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Harmonically Induced Representations of Solvable Lie Groups*

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1. INTRODUCTION

The "orbit method" for a Lie group G with Lie algebra $\mathfrak{g}$ associates unitary representations of G to suitable orbits of G on $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. In reasonable cases, these representations are generally obtained by some form of "induction," starting from a polarization $\mathfrak{h}$ for the orbit. The purpose of this paper is to study the representations obtained when $\mathfrak{h}$ is totally complex, not necessarily positive, and G is solvable (generally non-unimodular). We obtain both positive and negative results. On the positive side, we show that such "harmonically induced" representations are in principle computable, and we actually carry out the computations for one non-trivial example, $G = N_{2n+1} \rtimes \mathbb{R}_+^\times$, the semidirect product of the Heisenberg group N_{2n+1} of dimension $2n+1$ by a one-parameter group of dilations. (This G is the "AN-group" of the Iwasawa decomposition of $SU(n+1, 1)$, $n \geqslant 1$.) To some extent, these harmonically induced represen-

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tations behave as in the better-understood nilpotent [17, 19, 23] and semisimple [25, 26] cases. We also collect together in Section 2 a number of general vanishing and non-vanishing theorems valid for harmonically induced representations of completely arbitrary Lie groups. These results are not particularly original but they seem not to be widely known.

On the negative side, however, we find several new kinds of "pathology" not present in the more familiar unimodular cases. The Lie algebra cohomology groups associated to "formal harmonic spaces" turn out to be generally infinite-dimensional and non-Hausdorff. This means that the calculation of harmonically induced representations cannot be reduced to purely algebraic calculations, and the spectral sequences for Lie algebra cohomology that are so useful in the semisimple and nilpotent cases turn out to be useless for our purposes in the solvable case. Hence it is necessary to make direct analytical calculations, which seem hopelessly difficult unless G has a particularly simple structure. Worse still, there is a new surprise in the theory: it is possible for a harmonically induced representation to be not only non-irreducible but even of infinite multiplicity. This implies that the restriction of unimodularity in the L^2-index theorem of Connes and Moscovici [6] is not just a technical convenience—in fact, there can be no such theorem when G is non-unimodular. (This initially came as a bit of a shock, since one of us had spent several months with Henri Moscovici in 1979, trying in vain to extend the Connes–Moscovici results, if only in a weak way, to the case of non-unimodular solvable groups!)

From one point of view, then, our results consist basically of counterexamples. However, conversations with Nolan Wallach, Dan Barbasch, and David Vogan have shown us that our results for the "AN-group" of $SU(n+1, 1)$ fit in very nicely with the general theory of the discrete series on semisimple groups associated with hermitian symmetric spaces, and in fact could also have been obtained this way. A brief discussion of this other method of proof is given in Section 4. However, complete details, together with a more general analysis of the restriction of unitary representations of semisimple Lie groups to a maximal split solvable subgroup, will be the subject of another paper (with different authors).

Let us say a few words about our notations. Lie groups are denoted by capital Roman letters, and their Lie algebras by the corresponding lowercase Gothic letters. If V is a real vector space, V^* denotes its dual and $V_{\mathbb{C}}$ its complexification. Theorems, propositions, definitions, remarks, and formulas are all numbered with a single numbering system, consecutively in each section. Our main result is Theorem 3.8, but this depends on concepts developed in Section 2 and the earlier part of Section 3. Another proof of the same theorem is outlined in Section 4. Theorems 2.8, 2.15, and 2.20, though really only translations to our context of known results, may be of independent general interest.

2. The General Theory of Holomorphic and Harmonic Induction

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and $f \in \mathfrak{g}^*$ is an *integral* linear functional, i.e., there exists a unitary character χ of the stabilizer $G(f)$ of f, with $d\chi = if$. Let $\mathfrak{h}$ be an *invariant polarization* for f, which means that:

(2.1) $\mathfrak{h}$ is a (complex) Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and $f|_{[\mathfrak{h},\mathfrak{h}]} = 0$ (we extend f from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$ by complex linearity).

(2.2) $\mathfrak{h}$ is maximal isotropic for the alternating bilinear form $B_f = f([\cdot, \cdot])$, i.e., (2.1) holds and

$$\dim_{\mathbb{C}} \mathfrak{h} = (\dim \mathfrak{g}(f) + \dim \mathfrak{g})/2.$$

(2.3) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$, necessarily of the form $\mathfrak{e}_{\mathbb{C}}$, for some Lie subalgebra $\mathfrak{e}$ of $\mathfrak{g}$.

(2.4) ("Invariance") $\mathrm{Ad}_{\mathfrak{g}_{\mathbb{C}}}(G(f))$ maps $\mathfrak{h}$ into itself. We shall also need one other technical condition which is satisfied in all conditions of practical interest, namely,

(2.5) The groups $D = G(f)\exp(\mathfrak{h} \cap \mathfrak{g})$ and $E = G(f)\exp(\mathfrak{e})$ are closed in G.

The character χ is not necessarily uniquely determined by f, although it will be if $G(f)$ is connected (and in this case (2.4) is automatic). In any event, a choice of χ determines an induced line bundle $\mathscr{L}_\chi$ on $G/G(f)$, and from this line bundle, together with the extra structure given by the polarization $\mathfrak{h}$, one would like to construct a unitary representation of G, preferably irreducible. (*Caution*: Even when this is possible, the unitary representation so obtained will not generally be associated, in the sense of [9], to the coadjoint orbit of f. Instead, it will belong to the orbit of an admissible, but not necessarily integral, functional obtained from f by a certain shift. But for present purposes we do not need to worry about this point.)

When $\mathfrak{h}$ is *real* (i.e., $\mathfrak{h} = \bar{\mathfrak{h}}$), it is clear how to proceed: one extends χ to a character of $D = E = G(f)\exp(\mathfrak{g} \cap \mathfrak{h})$ and takes the unitarily induced representation of G. In the presence of the "Pukanszky condition," one expects the induced representation to be irreducible, and in any event it is certainly non-zero.

We shall be concerned, therefore, with the opposite case which occurs when $\mathfrak{h}$ is *totally complex*, i.e., $\mathfrak{e} = \mathfrak{g}$. The general case reduces to this anyway, since assuming by (2.5) that $E = G(f)\exp(\mathfrak{e})$ is closed in G, the normal procedure for obtaining a unitary representation of G is first to construct a unitary representation of E, then to induce unitarily. Of course,

$\mathfrak{h}$ is a totally complex polarization for $f|_{\mathfrak{e}}$. So we assume henceforth that $\mathfrak{h}$ is totally complex, i.e., (2.3) is replaced by

$$(2.3') \quad \mathfrak{h} + \bar{\mathfrak{h}} = \mathfrak{g}_{\mathbb{C}}.$$

Then $D = G(f)$ and $E = G$ so that (2.5) is automatic. Also $\mathfrak{h}$ defines a left-invariant almost-complex structure on the complexified tangent bundle of $G/G(f)$ which is closed under Lie brackets (since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$), hence integrates to a G-invariant complex structure on $G/G(f)$. We choose this structure so that $\mathfrak{h}$ corresponds to the *anti*-holomorphic tangent vectors. Then because of condition (2.1), the line bundle $\mathscr{L}_\chi$ acquires a G-invariant holomorphic structure.

DEFINITION 2.6. Assume the polarization $\mathfrak{h}$ satisfies (2.1), (2.2), and (2.3′) above. Define a sesquilinear form S_f on $\mathfrak{h}$ by

$$S_f(x, y) = if([x, \bar{y}]) = iB_f(x, \bar{y}).$$

It is easy to see that $S_f(y, x) = \overline{S_f(x, y)}$ and that the radical of S_f is exactly $\mathfrak{g}(f)_{\mathbb{C}}$, the complexified Lie algebra of $G(f)$. Thus S_f defines a non-degenerate hermitian form on $\mathfrak{h}/\mathfrak{g}(f)_{\mathbb{C}}$.

