C_0^\infty(E^q) . \tag{2.23}$$

For, let us take $\alpha \in C_0^\infty(F^q)$, and let $\beta = \check\alpha$. Then, by the Plancherel formula

$$((\vartheta_b\omega)^\wedge, \alpha) = (\vartheta_b\omega, \beta) = (\omega, \bar\partial_b\beta) = (\hat\omega, \bar\partial_u\alpha) ;$$

this for all $\alpha \in C_0^\infty(F^q)$, so we must have $(\vartheta_b\omega)^\wedge = \vartheta_u\hat\omega$.

(iii) Let $\omega \in C_0^\infty(E^q)$. Then, by (2.23) and Lemma 2.18, $\hat\omega \in W^q(F)$, and $W^2(\hat\omega) = W^2(\omega)$. Thus the map $\omega \to \hat\omega$ extends to an isometry of $W^q(E)$ into $W^q(F)$. Since this isometry transports $\bar\partial_b$ and ϑ_b to $\bar\partial_u$ and ϑ_u, it takes $H^q(E)$ into $H^q(F)$.

(iv) this map is surjective. Let $\omega \in H^q(F)$. Then $\omega = \lim \omega_\nu$, $\omega_\nu \in C_0^\infty(F^q)$, with $\bar\partial_u\omega_\nu \to 0$, $\vartheta_u\omega_\nu \to 0$. By (i), $\omega_\nu = \hat\alpha_\nu$ with $(\bar\partial_b\alpha_\nu)^\wedge = \bar\partial_u\omega_\nu$, $(\vartheta_b\alpha_\nu)^\wedge = \vartheta_u\omega_\nu$. Since the correspondence $\omega \to \alpha$ is isometric in the W-norm, the $\{\alpha_\nu\}$ are also Cauchy, so $\alpha_\nu \to \alpha$ for some α, and $\bar\partial_b\alpha_\nu \to 0$, $\vartheta_b\alpha_\nu \to 0$. Thus $\alpha \in H^q(E)$, and $\hat\alpha = \omega$.

For the remainder of this and the next section we shall be concerned with an explicit determination of the spaces $H^q(F)$. First, we introduced the L^2-cohomology along the ξ-fibers of $R^{n*} \times C^m$, $\xi \in R^{n*}$.

Let $C^{0,q}$ represent the space of $C^\infty(0, q)$-forms on C^m. For $\xi \in R^{n*}$, introduce the ξ-norm

$$\|\Sigma' a_I d\bar u_I\|_\xi^2 = \sum_I \int_{C^m} |a_I(u)|^2 e^{-2Q_\xi(u,u)} du \; .$$

Now, we can apply the definitions 2.15–2.19 to the $\bar\partial$-complex $(C^{0,q}, \bar\partial)$ together with the ξ-norm. We shall let $H^q(\xi)$ refer to the associated L^2-cohomology space:

$$H^q(\xi) = \{\omega \in W^q(\xi);\ \bar\partial\omega = \vartheta_\xi\omega = 0\} \tag{2.24}$$

where W_ξ^q is the completion of $C^{0,q}$ in the norm

$$W_\xi^2(\omega) = \|\omega\|_\xi^2 + \|\bar\partial\omega\|_\xi^2 + \|\vartheta_\xi\omega\|_\xi^2 \; .$$

For $\omega \in L^q(F)$, $\omega = \Sigma' a_I d\bar u_I$ define ω_ξ by fixing ξ:

$$\omega_\xi(u) = \Sigma' a_I(\xi, u) d\bar u_I \; .$$

Then ω_ξ is defined and in $L^q(\xi)$ for almost all ξ.

