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A Poisson-Plancherel formula for semi-simple Lie groups

By MICHELE VERGNE

1. The Plancherel formula and the orbit method

The harmonic analysis on a Lie group G is strongly related to the Fourier analysis on its Lie algebra $\mathfrak{g}$, as shown by the orbit method. It is my desire to give an interpretation of the Plancherel formula for the group G in this context.

Let me first give the example of a torus T:

Let Γ be a lattice in a real vector space V and $T = V/\Gamma$ the torus. We identify the Lie algebra $\mathfrak{g}$ of T with V. For $f \in V^*$ we denote by $\chi_f(x) = e^{i(f,x)}$ the associated character of the abelian group V. Thus $\hat{T}$ is described as the dual lattice Γ^* of Γ in V^*, where $\Gamma^* = \{f \in V^*, \chi_f(x) = 1, \text{ for all } x \in \Gamma\}$. The Poisson summation formula is given by

$$\sum_{\gamma\in\Gamma} f(\gamma) = \sum_{\gamma^*\in\Gamma^*} \hat{f}(\gamma^*)$$

which we may rewrite as:

$$\sum_{\substack{\gamma\in\mathfrak{g}\\ \exp\gamma=1}} f(\gamma) = \sum_{t\in\hat{G}\subset\mathfrak{g}^*} \hat{f}(t).$$

We are able to generalize this formula to the case of a linear real semi-simple Lie group G.

When G is a semi-simple Lie group with finite center, the reduced dual $\hat{G}_r$ together with the Plancherel formula for G has been described by Harish-Chandra [6]. Let us first formulate some of the relevant data in the context of a general Lie group G, before stating a generalization of the Poisson summation formula.

Let G be a unimodular* Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual vector space to $\mathfrak{g}$. The group G operates on $\mathfrak{g}^*$ by the coadjoint action. Let

$$j(X) = \det\frac{e^{\operatorname{ad} X} - 1}{\operatorname{ad} X}$$

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be the Jacobian of the exponential map. Let $j^{1/2}$ be a square root of j analytic near 0 on $\mathfrak{g}$ such that $j^{1/2}(0) = 1$.

Let $\hat{G}_r$ be the reduced dual of the Lie group G. It was conjectured by Kirillov [12] that for a representation T in $\hat{G}_r$, there exists an orbit $\mathcal{O}_T$ of G in $\mathfrak{g}^*$ such that the following formula holds in a neighborhood of 0:

$$\operatorname{Tr}(T(\exp X))j(X)^{1/2} = \int_{\mathcal{O}_T} e^{i(\xi, X)}\, d\mu(\xi)$$

where $d\mu(\xi)$ is a G-invariant measure on the orbit $\mathcal{O}_T$. (This equality has to be interpreted as a distribution equality in a neighborhood of 0: for α a C^∞ function with small support near 0 in $\mathfrak{g}$, then:

$$\operatorname{Tr}\int_{\mathfrak{g}} T(\exp X)\alpha(X)j(X)^{1/2}\, dX = \int_{\mathcal{O}_T} \hat{\alpha}(\xi)\, d\mu(\xi).)$$

This formula has been proved in numerous cases, in particular in the semi-simple case by W. Rossmann [13] (see also [14]). This leads to a map $T \to \mathcal{O}_T$ from $\hat{G}_r$ to orbits of G in $\mathfrak{g}^*$, whose fiber is $X^{\text{irr}}(f)$, a certain subset of $\hat{G}(f)$ ($f \in \mathcal{O}_T$).* As clearly indicated by the case of compact groups, only orbits of G in $\mathfrak{g}^*$ satisfying certain integrality conditions are possibly associated to representations of the Lie group G. Thus Kirillov's formula relates characters of the Lie group G to Fourier transforms of certain "admissible" orbits of G in $\mathfrak{g}^*$. (We refer to [2] for the general definitions. I will give in Section 4 of this article the necessary definitions when $\mathfrak{g}$ is semi-simple.)

Can we also relate the Plancherel formula for G to Fourier analysis on these admissible orbits? I have conjectured in [15] the following Poisson-Plancherel formula:

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let e be the identity element of G. Let

$$E_G = \{x \in \mathfrak{g};\, \exp X = e\},$$
$$\mathfrak{g}^*_G = \{f \in \mathfrak{g}^*;\, f\ G - \text{admissible}\}.$$

Let us suppose (this is not an important restriction on $\mathfrak{g}$) that there exists an analytic function $j^{1/2}$ on $\mathfrak{g}$ such that $(j^{1/2})^2 = j$. Then we may conjecture: there exists a G-invariant measure dm on E_G and a G-invariant positive measure dm^* on $\mathfrak{g}^*_G$ such that the equality

$$\text{(C)} \qquad \int_{E_G} \varphi\, dm = \int_{\mathfrak{g}^*_G} (\varphi j^{1/2})^\wedge\, dm^*$$

holds for every Schwartz-function φ on $\mathfrak{g}$.

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*Under very general hypotheses, M. S. Khalgui has proved recently that

$$j(X)^{1/2}\operatorname{Tr} T_{f,\tau}(\exp X) = \dim \tau \int_{\mathcal{O}_T} e^{i(\xi, X)}\, d\beta(\xi),$$

where $d\beta(\xi)$ is the canonical Liouville measure on $\mathcal{O}_T$.

Of course, if φ has small support near 0, the first member of the formula reduces to $\varphi(0) = (\varphi \circ \mathrm{Log})(e)$. The second member, modulo Kirillov's conjecture, is an integral over $\mathfrak{g}_G^*/G$ of the characters $(\mathrm{Tr}\, T, \varphi \circ \mathrm{Log})$. Thus, if $P(T)$ is the Plancherel measure on $\hat{G}_r$, the quotient of the measure dm^* by the canonical measures of the orbits of G in $\mathfrak{g}_G^*$ should be the image of the measure $(\dim \tau)p(Tf, \tau)$ by the map $\hat{G}_r \to \mathfrak{g}_G^*/G$.

Let us say here that, although this conjecture was motivated by the Plancherel formula for G, it is formulated in purely geometrical terms and is independent of any knowledge of $\hat{G}$.

I will prove here a more precise version of this conjecture in the case of a linear semi-simple Lie group G (Theorem 3.5). As remarked before, although the terms of the precise form of the conjecture are prescribed by the explicit Plancherel theorem of Harish-Chandra, our proof is independent of the proof of the Plancherel formula. However, we relay on results of Harish-Chandra on properties of the invariant integral [4] and properties of the discrete series characters [5], throughout the paper.

The conjecture (C) is also true when G is solvable (M. Duflo–M. Vergne) or when G is a complex Lie group (M. Duflo) [3].

Let me indicate here the line of approach I will follow for the case where G is a real semi-simple Lie group. The set E_G is a countable union of elliptic orbits of G in $\mathfrak{g}$. The determination of discrete series characters by R. Herb [7], [10], leads, modulo W. Rossmann's theorem [13], to an explicit expression of the Fourier transforms of elliptic orbits. One of the main results of this article is an identity relating the Plancherel functions to discrete series constants. We feel this identity is important in itself (Theorem 2.15). That such an identity should exist was suggested by M. Duflo, as well as the idea of introducing characters of the center of G. I would like to thank him for many discussions of this work.

I have used extensively the method of R. Herb of 2-structures, her results on discrete series constants and have clearly plagiarized her approach to the Plancherel formula to obtain this Poisson formula. Many ideas and technical comments herein are owed to her.

I would like also to thank G. Heckman for discussions, and the referee for many helpful comments.

2. The Plancherel function and characters of discrete series

Let Φ be a root system on a vector space $\mathfrak{A}$. We denote by $W = W(\Phi)$ the Weyl group of the system Φ. Let us choose a positive system Φ^+ for Φ. We set $\rho = 1/2\Sigma_{\alpha\in\Phi^+}\alpha$.

We consider the Plancherel function P on $\mathfrak{A}_C$ defined by:

2.1.
$$P(h, \Phi^+) = \prod_{\alpha\in\Phi^+} \frac{\mathrm{sh}\left(\frac{\alpha}{2}, h\right)}{\mathrm{ch}\left(\frac{\alpha}{2}, h\right)}$$

i.e.
$$P = \prod_{\alpha\in\Phi^+} \frac{e^{\alpha/2} - e^{-\alpha/2}}{e^{\alpha/2} + e^{-\alpha/2}}.$$

The following transformation property of P is clear:

2.2.
$$P(w \cdot h) = \varepsilon(w) P(h) \quad \text{for } w \in W.$$

Following R. Herb, we will express $P(h, \Phi^+)$ for any root system Φ as a sum of functions P associated to systems of type A_1 or B_2.

We first recall the notion of 2-structure for Φ [9].

2.3. We say a root system $\varphi \subset \Phi$ is a 2-structure for Φ if

1) all simple factors of φ are of type A_1 or B_2;

2) if $\varphi^+ = \Phi^+ \cap \varphi$, then $\{w \in W(\Phi);\ w\varphi^+ = \varphi^+\}$ contains no elements of determinant -1.

(Clearly 2) is independent of the choice of Φ^+.)

Let $\mathcal{G}(\Phi)$ denote the set of all 2-structure for Φ. We have the following:

2.4. Theorem (R. Herb) [9], [10]).

a) $[\mathcal{G}(\Phi)]$ *is odd.*

b) *Any two elements of* $\mathcal{G}(\Phi)$ *are conjugate by* W.

Remark: φ is not assumed to be of the same rank as Φ.

For $\varphi \in \mathcal{G}(\Phi)$, we set $\varphi^+ = \Phi^+ \cap \varphi$.

2.5. Lemma [9]. *There exists a unique function* $\varepsilon(\varphi)$ *on* $\mathcal{G}(\Phi)$ *such that*

1) $\varepsilon(w \cdot \varphi) = \varepsilon(w)\varepsilon(\varphi)$, *if* $w \cdot \varphi^+ \subset \Phi^+$.

2) $\Sigma_{\varphi\in\mathcal{G}(\Phi)}\varepsilon(\varphi) = 1$.

Proof. Let φ_0 be an element of $\mathcal{G}(\Phi)$. For $\varphi \in \mathcal{G}(\Phi)$ there exists an element $w \in W$ such that $\varphi^+ = w \cdot \varphi_0^+$; by condition 2.3.2), $\varepsilon(w)$ is independent of the choice of w. Thus we define $\varepsilon(\varphi) = \varepsilon(w)c_0$, with c_0 to be determined. Now $\Sigma\varepsilon(\varphi)$ is not zero, as the $\varepsilon(w)$ are equal to ± 1 and $[\mathcal{G}(\Phi)]$ is odd. Thus we may normalize c_0 such that $\Sigma\varepsilon(\varphi) = 1$.

2.6. Theorem (R. Herb) [8].

$$P(h, \Phi^+) = \sum_{\varphi\in\mathcal{G}(\Phi)} \varepsilon(\varphi) P(h, \varphi^+).$$

Proof. Let us first remark that the right-hand side is a W-anti-invariant function on $\mathfrak{A}$: in fact

$$\begin{aligned}\varepsilon(\varphi)P(w^{-1}h, \varphi^+) &= \varepsilon(\varphi)P(h, w\cdot\varphi^+)\\ &= \varepsilon(\varphi)\varepsilon(s)P(h,(w\cdot\varphi)^+)\\ &\qquad \text{for } (w\cdot\varphi)^+ = sw\varphi^+ \quad \text{with } s\in W(w\cdot\varphi)\\ &= \varepsilon(w)\varepsilon(w\cdot\varphi)P(h,(w\cdot\varphi)^+)\\ &\qquad \text{by the property 2.5.1).}\end{aligned}$$

Let

$$D(h,\Phi^+) = \prod_{\alpha\in\Phi^+}\operatorname{ch}\left(\frac{\alpha}{2},h\right) = \prod_{\alpha\in\Phi^+}\tfrac{1}{2}\left(e^{(\alpha/2,h)} + e^{-(\alpha/2,h)}\right).$$

$D(h, \Phi^+)$ is a W-invariant function on $\mathfrak{A}$.

Let

$$N(h,\varphi,\Phi^+) = \prod_{\alpha\in\varphi^+}\operatorname{sh}\left(\frac{\alpha}{2},h\right)\prod_{\beta\in\Phi^+-\varphi^+}\operatorname{ch}(\beta/2,h).$$

With a common denominator, the right hand side is

$$\frac{\sum_{\varphi}\varepsilon(\varphi)N(h,\varphi,\Phi^+)}{D(h,\Phi^+)}.$$

As the function $\Sigma_\varphi\varepsilon(\varphi)N(h,\varphi,\Phi^+)$ is W-anti-invariant, it is divisible by $\Pi_{\alpha\in\Phi^+}(e^{\alpha/2} - e^{-\alpha/2})$. In view of the coefficient of the term e^{ρ}, the coefficient of divisibility is seen to be $\Sigma_\varphi\varepsilon(\varphi) = 1$, Q.E.D.