The polarization is said to be *positive* if and only if the form S_f is positive definite on $\mathfrak{h}/\mathfrak{g}(f)_{\mathbb{C}}$ (or positive semidefinite on $\mathfrak{h}$). More generally, the *negativity index* $q(\mathfrak{h}, f)$ is defined to be the dimension of a maximal subspace of $\mathfrak{h}/\mathfrak{g}(f)_{\mathbb{C}}$ on which S_f is negative definite. Clearly $\mathfrak{h}$ is positive if and only if $q(\mathfrak{h}, f) = 0$. Positivity of the polarization $\mathfrak{h}$ corresponds geometrically to positivity of the line bundle $\mathscr{L}_\chi$ for the holomorphic structure defined by $\mathfrak{h}$, or to $G/G(f)$ being a Kähler manifold (for the complex structure defined by $\mathfrak{h}$ and the symplectic structure defined by B_f).

2.7. The simplest way to try to construct a representation of G from the pair $(\chi, \mathfrak{h})$, where χ is a character of $G(f)$ with differential if and $\mathfrak{h}$ is an invariant totally complex polarization for f, is the process known as *holomorphic induction.* As is well known, this process is sufficient for the "geometric realization" of the irreducible representations of connected type I solvable Lie groups [4], of compact connected Lie groups, and, more generally, of connected type I Lie groups with cocompact radical [15]. However, it does not suffice for the geometric realization of the discrete series of semisimple Lie groups (except in the exceptional case of groups locally isomorphic to a product of copies of $SL(2, \mathbb{R})$)—this was the principal motivation for the study of harmonic induction, which we shall define later.

The holomorphically induced representation obtained from $(\chi, \mathfrak{h})$, which we shall denote $\pi^0(\chi, \mathfrak{h})$ (or $\pi^0(f, \mathfrak{h})$ in case $G(f)$ is connected), is defined to be the unitary representation of G by left translation on the Hilbert space

$$\begin{aligned}\mathscr{H}^0(\chi, \mathfrak{h}) &= \{L^2 \text{ holomorphic sections of } \mathscr{L}_\chi\}\\ &= \Big\{\varphi \in C^\infty(G) : \varphi(gd) = \chi(d)^{-1}\,\varphi(g) \text{ for all } g \in G,\ d \in G(f);\\ &\qquad R(X)\,\varphi + if(X)\,\varphi = 0 \text{ for all } X \in \mathfrak{h}; \text{ and } \int_{G/G(f)} |\varphi|^2\, d\mu < \infty\Big\}.\end{aligned}$$

Here we use the notation

$$(R(X)\,\varphi)(s) = \frac{d}{dt}\bigg|_{t=0} \varphi(s \exp tX) \text{ for } X \in \mathfrak{g},\ s \in G,\ \varphi \in C^\infty(G),$$

and extend R to $\mathfrak{g}_{\mathbb{C}}$ by complex linearity. For $\varphi \in \mathscr{H}^0(\chi, \mathfrak{h})$, thought of as a function on G, $|\varphi|$ is constant on left cosets of $G(f)$, hence $\int_{G/G(f)} |\varphi|^2\, d\mu$ makes sense if $d\mu$ is a G-invariant measure on $G/G(f)$. (Such a measure exists and is unique up to a scalar multiple, since we may pull back the canonical measure on the coadjoint orbit $G \cdot f$.) For purposes of defining π^0, it would suffice to consider the abstractly defined Hilbert space completion of the above space of analytic functions; however, the space is already complete since the minimal and maximal domains of the $\bar{\partial}$ operator coincide.

A famous theorem of Blattner [5, Corollaries to Theorem 4] and of Kobayashi [13][1], the idea of which may be traced back to Harish-Chandra [11], asserts that if G is connected and $\mathscr{H}^0(\chi, \mathfrak{h}) \neq 0$, then $\pi^0(\chi, \mathfrak{h})$ is automatically irreducible. Unfortunately this is of little help if one can't decide when $\mathscr{H}^0(\chi, \mathfrak{h}) \neq 0$, so it is quite useful to know the following necessary condition. This result has been part of the folk literature for over a decade, and implicitly it plays an important role in motivating the study of positive polarizations in [4]. However, we have never seen this theorem in print, and since it is not as well known as it should be, we include the proof here. No claim is made for originality; in fact the proof is merely an adaptation of the argument of [11, Lemma 18] to a more general setting.

THEOREM 2.8. *Let G be any Lie group and f, $\mathfrak{h}$, χ as above. Assume that $\mathscr{H}^0(\chi, \mathfrak{h}) \neq 0$. Then the polarization $\mathfrak{h}$ is positive (for f).*

[1] Kobayashi's theorem at first seems different from Blattner's, since it is formulated in terms of holomorphic n-forms rather than holomorphic sections. But one can easily pass from one to the other (changing the bundle, of course) if $G/G(f)$ has a G-invariant hermitian metric. We return to this point later.

Proof. The salient feature here is that $\mathscr{H}^0(\chi, \mathfrak{h})$ has a "reproducing kernel." More specifically, if $\mathscr{H} = \mathscr{H}^0(\chi, \mathfrak{h}) \neq 0$, then this space consists of real-analytic functions on G, and by the Cauchy estimates, the functionals that evaluate such functions at points of G are continuous (for the Hilbert space topology). Then since any Hilbert space may be identified with its dual, there exists $\varphi_0 \in \mathscr{H}$ such that for all $\varphi \in \mathscr{H}$,

$$\langle \varphi, \varphi_0 \rangle = \varphi(1),$$

where 1 is the identity element of G. Furthermore, $\mathscr{H} \neq 0$ implies that $\varphi_0 \neq 0$ (since if $\varphi \neq 0$, some translate of φ doesn't vanish at 1).

Let $\mathscr{H}_\infty$ denote the space of C^∞-vectors for $\pi^0(\chi, \mathfrak{h})$ in $\mathscr{H}$. Then $\varphi_0 \in \mathscr{H}_\infty$ (in fact, φ_0 is even an analytic vector), since for $\varphi \in \mathscr{H}$ and $s \in G$, we have

$$\langle \pi(s)\, \varphi_0, \varphi \rangle = \langle \varphi_0, \pi(s^{-1})\, \varphi \rangle = \overline{(\pi(s^{-1})\, \varphi)(1)} = \overline{\varphi(s)},$$

hence $s \mapsto \pi(s)\, \varphi_0$ is weakly real-analytic.

Now let $X \in \mathfrak{g}$, $\varphi \in \mathscr{H}_\infty$. We have

$$\begin{aligned} \langle d\pi(X)\, \varphi, \varphi_0 \rangle &= \frac{d}{dt}\bigg|_{t=0} \varphi((\exp - tX)\, 1) = -(R(X)\, \varphi)(1) \\ &= -\langle R(X)\, \varphi, \varphi_0 \rangle, \end{aligned}$$

hence for $X \in \mathfrak{h}$, we have (denoting by $d\pi$ also the complexification of the derived representation of $\mathfrak{g}$)

$$\begin{aligned} \langle d\pi(X)\, \varphi, \varphi_0 \rangle &= -\langle R(X)\, \varphi, \varphi_0 \rangle \\ &= if(X) \langle \varphi, \varphi_0 \rangle \qquad (\text{since } R(X)\, \varphi + if(X)\, \varphi = 0) \\ &= \langle \varphi, -\overline{if(X)}\, \varphi_0 \rangle, \end{aligned}$$

hence (since $d\pi$ is skew-adjoint on the real Lie algebra)

$$\langle \varphi, -d\pi(\bar{X})\, \varphi_0 \rangle = \langle d\pi(X)\, \varphi, \varphi_0 \rangle = \langle \varphi, -\overline{if(X)}\, \varphi_0 \rangle.$$

Since φ was arbitrary in $\mathscr{H}_\infty$, this shows

$$d\pi(\bar{X})\, \varphi_0 = \overline{if(X)}\, \varphi_0 \qquad \text{for } X \in \mathfrak{h}. \tag{2.9}$$

Again let $X \in \mathfrak{h}$. We have

$$\begin{aligned} 0 \leqslant \langle d\pi(X)\, \varphi_0, d\pi(X)\, \varphi_0 \rangle &= \langle -d\pi(\bar{X})\, d\pi(X)\, \varphi_0, \varphi_0 \rangle \\ &= \langle -d\pi(X)\, d\pi(\bar{X})\, \varphi_0, \varphi_0 \rangle + \langle d\pi([X, \bar{X}])\, \varphi_0, \varphi_0 \rangle. \end{aligned} \tag{2.10}$$