2.25. PROPOSITION. *For $\omega \in H^q(F)$, $\omega_\xi \in H^q(\xi)$ for almost all ξ.*

Proof. The following facts, for $\omega \in C^\infty(F^q)$, are easily verified:

$$\begin{aligned} \|\omega\|_u^2 &= \int_{R^{n*}} \|\omega_\xi\|_\xi^2 d\xi \; , \\ \bar\partial\omega_\xi &= (\bar\partial_u\omega)_\xi,\ \vartheta_\xi\omega_\xi = (\vartheta_u\omega)_\xi \; . \end{aligned} \tag{2.26}$$

Since $\omega \in H^q(F)$, we can find a sequence $\omega_\nu \in C_0^\infty(F^q)$ such that $\omega_\nu \to \omega$, $\bar\partial_u\omega_\nu \to 0$, $\vartheta_u\omega_\nu \to 0$ in $L^q(F)$. Replace $\{\omega_\nu\}$ by a subsequence converging so fast that

$$\begin{aligned} \sum_\nu \|\omega_\nu - \omega_{\nu-1}\|_u^2 &= \int_{R^{n*}} \sum_\nu \|\omega_{\nu,\xi} - \omega_{\nu-1,\xi}\|^2 d\xi < \infty \\ \sum_\nu \|\bar\partial_u\omega_\nu\|^2 &= \int_{R^{n*}} \sum_\nu \|\bar\partial\omega_{\nu,\xi}\|_\xi^2 d_\xi < \infty \\ \sum_\nu \|\vartheta_u\omega_\nu\|_u^2 &= \int_{R^{n*}} \sum_\nu \|\vartheta_\xi\omega_{\nu,\xi}\|^2 d\xi < \infty \; . \end{aligned}$$

Then, for almost all ξ, the series being integrated on the right are all finite. For such a ξ, we will have the first series telescoping and the general term of the other series tending to zero. Thus $\{\omega_{\nu,\xi}\}$ converges with $\bar\partial\omega_{\nu,\xi} \to 0$, $\vartheta_\xi\omega_{\nu,\xi} \to 0$ in $L^q(\xi)$. Thus $\lim \omega_{\nu,\xi}$ is in $H^q(\xi)$, but for almost all ξ, $\lim \omega_{\nu,\xi} = \omega_\xi$.

3. Computation of $H^p(\xi)$. First, we summarize the situation of the preceding section. Q is a nondegenerate C^n-valued hermitian form on C^m. For $\xi \in R^{n*}$, we introduce the scalar hermitian form

$$Q_\xi(u, v) = \langle \xi, Q(u, v) \rangle \; .$$

3.1. DEFINITION. Let $U = \{\xi \in R^{n*};\ Q_\xi$ is nondegenerate$\}$.

Our basic hypothesis is that $U = \varnothing$; in this case $R^{n*} - U$ has measure zero. Let $\langle \mid \rangle$ represent the Euclidean inner product on C^m. For $\xi \in U$, define the operator A_ξ by

$$\langle A_\xi u \mid v \rangle = Q_\xi(u, v) .$$

Since Q_ξ is hermitian, A_ξ is self-adjoint, so C^m has an orthonormal basis of eigenvectors of A_ξ. If $u_1 = u_1(\xi), \cdots, u_m = u_m(\xi)$ are linear forms dual to such a basis and $\lambda_1, \cdots, \lambda_m$ are the corresponding eigenvalues, we compute that

$$Q_\xi(u, v) = \Sigma \lambda_i u_i \bar{v}_i .$$

Now the λ_i are real and since Q is nondegenerate no λ_i is zero. Reordering, we can find positive numbers $\mu_1, \cdots, \mu_m$ such that

$$Q_\xi(u, v) = \sum_{i=1}^{q} \mu_i^2 u_i \bar{v}_i - \sum_{i=q+1}^{m} \mu_i^2 u_i \bar{v}_i . \tag{3.2}$$

The number q is determined by Q_ξ, it is the dimension of a maximal space to which Q_ξ restricts as an inner product.