In particular we will use the following trigonometric formula (which is a special case of 2.6, for $\Phi = A_2$!):

2.7 $$\operatorname{th}\alpha\operatorname{th}\beta\operatorname{th}(\alpha+\beta) = \operatorname{th}\alpha + \operatorname{th}\beta - \operatorname{th}(\alpha+\beta).$$

2.8. We introduce the following lattices in $\mathfrak{A}$:

$$R_Z^\Phi = \{h\in\mathfrak{A};\,(\alpha,h)\in 2\pi\mathbf{Z},\,\alpha\in\Phi\};$$

$$R_E^\Phi = \sum_{\alpha\in\Phi}\mathbf{Z}2\pi H_\alpha, \text{ where the } H_\alpha\text{'s are the coroots.}$$

Clearly $R_E^\Phi \subset R_Z^\Phi$, and if $\xi\in R_Z^\Phi$, $w\in W$, $w\cdot\xi - \xi\in R_E^\Phi$.

2.9. We say that Φ is spanned by strongly orthogonal roots, if there exists Σ a subset of Φ composed of strongly orthogonal roots (i.e. if $\alpha, \beta\in\Sigma$, then

$\alpha \pm \beta \notin \Phi$) such that $\mathfrak{A} = \sum_{\xi \in \Sigma} \mathbf{R} H_\xi$. Such a Φ is of type A_1, B_n, C_n, D_{2n}, E_7, E_8, F_4 or G_2. Let us introduce

$$\Gamma_\Phi = \tfrac{1}{2} R_E^\Phi = \sum_{\alpha \in \Phi} \mathbf{Z} \pi H_\alpha .$$

2.10. LEMMA. *If Φ is spanned by strongly orthogonal roots, then*

$$R_Z^\Phi \subset \Gamma_\Phi .$$

Proof. Assume $h \in R_Z^\Phi$, say $h = \sum_{\xi \in \Sigma} c_\xi H_\xi$, with $c_\xi \in \mathbf{R}$. By strong orthogonality, $(\xi, h) = 2c_\xi$. But also $(\xi, h) \in 2\pi \mathbf{Z}$, by definition of R_Z^Φ. Thus $c_\xi \in \pi \mathbf{Z}$, and $h \in \Gamma_\Phi$.

The study of averaged discrete series characters [5], [7] leads to the definition of functions $\bar{c}(h, \Phi^+)$ on $\mathfrak{A}$ with the following properties:

2.11. 1) $\bar{c}(h, \Phi^+) = 0$, if Φ^+ is not spanned by strongly orthogonal roots.

2) $\bar{c}(w \cdot h, w\Phi^+) = \bar{c}(h, \Phi^+)$ for $w \in W$.

3) The function $h \to \bar{c}(h, \Phi^+)$ is constant on each connected component of $\mathfrak{A}' = \{h;\, \alpha(h) \neq 0,\ \alpha \in \Phi\}$.

4) For every $\lambda \in \mathfrak{A}^*$, the function

$$h \to \sum_{w \in W} \bar{c}(w \cdot h, \Phi^+) e^{-(w \cdot h, \lambda)}$$

is continuous on $\mathfrak{A}$.

5) $\bar{c}(h, \Phi^+) = 0$ if $h \notin \overline{\sum_{\alpha \in \Phi^+} \mathbf{R}^+ H_\alpha}$.

6) Let α be a simple root of Φ^+,

$$\Phi_\alpha = \{\beta \in \Phi, (\beta, \alpha) = 0\},$$
$$\Phi_\alpha^+ = \Phi_\alpha \cap \Phi^+ ,$$
$$\mathfrak{A}_\alpha = \sum_{\beta \in \Phi_\alpha} \mathbf{R} H_\beta .$$

Let p_α be the orthogonal projection of $\mathfrak{A}$ onto $\mathfrak{A}_\alpha$, then for $h \in \mathfrak{A}'$,

$$\bar{c}(h, \Phi^+) + \bar{c}(s_\alpha h, \Phi^+) = 2\bar{c}(p_\alpha h, \Phi_\alpha^+).$$

7) If $\Phi = \{\alpha, -\alpha\}$ is of type A_1

$$\begin{aligned} \bar{c}(hH_\alpha, \alpha) &= 2 \quad \text{if } h > 0, \\ &= 0 \quad \text{if } h < 0, \\ &= 1 \quad \text{if } h = 0. \end{aligned}$$

Formulas for the "constants" $\bar{c}(h, \Phi^+)$ are given by R. Herb [7], [10]: if Φ is a system spanned by strongly orthogonal roots, if $\varphi \in \mathcal{G}(\Phi)$, then φ is of the same rank as Φ and we have the following:

2.12. THEOREM (R. Herb) [9]. *Let Φ be a system spanned by strongly orthogonal roots; then:*

$$\bar{c}(h, \Phi^+) = \sum_{\varphi \in \mathcal{G}(\Phi)} \varepsilon(\varphi)\bar{c}(h, \varphi^+).$$

We will also need the following result, which can be verified case by case.

2.13. Let Φ be a root system spanned by strongly orthogonal roots, then: for $\xi \in R_Z^\Phi$, $\varphi \in \mathcal{G}(\Phi)$, $e^{i(\rho_{\varphi^+}, \xi)} = e^{i(\rho_{\Phi^+}, \xi)}$.

Let χ be a character of Γ_Φ trivial on R_E^Φ, i.e.

$$\chi(\pi H_\alpha) = \pm 1, \quad \text{for all } \alpha \in \Phi.$$

We define:

$$p(\lambda, \chi, \Phi^+) = \prod_{\alpha \in \Phi^+} \frac{\operatorname{sh}(\pi\lambda, H_\alpha)}{\operatorname{ch}(\pi\lambda, H_\alpha) + (\chi, \pi H_\alpha)}.$$

We denote by $\hat{\Gamma}_\Phi$ the set of characters of Γ_Φ.

2.14. *Remark.* Let μ be a linear function on $\mathfrak{A}$ such that

$$(\chi, \pi H_\alpha) = e^{i\pi(\mu, H_\alpha)} \qquad (\text{with } (\mu, H_\alpha) \in \mathbf{Z});$$

then

$$p(\lambda, \chi, \Phi^+) = \prod_{\alpha \in \Phi^+} \operatorname{th}\left(\frac{(\pi(\lambda + i\mu), H_\alpha)}{2}\right).$$

Proof. This follows from the equalities

$$\frac{\operatorname{sh} t}{\operatorname{ch} t + 1} = \operatorname{th} t/2,$$

$$\frac{\operatorname{sh} t}{\operatorname{ch} t - 1} = \operatorname{coth} t/2.$$

2.15. THEOREM. *Let Φ be a root system spanned by strongly orthogonal roots. Let χ be a character of the lattice R_Z^Φ trivial on R_E^Φ. Let*

$$\mathcal{F}_{\Phi^+} = \{\lambda \in \mathfrak{A}^*; (\lambda, H_\alpha) > 0, \quad \text{for all } \alpha \in \Phi^+\};$$

then:

$$\frac{1}{[\Gamma_\Phi : R_Z^\Phi]} \sum_{\substack{\delta \in \hat{\Gamma}_\Phi \\ \delta|R_Z^\Phi = \chi}} p(\lambda, \delta, \Phi^+) = \sum_{\xi \in R_Z^\Phi} \bar{c}(\xi, \Phi^+) e^{i(\rho, \xi)} \chi(\xi) e^{-(\xi, \lambda)}$$

as an equality between analytic functions on $\mathcal{F}_{\Phi^+}$.

Proof. We will reduce our computation to case A_1 or B_2. By Remark 2.14, it is sufficient to consider the case where $\chi \equiv 1$.

Let us first consider case A_1: Let

$$\mathfrak{A} = \mathbf{R}e_1, \qquad \Phi^+ = e_1^*, \qquad \lambda = \lambda e_1^*, \qquad \lambda > 0.$$

As $R_Z^\Phi = \mathbf{Z}(2\pi e_1)$, $\rho(2\pi e_1) = \pi$, the right-hand side, for $\chi \equiv 1$, is the absolutely convergent series:

$$1 + 2\Big(\sum_{n>0}(-1)^n e^{-2\pi n\lambda}\Big) = 2\sum_{n\geq 0}(-1)^n e^{-2\pi n\lambda} - 1$$
$$= \frac{2}{1+e^{-2\pi\lambda}} - 1 = \frac{1-e^{-2\pi\lambda}}{1+e^{-2\pi\lambda}} = \operatorname{th}\pi\lambda.$$

We now consider case B_2. Let $\mathfrak{A} = \mathbf{R}e_1 + \mathbf{R}e_2$,

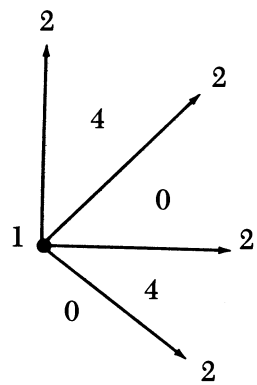
$$\Phi^+ = (e_1^* - e_2^*, e_1^* + e_2^*, e_1^*, e_2^*);$$

then

$$R_Z^\Phi = \mathbf{Z}2\pi e_1 + \mathbf{Z}2\pi e_2,$$
$$\Gamma_\Phi = \mathbf{Z}\pi(e_1 + e_2) + \mathbf{Z}\pi(e_1 - e_2),$$
$$\rho = \tfrac{1}{2}(3e_1^* + e_2^*).$$

It is easy to deduce from the relations 2.11 the values of the averaged discrete series constants $\bar{c}(h, \Phi^+)$. We obtain:

$$\bar{c}(xe_2 + y(e_1+e_2)) = 4 \quad \text{if } x>0,\ y>0,$$
$$= 2 \quad \text{if} \begin{cases} x>0 \\ y=0 \end{cases} \text{or} \begin{cases} x=0 \\ y>0, \end{cases}$$
$$= 1 \quad \text{if } x=y=0;$$
$$\bar{c}(xe_1 + y(e_1-e_2)) = 4 \quad \text{if } x>0,\ y>0,$$
$$= 2 \quad \text{if} \begin{cases} x>0 \\ y=0 \end{cases} \text{or} \begin{cases} x=0 \\ y>0, \end{cases}$$
$$= 1 \quad \text{if } x=y=0;$$



$\bar{c} = 0$ if not in one of the preceding sets.

Let us consider a character δ of Γ_Φ trivial on R_Z. Thus $\delta(\pi(e_1+e_2)) = \delta(\pi(e_1-e_2)) = \varepsilon = \pm 1$. Let $\lambda = \lambda_1 e_1^* + \lambda_2 e_2^*$. The first member M of our equality is given by:

$$\frac{1}{2}\operatorname{th}\pi\lambda_2 \operatorname{th}\pi\lambda_1 \operatorname{th}\frac{\pi(\lambda_1+\lambda_2)}{2}\operatorname{th}\frac{\pi(\lambda_1-\lambda_2)}{2}$$
$$+\frac{1}{2}\operatorname{th}\pi\lambda_2 \operatorname{th}\pi\lambda_1 \operatorname{coth}\frac{\pi(\lambda_1+\lambda_2)}{2}\operatorname{coth}\frac{\pi(\lambda_1-\lambda_2)}{2}$$

By 2.7

$$\operatorname{th}\pi\lambda_1 \operatorname{th}\pi\frac{\lambda_1+\lambda_2}{2}\operatorname{th}\pi\frac{\lambda_1-\lambda_2}{2} = \operatorname{th}\pi\left(\frac{\lambda_1+\lambda_2}{2}\right) + \operatorname{th}\pi\left(\frac{\lambda_1-\lambda_2}{2}\right) - \operatorname{th}\pi\lambda_1,$$

$$\operatorname{th}\pi\lambda_1 \coth\pi\frac{\lambda_1+\lambda_2}{2}\coth\pi\frac{\lambda_1-\lambda_2}{2} = \coth\pi\frac{\lambda_1+\lambda_2}{2} + \coth\pi\frac{\lambda_1-\lambda_2}{2} - \operatorname{th}\pi\lambda_1.$$

Using the formula $\operatorname{th} t + \coth t = 2\coth 2t$, we may rewrite M as

$$M = \operatorname{th}\pi\lambda_2 \coth\pi(\lambda_1+\lambda_2) + \operatorname{th}\pi\lambda_2\coth\pi(\lambda_1-\lambda_2) - \operatorname{th}\pi\lambda_1\operatorname{th}\pi\lambda_2.$$

Let c_1 be the function on $\mathfrak{A}$ given by

$$c_1(xe_2 + y(e_1+e_2)) = \bar{c}(x)\bar{c}(y) \quad \text{with } \bar{c}(x)\begin{cases} = 2 & \text{if } x > 0, \\ = 1 & \text{if } x = 0, \\ = 0 & \text{if } x < 0, \end{cases}$$

Then, by the preceding calculation on A_1, we see that

$$\operatorname{th}\pi\lambda_2\coth\pi(\lambda_1+\lambda_2) = \sum_{\xi\in R_Z} c_1(\xi)e^{i(\rho,\xi)}e^{-(\lambda,\xi)}.$$

Similarly, we consider c_2 defined by

$$c_2(xe_2 + y(e_1-e_2)) = \bar{c}(x)\bar{c}(y),$$

and $c_3(xe_1 + ye_2) = \bar{c}(x)\bar{c}(y)$.