However, $\mathfrak{h}$ is totally complex, so we can write $[X, \bar{X}] = Y_1 + \bar{Y}_2$ with $Y_1, Y_2 \in \mathfrak{h}$. Then

$$\begin{aligned}\langle d\pi([X, \bar{X}])\, \varphi_0, \varphi_0\rangle &= \langle d\pi(Y_1)\, \varphi_0, \varphi_0\rangle + \langle d\pi(\bar{Y}_2)\, \varphi_0, \varphi_0\rangle \\ &= -\langle \varphi_0, d\pi(\bar{Y}_1)\, \varphi_0\rangle + \langle d\pi(\bar{Y}_2)\, \varphi_0, \varphi_0\rangle \\ &= -\langle \varphi_0, \overline{if(Y_1)}\, \varphi_0\rangle + \langle \overline{if(Y_2)}\, \varphi_0, \varphi_0\rangle \qquad \text{by (2.9)} \\ &= if(Y_1)\langle \varphi_0, \varphi_0\rangle + \overline{if(Y_2)}\langle \varphi_0, \varphi_0\rangle \\ &= if([X, \bar{X}])\langle \varphi_0, \varphi_0\rangle, \end{aligned} \tag{2.9}$$

and in fact

$$\langle d\pi(Y)\, \varphi_0, \varphi_0\rangle = if(Y)\langle \varphi_0, \varphi_0\rangle \qquad \text{for any } Y \in \mathfrak{g}_{\mathbf{C}}. \tag{2.11}$$

Substituting in (2.10), we obtain

$$\begin{aligned} 0 &\leqslant \langle d\pi(\bar{X})\, \varphi_0, d\pi(\bar{X})\, \varphi_0\rangle + if([X, \bar{X}])\langle \varphi_0, \varphi_0\rangle \\ &= \{|f(X)|^2 + if([X, \bar{X}])\}\, \langle \varphi_0, \varphi_0\rangle. \end{aligned}$$

Since $\varphi_0 \neq 0$, this shows

$$|f(X)|^2 + if([X, \bar{X}]) \geqslant 0 \qquad \text{for all } X \in \mathfrak{h}. \tag{2.12}$$

In case $f|_{\mathfrak{g}(f)}$ is not identically zero, this finishes the argument, since we may apply (2.12) to a vector subspace V of $\mathfrak{h}$ such that $V \subseteq \ker f$ and $\mathfrak{h} = \mathfrak{g}(f) \oplus V$ (as vector spaces).

Otherwise, (2.12) at least shows S_f is positive on $\mathfrak{h} \cap \ker f$, which will be of codimension 1 in $\mathfrak{h}$. Choose $X \in \mathfrak{h}$ with $f(X) = 1$. Using (2.9), (2.10), and (2.11), we have

$$\begin{aligned} &\begin{bmatrix} \langle d\pi(X)\, \varphi_0, d\pi(X)\, \varphi_0\rangle & \langle d\pi(X)\, \varphi_0, d\pi(\bar{X})\, \varphi_0\rangle \\ \langle d\pi(\bar{X})\, \varphi_0, d\pi(X)\, \varphi_0\rangle & \langle d\pi(\bar{X})\, \varphi_0, d\pi(\bar{X})\, \varphi_0\rangle \end{bmatrix} \\ &= \begin{bmatrix} \{1 + if([X, \bar{X}])\}\langle \varphi_0, \varphi_0\rangle & \langle d\pi(X)\, \varphi_0, i\varphi_0\rangle \\ \langle i\varphi_0, d\pi(X)\, \varphi_0\rangle & \langle \varphi_0, \varphi_0\rangle \end{bmatrix} \\ &= \begin{bmatrix} 1 + if([X, \bar{X}]) & 1 \\ 1 & 1 \end{bmatrix} \langle \varphi_0, \varphi_0\rangle. \end{aligned}$$

Since $\langle \varphi_0, \varphi_0\rangle > 0$ and $<, >$ must be positive definite on the linear span of $d\pi(X)\, \varphi_0$ and $d\pi(\bar{X})\, \varphi_0$, the determinant of

$$\begin{bmatrix} 1 + if([X, \bar{X}]) & 1 \\ 1 & 1 \end{bmatrix}$$

must be positive, i.e., $S_f(X, X) > 0$. This completes the proof in the second case. ∎

Remark 2.13. The converse of Theorem 2.8 is certainly false; positivity of $\mathfrak{h}$ for f does not imply $\mathscr{H}^0(\chi, \mathfrak{h}) \neq 0$. The actual conditions for non-vanishing of π^0 tend to be complicated (see [11, Theorem 3; 24, Theorems 4.26 and A.12; 10; 28]), and involve the detailed structure of the group G and the polarization $\mathfrak{h}$. Nevertheless, it is a general phenomenon, which is hard to make precise (but see Theorem 2.15 below), that π^0 will be non-zero provided $\mathfrak{h}$ is "sufficiently positive" for f, or that $\mathfrak{h}$ is positive not only for f but also for some shift of f involving the root structure of $\mathfrak{g}$ and $\mathfrak{h}$. In fact, if one is willing to make this shift slightly larger than necessary, non-vanishing results for π^0 of exponential groups become fairly easy [21]. We shall state a non-vanishing theorem of this type valid for arbitrary Lie groups.

Before formulating the partial converse of Theorem 2.8, it is useful to introduce one additional notion.

DEFINITION 2.14. If $\mathfrak{h}$ is a totally complex invariant polarization for $f \in \mathfrak{g}^*$ (satisfying (2.1), (2.2), (2.3′), and (2.4)), we say $\mathfrak{h}$ is a *metric polarization* if one has

(2.5′) $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{g}(f)} G(f)$ is compact.

This is equivalent to assuming there exists a G-invariant Riemannian metric on $G/G(f)$, or even a G-invariant hermitian metric (for the complex structure defined by $\mathfrak{h}$).

Note that if $\mathfrak{h}$ is positive, it is automatically metric, since $G(f)$ maps $\mathfrak{h}$ into itself by (2.4), and S_f is a $G(f)$-invariant hermitian inner product on $\mathfrak{h}/\mathfrak{g}(f)$ (which can be used to define an invariant hermitian metric on $G/G(f)$). In this case, as we said before, $G/G(f)$ even carries a G-invariant Kähler metric.

Very little is known about how to construct unitary representations of G from the totally complex polarization $\mathfrak{h}$ if condition (2.5′) is not satisfied. Theorem 2.8 shows that holomorphic induction yields nothing for non-metric polarizations, and "harmonic induction" can't be defined either. In fact, the only positive results we know of on non-metric polarizations are those of [22] and [20]. All the polarizations we consider in this paper will be metric.

A natural alternative to holomorphic induction is to consider L^2 holomorphic $(n, 0)$-forms with values in $\mathscr{L}_\chi$ (where $n = \dim_{\mathbb{C}} G/G(f)$) instead of L^2 holomorphic sections. This has the advantage that the L^2-condition has an intrinsic meaning even in the absence of a choice of a metric on $G/G(f)$ or on $\mathscr{L}_\chi$ (ω is L^2 if and only if $|\int \omega \wedge \bar{\omega}| < \infty$); however, this is not a great advantage for us here since one can always put a hermitian metric and connection on $\mathscr{L}_\chi$ having holomorphic curvature form associated to S_f ("Kostant's prequantization theorem" [14]), and as we remarked, $G/G(f)$ carries a canonical volume form. When $\mathfrak{h}$ is metric (but not in general), $G(f)$ acts isometrically on $\Lambda^n\mathfrak{h}$ and on $\Lambda^n\bar{\mathfrak{h}}^*$, hence the holomorphic line bundle $\mathscr{K}$ of holomorphic $(n, 0)$ forms on $G/G(f)$ admits a G-invariant hermitian metric. Using this metric, one may identify L^2 holomorphic sections of $\mathscr{L}_\chi$ with L^2 holomorphic $(n, 0)$-forms with values in $\mathscr{L}_\chi \otimes \mathscr{K}^{-1}$ (and L^2 holomorphic $(n, 0)$-forms with values in $\mathscr{L}_\chi$ with L^2 holomorphic sections of $\mathscr{L}_\chi \otimes \mathscr{K}$), so the Blattner and Kobayashi irreducibility theorems are equivalent in this case.

Now we can state the partial converse of Theorem 2.8.