3.3. DEFINITION. $U_q = \{\xi \in U;\ Q_\xi$ has the form (3.2)$\}$.

3.4. PROPOSITION. *For each $\xi \in U_q$, we can find an orthonormal coordinate set for C^m, $u_1, \cdots, u_m$, so that* (3.2) *holds. The correspondence $\xi \longrightarrow (u_1, \cdots, u_m)$ can be chosen* (locally) *so as to depend smoothly on ξ.*

The proposition is clear. Now, we shall fix a $\xi \in U_q$, and, to keep the notation clear we shall suppress reference to this ξ, denoting

$$\phi(u) = Q_\xi(u, u) = \sum_{i=1}^{q} \mu_i^2 |u_i|^2 - \sum_{i=q+1}^{m} \mu_i^2 |u_i|^2 .$$

We will now compute the cohomology spaces $H^q(\xi)$ following the notation and ideas of Hörmander [7].

As in §2, $C^{0,q}$ is the space of smooth q-forms defined on C^m; $C_0^{0,q}$, those of compact support. We consider the Hilbert space norm on $C^{0,p}$, for $\omega = \Sigma' a_I d\bar{u}_I$

$$\|\omega\|^2 = \sum_I \int_{C^m} |a_I|^2 e^{\phi} du . \tag{3.5}$$

This expression is valid for ω so represented in terms of any orthonormal coordinate set $u_1, \cdots, u_m$. Let, for f a smooth function

$$\begin{aligned} &\partial_j f = \frac{\partial f}{\partial u_j},\ \bar{\partial}_j f = \frac{\partial f}{\partial \bar{u}_j}, \\ (3.6)\qquad &\vartheta_j f = e^{-\phi}\partial_j(e^{\phi} f) = \partial_j\phi\cdot f + \partial_j f \\ &\bar{\vartheta}_j f = e^{-\phi}\bar{\partial}_j(e^{\phi} f) = \bar{\partial}_j\phi\cdot f + \bar{\partial}_j f\ . \end{aligned}$$

Thus,

$$(3.7)\qquad [\bar{\partial}_j, \vartheta_k] = \bar{\partial}_j\vartheta_k - \vartheta_k\bar{\partial}_j = \partial_j^k\lambda_j\ .$$

Furthermore, if either f or g is compactly supported

$$(8.3)\qquad \int_{C^m}(\partial_j f)ge^{\phi}du = -\int_{C^m} f(\vartheta_j g)e^{\phi}du$$

and similarly for the barred operators. Now, for $\omega = \Sigma' a_I d\bar{u}_I$ a q-form we have

$$(3.9)\qquad \bar{\partial}\omega = \sum_I{}' \sum_{j=1}^m \bar{\partial}_j a_I d\bar{u}_j \wedge d\bar{u}_I\ ,$$

$$(3.10)\qquad \vartheta\omega = \sum_I{}' \sum_{j=1}^m \vartheta_j(a_{jI})d\bar{u}_I$$

where ϑ is the formal adjoint of $\bar{\partial}$. (Here the $'$ refers to the summation convention introduced in the preceding section.) Finally, we shall need two fundamental identities. First, if f is smooth and compactly supported,

$$(3.11)\qquad \int_{C^m}|\vartheta_j f|^2 e^{\phi}du - \int_{C^m}|\bar{\partial}_j f|^2 e^{\phi}du + \lambda_j\int_{C^m}|f|^2 e^{\phi}du = 0\ .$$

This follows from applying (3.8) to (3.7) in its integrated form:

$$\lambda_j \int |f|^2 e^{\phi}du = \int [\bar{\partial}_j, \vartheta_j]f\cdot\bar{f}e^{\phi}du\ .$$