Then

$$M = \sum_{\xi\in R_Z}(c_1(\xi)+c_2(\xi)-c_3(\xi))e^{i(\rho,\xi)}e^{-(\lambda,\xi)}.$$

It is easily verified that $c_1(\xi)+c_2(\xi)-c_3(\xi) = \bar{c}(\xi,\Phi^+)$, proving thus Theorem 2.15 for case B_2. (We wish to thank G. Heckman, who offered us an alternate proof of this formula for B_2).

We now use the notion of 2-structures to prove Theorem 2.15. Let $\varphi\subset\Phi$ be a 2-structure; we have the inclusions:

$$R_Z^{\Phi}\subset R_Z^{\varphi}\subset\Gamma_\varphi\subset\Gamma_\Phi.$$

Using 2.6 and 2.14, we have

$$p(\lambda,\delta,\Phi^+) = \sum_{\varphi\in\mathcal{S}(\Phi)}\varepsilon(\varphi)p(\lambda,\delta,\varphi^+)$$

where we still denote by δ the restriction of $\delta \in \hat{\Gamma}_\Phi$ to Γ_φ. But

$$\frac{1}{[\Gamma_\Phi : R_Z^\Phi]} \sum_{\substack{\delta \in \hat{\Gamma}_\Phi \\ \delta | R_Z^\Phi = \chi}} p(\lambda, \delta, \varphi^+) = \frac{1}{[\Gamma_\varphi : R_Z^\Phi]} \sum_{\substack{\delta \in \hat{\Gamma}_\varphi \\ \delta | R_Z^\Phi = \chi}} p(\lambda, \delta, \varphi^+)$$

$$(*) \qquad = [R_Z^\varphi : R_Z^\Phi]^{-1} [\Gamma_\varphi : R_Z^\varphi]^{-1} \sum_{\substack{\mu \in \hat{R}_Z^\varphi \\ \mu | R_Z^\Phi = \chi}} \sum_{\substack{\delta \in \hat{\Gamma}_\varphi \\ \delta | R_Z^\varphi = \mu}} p(\lambda, \delta, \varphi^+).$$

As φ is the product of factors of type B_2 or A_1,

$$[\Gamma_\varphi : R_Z^\varphi]^{-1} \sum_{\substack{\delta \in \hat{\Gamma}_\varphi \\ \delta | R_Z^\varphi = \mu}} p(\lambda, \delta, \varphi^+) = \sum_{\xi \in R_Z^\varphi} \bar{c}(\xi, \varphi^+) e^{-i(\rho_\varphi, \xi)} \mu(\xi) e^{-(\lambda, \xi)}.$$

But, if $\xi \in R_Z^\varphi$,

$$[R_Z^\varphi : R_Z^\Phi]^{-1} \sum_{\substack{\mu \in \hat{R}_Z^\varphi \\ \mu | R_Z^\Phi = \chi}} \mu(\xi) = 0 \quad \text{if } \xi \notin R_Z^\Phi,$$

$$= \chi(\xi) \quad \text{if } \xi \in R_Z^\Phi.$$

Thus $(*)$ becomes

$$\sum_{\xi \in R_Z^\Phi} \bar{c}(\xi, \varphi^+) e^{-i(\rho_\varphi, \xi)} \chi(\xi) e^{-(\lambda, \xi)}$$

and the left-hand side of formula 2.15 is equal to

$$\sum_{\xi \in R_Z^\Phi} \sum_{\varphi \in \mathcal{G}(\Phi)} \varepsilon(\varphi) \bar{c}(\xi, \varphi^+) e^{-i(\rho_\varphi, \xi)} \chi(\xi) e^{-(\lambda, \xi)}.$$

Our theorem then follows from 2.12, as we have

$$e^{i(\rho_{\varphi^+}, \xi)} = e^{i(\rho_{\Phi^+}, \xi)}, \quad \text{if } \xi \in R_Z^\Phi. \qquad \text{Q.E.D.}$$

3. The Poisson-Plancherel formula

Let $\mathfrak{g}$ be a real, simple Lie algebra. Let $\mathfrak{g}_C$ be the complexification of the Lie algebra $\mathfrak{g}$ and G_C the simply connected Lie group with Lie algebra $\mathfrak{g}_C$. Let G be the real, analytic subgroup of G_C with Lie algebra $\mathfrak{g}$. Let Z be the center of G, Z_C the center of G_C, e the identity element of the group G_C.

We consider the exponential map exp: $\mathfrak{g}_C \to G_C$ and define:

3.1. $$E = \{X \in \mathfrak{g}_C, \exp X = e\},$$

3.2. $$E_{Z_C} = \{X \in \mathfrak{g}_C, \exp X \in Z_C\},$$

3.3. $$E_Z = \{X \in \mathfrak{g}, \exp X \in Z\}.$$

Let $\mathfrak{h}_{\mathbf{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Let $\Delta = \Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be the system of roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$. For $\alpha \in \Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, let $H_\alpha \in [\mathfrak{g}_{\mathbf{C}}(\alpha), \mathfrak{g}_{\mathbf{C}}(-\alpha)]$ be such that $(\alpha, H_\alpha) = 2$. Δ is a root-system on the vector space $\sum_{\alpha\in\Delta}\mathbf{R}H_\alpha$. We have, in the notations of 2.8, this standard lemma:

3.3. LEMMA. a) $E \cap \mathfrak{h}_{\mathbf{C}} = \sum_{\alpha\in\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})} 2i\pi\mathbf{Z}H_\alpha = iR_E^\Delta$.
b) $E_{Z_{\mathbf{C}}} \cap \mathfrak{h}_{\mathbf{C}} = \{X \in \mathfrak{h}_{\mathbf{C}}, (\alpha, X) \in 2i\pi\mathbf{Z}$, *for every* $\alpha \in \Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})\}$
$= iR_{\mathbf{Z}}^\Delta$.

For $\alpha \in \Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, we denote $\gamma_\alpha = \exp(i\pi H_\alpha)$. Using the map φ_α from $\mathrm{SL}(2, \mathbf{C})$ into $G_{\mathbf{C}}$ associated to the root α, we have $\gamma_\alpha = \exp \pi(X_\alpha - X_{-\alpha}) = \varphi_\alpha\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let θ be the corresponding involution. Let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$.

Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. We have $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. The sets $E \cap \mathfrak{h}$ and $E_{Z_{\mathbf{C}}} \cap \mathfrak{h}$ are lattices in $\mathfrak{t}$. The sets $E \cap \mathfrak{h}_{\mathbf{C}}$ and $E_{Z_{\mathbf{C}}} \cap \mathfrak{h}_{\mathbf{C}}$ are lattices in $\mathfrak{t} + i\mathfrak{a}$. We denote by p the projection of $\mathfrak{h}_{\mathbf{C}}$ to $\mathfrak{a}_{\mathbf{C}}$.

Let $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be the root system of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$,

$\Delta_R(\mathfrak{g}, \mathfrak{h})$ be the subsystem of real roots ($\alpha \in \Delta_R(\mathfrak{g}, \mathfrak{h})$,

$$\begin{cases} \text{if } \alpha \text{ takes real values on } \mathfrak{h} \text{ or equivalently,} \\ \text{if } \alpha \text{ vanishes on } \mathfrak{t}), \end{cases}$$

$\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be a positive system for $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$,

$$\Delta_{\mathbf{R}}^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}) \cap \Delta_{\mathbf{R}}(\mathfrak{g}, \mathfrak{h}),$$

$$\rho = \tfrac{1}{2} \sum_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})} \alpha,$$

$$\rho_{\mathbf{R}} = \tfrac{1}{2} \sum_{\alpha\in\Delta_{\mathbf{R}}^+(\mathfrak{g}, \mathfrak{h})} \alpha,$$

$$\pi(h) = \prod_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})} (\alpha, h),$$

$$\pi_{\mathbf{R}}(h) = \prod_{\alpha\in\Delta_{\mathbf{R}}^+(\mathfrak{g}, \mathfrak{h})} (\alpha, h),$$

$\Delta_I(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be the subsystem of imaginary roots ($\alpha \in \Delta_I(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ if α vanishes on $\mathfrak{a}$),

$\Delta_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be the set of complex roots (i.e. those which are neither real nor imaginary).

$W_{\mathbf{R}}$ be the group generated by the reflections s_α, $\alpha \in \Delta_{\mathbf{R}}$.

Let H be the centralizer of $\mathfrak{h}$ in G, $N(G, \mathfrak{h})$ be the normalizer of $\mathfrak{h}$ in G, $W(G, \mathfrak{h}) = N(G, \mathfrak{h})/H$. We have $H = TA$, with $T = H \cap K$, $A = \exp \mathfrak{a}$. The group $W(G, \mathfrak{h})$ leaves T and A stable. The group T is not necessarily connected. For $\alpha \in \Delta_{\mathbf{R}}(\mathfrak{g}, \mathfrak{h})$ the element $\gamma_\alpha = \exp(i\pi H_\alpha) = \exp(\pi(X_\alpha - X_{-\alpha}))$ is in T. Let T^0 be the connected component of T. We consider the lattices $\Gamma_{\Delta_{\mathbf{R}}}$, $R_Z^{\Delta_{\mathbf{R}}}$, $R_E^{\Delta_{\mathbf{R}}}$ (2.8–2.9) of the subspace $\Sigma_{\alpha\in\Delta_{\mathbf{R}}}\mathbf{R}H_\alpha$ of $\mathfrak{a}$.

3.4. LEMMA. *The map* $\xi \to \exp(i\xi)$ *defines an isomorphism of*
a) $\Gamma_{\Delta_{\mathbf{R}}}/p(R_E^\Delta) \cap \Gamma_{\Delta_{\mathbf{R}}}$ *with* T/T^0
b) $\Gamma_{\Delta_{\mathbf{R}}}/p(R_Z^\Delta) \cap \Gamma_{\Delta_{\mathbf{R}}}$ *with* T/T^0Z.

Proof. a) follows from the fact that T/T^0 is generated by the elements γ_α (see [16]). For b), we remark that if $u = t + \xi$ with $u \in R_Z^\Delta$, $\xi \in \Gamma_{\Delta_{\mathbf{R}}}$, $t \in i\mathfrak{t}$, then $\exp(iu) \in G \cap Z_{\mathbf{C}} = Z$.

3.5 LEMMA. *Let* $w \in W_{\mathbf{R}}$. *The character* $\xi \to e^{i(w\cdot\rho_{\mathbf{R}} - \rho_{\mathbf{R}}, \xi)}$ *of* $\Gamma_{\Delta_{\mathbf{R}}}$ *is trivial on* $p(R_Z^\Delta) \cap \Gamma_{\Delta_{\mathbf{R}}}$.

Proof. As every real root vanishes on $\mathfrak{t}$, $p(R_Z^\Delta) \cap \Gamma_{\Delta_{\mathbf{R}}} \subset R_Z^{\Delta_{\mathbf{R}}}$. But if $\xi \in R_Z^{\Delta_{\mathbf{R}}}$, $w \in W_{\mathbf{R}}$, $w \cdot \xi - \xi \in R_E^{\Delta_{\mathbf{R}}}$ and the lemma follows.