THEOREM 2.15. *Assume f, χ, and $\mathfrak{h}$ are as above (i.e., f is integral, $d\chi = if$, and $\mathfrak{h}$ satisfies* (2.1), (2.2), (2.3′), *and* (2.4)) *and $\mathfrak{h}$ is positive. Then for N sufficiently large (depending perhaps on G, f, and $\mathfrak{h}$), $\mathscr{H}^0(\chi^N, \mathfrak{h}) \neq 0$. In particular, if $G(f)$ is connected, $\pi^0(Nf, \mathfrak{h}) \neq 0$ for N sufficiently large.*

Proof. Modulo the conversion between holomorphic sections and holomorphic $(n, 0)$-forms discussed above, this is just a special case of [27, Lemma I-B]. In other words, this amounts to the generalization of a famous theorem of Kodaira to the case of non-compact complete Kähler manifolds, using L^2 estimates for the $\bar{\partial}$ operator. See also [7] for much of this theory. ∎

Remark 2.16. In the statements above, the hypotheses that $\mathfrak{h}$ be $G(f)$-invariant and that there exist a character of $G(f)$ with differential if were not strictly necessary. Everything we have said (including the Blattner–Kobayashi Theorem, Theorem 2.8, and Theorem 2.15) applies just as well if we substitute for $G(f)$ any open subgroup D_1 of $G(f)$ and require only that $\mathfrak{h}$ be D_1-invariant and that there exists a character of D_1 with differential if. (This is only integrality in the weakest sense if we take $D_1 = G(f)_0$.)

This appears to have a surprising consequence. Suppose $\mathfrak{h}$ is positive and D_1-invariant and f is D_1-integral, and suppose D_2 is an open subgroup of finite index in D_1. Let χ_1 be a character of D_1 with differential if, and let χ_2 be the restriction of χ_1 to D_2. By Theorem 2.15, for N sufficiently large, $\mathscr{H}^0(\chi_1^N, \mathfrak{h})$ and $\mathscr{H}^0(\chi_2^N, \mathfrak{h})$ are both non-zero, and in fact by the proof in [27], they "separate points" in G/D_1 and G/D_2, respectively. It is also clear that $\mathscr{H}^0(\chi_1^N, \mathfrak{h}) \subseteq \mathscr{H}^0(\chi_2^N, \mathfrak{h})$; if $D_2 \neq D_1$, then $\mathscr{H}^0(\chi_1^N, \mathfrak{h})$ cannot separate points in G/D_2 and so the inclusion is strict. However, by the Blattner–Kobayashi Theorem, if G is connected, both $\pi^0(\chi_1^N, \mathfrak{h})$ and

$\pi^0(\chi_2^N, \mathfrak{h})$ are irreducible! This is a contradiction, hence D_1 can have no open proper subgroups of finite index. This proves

THEOREM 2.17. *Suppose G is a connected Lie group and $f \in \mathfrak{g}^*$ is $(G(f)-)$ integral and admits an invariant totally complex positive polarization. Then $G(f)$ is connected.*

Proof. If $G(f)/G(f)_0$ has a non-trivial finite subgroup, let D_1 be its inverse image in $G(f)$ and let $D_2 = G(f)_0$. If $G(f)/G(f)_0$ is non-trivial but has all elements of infinite order, let D_1 be the inverse image of a cyclic subgroup and let D_2 be a subgroup of finite index. Either way, the argument above gives a contradiction. ∎

Remark 2.18. Theorem 2.17 reflects several well-known facts in Lie group structure theory. Coadjoint orbits of connected compact groups, or regular elliptic orbits of connected semisimple groups, always have connected stabilizers. In fact if we strengthen the "metric polarization" condition (2.5′) to state merely

(2.5″) $\operatorname{Ad} G(f)$ is compact, i.e., $G(f)$ is compact modulo the center of G,

then $\pi^0(f, \mathfrak{h})$ if non-zero will be square-integrable modulo the center of G, and connectedness of $G(f)$ reflects the structure theory for groups with such representations [2, 3].

So far we have only considered holomorphic induction. If $\mathfrak{h}$ is an invariant metric polarization for f which is not necessarily positive, one can more generally define "harmonically induced" representations in the sense of [25, 17–19, 12, etc.].

DEFINITION 2.19. Suppose G is any Lie group, $f \in \mathfrak{g}^*$ is integral with associated character χ, and $\mathfrak{h}$ is an invariant totally complex metric polarization for f, i.e., satisfies (2.1), (2.2), (2.3′), (2.4), and (2.5′). The *harmonically induced* representations $\pi^k(\chi, \mathfrak{h})$ (or $\pi^k(f, \mathfrak{h})$ if $G(f)$ is connected), $0 \leqslant k \leqslant n = \dim_{\mathbb{C}} G/G(f)$, are defined to be the unitary representations of G by left translation on the Hilbert spaces

$$\mathscr{H}^k(\chi, \mathfrak{h}) = \{L^2\ (0, k)\text{-forms } \omega \text{ with values in } \mathscr{L}_\chi \text{ satisfying } \bar\partial\omega = 0, \bar\partial^*\omega = 0\},$$

where $\bar\partial$ is defined with respect to the complex structure defined by $\mathfrak{h}$, and $\bar\partial^*$, its formal adjoint, is defined using a G-invariant hermitian metric on $G/G(f)$, which we fix once and for all. The metric also defines the notion of an L^2 form. This metric is not unique, but by [12], the equivalence class of $\pi^k(\chi, \mathfrak{h})$ is independent of the metric chosen. Once again (see [18] and [12] for details), ellipticity of the $\bar{\partial}$-complex ensures that we may work only with smooth forms and still get a complete Hilbert space; all distributional solutions of the equations $\bar{\partial}\omega = 0$, $\bar{\partial}^*\omega = 0$ are C^∞ by elliptic regularity. Also there are no domain problems for our unbounded operators since the minimal and maximal domains of $\bar{\partial}$ coincide.

Formal curvature calculations suggest that just as $\pi^0(\chi, \mathfrak{h})$ tends to be non-zero exactly when $\mathfrak{h}$ is positive, $\pi^k(\chi, \mathfrak{h})$ should be non-zero exactly when $k = q(\mathfrak{h}, f)$ (recall the definition of the negativity index in 2.6). Furthermore, one would again hope $\pi^k(\chi, \mathfrak{h})$ is irreducible, or at least finitely decomposable, when it is non-zero. In fact, both of these statements are false in general, although they are true under some circumstances. For instance, as an accompaniment to Theorem 2.15, one has the following "vanishing theorem."

THEOREM 2.20. *With f, χ, and $\mathfrak{h}$ as above and $\mathfrak{h}$ positive (i.e., the same hypotheses as in Theorem 2.15), for N sufficiently large,*

$$\mathscr{H}^k(\chi^N, \mathfrak{h}) = 0$$

for all $k \geqslant 1$.

Proof. See [27] or [7]. A more group-theoretic formulation of what is essentially the same argument may be found in [16]; the restriction of nilpotence of G there is not particularly important. Results analogous to Theorems 2.15 and 2.20 are also proved in [8]. ∎

More generally, the same sorts of arguments would appear to give the following result. We record this as a "statement" rather than a "theorem" since we have not checked all the details in the non-positive case, except to the extent that they may be found in [16].

STATEMENT 2.21. *If G is any Lie group, $f \in \mathfrak{g}^*$ is integral, χ is a character of $G(f)$ with differential if, and $\mathfrak{h}$ is an invariant totally complex metric polarization for f (satisfying* (2.1), (2.2), (2.3′), (2.4), *and* (2.5′)), *then there exists a positive integer N_0 (possibly depending on all the data) such that for $N > N_0$, one has*

$$\pi^k(\chi^N, \mathfrak{h}) = 0 \qquad \text{for } k \neq q(\mathfrak{h}, f),$$
$$\pi^{q(\mathfrak{h}, f)}(\chi^N, \mathfrak{h}) \neq 0.$$

Two questions that we have not been able to answer are the following.

Open Problems

(2.22) Under the above hypotheses, can it ever happen that $\pi^k(\chi, \mathfrak{h})$ is non-zero for two distinct values of k? We see no obvious reason why not, but at the same time we know of no such examples.

(2.23) Under the hypotheses of (2.21), is $\pi^{q(\mathfrak{h},f)}(\chi^N, \mathfrak{h})$ *irreducible* for N sufficiently large?