By direct computation we obtain, for $\omega = \Sigma' a_I d\bar{u}_I \in C_0^{0,p}$,

$$\begin{aligned} &||\bar{\partial}\omega||^2 + ||\vartheta\omega||^2 \\ &= \sum_{K=q-1}{}' \sum_{j,l} \int_{C^m}(\vartheta_j a_{jK}\overline{\vartheta_l a_{lK}} - \bar{\partial}_j a_{jK}\overline{\bar{\partial}_l a_{lK}})e^{\phi}du \\ &\quad + \sum_{I,j}{}' \int_{C^m}|\bar{\partial}_j a_I|^2 e^{\phi}du\ . \end{aligned}$$

Using the above integration-by-parts formula on the first term on the right, this becomes

$$(3.12)\qquad ||\bar{\partial}\omega||^2 + ||\vartheta\omega||^2 = \sum_I{}'\sum_j \int |\bar{\partial}_j a_I|^2 e^{\phi}du - \sum_K{}'\sum_j \lambda_j \int |a_{jK}|^2 e^{\phi}du$$

(These are respectively the analogues of (2.1.8)$'$ and (2.1.13) of [7].)

Let $c = \min |\lambda_i| > 0$.

3.13. LEMMA. *Let N be the multi index $(1, 2, \cdots, q)$. Then, for $\omega = \Sigma' a_I d\bar{u}_I \in C_0^{0,p}$, we have*

$$\|\bar{\partial}\omega\|^2 + \|\vartheta\omega\|^2 \geqq \sum_{I \neq N}' c \int |a_I|^2 e^{\phi} du + \sum_I' \left(\sum_{j=1}^{q} \int |\vartheta_j a_I|^2 e^{\phi} du + \sum_{j=q+1}^{m} \int |\bar{\partial}_j a_I|^2 e^{\phi} du \right).$$

Proof. Let us adopt the notation $\lambda_I = \sum_{j \in I} \lambda_j$. Note that for $I \neq N$, $\lambda_N - \lambda_I \geqq c > 0$. We rewrite (3.12) as

$$\|\bar{\partial}\omega\|^2 + \|\vartheta\omega\|^2 \geqq \sum_I' \left(\sum_j \int |\bar{\partial}_j a_I|^2 e\phi du - \lambda_I \int |a_I|^2 e^{\phi} du \right). \tag{3.14}$$

We treat each term individually.

$$\sum_j \int |\bar{\partial}_j a_I|^2 e^{\phi} du - \lambda_I \int |a_I|^2 e^{\phi} du$$
$$= \sum_j \int |\bar{\partial}_j a_I|^2 e^{\phi} du - \lambda_N \int |a_I|^2 e^{\phi} du + (\lambda_N - \lambda_I) \int |a_I|^2 e^{\phi} du .$$

Applying (3.11) to the second term (note $\lambda_N = \lambda_1 + \cdots + \lambda_q$), we obtain

$$= \sum_j \int |\bar{\partial}_j a_I|^2 e^{\phi} du + \sum_{j=1}^{q} \left(\int |\vartheta_j f|^2 e^{\phi} du - \int |\bar{\partial}_j a_I|^2 e^{\phi} du \right) + (\lambda_N - \lambda_I) \int |a_I|^2 e^{\phi} du$$
$$= (\lambda_N - \lambda_I) \int |a_I|^2 e^{\phi} du + \sum_{j=1}^{q} \int |\vartheta_j f|^2 e^{\phi} du + \sum_{j=q+1}^{m} \int |\bar{\partial}_j f|^2 e^{\phi} du .$$

If $I = N$, the first term drops out; otherwise it dominates $c \int |a_I|^2 e^{\phi} du$. The lemma is proven.

Now, we recall that W^p is defined as the Hilbert space completion of those $\omega \in C^{0,p}$ such that

$$W^2(\omega) = \|\omega\|^2 + \|\bar{\partial}\omega\|^2 + \|\vartheta\omega\|^2 < \infty$$

in this W-norm. $H^p = H^p(\xi) = \ker \bar{\partial} \cap \ker \vartheta$. The relevance of the above estimate is that it holds on W^p, because $C_0^{0,p}$ is dense in W^p as we now prove.