Let b be a character of H, $\log b \in i\mathfrak{h}^*$ its differential. We write also $b = \mu \otimes e^{i\nu}$, where μ is a character of T and ν a real linear form on $\mathfrak{a}$. We define

$$3.6. \qquad P(b, \Delta_{\mathbf{R}}^+) = \prod_{\alpha\in\Delta_{\mathbf{R}}^+} \frac{\operatorname{sh}(\pi\nu, H_\alpha)}{\operatorname{ch}(\pi\nu, H_\alpha) + \mu(\gamma_\alpha)(-1)^{(\rho_{\mathbf{R}}, H_\alpha)}}.$$

Let us consider $W_{\mathbf{R}}$ as a subgroup of $W(G, \mathfrak{h})$. Let $W(\Delta_{\mathbf{R}}^+)$ be the stabilizer of the set $\Delta_{\mathbf{R}}^+$ in $W(G, \mathfrak{h})$. It is clear that $W(G, \mathfrak{h}) = W_{\mathbf{R}} \cdot W(\Delta_{\mathbf{R}}^+)$. From 3.4 b) and 3.5, for $w \in W(G, \mathfrak{h})$, we can consider $\delta_w(\exp i\xi) = e^{i(w\cdot\rho_{\mathbf{R}} - \rho_{\mathbf{R}}, \xi)}$ as a character of T/T^0Z. We define also $\varepsilon_{\mathbf{R}}(w)$ on $W(G, \mathfrak{h})$ by $w \cdot \pi_{\mathbf{R}} = \varepsilon_{\mathbf{R}}(w)\pi_{\mathbf{R}}$. The transformation property for P under $W_{\mathbf{R}}$ is the following:

$$3.7. \qquad P\big(w \cdot (\mu \otimes e^{i\nu}), \Delta_{\mathbf{R}}^+\big) = \varepsilon_{\mathbf{R}}(w) P\big(\mu\delta_w \otimes e^{i\nu}, \Delta_{\mathbf{R}}^+\big).$$

We will identify freely $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{h}$ with $\mathfrak{h}^*$ by the Killing form. We define the Fourier transform on $\mathfrak{g}$ and $\mathfrak{h}$ by:

$$(F_{\mathfrak{g}}\Phi)(x) = \int e^{i(x,y)}\Phi(y)\,dy,$$

$$(F_{\mathfrak{h}}\Phi)(t) = \int e^{i(t,s)}\Phi(s)\,ds.$$

The Lebesgue measures on $\mathfrak{g}$ and $\mathfrak{h}$ are normalized so that $(F_{\mathfrak{g}}^2\Phi)(x) = \Phi(-x), (F_{\mathfrak{h}}^2\Phi)(t) = \Phi(-t)$ for rapidly decreasing functions Φ. We normalize $d\dot{g}$

on G/H such that, for φ a function on $\mathfrak{g}$ supported on $G \cdot \mathfrak{h}$,

$$\int \varphi(x)\, dx = \int |\pi(h)|^2 \left(\int_{G/H} \varphi(g \cdot h) d\dot{g} \right) dh.$$

We recall that each orbit $\Omega = G \cdot f$ of G in $\mathfrak{g}^*$ carries a canonical measure μ_Ω: First define a skew form B_f on the tangent space $T_f(\Omega) = \mathfrak{g} \cdot f$ on Ω at f by $B_f(X \cdot f, Y \cdot f) = (f, [X, Y])$. As this form is non-degenerate, one can define a volume element on $T_f(\Omega)$ by assigning the volume

$$(2\pi)^{-\dim \Omega/2} |(\det B_f(u_i, u_j))|^{1/2}$$

to the parallelepiped spanned by a basis u_i of $T_f\Omega$. The measure obtained this way on Ω is the G-invariant measure μ_Ω. It is easy to see [13] that if h is regular in $\mathfrak{h}$ and $\Omega = G \cdot h$

$$\int_\Omega \varphi(\xi)\, d\mu_\Omega(\xi) = |W(G, \mathfrak{h})|\, |\pi(h)| \int_{G/H} \varphi(g \cdot h)\, d\dot{g}.$$

We define the Kirillov distribution K_Ω on $\mathfrak{g}$ by

$$(K_\Omega, \varphi) = \int_\Omega (F\mathfrak{g}\varphi)(\xi)\, d\mu_\Omega(\xi),$$

for a closed orbit Ω of G in $\mathfrak{g}^*$. If $\lambda \in \mathfrak{h}^*$, we note that $(K_\lambda, \varphi) = (K_{G\cdot\lambda}, \varphi)$. We denote by $\hat{H}$ the dual of the abelian group H. We denote by db a Haar measure on $\hat{H}$ (an appropriate normalization of db will be computed in the proof, in an explicit form, in the case where rank of G = rank of K). Let χ be a character of the center Z of G. We set

$$\hat{H}(\chi) = \{ b \in \hat{H};\, b \,|\, Z = \chi \}.$$

Let $\mathfrak{b}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. We have $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$, with $\mathfrak{b}_0 = \mathfrak{b} \cap k$. Let B be the Cartan subgroup of G corresponding to $\mathfrak{b}$. Let us choose a positive system $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ for $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. For h regular in $\mathfrak{h}$, we define

$$(M\varphi)(h) = \pi(h) \int_{G/B} \varphi(g \cdot h)\, d\dot{g}.$$

Let D_π be the constant coefficient differential operator on $\mathfrak{b}$ corresponding to π. It is known [4] that the function $h \to (D_\pi Mf)(h)$ extends to a continuous function on $\mathfrak{b}$ (and is independent of the order on $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$). (We recall that with the preceding normalizations $f(0) = (-1)^{m_\mathfrak{g}}(D_\pi Mf)(0)$ with $m_\mathfrak{g}$ = 1/2 (dim $\mathfrak{g}/k$-rank of $\mathfrak{g}$ + rank of k) [4].)

We denote by Car $\mathfrak{g}$ the set of conjugacy classes of Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ under G. We will prove the following:

3.8. THEOREM. *Let χ be a character of the center Z of G; then for every $f \in \mathcal{S}(\mathfrak{g})$*

$$(-1)^{m_\mathfrak{g}} \sum_{\xi \in E_Z \cap \mathfrak{b}} \chi(\exp \xi)(D_\pi Mf)(\xi) = \sum_{\mathfrak{h} \in \operatorname{Car} \mathfrak{g}} \int_{b \in \hat{H}(\chi)} |P(b)\pi(\log b)| \left(K_{(\log b/i)}, f\right) db.$$

Proof. We will first prove Theorem 3.8 when rank of $\mathfrak{g}$ = rank of k. Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in k. As all the roots of $\Delta(\mathfrak{g}_\mathbf{C}, \mathfrak{b}_\mathbf{C})$ take imaginary values on $\mathfrak{b}$, $\mathfrak{b}_\mathbf{C} \cap E_{Z_\mathbf{C}} = \mathfrak{b} \cap E_Z$. In particular $Z_\mathbf{C} = \exp(\mathfrak{b}_\mathbf{C} \cap E_{Z_\mathbf{C}}) = \exp(\mathfrak{b} \cap E_Z) = Z$. Let $W_\mathbf{C}$ be the Weyl group associated to $\Delta(\mathfrak{g}_\mathbf{C}, \mathfrak{b}_\mathbf{C})$; then $W_\mathbf{C}$ leaves $\mathfrak{b}$ stable. We consider, as in R. Herb [7], the averaged orbital integral

$$(\overline{M}f)(h) = \sum_{w \in W_\mathbf{C}} \varepsilon(w)(Mf)(w \cdot h).$$

As our lattice $E_Z \cap \mathfrak{b}$ as well as the character $\xi \to \chi(\exp \xi)$ are invariant by $W_\mathbf{C}$, we may rewrite the first term of our sum as

$$(-1)^{m_\mathfrak{g}}[W_\mathbf{C}]^{-1} \sum_{\xi \in E_Z \cap \mathfrak{b}} \chi(\exp \xi)(D_\pi \overline{M}f)(\xi).$$

Let us compute the Fourier transform of the distribution $f \to (\overline{M}f)(h)$, for h regular. By Harish-Chandra, we have:

$$(\overline{M}f)(h) = \int_{\mathfrak{g}^*} \bar{\theta}(h, \lambda)(F_\mathfrak{g} f)(\lambda)\, d\lambda$$

where $\lambda \to \bar{\theta}(h, \lambda)$ is a locally-integrable G-invariant function on $\mathfrak{g}^*$, which is analytic on the set of regular elements of $\mathfrak{g}^*$. We thus may write:

$$(\overline{M}f)(h) = \sum_{\mathfrak{h} \in \operatorname{Car} \mathfrak{g}} \int_{\mathfrak{h}^*} \bar{\theta}(h, \lambda) |\pi(\lambda)|^2 \left(\int_{G/H} (F_\mathfrak{g} f)(g \cdot \lambda)\, d\dot{g} \right) d\lambda.$$

Let us consider $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ a θ-stable Cartan subalgebra of $\mathfrak{g}$. We choose an element $y \in G_\mathbf{C}$ such that $y(\mathfrak{b}) = \mathfrak{t} \oplus i\mathfrak{a}$. We define $\Delta = \Delta(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$, $\Delta^+(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C}) = y(\Delta^+(\mathfrak{g}_\mathbf{C}, \mathfrak{b}_\mathbf{C}))$. From the condition rank of $\mathfrak{g}$ = rank of $\mathfrak{k}$, $\mathfrak{a}^*$ is generated by strongly orthogonal roots of $\Delta_\mathbf{R}$ and in particular $\Delta_\mathbf{R}$ is a root system on $\mathfrak{a}$. Let $\mathcal{F}^+ \subset \mathfrak{a}^*$ be the positive chamber of $\Delta_\mathbf{R}^+$. The group $W_\mathbf{R}$ operates on $\mathfrak{h}$ (leaving $\mathfrak{t}$ fixed). The function $\lambda \to \bar{\theta}(h, \lambda)$ is invariant by $W_\mathbf{R}$; thus it is enough to determine $\bar{\theta}(h, \lambda)$ when λ belongs to the fundamental domain $\mathfrak{t}^* \oplus \mathcal{F}^+$ for the action of $W_\mathbf{R}$ on $\mathfrak{h}$. For such a λ, we have

$$\bar{\theta}(h, \lambda) = i^{\#\Delta_+}(-1)^{m_\mathfrak{g}} \sum_{w \in W_\mathbf{C}} \frac{\varepsilon(w)\bar{c}(p(iywh), \Delta_\mathbf{R}^+)e^{-i(ywh, \lambda)}}{\pi(\lambda)}$$

for h regular in $\mathfrak{b}$, from results of Harish-Chandra [5] and Rossmann's theorem [13], [14], with the constants $\bar{c}$ satisfying the relations 2.11. By differentiation we obtain

3.9
$$(-1)^{m_{\mathfrak{g}}}(D_\pi \bar{M} f)(h) = \sum_{\mathfrak{h} \in \mathrm{Car}\, \mathfrak{g}} [W_{\mathbf{R}}] \int_{\mathfrak{t}^* \oplus \mathfrak{F}^+} \bar{R}(h, \lambda) \,|\pi(\lambda)|^2 \times \left(\int_{G/H} (F_{\mathfrak{g}} f)(g \cdot \lambda) d\dot{g} \right) d\lambda$$

with

$$\bar{R}(h, \lambda) = \sum_{w \in W_{\mathbf{C}}} \bar{c}(p(iywh), \Delta_{\mathbf{R}}^+) e^{-i(ywh, \lambda)}$$

for λ belonging to $\mathfrak{t}^* \oplus \mathfrak{F}^+$. From condition 2.11. 4) on the constants $\bar{c}$, the equality 3.9 is valid for any $h \in \mathfrak{b}$. Let, for $f \in \mathcal{S}(\mathfrak{g})$,

$$(d_{\mathfrak{h}}, f) = [W_{\mathbf{C}}]^{-1} \sum_{\xi \in E_Z \cap \mathfrak{b}} \chi(\exp \xi) \int_{\mathfrak{t}^* \oplus \mathfrak{F}^+} \bar{R}(\xi, \lambda) \,|\pi(\lambda)|\, (K_\lambda, f)\, d\lambda .$$

The left-hand side of 3.8 is equal to

$$\sum_{\mathfrak{h} \in \mathrm{Car}\, \mathfrak{g}} [W(G, \mathfrak{h})]^{-1} [W_{\mathbf{R}}] (d_{\mathfrak{h}}, f).$$