3. Harmonically Induced Representations of Certain Solvable Lie Groups

In order to obtain more precise results than those in Section 2 above, it is necessary to make some assumptions about the groups G, linear functionals f, and polarizations $\mathfrak{h}$ which we will consider. To provide a reasonable motivation for our calculations, we first remind the reader of the results in the semisimple and nilpotent cases.

Although, historically, the situation for semisimple groups was worked out before that for nilpotent groups, we begin with the nilpotent case since the results are easier to state. Various cases of the following theorem were first worked out by Carmona, Satake, Okamoto, and Moscovici and Verona. The result is as pretty as one could hope for.

THEOREM 3.1 [19, 23]. *Let G be a connected nilpotent Lie group, $f \in \mathfrak{g}^*$ an integral linear functional. (In this case, $G(f)$ is always connected.) Suppose f admits a metric totally complex polarization $\mathfrak{h}$. Then $\pi^k(f, \mathfrak{h}) = 0$ for $k \neq q(\mathfrak{h}, f)$, and $\pi^{q(\mathfrak{h},f)}(f, \mathfrak{h})$ is the irreducible unitary representation π_f associated by the Kirillov correspondence to the orbit of f.*

The semisimple case is similar except for a certain shift which already occurs in the compact case (Borel–Weil–Bott Theorem). Once again, many special cases of the following theorem were worked out before the definitive result was proved by Schmid. Some of the names involved were Borel and Weil, Bott, Kostant, Okamoto and Narasimhan, Hotta, and Parthasarathy. The hypothesis that G be a linear group, which one finds in Schmid's paper, is not in fact necessary.

THEOREM 3.2. [25, 26]. *Let G be a connected semisimple Lie group, $f \in \mathfrak{g}^*$ such that $G(f)$ is a Cartan subgroup of G which is compact modulo the center (hence automatically connected). Suppose f is integral, and let $\mathfrak{h}$ be a totally complex polarization for f. (Such an $\mathfrak{h}$ always exists and will be a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$.) Let $\rho = (i/2)\,\Sigma\alpha$, where α runs over the roots of $\mathfrak{g}(f)_{\mathbb{C}}$ in $\mathfrak{h}$. (The factor of i is inserted so as to have ρ in the real dual of $\mathfrak{g}(f)$.) Then if $f + \rho$ is singular, $\pi^k(f, \mathfrak{h}) = 0$ for all k. Otherwise, $\pi^k(f, \mathfrak{h})$ is* *non-zero exactly when* $k = q(\mathfrak{h}, f+\rho)$; *for this value of* k, $\pi^k(f, \mathfrak{h})$ *is the irreducible discrete series representation of* G *attached (via the Harish–Chandra parametrization) to the orbit of* $f+\rho$.

Remarks 3.3. (a) Note that for f "sufficiently large," i.e., for large multiples f of any f_0 satisfying the hypothesis of (3.2), $f+\rho$ will lie in the same Weyl chamber as f, so that one may substitute $q(\mathfrak{h}, f)$ for $q(\mathfrak{h}, f+\rho)$. However, $q(\mathfrak{h}, f)$ and $q(\mathfrak{h}, f+\rho)$ may differ for f "close to the walls." This phenomenon is in accord with (2.15), (2.20), and (2.21) above.

(b) Actually, in the above theorem, it is not really necessary to suppose the stabilizer of f is a Cartan subgroup. Essentially the same statement holds whenever one induces harmonically from a unitary character of an Ad-compact Cartan subgroup, even if the stabilizer of the character is larger.

(c) There is an analogue of the above results valid for extensions of a nilpotent Lie group by a reductive one, obtained essentially by combining Theorems 3.1 and 3.2 [23, Theorem 4.8]. Thus one has fairly precise results on harmonic induction for all locally algebraic unimodular Lie groups having square-integrable representations [1]. See [23, Sect. 4] for further discussion.

Since the theory of harmonic induction from a compact subgroup of a unimodular Lie group is largely complete, we wish now to attack the non-unimodular case. Our methods would in principle work in much greater generality, but for reasons that will become apparent shortly, the calculations involved rapidly get out of hand except for the simplest groups. We therefore restrict attention to the same sort of groups considered in [24]. In other words, we assume for the rest of this section that the following holds.

HYPOTHESIS 3.4. *G is a connected, simply connected, locally algebraic Lie group, completely solvable over $\mathbb{R}$ (hence an exponential solvable group). We also assume G has at least one open orbit in $\mathfrak{g}^*$.*

The open-orbit assumption guarantees that $\hat{G}_d$, the set of equivalence classes of square-integrable irreducible representations (or "discrete series") of G, is non-empty. This coupled with the algebraicity implies that the Plancherel measure class of G is supported on $\hat{G}_d$, and that this is a finite set. (In fact, the union of the open orbits of G in $\mathfrak{g}^*$ is Zariski-open, and the complex adjoint group of G acts transitively on this open dense set.) We shall always assume that $f \in \mathfrak{g}^*$ is chosen in one of the open orbits, hence that $G(f)$ is trivial, and that f admits a totally complex polarization $\mathfrak{h}$. Obviously, conditions (2.4) and (2.5′) are automatic, so that $\pi^k(f, \mathfrak{h})$ is well defined for $0 \leqslant k \leqslant \dim_{\mathbb{C}} G \cdot f = (\dim G)/2$. We remind the reader that large numbers of interesting groups satisfying these conditions may be obtained by taking the "AN-group" of the Iwasawa decomposition KAN of the automorphism group of a hermitian symmetric space of non-compact type. In this case there is a "standard" choice of $\mathfrak{h}$ corresponding to the complex structure on the symmetric domain. For details and the structure theory of a larger class of solvable groups satisfying our conditions, see [24, Sections 3–4].

THEOREM 3.5. *Let G satisfy (3.4) and suppose $f \in \mathfrak{g}^*$ lies in an open orbit and $\mathfrak{h}$ is a totally complex polarization for f. Then $\pi^k(f, \mathfrak{h})$ $(0 \leqslant k \leqslant n = \dim_{\mathbb{C}} G \cdot f)$ decomposes G-equivariantly as follows:*

$$\mathscr{H}^k(f, \mathfrak{h}) \cong \bigoplus_{\pi \in \hat{G}_d} \mathscr{H}_\pi \otimes \mathscr{H}^k(\check{\pi}, f, \mathfrak{h}).$$

Here $\mathscr{H}^k(\check{\pi}, f, \mathfrak{h})$, the kth *formal harmonic space of $\check{\pi}$ with respect to f and $\mathfrak{h}$, may be variously described as*

$$\begin{aligned}\mathscr{H}^k(\check{\pi}, f, \mathfrak{h}) &= \{\Phi \in \mathscr{H}_{\check{\pi}} \otimes \Lambda^k\mathfrak{h}^* : \Phi \perp (\delta + \delta^*)((\mathscr{H}_{\check{\pi}})_\infty \otimes \Lambda^*\mathfrak{h}^*)\} \\ &\cong (\ker \delta) \cap (\ker \delta^*) \qquad \text{in } (\mathscr{H}_{\check{\pi}})_\infty \otimes \Lambda^k\mathfrak{h}^* \\ &\cong \bar{H}^k(\mathfrak{h}, (\mathscr{H}_{\check{\pi}})_\infty \otimes \mathbb{C}_{if + \Delta/2}).\end{aligned}$$

We have used the following notation: $\mathscr{H}_\pi$ is the Hilbert space on which π acts, $\check{\pi}$ is the contragredient representation on $\mathscr{H}_{\check{\pi}}$, $(\mathscr{H}_{\check{\pi}})_\infty$ is the Fréchet space of C^∞-vectors for $\check{\pi}$, and $\Delta = \operatorname{tr} \operatorname{ad}$ is the differential of the modular function of G. Also

$$\delta\colon M \otimes \Lambda^j\mathfrak{h}^* \to M \otimes \Lambda^{j+1}\mathfrak{h}^*$$

is the coboundary operator for the standard complex that computes the Lie algebra cohomology of an $\mathfrak{h}$-module M, applied here to $M = (\mathscr{H}_{\check{\pi}})_\infty \otimes \mathbb{C}_{if+\Delta/2}$ (the second factor being a one-dimensional module on which $\mathfrak{h}$ acts by the indicated linear functional). δ^* is the formal adjoint of δ, and $\bar{H}^k(\mathfrak{h}, M)$ is the "Hausdorffification" of the Lie algebra cohomology $H^k(\mathfrak{h}, M)$, i.e.,

$$\begin{aligned}\bar{H}^k(\mathfrak{h}, M) = \ker(\delta\colon M \otimes \Lambda^k\mathfrak{h}^* &\to M \otimes \Lambda^{k+1}\mathfrak{h}^*)/ \\ \operatorname{cl} \operatorname{im}(\delta\colon M \otimes \Lambda^{k-1}\mathfrak{h}^* &\to M \otimes \Lambda^k\mathfrak{h}^*).\end{aligned}$$

Note that δ is a continuous linear map from one Fréchet space to another, but a priori, its image may not be closed.