3.15. LEMMA. *$C_0^{0,p}$ is dense in W^p in the W-norm.*

Proof. Let h be as introduced in Lemma 2.18, and let $h_\nu(u) = h(|u|/2^\nu)$. Suppose $\omega \in C^{0,p}$ has finite W-norm. Let $\omega_\nu = h_\nu \cdot \omega$. We shall show that $\omega_\nu \to \omega$ in the W-norm, or, what is the same,

$$\omega_\nu \longrightarrow \omega,\ \bar{\partial}\omega_\nu \longrightarrow \bar{\partial}\omega,\ \vartheta\omega_\nu \longrightarrow \vartheta\omega\ . \tag{3.16}$$

First of all, since $h_\nu \to 1$ boundedly we can conclude that $h_\nu \cdot \theta \to \theta$ in L^2, for any square integrable form θ. Now, form formulae (3.9) and (3.10) we easily conclude that

$$\begin{aligned} \bar{\partial}(h_\nu\omega) &= h_\nu\bar{\partial}\omega + \sum_{I,j}{}' \frac{\partial h_\nu}{\partial \bar{u}_j} a_I d\bar{u}_j \wedge d\bar{u}_I \\ \vartheta(h_\nu\omega) &= h_\nu\vartheta\omega + \sum_{I,j}{}' \frac{\partial h_\nu}{\partial u_j} a_{jI} d\bar{u}_I\ . \end{aligned} \tag{3.17}$$

It remains only to show that the last terms in (3.17) tend to zero as $\nu \to \infty$. Each term is a fixed linear combination of terms of the form $(D \cdot h_\nu)a$, where D is a constant coefficient first order operator, and a is a typical coefficient of ω. Now, the $(D \cdot h_\nu)$ are uniformly bounded and have disjoint supports, so $\Sigma(D \cdot h_\nu)^2$ is bounded. Thus $(\sum_\nu D \cdot h_\nu)^2 |a|^2$ is integrable, so the general term tends to zero in L^1. Thus the last term in (3.17) tends to zero in L^2, so the lemma is proven.

3.18. THEOREM. (1) *For* $\xi \in U_q$, *we have* $H^p(\xi) = \{0\}$ *for* $p \neq q$. (2) *Let* $u_1, \cdots, u_m$ *be the basis of* C^m *found in Proposition* 3.4, *and let* $v_1 = \mu_1\bar{u}_1, \cdots, v_q = \mu_q\bar{u}_q, v_{q+1} = \mu_{q+1}u_{q+1}, \cdots, v_m = \mu_m u_m$. *Then*

$$\begin{aligned} H^q(\xi) = \Big\{\omega &= f(v)\exp\Big(-\sum_{i=1}^{q} |v_i|^2\Big) d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_q\ , \\ &\text{where } f \text{ is holomorphic and} \\ \|\omega\|^2 &= \frac{1}{(\mu_1 \cdots \mu_m)} 2 \int |f|^2 e^{-\|v\|^2} dv < \infty \Big\}\ . \end{aligned} \tag{3.19}$$

Proof. Let $\omega \in H^p(\xi)$, $\omega = \Sigma' a_I d\bar{u}_I$. By the preceding lemma there is a sequence $\{\omega_\nu\} \subset C_0^{0,p}$ such that $\omega_\nu \to \omega$ in L^p and $\bar{\partial}\omega_\nu \to 0$, $\vartheta\omega_\nu \to 0$ in L^p. By the estimate in Lemma 3.13 we conclude that, for $\omega_\nu = \Sigma' a_{I,\nu} d\bar{u}_I$, $a_{I,\nu} \to a_I$, and

(a) for $I \neq N = \{1, \cdots, q\}$, $a_{I,\nu} \longrightarrow 0$,

(b) for $j > q$, $\dfrac{\partial a_{N,\nu}}{\partial \bar{u}_j} \longrightarrow 0$ in L^1_{loc} ,

(c) for $j \leq q$, $\dfrac{\partial}{\partial u_j}(e^\phi a_{N,\nu}) \longrightarrow 0$ in L^1_{loc} .