Let us consider $y(E_{Z_{\mathbf{C}}} \cap \mathfrak{b}_{\mathbf{C}}) = E_{Z_{\mathbf{C}}} \cap \mathfrak{h}_{\mathbf{C}} = iR_Z^\Delta$. As $E_Z \cap \mathfrak{b} = E_{Z_{\mathbf{C}}} \cap \mathfrak{b}_{\mathbf{C}}$ is stable by $W_{\mathbf{C}}$ and $\exp(y\xi) = \exp \xi$ for $\xi \in E_{Z_{\mathbf{C}}}$, we obtain:
3.10

$$(\bar{d}_{\mathfrak{h}}, f) = \sum_{\xi \in E_{Z_{\mathbf{C}}} \cap \mathfrak{h}_{\mathbf{C}}} \chi(\exp \xi) \int_{\mathfrak{t}^* \oplus \mathfrak{F}^+} \bar{c}(p(i\xi), \Delta_{\mathbf{R}}^+) e^{-i(\xi, \lambda)} \,|\pi(\lambda)|\, (K_\lambda, f)\, d\lambda .$$

If rank of $\mathfrak{g}$ = rank of $\mathfrak{k}$, Theorem 3.8 will then follow from

3.11. PROPOSITION.

$$[W(G, \mathfrak{h})]^{-1} [W_{\mathbf{R}}] (d_{\mathfrak{h}}, f) = \int_{b \in \hat{H}(\chi)} |P(b, \Delta_{\mathbf{R}}^+)| \,|\pi(\log b)|\, (K_{\log b / i}, f)\, db .$$

Proof. $\Delta_{\mathbf{R}}$ is a system of roots on $\mathfrak{a}$. We have $E_{Z_{\mathbf{C}}} \cap \mathfrak{h}_{\mathbf{C}} = iR_Z^\Delta$ and $p(R_Z^\Delta) \subset R_Z^{\Delta_{\mathbf{R}}}$. We will compute $(d_{\mathfrak{h}}, f)$ by summing first the ξ in $E_{Z_{\mathbf{C}}} \cap \mathfrak{h}_{\mathbf{C}}$ with $p(i\xi)$ fixed in $p(R_Z^\Delta)$, then summing up the result over $p(R_Z^\Delta)$. Let us fix $\xi_1 = p(i\xi)$ in $p(R_Z^\Delta)$. The set of elements ξ of $E_{Z_{\mathbf{C}}} \cap \mathfrak{h}_{\mathbf{C}}$ such that $p(i\xi) = \xi_1$ is

$\xi + E_Z \cap \mathfrak{t}$. We compute

3.12.

$$\sum_{\gamma \in E_Z \cap \mathfrak{t}} \chi(\exp(\xi+\gamma)) \int_{\mathfrak{t}^* \oplus \mathcal{F}^+} \bar{c}(p(i(\xi+\gamma)), \Delta_R^+) e^{-i(\xi+\gamma,\lambda)} |\pi(\lambda)| (K_\lambda, f)\, d\lambda$$

$$= \chi(\exp \xi) \bar{c}(\xi_1, \Delta_R^+) \sum_{\gamma \in E_Z \cap \mathfrak{t}} \chi(\exp \gamma) \int_{\mathfrak{t}^* \oplus \mathcal{F}^+} e^{-i(\gamma,\lambda)} e^{-i(\xi,\lambda)} |\pi(\lambda)| (K_\lambda, f)\, d\lambda.$$

Let us denote by $\varphi(\lambda)$ the function $|\pi(\lambda)|(K_\lambda, f)$. This is a continuous function of $\lambda(\lambda \to (K_\lambda, f)$ has its discontinuities on the set $\pi(\lambda) = 0$, with limits in each chamber [4]). Let

$$\mathfrak{t}^*(\chi) = \{u \in \mathfrak{t}^*, e^{i(u,\gamma)} = \chi(\exp \gamma) \quad \text{for all } \gamma \in E_Z \cap \mathfrak{t}\}$$

and $v(E_Z \cap \mathfrak{t})$ be the measure of a fundamental domain for the lattice $E_Z \cap \mathfrak{t}$ (for the self-dual measure on $\mathfrak{t}^*$). Then, by the Poisson summation formula, 3.12 is equal to

$$\chi(\exp \xi) \bar{c}(\xi_1, \Delta_R^+) v(E_Z \cap \mathfrak{t}) \sum_{u \in \mathfrak{t}^*(\chi)} \int_{\nu \in \mathcal{F}^+} e^{-i(u+\nu,\xi)} \varphi(u+\nu)\, d\nu$$

where now $d\nu$ is the self-dual measure on $\mathfrak{a}^*$. For $u \in \mathfrak{t}^*(\chi)$, the character $\xi \to e^{-i(u,\xi)}\chi(\exp \xi)$ of $E_{Z_C} \cap \mathfrak{h}_C$ is trivial on $E_Z \cap \mathfrak{t}$; thus

$$\delta_{u,\chi}(\xi_1) = e^{-i(u,\xi)} \chi(\exp \xi) \quad \text{for } p(i\xi) = \xi_1$$

is well defined. We thus obtain

$$(d_{\mathfrak{h}}, f) = v(E_Z \cap \mathfrak{t}) \sum_{u \in t^*(\chi)} \sum_{\xi \in p(R_Z^\Delta)} \delta_{u,\chi}(\xi) \bar{c}(\xi, \Delta_R^+) \int_{\nu \in \mathcal{F}^+} e^{-(\nu,\xi)} \varphi(u+\nu)\, d\nu.$$

We compute, for $\nu \in \mathcal{F}^+$

$$\sum_{\xi \in p(R_Z^\Delta)} \bar{c}(\xi, \Delta_R^+) \delta_{u,\chi}(\xi) e^{-(\nu,\xi)}.$$

As $p(R_Z^\Delta)$ is contained in $R_Z^{\Delta_R}$, this is equal by 2.15, to:

$$\frac{1}{[\Gamma_{\Delta_R} : p(R_Z^\Delta)]} \sum_{\substack{\delta \in \hat{\Gamma}_{\Delta_R} \\ \delta | p(R_Z^\Delta) = \delta_{u,\chi}}} p(\nu, \delta e^{-i\rho_R}, \Delta_R^+).$$

We define $\hat{T}(\chi) = \{\mu$, character of T such that $\mu | Z = \chi\}$.

3.13. LEMMA. *Let* $u \in \mathfrak{t}^*(\chi)$ *and*

$$\hat{T}(u, \chi) = \{\mu \in \hat{T}(\chi), \mu(\exp t) = e^{i(u,t)} \quad \text{for } t \in \mathfrak{t}\},$$
$$\hat{\Gamma}_{\Delta_R}(\delta_{u,\chi}) = \{\delta \in \hat{\Gamma}_{\Delta_R}, \delta \mid p(R_Z^\Delta) = \delta_{u,\chi}\}.$$

The equation $\delta(\xi) = \mu(\exp i\xi)$, $\xi \in \Gamma_{\Delta_R}$ *is a bijection of* $\hat{T}(u, \chi)$ *with* $\hat{\Gamma}_{\Delta_R}(\delta_{u,\chi})$.

Proof. It is easy to verify that if $\mu \in \hat{T}(u, \chi)$ and $\delta(\xi) = \mu(\exp i\xi)$, then $\delta \in \hat{\Gamma}_{\Delta_R}(\delta_{u,\chi})$. The lemma follows then from 3.4 as, in our case,

$$p(R_Z^\Delta) \subset R_Z^{\Delta_R} \subset \Gamma_{\Delta_R}.$$

If $\delta(\xi) = \mu(\exp i\xi)$ for $\xi \in \Gamma_{\Delta_R}$, $\mu \in \hat{T}(u, \chi)$, then

$$p(\nu, \delta e^{-i\rho_R}, \Delta_R^+) = P(\mu \otimes e^{i\nu}, \Delta_R^+)$$

with P defined in 3.6. Thus

$$(d_{\mathfrak{h}}, f) = v(E_Z \cap \mathfrak{t})[\Gamma_{\Delta_R} : p(R_Z^\Delta)] \times \sum_{u \in \mathfrak{t}^*(\chi)} \sum_{u \in \hat{T}(u,\chi)} \int_{\nu \in \mathfrak{F}^+} P(\mu \otimes e^{i\nu}, \Delta_R^+) \varphi(u + \nu)\, d\nu.$$

Let us now compute the right-hand side of 3.11. When b describes $\hat{H}(\chi)$, $\log b/i$ describes $\mathfrak{t}^*(\chi) \oplus \mathfrak{a}^*$. Thus the right-hand side of 3.11 is

$$c \sum_{u \in \mathfrak{t}^*(\chi)} \int_{\nu \in \mathfrak{a}^*} \Bigg(\sum_{\substack{b \in \hat{H}(\chi) \\ \log b/i = u + \nu}} |P(b)| \Bigg) \varphi(u + \nu)\, d\nu,$$

with c a positive constant to be determined.

Now, from 3.7,

$$\sum_{\substack{b \in \hat{H}(\chi) \\ \log b/i = u + \nu}} |P(b)|$$

is invariant under W_R and $P(b)$ is positive when $\nu \in \mathfrak{F}^+$. Thus, if we use the W_R-invariance, the right-hand side is

$$c[W_R] \sum_{u \in \mathfrak{t}^*(\chi)} \int_{\nu \in \mathfrak{F}^+} \Bigg(\sum_{\substack{b \in \hat{H}(\chi) \\ \log b/i = u + \nu}} P(b) \Bigg) \varphi(u + \nu)\, d\nu,$$

which proves 3.8 provided that

$$c = W(G, \mathfrak{h})^{-1} v(E_Z \cap \mathfrak{t})[\Gamma_{\Delta_R} : p(R_Z^\Delta)]^{-1}.$$

Thus our theorem is proved when rank of $\mathfrak{g}$ = rank of k.

Let now $\mathfrak{g}$ be an arbitrary semi-simple Lie algebra. Let $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$ be the fundamental θ-stable Cartan subalgebra and B the corresponding Cartan subgroup. Let $\mathfrak{g}_0$ be the centralizer of $\mathfrak{b}_1$. Then $\mathfrak{g}_0 = \mathfrak{M}_0 \oplus \mathfrak{b}_1$, with $\mathfrak{M}_0$ a reductive Lie algebra admitting $\mathfrak{b}_0$ as a compact Cartan subalgebra. Let G_0 be the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. We choose a positive system $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ such that if $\alpha \in \Delta^+$ and $\alpha \mid \mathfrak{b}_1 \neq 0$, then $\bar{\alpha} \in \Delta^+$. We set

$$\Delta^+((\mathfrak{g}_0)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}}) = \Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}}) \cap \Delta((\mathfrak{g}_0)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}}).$$

Let $\pi_{\mathfrak{g}}(h), \pi_{\mathfrak{g}_0}(h)$ be the corresponding functions on $\mathfrak{b}$ and D, D_0 the constant coefficient operators associated to $\pi_{\mathfrak{g}}, \pi_{\mathfrak{g}_0}$. We set

$$\pi^+(h) = \prod_{\substack{\alpha \in \Delta^+ \\ \alpha|\mathfrak{b}_1 \neq 0}} (\alpha, h).$$

Let us remark that $\pi^+(h)$ is a positive function.

For $f \in \mathcal{S}(\mathfrak{g})$ and $h \in \mathfrak{b}$ regular with respect to $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$,

$$(Mf)(h) = \pi_{\mathfrak{g}}(h) \int_{G/B} f(g \cdot h)\, d\dot{g} = \int_{\mathfrak{g}} \theta(h, \lambda, \mathfrak{g})(F_{\mathfrak{g}} f)(\lambda)\, d\lambda.$$

For $f \in \mathcal{S}(\mathfrak{g}_0)$ and $h \in \mathfrak{b}$ regular with respect to $\Delta((\mathfrak{g}_0)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$,

$$(M_0 f)(h) = \pi_{\mathfrak{g}_0}(h) \int_{G_0/B} f(g_0 \cdot h)\, d\dot{g}_0 = \int_{\mathfrak{g}_0} \theta(h, \lambda, \mathfrak{g}_0)(F_{\mathfrak{g}_0} f)(\lambda)\, d\lambda.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$; then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathrm{Car}(\mathfrak{g}) = \mathrm{Car}(\mathfrak{g}_0)$ (see [16]). Let $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, $\Delta^+((\mathfrak{g}_0)_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be positive systems for $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, $\Delta((\mathfrak{g}_0)_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ and π, π_0 be the corresponding functions on $\mathfrak{h}$.