Proof. Since $\mathscr{H}^k(f, \mathfrak{h})$ is a G-invariant subspace of $L^2(G) \otimes \Lambda^k\mathfrak{h}^*$, it is obviously quasi-contained in the left regular representation of G, and so we have a G-equivariant Plancherel decomposition

$$\mathscr{H}^k(f, \mathfrak{h}) \cong \bigoplus_{\pi \in \hat{G}_d} \mathscr{H}_\pi \otimes \operatorname{Hom}_G(\mathscr{H}_\pi, \mathscr{H}^k(f, \mathfrak{h})).$$

We now follow the method of Schmid [25] for analyzing the space of intertwining operators

$$\operatorname{Hom}_G(\mathscr{H}_\pi, \mathscr{H}^k(f, \mathfrak{h})),$$

as extended to the case of more general groups by Moscovici and Verona [17], Penney [18], and Hersant [12]. The one thing that is new in our case is the adjustment needed to take into account the non-unimodularity of G. This results from the fact that holomorphic sections φ of $\mathscr{L}_\chi$ are defined (recall (2.7)) by the conditions

$$\varphi \in C^\infty(G),\ R(X)\,\varphi + if(X)\,\varphi = 0 \text{ for } X \in \mathfrak{h},$$

whereas when we make the Plancherel identification

$$L^2(G) \cong \bigoplus_{\pi \in \hat{G}_d} \mathscr{H}_\pi \otimes \mathscr{H}_{\check{\pi}} \qquad \text{(as unitary representations of } G \times G\text{)},$$

the differential of the action of G on $\mathscr{H}_{\check{\pi}}$ corresponds not to R but to $R - \Delta/2$. The rest of the details, including the coincidence of the three possible definitions of $\mathscr{H}^k(\check{\pi}, f, \mathfrak{h})$, are in [12, Sect. 5]. ∎

Remark 3.6. In the semisimple and nilpotent cases, the Lie algebra cohomology groups analogous to our $H^k(\mathfrak{h}, (\mathscr{H}_{\check{\pi}})_\infty \otimes \mathbb{C}_{if + \Delta/2})$ were computable with the help of the Hochschild–Serre spectral sequence, and turned out to be finite-dimensional, hence a fortiori already Hausdorff (by the closed graph theorem). This is not always the case here, as the following example illustrates.

EXAMPLE 3.7. Let G be the "$ax+b$" group with Lie algebra spanned by X and Y, where $[X, Y] = Y$. Let X^*, Y^* be the dual basis of $\mathfrak{g}^*$ and suppose $f = \alpha Y^*$, $\alpha \neq 0$, $\mathfrak{h} = \mathbb{C}(X + iY)$. Note that $\Delta = X^*$ in this case.

There are two open orbits of G on $\mathfrak{g}^*$, and the corresponding representations in $\hat{G}_d$ are π_+ and π_-, both acting on $L^2(\mathbb{R})$, where

$$d\pi_\pm(X) = d/dt,$$

$$d\pi_\pm(Y) = \pm ie^t$$

on suitable domains of C^∞ functions. Note that $\check{\pi}_+ \cong \pi_-$, $\check{\pi}_- \cong \pi_+$. Thus on

$$M = (\mathscr{H}_{\check{\pi}_\pm})_\infty \otimes \mathbb{C}_{if+\Delta/2} \subsetneq C^\infty(\mathbb{R}) \cap L^2(\mathbb{R}),$$

$X + iY \in \mathfrak{h}$ acts by the differential operator

$$D = \frac{d}{dt} \pm e^t + \frac{1}{2} - \alpha.$$

This has formal adjoint

$$D^* = -\frac{d}{dt} \pm e^t + \frac{1}{2} - \alpha.$$

A simple calculation shows that the kernels of D and D^* in $C^\infty(\mathbb{R})$ are spanned (respectively) by the functions

$$t \mapsto \exp((\alpha - \tfrac{1}{2})\, t \mp e^t), \qquad \exp(-(\alpha - \tfrac{1}{2})\, t \pm e^t).$$

Thus

$$(\ker D) \cap L^2(\mathbb{R}) \neq 0 \Leftrightarrow M = (\mathscr{H}_{\check{\pi}_+})_\infty \otimes \mathbb{C}_{if+\Delta/2} \text{ and } \alpha > \tfrac{1}{2},$$
$$(\ker D^*) \cap L^2(\mathbb{R}) \neq 0 \Leftrightarrow M = (\mathscr{H}_{\check{\pi}_-})_\infty \otimes \mathbb{C}_{if+\Delta/2} \text{ and } \alpha < \tfrac{1}{2}.$$

Applying Theorem 3.5, we see

$$\begin{aligned} \mathscr{H}^k(f, \mathfrak{h}) &\cong \pi_+ && \text{if } \alpha > \tfrac{1}{2},\ k = 0, \\ &\cong 0 && \text{if } \alpha = \tfrac{1}{2}, \text{ or if } \alpha > \tfrac{1}{2} \text{ and } k = 1, \text{ or if } \alpha < \tfrac{1}{2} \text{ and } k = 0, \\ &\cong \pi_- && \text{if } \alpha < \tfrac{1}{2},\ k = 1. \end{aligned}$$

For comparison, note that

$$if([X + iY, X - iY]) = 2\alpha,$$

so that

$$\begin{aligned} q(\mathfrak{h}, f) &= 0 && \text{if } \alpha > 0, \\ &= 1 && \text{if } \alpha < 0. \end{aligned}$$

In accordance with (2.21), $\pi^k(f, \mathfrak{h})$ is non-zero exactly when $k = q(\mathfrak{h}, f)$, provided that $|\alpha| > \frac{1}{2}$. However, for α "small" there is some anomalous behavior. Furthermore, although in this case $\bar{H}^1(\mathfrak{h}, M)$ always has dimension 0 or 1, $H^1(\mathfrak{h}, M)$ may be infinte-dimensional. To see this, let us take $\alpha = \frac{1}{2}$ and show that

$$\left(\frac{d}{dt} - e^t\right)((\mathscr{H}_{\pi_+})_\infty)$$

is not closed and is of infinite codimension in $(\mathscr{H}_{\pi_+})_\infty$. First note that $(\mathscr{H}_{\pi_+})_\infty$ consists of C^∞ functions φ such that φ and all its derivatives lie in $L^2(\mathbb{R})$ and have very rapid (exponential) decrease on the *right* half-line. Thus $(\mathscr{H}_{\pi_+})_\infty$ contains an infinite-dimensional subspace of functions $\varphi(t)$ that look like const/t for t negative. Since $D = d/dt - e^t$ has zero kernel in $L^2(\mathbb{R})$, any solution of $D\psi = \varphi$ must be of the form

$$\psi(t) = e^{e^t} \int_{-\infty}^{t} \varphi(s)\, e^{-e^s}\, ds.$$

If $\varphi(t) \sim |t|^{-1}$ for $t < \infty$, then $\psi(t) \sim \log |t|$, and so ψ is not in L^2. ∎

We are now ready for our main calculation, which involves another rather special group. For the moment the example may seem somewhat ad hoc, though we found it very instructive. Our methods will be strictly classical for the moment, although we shall give another interpretation of the theorem in Section 4.