From (a) we conclude that $a_I = 0$ for $I \neq N$. Thus (1) is proven, and for $\cdot = q$, we have $\omega = a d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_q$ where $a = \lim a_\nu$ with

$$\frac{\partial a_\nu}{\partial \bar{u}_j} \longrightarrow 0,\ j > q,\ \frac{\partial e^{\phi} a_\nu}{\partial u_j} \longrightarrow 0,\ j \leqq q$$

in L^1_{loc}. Thus $f(u) = a(u) \exp(\sum_{i=1}^q \mu_i^2 |u_i|^2)$ is a weak solution of

$$\partial_j f = 0,\ 1 \leqq j \leqq q,\ \bar{\partial}_j f = 0,\ q+1 \leqq j \leqq n\ .$$

By the regularity theorem for the Cauchy-Riemann equations, it follows that f is holomorphic in $\bar{u}_1, \cdots, \bar{u}_q, u_{q+1}, \cdots, u_m$ and

$$\int |f(u)|^2 \exp\Big(-\sum_{i=1}^m \mu_i^2 |u_i|^2\Big) du = \int |a|^2 e^{\phi} du = ||\omega||^2\ .$$

This is, up to the desired change of variable, what was to be proved.

The preceding results tell us that the fibration $H^q(\xi) \rightarrow \xi$ is a locally trivial bundle of Hilbert spaces, with generic fiber naturally isomorphic to

$$H_0 = \Big\{ f \in \mathcal{O}(C^m);\ \int_{C^m} |f(v)|^2 e^{-||v||^2} dv < \infty \Big\}\ . \tag{3.20}$$

We want to observe that $H^q(F)$ is a space of square integrable sections on U_q of this bundle.

3.21. THEOREM. *Let $S^q(F)$ be the space of C^∞ sections of F^q over U_q such that, for all $\xi \in U_q$, $\omega_\xi \in H^q(\xi)$ and*

$$||\omega||^2 = \int_{U_q} ||\omega_\xi||^2 d\xi < \infty\ . \tag{3.22}$$

Then $H^q(F)$ is the completion of $S^q(F)$ in this norm.

Proof. By (2.26), for such $\omega \in S^q(F)$ we have $||\omega||_u^2 = ||\omega||^2$, $\bar{\partial}_u \omega = \vartheta_u \omega = 0$, and so $S^q(F)$ is isometric to a subspace of $H^q(F)$. We have to show that $S^q(F)$ is dense.

Let $\omega \in H^q(F)$. By Proposition 2.25, $\omega_\xi \in H^q(\xi)$ for almost all $\xi \in U$, so ω is supported in U_q. Fix $\xi_0 \in U_q$, and let N be a neighborhood of ξ_0 such that we can find smooth functions $u_1(\xi, u), \cdots, u_n(\xi, u)$ defined on $N \times C^m$ such that

(a) for all ξ, $u_1(\xi, u), \cdots, u_n(\xi, u)$ form an orthonormal coordinate set for C^m,

(b) $Q_\xi(u, u) = \sum_{i=1}^q \mu_i(\xi)^2 |u_i(\xi, u)|^2 - \sum_{i=q+1}^m \mu_i(\xi)^2 |u_i(\xi, u)|^2$. Let $\Omega_\xi = \exp(-\sum_1^q \mu_i^2 |u_i|^2) d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_q$. Let $d(\xi) = [\mu_1(\xi) \cdots \mu_n(\xi)]^{-2}$, $v_1 = \mu_1 \bar{u}_1, \cdots, v_q = \mu_q \bar{u}_q, v_{q+1} = \mu_1 u_{q+1}, \cdots, v_m = \mu_m u_m$. Then, for almost all $\xi \in N$,

$$\omega(\xi, u) = f(\xi, v) \Omega_\xi\ ,$$

and

$$\|\omega|_N\|^2 = \int_N \left[\int_{C^m} |f(\xi, v)|^2 e^{-\|v\|^2} dv\right] d(\xi) d\xi .$$

The proof of Theorem 2.26 of [10] applies on the right, to show that f can be approximated by functions of the form $\sum_{k=1}^K l_k(\xi)P_k(u)$, where $l_k \in C_0^\infty(N)$ and P_k is a polynomial.