3.14. Lemma. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$; then for $\lambda \in \mathfrak{h}$,*

$$\theta(h, \lambda, \mathfrak{g}) = [W(G, \mathfrak{h})]^{-1} \sum_{w \in W(G, \mathfrak{h})} \frac{\theta(h, w \cdot \lambda, \mathfrak{g}_0)\,|\pi_0(w \cdot \lambda)|}{|\pi(\lambda)|}.$$

Proof. We have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{N}^+ \oplus \mathfrak{N}^-$, with

$$(\mathfrak{N}^+)_{\mathbf{C}} = \sum_{\substack{\alpha \in \Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}}) \\ \alpha|\mathfrak{b}_1 \neq 0}} \mathfrak{g}_{\mathbf{C}}(\alpha).$$

We denote by $f \to f_{\mathfrak{N}^+}$ the map from $\mathcal{S}(\mathfrak{g})$ to $\mathcal{S}(\mathfrak{g}_0)$ given by

$$(f_{\mathfrak{N}^+})(X) = \int_{\mathfrak{N}^+} f(X + U)\, dU, \quad \text{for } X \in \mathfrak{g}_0.$$

It follows from [4] that $(F_{\mathfrak{g}} f)_{\mathfrak{N}^+} = F_{\mathfrak{g}_0}(f_{\mathfrak{N}^+})$.

Let N^+ be the analytic subgroup of G with Lie algebra $\mathfrak{N}^+$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$, H the corresponding Cartan subgroup of G_0. For φ a function on G/H, we have the integral formula

$$\int_{G/H} \varphi(g)\, d\dot{g} = \oint_{G/G_0N^+} \int_{\mathfrak{N}^+} \int_{G_0/H} \varphi(g \exp U g_0)\, d\dot{g}_0\, dU\, d\dot{g}$$

(see for example [1, Ch. 5] for the notation). In particular, if λ is a regular element of $\mathfrak{h}$ with respect to $\Delta(\mathfrak{g}_C, \mathfrak{h}_C)$, and $\varphi \in \mathcal{S}(\mathfrak{g})$, the following formula holds:

$$\int_{G/H} \varphi(g \cdot \lambda)\, d\dot{g} = \oint_{G/G_0N^+} \int_{\mathfrak{N}^+} \int_{G_0/H} \varphi(g \exp U g_0 \cdot \lambda)\, d\dot{g}_0\, dU\, d\dot{g}.$$

The map $U \to V$ given by $(\exp U)\, g_0 \cdot \lambda = g_0 \cdot \lambda + V$ is a diffeomorphism from $\mathfrak{N}^+$ to $\mathfrak{N}^+$. Calculating the Jacobian of this transformation, we obtain

$$\int_{G/H} \varphi(g \cdot \lambda)\, d\dot{g} = \oint_{G/G_0N^+} \left| \frac{\pi_0(\lambda)}{\pi(\lambda)} \right| \int_{G_0/H} \int_{\mathfrak{N}^+} \varphi(g \cdot (g_0 \cdot \lambda + V))\, dV\, d\dot{g}_0\, d\dot{g}.$$

We denote by φ^g the function $(\varphi^g)(X) = \varphi(g \cdot X)$; the preceding equality reads:

$$3.15. \quad \int_{G/H} \varphi(g \cdot \lambda)\, d\dot{g} = \oint_{G/G_0N^+} \frac{|\pi_0(\lambda)|}{|\pi(\lambda)|} \left(\int_{G_0/H} (\varphi^g)_{\mathfrak{N}^+}(g_0 \cdot \lambda)\, d\dot{g}_0 \right) d\dot{g}.$$

We calculate

$$\begin{aligned}(Mf)(h) &= \pi(h) \int_{G/B} f(g \cdot h)\, d\dot{g} \\ &= \pi_0(h)\pi_+(h) \oint_{G/G_0N^+} \frac{|\pi_0(h)|}{|\pi(h)|} \left(\int_{G_0/B} (\varphi^g)_{\mathfrak{N}^+}(g_0 \cdot h)\, d\dot{g}_0 \right) d\dot{g}\end{aligned}$$

using 3.15. As $\pi_+(h)$ is positive, $\pi_+(h) = \dfrac{|\pi(h)|}{|\pi_0(h)|}$. Thus

$$\begin{aligned}(Mf)(h) &= \oint_{G/G_0N^+} \pi_0(h) \left(\int_{G_0/B} (\varphi^g)_{\mathfrak{N}^+}(g_0 \cdot h)\, d\dot{g}_0 \right) d\dot{g} \\ &= \oint_{G/G_0N^+} M_0((\varphi^g)_{\mathfrak{N}^+})(h)\, d\dot{g} \\ &= \oint_{G/G_0N^+} \left(\int_{\mathfrak{g}_0} \theta(h, \lambda, \mathfrak{g}_0) F_{\mathfrak{g}_0}((\varphi^g)_{\mathfrak{N}^+})(\lambda)\, d\lambda \right) d\dot{g} \\ &= \oint_{G/G_0N^+} \int_{\mathfrak{g}_0} \theta(h, \lambda, \mathfrak{g}_0) \left((F_{\mathfrak{g}}\varphi)^g \right)_{\mathfrak{N}^+}(\lambda)\, d\lambda\, d\dot{g}\end{aligned}$$

as $F_{\mathfrak{g}_0}((\varphi^g)_{\mathfrak{N}^+}) = \left(F_{\mathfrak{g}}(\varphi^g)\right)_{\mathfrak{N}^+} = \left((F_{\mathfrak{g}}\varphi)^g\right)_{\mathfrak{N}^+}$,

$$= \oint_{G/G_0N^+} \sum_{\mathfrak{h}\in \mathrm{Car}\,\mathfrak{g}_0} \int_{\lambda\in\mathfrak{h}} \theta(h,\lambda,\mathfrak{g}_0)\,|\pi_0(\lambda)|^2 \int_{G_0} (F_{\mathfrak{g}}\varphi)^{g}_{\mathfrak{N}^+}(g_0\cdot\lambda)\,d\dot g_0\,d\lambda\,d\dot g$$

$$= \sum_{\mathfrak{h}\in \mathrm{Car}\,\mathfrak{g}_0} \int \theta(h,\lambda,\mathfrak{g}_0)\,|\pi_0(\lambda)|^2 \left(\oint_{G/G_0N^+}\int_{G_0} (F_{\mathfrak{g}}\varphi)^{g}_{\mathfrak{N}^+}(g_0\cdot\lambda)\,d\dot g_0\,d\dot g\right) d\lambda$$

$$= \sum_{\mathfrak{h}\in \mathrm{Car}\,\mathfrak{g}_0} \int \theta(h,\lambda,\mathfrak{g}_0)\frac{|\pi_0(\lambda)|}{|\pi(\lambda)|}\,|\pi(\lambda)|^2 \int_{G/H} (F_{\mathfrak{g}}\varphi)(g\cdot\lambda)\,d\dot g\,d\lambda \quad \text{by 3.15.}$$

The function $\lambda \to |\pi(\lambda)|^2 \int_{G/H}(F_{\mathfrak{g}}\varphi)(g\cdot\lambda)\,d\dot g$ is invariant by $W(G,\mathfrak{h})$; thus

$$(Mf)(h) = \sum_{\mathfrak{h}\in \mathrm{Car}\,\mathfrak{g}_0} \int [W(G,\mathfrak{h})]^{-1} \sum_{w\in W(G,\mathfrak{h})} \theta(h, w\cdot\lambda, \mathfrak{g}_0) \frac{|\pi_0(w\cdot\lambda)|}{|\pi(\lambda)|}\,|\pi(\lambda)|^2\left(\int_{G/H}(F_{\mathfrak{g}}\varphi)(g\cdot\lambda)\,d\dot g\right) d\lambda.$$

The function $\lambda \to \sum_{w\in W(G,\mathfrak{h})}\theta(h, w\cdot\lambda,\mathfrak{g}_0)\dfrac{|\pi_0(w\cdot\lambda)|}{|\pi(\lambda)|}$ can now be extended to a G-invariant function on $G\cdot\mathfrak{h}$ and we obtain our lemma.

Let $\mathfrak{h}\in \mathrm{Car}\, G_0$, $\mathfrak{h} = \mathfrak{t}\oplus\mathfrak{a}$. Let $\Delta^0_{\mathbf{R}} = \Delta_{\mathbf{R}}(\mathfrak{g}_0,\mathfrak{h})$, $\Delta_{\mathbf{R}} = \Delta_{\mathbf{R}}(\mathfrak{g},\mathfrak{h})$, $\mathcal{F}_0^+$ be the positive chamber of $\mathfrak{a}^*$ with respect to $(\Delta^0_{\mathbf{R}})^+$ and $\mathcal{F}^+$ be the positive chamber of $\mathfrak{a}^*$ with respect to $\Delta^+_{\mathbf{R}}$. As $\lambda\to\theta(h,\lambda,\mathfrak{g}_0)\,|\pi_0(\lambda)|$ is invariant under $W(\Delta^0_{\mathbf{R}})$, we have, for $p(\lambda)\in\mathcal{F}^+$,

$$\theta(h,\lambda,\mathfrak{g}) = [W(G,\mathfrak{h})]^{-1}[W(\Delta^0_{\mathbf{R}})] \sum_{\substack{w\in W(G,\mathfrak{h})\\ w(\mathcal{F}^+)\subset\mathcal{F}_0^+}} \frac{\theta(h,w\cdot\lambda,\mathfrak{g}_0)\,|\pi_0(w\cdot\lambda)|}{|\pi(\lambda)|}.$$

We introduce $W_0^{\mathbf{C}}$, the Weyl group associated to the root system $\Delta((\mathfrak{g}_0)_{\mathbf{C}},\mathfrak{b}_{\mathbf{C}})$, and

$$(\overline{M}f)(h) = \sum_{w_0\in W_0^{\mathbf{C}}} \varepsilon(w_0)(Mf)(w_0\cdot h)$$

$$= \int \bar\theta\,(h,\lambda,\mathfrak{g})(F_{\mathfrak{g}}f)(\lambda)\,d\lambda,$$

$$(\overline{M}_0 f)(h) = \sum_{w_0\in W_0^{\mathbf{C}}} \varepsilon(w_0)(M_0 f)(w_0\cdot h)$$

$$= \int_{\mathfrak{g}_0} \bar\theta\,(h,\lambda,\mathfrak{g}_0)(F_{\mathfrak{g}_0}f)(\lambda)\,d\lambda,$$

with

$$\bar\theta\,(h,\lambda,\mathfrak{g}) = \sum_{w_0\in W_0^{\mathbf{C}}} \varepsilon(w_0)\theta(w_0\cdot h,\lambda,\mathfrak{g})$$

$$\bar\theta\,(h,\lambda,\mathfrak{g}_0) = \sum_{w_0\in W_0^{\mathbf{C}}} \varepsilon(w_0)\theta(w_0\cdot h,\lambda,\mathfrak{g}_0).$$

Let y be an element of $(G_0)_{\mathbf{C}}$ such that $y(\mathfrak{b}_{\mathbf{C}}) = \mathfrak{h}_{\mathbf{C}}$. We have seen that for $p(\lambda) \in \mathfrak{F}_0^+$,

$$i^{-\#\Delta_0^+}(-1)^{m_{\mathfrak{g}_0}}\bar{\theta}(h, \lambda, \mathfrak{g}_0) = \sum_{w_0 \in W_0^{\mathbf{C}}} \frac{\varepsilon(w_0)\bar{c}\left(p(iyw_0h), (\Delta_{\mathbf{R}}^0)^+\right)e^{-i(yw_0h, \lambda)}}{\pi_0(\lambda)}.$$

Thus for $p(\lambda) \in \mathfrak{F}^+$,

$$i^{-\#\Delta_0^+}(-1)^{m_{\mathfrak{g}_0}}\bar{\theta}(h, \lambda, \mathfrak{g})\left[W(\Delta_{\mathbf{R}}^0)\right]^{-1}[W(G, \mathfrak{h})] = \sum_{\substack{w \in W(G, \mathfrak{h}) \\ w(\mathfrak{F}^+) \subset \mathfrak{F}_0^+}} \sum_{w_0 \in W_0^{\mathbf{C}}} \frac{\varepsilon(w_0)\bar{c}\left(p(iyw_0h), (\Delta_{\mathbf{R}}^0)^+\right)e^{-i(yw_0h, w\lambda)}|\pi_0(w \cdot \lambda)|}{|\pi(\lambda)|\pi_0(w \cdot \lambda)},$$

and, for D the differential operator (with respect to h) associated to $\pi_{\mathfrak{g}}$ on $\mathfrak{b}$,

$$i^{-(\#\Delta_0^+)}(-1)^{m_{\mathfrak{g}_0}}D \cdot \bar{\theta}(h, \lambda, \mathfrak{g})\left[W(\Delta_{\mathbf{R}}^0)\right]^{-1}[W(G, \mathfrak{h})] = i^{-(\#\Delta^+)} \sum_{\substack{w \in W(G, \mathfrak{h}) \\ w(\mathfrak{F}^+) \subset \mathfrak{F}_0^+}} \sum_{w_0 \in W_0^{\mathbf{C}}} \bar{c}\left(p(iyw_0h), (\Delta_{\mathbf{R}}^0)^+\right)e^{-i(yw_0h, w\lambda)\delta(w)}$$

with $\delta(w) = \dfrac{\pi(w \cdot \lambda)}{\pi_0(w \cdot \lambda)} \dfrac{|\pi_0(w \cdot \lambda)|}{|\pi(\lambda)|}$. Let

$$\pi_{\mathbf{R}}(\lambda) = \prod_{\alpha \in \Delta_{\mathbf{R}}^+} (\alpha, \lambda),$$

$$\pi_{\mathbf{R}}^0(\lambda) = \prod_{\alpha \in (\Delta_{\mathbf{R}}^0)^+} (\alpha, \lambda),$$

$\varepsilon_{\mathbf{R}}(w)$ be such that $\pi_{\mathbf{R}}(w \cdot \lambda) = \varepsilon_{\mathbf{R}}(w)\pi_{\mathbf{R}}(\lambda)$ for $w \in W(G, \mathfrak{h})$.