THEOREM 3.8. *Let $G = N_{2n+1} \rtimes \mathbb{R}_+^\times$ be the semidirect product of the Heisenberg group N_{2n+1} of dimension $2n+1$, $n \geqslant 1$, by a one-parameter group of dilations. More explicitly, G is the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ spanned by the elements $X_1,..., X_n$, $Y_1,..., Y_n$, H, Z, where*

$$[X_j, Y_k] = \delta_{jk} Z, \qquad [H, X_j] = X_j,$$
$$[H, Y_j] = Y_j, \qquad \text{and} \qquad [H, Z] = 2Z.$$

Let $f = Z^*$ with $\alpha \neq 0$, where $X_1^*,..., X_n^*$, $Y_1^*,..., Y_n^*$, H^*, Z^* is the dual basis of $\mathfrak{g}^*$, and let $\mathfrak{h}$ be the totally complex polarization for f spanned by the elements

$$J_0 = H + iZ,$$
$$J_j = X_j + iY_j, \qquad j = 1,..., n.$$

Let us designate the representations in $\hat{G}_d$ by π_+ and π_-, where both representations act on $L^2(\mathbb{R}^{n+1})$ and have differentials given by the formulae

$$d\pi_\varepsilon(H) = \frac{\partial}{\partial t_0},$$

$$d\pi_\varepsilon(X_j) = e^{t_0}\frac{\partial}{\partial t_j}, \qquad 1 \leqslant j \leqslant n,$$

$$d\pi_\varepsilon(Y_j) = i\varepsilon e^{t_0} t_j, \qquad 1 \leqslant j \leqslant n,$$

$$d\pi_\varepsilon(Z) = i\varepsilon e^{2t_0}$$

on suitable function spaces. Then the representations $\pi^k(f, \mathfrak{h})$ are given as follows:

(a) *If $\alpha \in \{1, 2,..., n+1\}$, all of the $\mathscr{H}^k(f, \mathfrak{h})$ are zero.*

(b) *Otherwise, $\mathscr{H}^k(f, \mathfrak{h})$ is non-zero for exactly one value of k, namely,*

$$\begin{aligned} k &= 0 && \text{if } \alpha > n+1, \\ k &= 1 && \text{if } n < \alpha < n+1, \\ &\;\;\cdots \\ k &= n && \text{if } 1 < \alpha < 2, \\ k &= n-1 && \text{if } \alpha < 1. \end{aligned}$$

(c) *In case* (b), *we have* (*for the above value of k*)

$$\begin{aligned} &\pi^k(f, \mathfrak{h}) \cong \pi_+ && \text{if } \alpha > n+1, \\ &\pi^k(f, \mathfrak{h}) \cong \pi_- && \text{if } \alpha < 1, \\ &\pi^k(f, \mathfrak{h}) \cong \infty \cdot \pi_+ \oplus \infty \cdot \pi_- \cong \text{regular rep. of } G \\ &&& \text{if } 1 < \alpha < n+1 \text{ and } \alpha \text{ is not integral.} \end{aligned}$$

Remark 3.9. Before we begin the proof, a few comments are in order. First of all, (3.7) is really a degenerate case of the above ($n = 0$), with the change of notation $H = 2X$, $Z = 2Y$, $Y^* = 2Z^*$. Second, one sees as before that

$$\begin{aligned} q(\mathfrak{h}, f) &= 0 && \text{if } \alpha > 0, \\ &= n+1 && \text{if } \alpha < 0. \end{aligned}$$

Thus the results are again consistent with (2.21) for $|\alpha| > n+1$, although the anomalous behavior for $|\alpha|$ small is now much more marked. One also finds as in (3.7) the pathology of non-Hausdorff Lie algebra cohomology.

Proof of (3.8). As before we use (3.5) and calculate $\mathscr{H}^k(\check{\pi}_\varepsilon, f, \mathfrak{h})$ for all possible values of k and for $\varepsilon = \pm 1$. It is the interpretation of the formal harmonic space as $(\ker \delta) \cap (\ker \delta^*)$ which is most effective here. The calculation when $k=0$ or $k=n+1$ is fairly simple; all the other cases are much harder but ultimately come down to the same equations, so we shall discuss the case $k=1$ in detail and then indicate what modifications are needed to treat the cases $1<k<n+1$.

A simple calculation gives $\Delta = (2n+2)\, H^*$. Thus on the module $M = (\mathscr{H}_{\check{\pi}_\varepsilon})_\infty \otimes \mathbb{C}_{if+\Delta/2} = (\mathscr{H}_{\pi_{-\varepsilon}})_\infty \otimes \mathbb{C}_{i\alpha Z^* + (n-1)H^*} \subsetneq C^\infty(\mathbb{R}^{n+1})$, the action of $\mathfrak{h}$ is given by

$$\begin{aligned} J_0 &\mapsto \frac{\partial}{\partial t_0} + \varepsilon e^{2t_0} + (n+1-\alpha), \\ J_j &\mapsto e^{t_0}\left(\frac{\partial}{\partial t_j} - \varepsilon t_j\right), \qquad j = 1,..., n. \end{aligned} \tag{3.10}$$

We omit discussion of domains and proceed formally, since by the theorem in [12] that the minimal and maximal domains of $(\delta+\delta^*)^2$ coincide, any form in $(\ker \delta) \cap (\ker \delta^*)$ which is C^∞ and L^2 will in fact be in $\mathscr{H}_\infty \otimes \Lambda^* \mathfrak{h}^*$.

First note that $\ker J_j$ (in L^2), for $j \geqslant 1$, will be non-zero if and only if $\varepsilon > 0$. Thus

$$\mathscr{H}^0(\check{\pi}_-, \alpha Z^*, \mathfrak{h}) = 0 \qquad \text{for all } \alpha,$$

and

$$\begin{aligned} &\dim \mathscr{H}^0(\check{\pi}_+, \alpha Z^*, \mathfrak{h}) = 1 \\ &\quad \Leftrightarrow (\ker J_0) \cap (\ker J_1) \cap \cdots \cap (\ker J_n) \neq (0) \quad (\text{in } L^2) \\ &\quad \Leftrightarrow \exp[(\alpha - n - 1)\, t_0 - \tfrac{1}{2} e^{t_0} - \tfrac{1}{2}(t_1^2 + \cdots t_n^2)] \\ &\qquad \text{is an } L^2 \text{ function of } (t_0, t_1, ..., t_n) \\ &\quad \Leftrightarrow \alpha > n+1. \end{aligned}$$

Hence

$$\begin{aligned} \mathscr{H}^0(f, \mathfrak{h}) &\cong 0 \qquad \text{if } \alpha \leqslant n+1, \\ &\cong \pi_+ \qquad \text{if } \alpha > n+1. \end{aligned}$$

The calculation of $\mathscr{H}^{n+1}(f, \mathfrak{h})$ is similar, although one must remember that the formula for δ^* in top degree involves a term that depends on the brackets in $\mathfrak{h}$. (Our $\mathfrak{h}$ is *not* commutative, unlike the situation in (3.7). Instead, $[J_0, J_j] = J_j$ for $1 \leqslant j \leqslant n$.) The formal adjoints of the operators corresponding to $J_0,..., J_n$ are

$$J_0^* = -\frac{\partial}{\partial t_0} + \varepsilon e^{2t_0} + (n+1-\alpha),$$
$$J_j^* = -e^{t_0}\left(\frac{\partial}{\partial t_j} - \varepsilon t_j\right), \qquad j = 1,\dots, n. \tag{3.11}$$

Thus $\ker J_j^*$ (in L^2), for $j \geqslant 1$, is non-zero if and only if $\varepsilon < 0$, and

$$\mathscr{H}^{n+1}(\check{\pi}_+, \alpha Z^*, \mathfrak{h}) = 0 \qquad \text{for all } \alpha;$$

$$\dim \mathscr{H}^{n+1}(\check{\pi}_-, \alpha Z^*, \mathfrak{h}) = 1 \qquad \text{(the only possibility aside from 0)}$$
$$\Leftrightarrow (\ker(J_0 - n)^*) \cap (\ker J_1^*) \cap \cdots \cap (\ker J^*) \neq (0) \qquad (\text{in } L^2)$$
$$\Leftrightarrow \exp[(1-\alpha)\, t_0 - \tfrac{1}{2} e^{2t_0} - \tfrac{1}{2}(t_1^2 + \cdots + t_n^2)]$$
$$\text{is an } L^2 \text{ function of } (t_0, t_1, \dots, t_n)$$
$$\Leftrightarrow \alpha < 1.$$

