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On Rossmann's Character Formula for Discrete Series

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In a recent article in this journal, W. Rossmann [3] has proved a formula conjectured by Kirillov connecting characters of the representations of semi-simple Lie groups occurring in the Plancherel formula with Fourier transforms of the measures on orbits in the coadjoint representation. Rossmann's formula has considerable conceptual importance, as it established, via the orbit method, the connection between the representation theory of general Lie groups and the cabalistic study of representations of semi-simple Lie groups.

I give here a simple proof of Rossmann's basic theorem, which relates the Fourier transforms on $\mathfrak{g}$ and on a Cartan subalgebra of compact type.

1. Let V be a real vector space with a non-degenerate symmetric form B. I consider the multiplication operator m_B given by $(m_B \cdot f)(x) = B(x,x) f(x)$, and Δ_B the corresponding Laplace operator $(\Delta_B \cdot f)(x) = \left(B\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) \cdot f\right)(x)$. Denoting the harmonic oscillator $i(m_B - \Delta_B)$ as J_B, $\left(\text{if } V \text{ is one dimensional, } J_B = i\left(x^2 - \frac{\partial}{\partial x^2}\right)\right)$, then the Fourier transform $(F_B f)(x) = \int e^{iB(x,y)} f(y)\,dy$ belongs (up to a scalar multiple) to the one parameter group of unitary transformations $e^{it(m_B - \Delta_B)}$. The basic idea of our proof will be referred as the harmonic principle: if an operator M commutes with m_B and Δ_B, then M commutes with F_B.

Let us specify the relation of F_B with J_B. We consider m_B and Δ_B as differential operators on V. Let $E = \sum x_i \frac{\partial}{\partial x_i}$ be the Euler operator on V. The following relations are immediate:

$$[m_B, \Delta_B] = -4(E + \tfrac{1}{2}\dim V),$$
$$[E, m_B] = 2 m_B,$$
$$[E, \Delta_B] = -2\Delta_B.$$

Hence $X = i m_B$, $Y = i\Delta_B$ and $H = E + \frac{1}{2}\dim V$ form a Lie algebra of differential operators on V isomorphic to $\mathscr{S}\mathscr{L}(2)$. There exists a unitary representation L_B, that we will call the harmonic representation, of the double covering group $\widetilde{SL}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ which exponentiates this representation. In particular if $f\in\mathscr{S}(V)$, the Schwartz space of V, then f is a C^∞ vector for the representation L_B and the action of dL_B on $\mathscr{S}(V)$ coincides with the given differential operators. (If B is not definite, the space of C^∞ vectors is actually much larger than $\mathscr{S}(V)$.)

We have:

$$\left(L_B\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\cdot f\right)(x)=e^{itB(x,x)}f(x),\qquad f\in L^2(V)$$

$$\left(L_B\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}f\right)(x)=a^{\frac{\dim V}{2}}f(a\cdot x),\qquad a>0$$

as follows from differentiation. Let us consider the element $J_0=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ of the Lie algebra $\mathscr{S}\mathscr{L}(2)$. The compact one-parameter subgroup $\exp tJ_0$ calculated in the metaplectic group covers the one-parameter subgroup $\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$ of $SL(2,\mathbb{R})$. In particular $\left(\exp\frac{\pi}{2}J_0\right)$ covers the element $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ of $SL(2,\mathbb{R})$. Let us normalize the Lebesgue measure dy such that $(F_B f)(x)=\int e^{iB(x,y)}f(y)\,dy$ is unitary. Let $p-q=$signature of B and $\gamma_B=e^{\frac{i\pi}{4}(p-q)}$. Then

$$L_B\left(\exp\frac{\pi}{2}J_0\right)=\gamma_B F_B.$$

(This can be seen by explicit diagonalization of $x^2-\frac{\partial}{\partial x^2}$ over the Hermite functions or, in a more enlightening way, by considering the Harmonic representation L_B as group of automorphisms of the canonical commutation relations, as introduced by Segal, Shale, Weil, see [4, 6].)