3.16. LEMMA. *For $\lambda \in \mathfrak{F}^+$ and $w \in W(G, \mathfrak{h})$ such that $w(\mathfrak{F}^+) \subset \mathfrak{F}_0^+$, $\delta(w) = \varepsilon_{\mathbf{R}}(w)$.*

Proof. We have

$$\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}}) = \Delta_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}) \cup \Delta_I(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}}) \cup \Delta_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}}).$$

We may choose a positive system $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ such that if α is a complex positive root, then $\bar{\alpha}$ is also positive (and is different from α). Thus $\prod_{\alpha \in \Delta_{\mathbf{C}}^+}(\alpha, \lambda) > 0$. Now, if α is an imaginary root, α vanishes on $\mathfrak{a}$ hence a fortiori on $\mathfrak{b}_1$ and α is in $\Delta_I((\mathfrak{g}_0)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. Thus we see that

$$\frac{\pi(\lambda)}{\pi_0(\lambda)} = c(\lambda)\frac{\pi_{\mathbf{R}}(\lambda)}{\pi_{\mathbf{R}}^0(\lambda)}$$

with $c(\lambda) > 0$. As $w \cdot \lambda \in \mathfrak{F}_0^+$ implies $\pi_{\mathbf{R}}^0(w \cdot \lambda) > 0$, the lemma follows.

We remark also that

$$i^{-(\# \Delta^+)} i^{(\# \Delta_0^+)} (-1)^{m_{\mathfrak{g}_0}} = (-1)^{m_{\mathfrak{g}}} \qquad ([4]).$$

Thus for $p(\lambda) \in \mathcal{F}^+$

3.16. $$(-1)^{m_{\mathfrak{g}}} (D \cdot \bar{\theta}\,(h, \lambda, \mathfrak{g})) = c \sum_{\substack{w \in W(G, \mathfrak{h}) \\ w(\mathcal{F}^+) \subset \mathcal{F}_0^+}} \varepsilon_{\mathrm{R}}(w) \sum_{w_0 \in W_0^{\mathrm{C}}} \bar{c}\big(p(iyw_0h), (\Delta_{\mathrm{R}}^0)^+\big) e^{-i(yw_0h,\, w\lambda)}$$

with $c = [W(\Delta_{\mathrm{R}}^0)][W(G, \mathfrak{h})]^{-1}$.

Let Z_0 be the center of G_0, then $Z \subset Z_0$. We consider

$$E_Z = \{X \in \mathfrak{g},\ \exp X \in Z\},$$
$$E_{Z_0} = \{X \in \mathfrak{g}_0,\ \exp X \in Z_0\}.$$

We have $E_Z \cap \mathfrak{b} = E_Z \cap \mathfrak{b}_0 \subset E_{Z_0} \cap \mathfrak{b}_0$.

Let χ be a character of the center Z of G. As $E_Z \cap \mathfrak{b}$ as well as $\xi \to \chi(\exp \xi)$ are stable under W_0^{C}, we may compute the left-hand side of 3.8, using (3.16), as:

$$(-1)^{m_{\mathfrak{g}}} [W_0^{\mathrm{C}}]^{-1} \sum_{\xi \in E_Z \cap \mathfrak{b}} \chi(\exp \xi) (D\overline{M}f)(\xi)$$
$$= \sum_{\mathfrak{h} \in \mathrm{Car}\, \mathfrak{g}_0} \sum_{\substack{w \in W_{\mathrm{R}}(G, \mathfrak{h}) \\ w(\mathcal{F}^+) \subset \mathcal{F}_0^+}} \varepsilon_{\mathrm{R}}(w) \sum_{\xi \in E_Z \cap \mathfrak{b}_0} \chi(\exp \xi) \int_{t^* \oplus \mathcal{F}^+} \bar{c}\big(p(iy\xi), (\Delta_{\mathrm{R}}^0)^+\big)$$
$$\times e^{-i(y\xi,\, w \cdot \lambda)} |\pi(\lambda)| (K_\lambda, f)\, d\lambda$$

where c is a positive constant.

Let H be the Cartan subgroup of G associated to $\mathfrak{h}$, H_0 the Cartan subgroup of G_0 associated to $\mathfrak{h}$. If $b \in \hat{H}$, b can be also considered as a character of H_0 by restriction and $P(b, (\Delta_{\mathrm{R}}^0)^+)$ is well defined. The group $W(G, \mathfrak{h})$ acts on $\hat{H}(\chi)$.

3.17. LEMMA.

$$\sum_{\xi \in E_Z \cap \mathfrak{b}} \chi(\exp \xi) \int_{t^* \oplus \mathcal{F}^+} \bar{c}\big(p(iy\xi), (\Delta_{\mathrm{R}}^0)^+\big) e^{-i(y\xi,\, w \cdot \lambda)} |\pi(\lambda)| (K_\lambda, f)\, d\lambda$$
$$= \int_{\substack{b \in \hat{H}(\chi) \\ p(\log b/i) \in \mathcal{F}^+}} P\big(w \cdot b, (\Delta_{\mathrm{R}}^0)^+\big) |(\pi, \log b)| \big(K_{\log b/i}, f\big)\, db.$$

Proof. This is proved in the same way as Proposition 3.11. It remains to prove that

$$\sum_{w \in W(G, \mathfrak{h})} \varepsilon_{\mathrm{R}}(w) P\big(w \cdot b, (\Delta_{\mathrm{R}}^0)^+\big) = P(b, \Delta_{\mathrm{R}}^+).$$

But if φ is a 2-structure for $\Delta^0_{\mathbf{R}}$, φ is a 2-structure for $\Delta_{\mathbf{R}}$ [11]. Thus this equality follows from 2.6, 2.4 b) and 2.6 (1).

4. The conjecture

Let $\mathfrak{g}$ be a semi-simple Lie algebra. We will now deduce our conjecture ([15]) from Theorem 3.5.

First we prove:

4.1. LEMMA. *Let $\mathfrak{g}$ be a real semi-simple Lie algebra. Let $j(X) = \det\left(\frac{e^{\operatorname{ad} X}-1}{\operatorname{ad} X}\right)$. There exists an analytic function $j^{1/2}$ on $\mathfrak{g}$ such that $(j^{1/2})^2 = j$, $j^{1/2}(0) = 1$.*

Proof. Let us consider the function $j_{\mathbf{C}}$ on $\mathfrak{g}_{\mathbf{C}}$ given by

$$j_{\mathbf{C}}(X) = \det{}_{\mathbf{C}}\left(\frac{e^{\operatorname{ad} X}-1}{\operatorname{ad} X}\right).$$

Then $j_{\mathbf{C}}$ is an analytic function on $\mathfrak{g}_{\mathbf{C}}$ such that the restriction of $j_{\mathbf{C}}$ to $\mathfrak{g}$ is equal to j. Let $\mathfrak{h}_{\mathbf{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$, $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ a positive system of roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$. We then have for $h \in \mathfrak{h}_{\mathbf{C}}$,

$$\begin{aligned} j_{\mathbf{C}}(h) &= \prod_{\alpha\in\Delta^+}\left(\frac{e^{(\alpha,h)}-1}{(\alpha,h)}\right)\cdot\prod_{\alpha\in\Delta^+}\left(\frac{e^{-(\alpha,h)}-1}{-(\alpha,h)}\right) \\ &= \prod_{\alpha\in\Delta^+}\left(\frac{e^{(\alpha/2,h)}-e^{-(\alpha/2,h)}}{(\alpha,h)}\right)^2. \end{aligned}$$

The function

$$\prod_{\alpha\in\Delta^+}\left(\frac{e^{\alpha/2}-e^{-\alpha/2}}{\alpha}\right)$$

is an analytic function on $\mathfrak{h}_{\mathbf{C}}$ invariant by the complex Weyl group. Thus by Chevalley's restriction theorem there exists an analytic function $j_{\mathbf{C}}^{1/2}$ on $\mathfrak{g}_{\mathbf{C}}$ such that $(j_{\mathbf{C}}^{1/2})^2 = j_{\mathbf{C}}$. The restriction of $j_{\mathbf{C}}^{1/2}$ to $\mathfrak{g}$ is a real analytic square root of j.

We now prove the following:

4.2. LEMMA. *Let $\xi \in R_Z \cap \mathfrak{b}$, let G_ξ be the stabilizer of ξ in G; then for $f \in \mathcal{S}(\mathfrak{g})$*

$$(D_\pi M(j^{1/2}f))(\xi) = c_\xi\int_{G/G_\xi} f(g\cdot\xi)\,d\dot{g}$$

when c_ξ is a constant.

Proof. This is true when $\xi = 0$ by Harish-Chandra's limit formula [4]. Let us consider ξ_0 an arbitrary element in $R_Z \cap \mathfrak{b} \subset \mathfrak{b}_0$. We consider

$$\Delta_0 = \{\alpha \in \Delta(\mathfrak{g}_C, \mathfrak{b}_C); (\alpha, \xi) = 0\},$$
$$\Delta_1 = \{\alpha \in \Delta(\mathfrak{g}_C, \mathfrak{b}_C), (\alpha, \xi) \neq 0\}.$$

Let us choose a positive system Δ^+ for $\Delta(\mathfrak{g}_C, \mathfrak{b}_C)$ such that $(\alpha, \xi) \geq 0$, for all $\alpha \in \Delta^+$, and set $\Delta_1^+ = \Delta^+ \cap \Delta_1$, $\Delta_0^+ = \Delta^+ \cap \Delta_0$. The Weyl group W_0 of the system Δ_0 leaves stable Δ_1^+.