For such an f, $f\Omega_\xi$ is in $S^q(F)$. Thus $\omega|_N$ is the closure of $S^q(F)$. Now, if we cover U_q by a locally finite collection of open sets $\{N_i\}$ of this type, then for any $\omega \in H^q(F)$ supported in N_i, ω is in the closure of $S^q(F)$. Let $\{\rho_i\}$ be a partition of unity subordinate to the cover $\{N_i\}$. It is easy to verify that, for $\omega \in H^q(F)$, $\rho_i\omega \in H^q(F)$ and $\omega = \sum_i \rho_i\omega$ in $W^q(F)$. Since each $\rho_i\omega$ is in the closure of $S^q(F)$, so also is ω.

4. Representations of $N(Q)$ on $H^q(\Sigma)$. Recall the group $N(Q)$ introduced at the beginning of §2 and its action by complex affine transformations on C^{n+m}, as given by (2.4). Since Σ is an orbit of $N(Q)$, and $N(Q)$ preserves the complex structure of C^{n+m}, it preserves the induced CR-structure on Σ. That is, for $n \in N(Q)$, the differential dn preserves the bundle A of holomorphic tangent vectors tangent to Σ. Since $E^q = \Lambda^q(A^*)$, there is induced an action of $N(Q)$ on $C^\infty(E^q)$ given by

$$(n\cdot\omega)(v_p) = \omega(dn^{-1}(v_p)) . \tag{4.1}$$

We can make this explicit, referring to the coordinates of §2:

$$\text{if} \quad \omega = \Sigma\, a_I d\bar u_I\,, \quad (n\cdot\omega)(p) = \Sigma a_I(n^{-1}\cdot p) d\bar u_I \tag{4.1$'$}$$

(the reason this is so simple is that the action of $N(Q)$ is by pure translation). Since $N(Q)$ preserves the measure $dxdu$ on Σ, this correspondence $\omega \to n\cdot\omega$ defines an isometry of $L^q(\Sigma)$, as defined in (2.13). Clearly $\bar\partial_b(n\cdot\omega) = n\cdot\bar\partial_b\omega$, so we also have, since ϑ_b is the formal adjoint of $\bar\partial_b$, $\vartheta_b(n\cdot\omega) = n\cdot\vartheta_b\omega$. Thus the action (4.1) induces an isometry of W^q preserving $H^q(\Sigma)$.

4.2. DEFINITION. Let ρ_q denote the unitary representation of $N(Q)$ on $H^q(\Sigma)$ induced by the action (4.1).

Now, we summarize the content of Theorem 3.19 as it applies to the representation ρ_q. First of all, the correspondence $\omega \to \hat\omega$ (as defined by (2.10) induced an isometry of $H^q(\Sigma)$ with $H^q(F)$ (Theorem 2.20), defined in terms of the $\bar\partial_u$-complex on $R^{n*} \times C^m$ We shall let $\tilde\rho_q$ represent the transport of ρ_q to $H^q(F)$ via this correspondence. Explicitly, $\tilde\rho_q$ is induced by this action of $N(Q)$ on $C_0^\infty(F^q)$:

$$n\cdot\hat\omega = (n\cdot\omega)\hat{}\,, \; n \in N(Q) .$$

Let us explicitly compute $\tilde{\rho}_q$. For $a \in C_0^\infty(E^0)$, and $n = (x_0, u_0)$ in $N(Q)$, we have