2. We will apply these general considerations to the situation of the invariant integral $M_{\mathfrak{t}}$ of Harish-Chandra. We follow the notations of Rossmann's article and its interpretation of Kirillov's formula: Rossmann regarded the Kirillov distribution $\int_{G\cdot f}\hat{\varphi}(g\cdot f)\,d\mu_f$ as a map from functions on $\mathfrak{g}$ to functions of the variable f varying in $\mathfrak{g}^*/G$ (parametrized here by the Cartan subalgebras):

Let $\mathfrak{g}$ be a reductive Lie algebra with a Cartan subalgebra $\mathfrak{t}$ of compact type. Let $(M_{\mathfrak{t}}\varphi)(s)=\pi(s)\int_G\varphi(g\cdot s)\,dg$ be the invariant integral. We consider $F_{\mathfrak{g}}$ the Fourier transform on $\mathfrak{g}$ with respect to a G-invariant form $B_{\mathfrak{g}}$ negative definite on $\mathfrak{k}$. It is then easy to see that Kirillov's formula for character of discrete series follows from Rossmann's theorem:

$$M_{\mathfrak{t}}F_{\mathfrak{g}}\varphi=\gamma F_{\mathfrak{t}}M_{\mathfrak{t}}\varphi\qquad\text{for }\varphi\in D(\mathfrak{g}_e).$$

3. We now give a proof of this theorem

We consider on $\mathfrak{g}$ the Casimir operator $\Delta_{\mathfrak{g}}$ corresponding to $B_{\mathfrak{g}}$. We have $\operatorname{sign} B_{\mathfrak{g}} = \dim \mathfrak{g}/\mathfrak{k} - \dim \mathfrak{k} = \dim \mathfrak{g} - 2\dim \mathfrak{k}$. We denote by $L_{\mathfrak{g}}$ the harmonic representation of the metaplectic group on $\mathscr{S}(\mathfrak{g})$. Similarly we denote by $B_{\mathfrak{t}}$ the restriction of the Killing form $B_{\mathfrak{g}}$ on $\mathfrak{t}$ ($\operatorname{sign} B_{\mathfrak{t}} = -\dim \mathfrak{t}$), by $\Delta_{\mathfrak{t}}$ the corresponding Laplace operator and by $L_{\mathfrak{t}}$ the corresponding harmonic representation of the metaplectic group on $\mathscr{S}(\mathfrak{t})$.

Let $P = MN$ be a parabolic subgroup of G of Lie algebra $\mathfrak{P} = \mathfrak{M} \oplus \mathfrak{N}$, with $\mathfrak{M}$ reductive. Let $f \in \mathscr{S}(\mathfrak{g})$. Following Harish-Chandra, we define $f_P \in \mathscr{S}(\mathfrak{M})$ by $f_P(Y) = \int_{\mathfrak{N}} f(Y+Z)\, dZ$ $(Y \in \mathfrak{M})$. We have:

1) $(B_{\mathfrak{g}} f)_P = B_{\mathfrak{M}} f_P$,

2) $(F_{\mathfrak{g}} f)_P = F_{\mathfrak{M}}(f_P)$.

1) follows immediately from the equality $B_{\mathfrak{g}}(Y+Z, Y+Z) = B_{\mathfrak{g}}(Y, Y)$ for $Y \in \mathfrak{M}$, $Z \in \mathfrak{N}$.

2) follows from the duality between $\overline{\mathfrak{N}}$ and $\mathfrak{N}$ (Lemma 1, p. 197, [2]).

Let ${}^0\mathscr{S}(\mathfrak{g})$ be the subspace of all f in $\mathscr{S}(\mathfrak{g})$ such that $f_P = 0$ for all $P \neq G$. Then ${}^0\mathscr{S}(\mathfrak{g})$ is a closed subspace of $\mathscr{S}(\mathfrak{g})$. As a function f in $D(\mathfrak{g}_e)$ is identically zero on the set of parabolic elements of $\mathfrak{g}$, the space ${}^0\mathscr{S}(\mathfrak{g})$ contains $D(\mathfrak{g}_e)$. The equalities 1) and 2) shows that ${}^0\mathscr{S}(\mathfrak{g})$ is stable under the operators $L_{\mathfrak{g}}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $F_{\mathfrak{g}}$. As the elements $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate the group $SL(2, \mathbb{R})$, we conclude:

Lemma. *${}^0\mathscr{S}(\mathfrak{g})$ is stable under the representation $L_{\mathfrak{g}}$ of $\widetilde{SL}(2, \mathbb{R})$.*

(I am thankful to the referee for suggesting the use of the canonical space ${}^0\mathscr{S}(\mathfrak{g})$.)