We write

$$\pi_0 = \prod_{\alpha \in \Delta_0^+} \alpha$$
$$\pi_1 = \prod_{\alpha \in \Delta_1^+} \alpha.$$

We consider D_0, D_1 the constant coefficient operators associated to π_0, π_1.

$$N_1 = \prod_{\alpha \in \Delta_1^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

Let $G_0 = G_{\xi_0}$ be the stabilizer in G of the point ξ_0 and $\mathfrak{g}_0$ its Lie algebra. For ξ a regular element in $\mathfrak{b}$, we can write

$$M(j^{1/2}f)(\xi) = \pi(\xi)\int_{G/G_0}\left(\int_{G_0}(j^{1/2}f)(gg_0\xi)\,d\dot{g}_0\right)d\dot{g}$$
$$= \pi(\xi)j^{1/2}(\xi)\int_{G/G_0}\left(\int_{G_0}f(gg_0\xi)\,d\dot{g}_0\right)d\dot{g}.$$

Thus, if for $g \in G$, f^g is the function on $\mathfrak{g}_0$ given by $(f^g)(u) = f(g \cdot u)$, we may rewrite

$$(Mj^{1/2}f)(\xi) = \int_{G/G_0}\pi_0(\xi)\pi_1(\xi)j^{1/2}(\xi)\left(\int_{G_0}(f^g)(g_0 \cdot \xi)\,d\dot{g}_0\right)d\dot{g}.$$

Now $\pi_1(\xi)j^{1/2}(\xi) = j_0^{1/2}(\xi)N_1(\xi)$ with

$$j_0^{1/2}(\xi) = \prod_{\alpha \in \Delta_0^+}\left(\frac{e^{\alpha/2} - e^{-\alpha/2}}{\alpha}\right),$$
$$N_1 = \prod_{\alpha \in \Delta_1^+}(e^{\alpha/2} - e^{-\alpha/2}).$$

The function $j_0^{1/2}$ is the j-function for the Lie algebra $\mathfrak{g}_0$. Let us remark that $N_1(\xi)$ is invariant under W_0; thus by Chevalley's theorem there exists a G_0-invariant function $\tilde{N}_1$ on $\mathfrak{g}_0$ such that $\tilde{N}_1 \mid \mathfrak{b} = N_1$. Denoting by M_0 the map M for

$\mathfrak{g}_0$, we have

$$\big(M(j^{1/2}f)\big)(\xi) = \int_{G/G_0} M_0\big(f^g j_0^{1/2}\tilde{N}_1\big)(\xi)\,d\dot{g},$$

$$D_\pi\big(M(j^{1/2}f)\big)(\xi) = \int_{G/G_0} D_0 D_1 M_0\big(f^g j_0^{1/2}\tilde{N}_1\big)(\xi)\,d\dot{g}.$$

As π_1 is W_0-invariant, there exists a G_0-invariant function $\tilde{\pi}_1$ on $\mathfrak{g}_0$, such that $\tilde{\pi}_1 \mid \mathfrak{h} = \pi_1$. Let $\tilde{D}_1$ be the constant coefficient G_0-invariant operator on $\mathfrak{g}_0$ corresponding to $\tilde{\pi}_1$. By one of the properties of the invariant integral M_0 [4], $M_0\tilde{D}_1 = D_1 M_0$ and we may write:

$$\big(D_\pi M(j^{1/2}f)\big)(\xi) = \int_{G/G_0} D_0 M_0\big(\tilde{D}_1 \cdot \big(f^g j_0^{1/2}\tilde{N}_1\big)\big)(\xi)\,d\dot{g}.$$

Now, as ξ_0 is in the center of the Lie algebra $\mathfrak{g}_0$ we have

$$\lim_{\xi\to\xi_0} (D_0 M_0 f)(\xi) = c_0 f(\xi_0).$$

Thus it remains to show that

$$\tilde{D}_1\big(f^g j_0^{1/2}\tilde{N}_1\big)(\xi_0) = c_0' f(g\cdot\xi_0).$$

Let $\tau(\xi_0)$ be the translation operator $(\tau(\xi_0)\cdot\alpha)(\xi) = \alpha(\xi+\xi_0)$. As $\tilde{D}_1$ commutes with translations, we compute

$$\big(\tilde{D}_1\cdot f\big)(\xi_0) \quad\text{by}\quad \big(\tilde{D}_1(\tau(\xi_0)\cdot f)(0)\big).$$

Now

$$\tilde{N}_1(\xi) = \prod_{\alpha\in\Delta_1^+} e^{(\alpha/2,\,\xi)} - e^{-(\alpha/2,\,\xi)}$$

$$= e^{(\rho_1,\,\xi)} \prod_{\alpha\in\Delta_1^+} \big(1 - e^{-(\alpha,\,\xi)}\big)$$

with

$$\rho_1 = \tfrac{1}{2}\sum_{\alpha\in\Delta_1^+} \alpha.$$

As $\xi_0 \in R_Z$, $\alpha(\xi_0) \in 2i\pi\mathbf{Z}$, for all $\alpha\in\Delta$ then

$$\tilde{N}_1(\xi+\xi_0) = e^{(\rho_1,\,\xi_0)}\tilde{N}_1(\xi) = e^{(\rho,\,\xi_0)}\tilde{N}_1(\xi).$$

Thus

$$\tilde{D}_1\cdot\big(\tau(\xi_0)\cdot\big(f^g j_0^{1/2}\tilde{N}_1\big)\big)(0) = e^{(\rho,\,\xi_0)}\big(\tilde{D}_1\cdot\tilde{N}_1\tau(\xi_0)\cdot\big(j_0^{1/2}f^g\big)\big)(0).$$

Now the function N_1, and thus $\tilde{N}_1$, has a zero of order $\#\,\Delta_1^+$ at 0; hence as $\tilde{D}_1$ is of order $\# \Delta_1^+$,

$$\begin{aligned}(\tilde{D}_1 \cdot \tilde{N}_1 f)(0) &= (\tilde{D}_1 \cdot \tilde{N}_1)(0)f(0)\\ &= (D_1 \cdot N_1)(0)f(0)\\ &= (\pi_1, \pi_1)f(0).\end{aligned}$$

As $j_0^{1/2}(\xi_0) = 1, (f^g)(\xi_0) = f(g \cdot \xi_0)$, we obtain our lemma.

Let $\mathfrak{g}$ be a semi-simple Lie algebra and G_1 be a linear group with Lie algebra $\mathfrak{g}$. We describe now the set $\mathfrak{g}^*_{G_1}$ of strongly G_1-admissible regular elements of $\mathfrak{g}^*$. The element f of $\mathfrak{g}^*$ is *strongly regular* if:

a) The stabilizer $\mathfrak{h}$ of f is a Cartan subalgebra of $\mathfrak{g}$.

b) The form $if + \rho$ exponentiates to a character $e^{(if+\rho)}$ of the connected subgroup $\exp \mathfrak{h} \subset G_1$.

Let $E_{G_1} = \{X \in \mathfrak{g};\ \exp X = e_{G_1}\}$. Let Z_1 be the reciprocal image of e_{G_1} in $G \subset G_{\mathbf{C}}$. Clearly $Z_1 \subset Z$, and $\xi \in \mathfrak{b} \cap E_{G_1}$ if and only if $\xi \in E_Z \cap \mathfrak{b}$ and $\exp \xi \in Z_1$. Thus, for $\varphi \in \mathcal{S}(\mathfrak{g})$,

4.3. $$\sum_{\substack{\chi \in \hat{Z}\\ \chi|Z_1 = 1}} \sum_{\xi \in E_Z \cap \mathfrak{b}} \chi(\exp \xi) e^{(\rho, \xi)} (D_\pi M j^{1/2} \varphi)(\xi) = [Z : Z_1] \sum_{\xi \in E_{G_1} \cap \mathfrak{b}} e^{(\rho, \xi)} (D_\pi M j^{1/2} \varphi)(\xi).$$

The set $E_{G_1} \cap \mathfrak{b}$, mod $W(G, \mathfrak{b})$ conjugacy, describes the set of conjugacy classes of elements of E_{G_1}. Thus by Lemma 4.2, formula 4.3 is of the form

$$\int_{E_{G_1}} \varphi(\xi)\, dm(\xi).$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and H be the corresponding Cartan subgroup of $G \subset G_{\mathbf{C}}$. Let ξ_ρ be the character of $H_{\mathbf{C}}$ given by $\xi_\rho(\exp H) = e^{(\rho, H)}$ for $H \in \mathfrak{h}_{\mathbf{C}}$. If b is a character of H such that $b \mid Z_1 = \xi_\rho$, it is clear that, if $\log b$ is regular, $\log b/i$ is a strongly regular G_1-admissible element of $\mathfrak{g}^*$. The set of elements $\log b$, for $b \mid Z_1 = \xi_\rho$, parametrizes, when H varies among Car G, the set of conjugacy classes of strongly regular G_1-admissible elements of $\mathfrak{g}^*$.

By Theorem 3.8, we obtain

4.4 $$\int_{E_{G_1}} \varphi\, dm = \sum_{\mathfrak{h} \in \mathrm{Car}\, G} \int_{\substack{b \in \hat{H}\\ b|Z_1 = \xi_\rho}} | P(b)(\pi, \log b) | \left(K_{\log b/i}, \varphi j^{1/2}\right) db$$

which we may rewrite as

$$\int_{E_{G_1}} \varphi\, dm = \int_{\mathfrak{g}^*_{G_1}} F_{\mathfrak{g}}(\varphi j^{1/2})\, dm^*.$$

Thus conjecture (C) holds for linear semi-simple Lie groups G.

Let us describe now the relation between the measure dm^* on $\mathfrak{g}^*_{G_1}$ and the Plancherel measure for G_1: Harish-Chandra parametrized the reduced dual $(\hat{G}_1)_r$ of G_1 by the set D of characters b of the different Cartan subgroups H of G, such that $b \mid Z_1 = \xi_\rho$ and $\log b$ is a regular element of $\mathfrak{h}^*$ (modulo the action of $W(G, \mathfrak{h})$) and obtained the Plancherel formula [6]:

$$\varphi(e) = \sum_{\mathfrak{h} \in \operatorname{Car} \mathfrak{g}} \int_{\substack{b \in \hat{H} \\ b|Z_1 = \xi_\rho}} | P(b)(\pi, \log b) | \operatorname{Tr}(\theta_b, \varphi)\, db,$$

when θ_b is the representation associated to b, by Harish-Chandra. The universal formula for character holds for semi-simple Lie groups and the map $T \to \mathcal{O}_T$ (see Section 1) is the map

$$\theta_b \to G \cdot \left(\frac{\log b}{i} \right).$$

The map $b \mapsto \log b/i$ provides a surjective map for $\hat{G}_r$ to $\mathfrak{g}^*_{G_1/G_1}$ with finite fibers. We thus see that the quotient of our measure dm^* by Kirillov's measures of the orbits is the image of the Plancherel measure on $\hat{G}_r$ by the map $T \to \mathcal{O}_T$.

For φ a function on $\mathfrak{g}$ supported in a small neighborhood of 0, the universal character formula is

$$\operatorname{Tr} \int_G \theta_b(g) \varphi(\log g)\, dg = \int_{G \cdot (\log b/i)} F_{\mathfrak{g}}(\varphi j^{1/2})(\xi)\, d\mu_{G \cdot (\log b/i)}(\xi).$$

Thus, for φ a function supported in a small neighborhood of 0 in $\mathfrak{g}$, the Plancherel formula for G implies that

$$\varphi(0) = \int_{\mathfrak{g}^*_{G_1}} F_{\mathfrak{g}}(\varphi j^{1/2})(\xi)\, dm^*(\xi),$$

and is equivalent to our Poisson-Plancherel formula for $(\mathfrak{g}_1, G)$. Obviously, this formula does not hold when the support of φ is large. It is remarkable that by our Poisson-Plancherel formula the sets $\mathfrak{g}^*_{G_1}$ and E_{G_1} play a similar role to dual lattices for the usual Poisson formula.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASS.

BIBLIOGRAPHY

[1] P. BERNAT, N. CONZE, M. DUFLO, M. LEVY-NAHAS, M. RAIS, P. RENOUARD, M. VERGNE, Représentations des groupes de Lie resolubles, DUNOD, Paris, 1972.

[2] M. DUFLO, Construction de gros ensembles de représentations unitaires irreductibles d'un groupe de Lie quelconque, OAGR Conference, 1980, INCREST, Bucharest, Roumania.

[3] ______, On a conjecture of M. Vergne, to appear.

[4] HARISH-CHANDRA, Some results on an invariant integral on a semi-simple Lie algebra, Ann. of Math. **80** (1964), 551–593.

[5] Harish-Chandra, Discrete series for semi-simple Lie groups I, Acta Math. **113** (1965), 241–318.

[6] ______, Harmonic analysis on real reductive groups III. The Maass-Selberg relations and the Plancherel formula, Ann. of Math. **104** (1976), 117–201.

[7] R. Herb, "Characters of averaged discrete series on semi-simple real Lie groups," Pacific Journal of Mathematics 80, 1979, 169–177.

[8] ______, Fourier inversion and the Plancherel theorem for semi-simple real Lie groups, to appear in Amer. J. Math.

[9] ______, Fourier inversion and the Plancherel theorem, *Non-Commutative Harmonic Analysis*, Luminy, 1980, Lecture Notes, Springer-Verlag # 880.

[10] ______, Discrete series characters and Fourier inversion on semi-simple real Lie groups, preprint, 1980.

[11] ______, Letter to the author, 17 October 1980.

[12] A. A. Kirillov, Character of unitary representations of Lie groups, Funct. Analysis Appl. **2-2** (1968), 40–55.

[13] W. Rossmann, Kirillov's character formula for reductive Lie groups, Invent. Math. **48** (1978), 207–220.

[14] M. Vergne, On Rossmann's character formula for discrete series, Invent. Math. **54** (1979), 11–14.

[15] ______, A Plancherel formula without group representations, OAGR Conference (1980), INCREST-Bucharest, Roumania.

[16] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups* II, Springer-Verlag, Berlin-Heidelberg-New York 1972.

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