$$\begin{aligned} n \cdot a(x, u) &= a((-x_0, -u_0)(x, u)) = a(x - x_0 - 2 \operatorname{Im} Q(u_0, u), u - u_0) , \\ n \cdot \hat{a}(\xi, u) &= (n \cdot a)\hat{}(\xi, u) = (\mathscr{F}_x(n \cdot a))(\xi, u)e^{Q_\xi(u,u)} \\ &= e^{-i\langle \xi, x_0 + 2\operatorname{Im} Q(u_0, u)\rangle}(\mathscr{F}_x a)(\xi, u - u_0)e^{Q_\xi(u,u)} \\ &= e^{-i\langle \xi, x_0\rangle}e^{-Q_\xi(u_0,u_0)}e^{2Q_\xi(u,u_0)}\hat{a}(\xi, u - u_0) . \end{aligned}$$

Thus

$$n \cdot \omega(\xi, u) = e^{-i\langle \xi, x_0\rangle}e^{-Q_\xi(u_0,u_0)}e^{2Q_\xi(u,u_0)}\omega(\xi, u - u_0) \tag{4.3}$$

for $n = (x_0, u_0)$ and $\omega \in C^\infty(F^q)$.

The content of Theorem 3.19 is that $H^q(F)$ can be realized as the space of square-integrable sections of the Hilbert fibration $H^q(\xi) \to \xi$ over U_q. From (4.3) we see that the action of $N(Q)$ is fiber-preserving. More precisely, we can freeze ξ in (4.3) and let it define an action on the space $C^{0,q}$ of q-forms on C^m:

$$(\rho(\xi)n)\omega(u) = e^{-i\langle \xi, x_0\rangle}e^{-Q_\xi(u_0,u_0)}e^{-2Q_\xi(u,u_0)}\omega(u - u_0) .$$

Since $Q_\xi(u, u_0)$ is holomorphic in u, this action commutes with $\bar{\partial}$. This action is isometric in the norm (3.5) (where $\phi(u) = Q_\xi(u, u)$), so there is induced an unitary representation $\rho(\xi)$ of $N(Q)$ on $H^q(\xi)$. Now Theorem 3.19 reads as follows.

4.4. THEOREM. $\rho_q \sim \tilde{\rho}_q \sim \int_{U_q} \oplus \rho(\xi) d\xi$.

Finally, we would like to point out that the representations $\rho(\xi)$ are those (in the case $n = 1$) found by Carmona [3]. They are irreducible, and we use Theorem 3.16 to see that. The coordinates $v_1(\xi), \cdots, v_m(\xi)$ found in that theorem are the coordinates produced by Ogden and Vagi [9] in their description of the Plancherel formula for the groups $N(Q)$. Theorem 3.16 describes the intertwining operator which intertwines $\rho(\xi)$ with their representation π_ξ. We can generalize their theorem.

4.5. THEOREM. *The representation* $\bigoplus \rho_q$ *of* $N(Q)$ *on* $\bigoplus H^q(E)$ *is isometric to a subrepresentation of the left regular representation on* $L^2(N(Q))$ *in which every irreducible* (except for a set of Plancherel measure zero) *occurs with multiplicity one.*

In the language of Auslander and Kostant, the vector bundle A of holomorphic tangent vectors tangent to Σ, arises from a Lie subalgebra $\mathfrak{h}$ of $\mu(Q)^c$. If $\mathfrak{z}$ is the center of $\mu(Q)$, then $\mathfrak{z}^C \oplus \mathfrak{h}$ is a polarization at ξ, for all $\xi \in \mathfrak{z}^*(\subset \mu(Q)^*)$ which is *positive* if and only if $\xi \in U_0$. If $\xi \in U_q$, $q \neq 0$, then the new coordinates of Theorem 3.16 relate to a positive polarization at ξ, and Theorem 3.16 exhibits the intertwining operator between the representations corresponding to these polarizations.

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