4. We recall now the fundamental properties of the invariant integral $M_{\mathfrak{t}}$ that we will use:

1) $M_{\mathfrak{t}}$ "intertwines" $B_{\mathfrak{g}}$ with $B_{\mathfrak{t}}$ and $\Delta_{\mathfrak{g}}$ with $\Delta_{\mathfrak{t}}$.

a) The commutation relation with $B_{\mathfrak{g}}$ is trivial to check as $B_{\mathfrak{g}}$ is G-invariant.

b) The commutation relation with $\Delta_{\mathfrak{g}}$ holds only in the following sense: if $\varphi \in \mathscr{S}(\mathfrak{g})$, then $M_{\mathfrak{t}} \Delta_{\mathfrak{g}} \varphi = \Delta_{\mathfrak{t}} M_{\mathfrak{t}} \varphi$ as C^∞ functions on the regular set $\mathfrak{t}_r$ of $\mathfrak{t}$. This is an elementary computation of the Casimir $\Delta_{\mathfrak{g}}$ modulo the left ideal of adjoint vector fields. (See for example p. 29 of [5].) But $M_{\mathfrak{t}} \varphi$ is usually not a C^∞ function on $\mathfrak{t}$. Hence we will need a deeper property of $M_{\mathfrak{t}}$:

2) $M_{\mathfrak{t}}$ defines a continuous mapping of ${}^0\mathscr{S}(\mathfrak{g})$ into $\mathscr{S}(t)$. (This follows from Theorem 2, p. 561 of [2].) Hence for f in ${}^0\mathscr{S}(\mathfrak{g})$, we have $M_{\mathfrak{t}} \Delta_{\mathfrak{g}} f = \Delta_{\mathfrak{t}} M_{\mathfrak{t}} f$ on the whole space $\mathfrak{t}$.

Rossmann's theorem is more appropriately rewritten as follows:

Theorem. $M_{\mathfrak{t}}(L_{\mathfrak{g}}(g \cdot f) = L_{\mathfrak{t}}(g) \cdot M_{\mathfrak{t}} f$ *for every* $f \in {}^0\mathscr{S}(\mathfrak{g})$, $g \in \widetilde{SL}(2, \mathbb{R})$.

(*Rossmann's equality then follows: for* $\sigma = \exp \frac{\pi}{2} J_0$, $L_{\mathfrak{g}}(\sigma) = \gamma_{\mathfrak{g}} F_{\mathfrak{g}}$, $L_{\mathfrak{t}}(\sigma) = \gamma_{\mathfrak{t}} F_{\mathfrak{t}}$ *and, as* $\dim \mathfrak{g}/\mathfrak{k}$ *is even,* $\gamma_{\mathfrak{t}} \gamma_{\mathfrak{g}}^{-1} = \gamma$.)

Proof. The commutation relations for $g=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$ are obvious. Thus we need (and are mainly interested in) only prove that:

$$M_t(L_g(\exp uJ_0))f=L_t(\exp uJ_0)M_t f, \quad \text{for } f\in{}^0\mathscr{S}(g),$$

$$\text{i.e.: } M_t(e^{iu(B_g-\Delta_g)}\cdot f)=e^{iu(B_t-\Delta_t)}M_t f, \quad \text{for every } u \text{ in } \mathbb{R}.$$

By differentiation, it is clear that this follows from the properties 1) and 2) of M_t. More precisely, let us consider the compact one-parameter subgroup $U=\exp uJ_0$ of the metaplectic group. We have to prove that for f in ${}^0\mathscr{S}(g)$, $M_t L_g(u)f=L_t(u)M_t f$ for every u in U. Let us develop f in eigenfunctions with respect to the action of $L_g(U)$. By the continuity property of M_t, it is then enough to prove it for an eigenvector. But for an eigenvector this is immediate.

My interest on character formula for discrete series representations was stimulated by discussing with D. Barbasch and V. Guillemin over the restriction problem from a representation of G to K. In particular D. Barbasch and D. Vogan had a different proof of Rossmann's formula based on the weak Blattner conjecture. I would like to thank them for communication of these earlier results. I would like to thank I. Segal for helpful discussions. I have also benefited from the interesting comments of the referee.

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