Hence

$$\mathscr{H}^{n+1}(f, \mathfrak{h}) \cong 0 \qquad \text{if } \alpha \geqslant 1,$$
$$\cong \pi_- \qquad \text{if } \alpha < 1.$$

Next we compute $\mathscr{H}^1(\check{\pi}_\pm, \alpha Z^*, \mathfrak{h})$. Now $M \otimes \Lambda^1 \mathfrak{h}^*$ consists of elements of the form $v_0 \otimes J_0^* + v_1 \otimes J_1^* + \cdots v_n \otimes J_n^*$. Since the only non-trivial brackets in $\mathfrak{h}$ involve J_0, one finds that the equations

$$\delta^*(v_0 \otimes J_0^* + \cdots + v_n \otimes J_n^*) = 0 \qquad \text{in } M,$$
$$\delta(v_0 \otimes J_0^* + \cdots + v_n \otimes J_n^*) = 0 \qquad \text{in } M \otimes \Lambda^2 \mathfrak{h}^*,$$

become

$$J_0^* \cdot v_0 + \cdots + J_n^* \cdot v_n = 0,$$
$$J_j \cdot v_k = J_k \cdot v_j, \qquad \text{if } j, k \geqslant 1, \tag{3.12}$$
$$J_0 \cdot v_k = J_k \cdot v_0 + v_k, \qquad \text{if } k \geqslant 1.$$

Here the operators J_j, J_j^* are given by (3.10) and (3.11). To simplify Eq. (3.12), it is useful to make the transformation

$$K_j = J_j \circ e^{-t_0},$$
$$K_0 = e^{-t_0} \frac{\partial}{\partial t_0} + \varepsilon e^{t_0} + (n-\alpha)\, e^{-t_0}, \tag{3.13}$$
$$K_j = \frac{\partial}{\partial t_j} + \varepsilon t_j, \qquad j = 1,\dots, n.$$

This converts the system (3.12) to the system

$$\begin{aligned} &K_0^* v_0 + \cdots K_n^* v_n = 0, \\ &\qquad K_j v_k = K_k v_j, \qquad j, k = 0,\ldots,n. \end{aligned} \tag{3.14}$$

Recall once again that we are looking for solutions with $v_0,\ldots, v_n \in L^2(\mathbb{R}^{n+1})$. Using the fact that we now have the variables separated (K_j involves only t_j), we may apply each K_j in turn to the first equation in (3.14), then use the second equation to obtain

$$\begin{aligned} K_0 K_0^* v_0 + K_1^* K_1 v_0 + \cdots + K_n^* K_n v_0 &= 0, \\ K_0^* K_0 v_1 + K_1 K_1^* v_1 + \cdots + K_n^* K_n v_1 &= 0, \\ K_0^* K_0 v_n + \cdots + K_{n-1}^* K_{n-1} v_n + K_n K_n^* v_n &= 0. \end{aligned} \tag{3.15}$$

Since the operators $K_j K_j^*$, $K_j^* K_j$ are all *formally* positive, one might think that this implies $v_0 = v_1 = \cdots = v_n = 0$. However, this is not necessarily the case since these operators are not necessarily essentially self-adjoint. For $j \neq 0$, $K_j K_j^*$ and $K_j^* K_j$ become the Hamiltonian operators for a quantum harmonic oscillator, which of course have a well-understood spectral theory. Thus one can see that if (3.15) has any L^2 solutions at all, it must have L^2 solutions with the variables separated, i.e., with each v_j a product of a function of t_0 and of Hermite functions in the remaining variables (since these are the eigenfunctions of the harmonic oscillator). Let $e_0, e_1, e_2,\ldots$ be the classical Hermite functions giving an orthogonal basis of $L^2(\mathbb{R})$. They satisfy

$$\begin{aligned} \left(\frac{d}{dt} + t\right) e_j(t) &= \sqrt{2j}\, e_{j-1}(t), \\ \left(\frac{d}{dt} + t\right)^* e_j(t) &= \sqrt{2(j+1)}\, e_{j+1}(t). \end{aligned} \tag{3.16}$$

Thus one finds that if the systems (3.14) and (3.15) have any L^2 solutions at all, one may take them to be of the following special type:

When $\varepsilon = +1$,

$$\begin{aligned} &v_0 = \varphi_0(t_0)\, e_r(t_1) \cdots e_r(t_n) \qquad \text{(for some fixed } r\text{)} \\ &v_j = \varphi_1(t_0)\, e_r(t_1) \cdots e_{r-1}(t_j) \cdots e_r(t_n), \qquad 1 \leqslant j \leqslant n, \\ &K_0 \varphi_1 = \sqrt{2r}\, \varphi_0, \qquad K_0^* \varphi_0 = -n\sqrt{2r}\, \varphi_1, \\ &K_0 K_0^* \varphi_0 = -2rn\varphi_0, \\ &K_0^* K_0 \varphi_1 = -2rn\varphi_1. \end{aligned} \tag{3.17}$$

When $\varepsilon = -1$,

$$
\begin{aligned}
v_0 &= \varphi_0(t_0)\, e_r(t_1) \cdots e_r(t_n) && \text{(for some fixed } r\text{)} \\
v_j &= \varphi_1(t_0)\, e_r(t_1) \cdots e_{r+1}(t_j) \cdots e_r(t_n), && 1 \leqslant j \leqslant n, \\
K_0 \varphi_1 &= -\sqrt{2(r+1)}\, \varphi_0, \qquad K_0^* \varphi_0 = n\sqrt{2(r+1)}\, \varphi_1, && (3.18) \\
K_0 K_0^* \varphi_0 &= -2(r+1)\, n\varphi_0, \\
K_0^* K_0 \varphi_1 &= -2(r+1)\, n\varphi_1.
\end{aligned}
$$

It remains to solve Eqs. (3.17) and (3.18) and to see when the solutions for φ_0 and φ_1 lie in L^2. For this purpose we make another change of variables that converts (3.17) and (3.18) to confluent hypergeometric equations, which can be solved by classical integral formulas. Thus let

$$
\begin{aligned}
\xi &= e^{2t_0}, \qquad \text{hence } J_0 = 2\xi \frac{\partial}{\partial \xi} + \varepsilon\xi + n + 1 - \alpha, \\
K_0 &= e^{-t_0}(J_0 - 1) = 2\xi^{1/2} \frac{\partial}{\partial \xi} + \varepsilon\xi^{1/2} + (n-\alpha)\, \xi^{-1/2}, \qquad (3.19) \\
K_0^* &= e^{-t_0} J_0^* = -2\xi^{1/2} \frac{\partial}{\partial \xi} + \varepsilon\xi^{1/2} + (n+1-\alpha)\, \xi^{-1/2}.
\end{aligned}
$$

To solve (3.17), the case $\varepsilon = +1$, we let

$$\varphi_1(t_0) = e^{-\xi/2} \xi^{(\alpha - n)/2} g(\xi), \qquad \text{where } \xi = e^{2t_0},$$

which gives for g "Kummer's equation"

$$\xi g''(\xi) + (b - \xi)\, g'(\xi) - a g(\xi) = 0, \tag{3.20}$$

where $b = \alpha - n$ and $a = (r+1)\, n/2$. We are interested in solutions with $\varphi_1 \in L^2(d\xi/\xi)$. Equation (3.20) has two linearly independent solutions, one which grows like e^{ξ} as $\xi \to \infty$ (which we can obviously discard), and the other given by the integral formula

$$g(\xi) = \int_0^\infty e^{-\xi s} s^{a-1} (1+s)^{b-a-1}\, ds$$

(convergent since $a > 0$). Then

$$\varphi_1(\xi) = e^{-\xi/2} \xi^{b/2} \int_0^\infty e^{-\xi s} s^{a-1} (1+s)^{b-a-1}\, ds \tag{3.21}$$

will die exponentially as $\xi \to \infty$. The behavior of φ_1 near $\xi = 0$ depends essentially on $b=\alpha-n$. If $b>1$, i.e., $\alpha>n+1$, then up to a constant, $\varphi_1(\xi)\sim\xi^{b/2}\xi^{1-b}=\xi^{(2-b)/2}$ and $\varphi_1(\xi)^2/\xi\sim\xi^{1-b}$, which is integrable in ξ if $b<2$. If $b=1$, i.e., $\alpha=n+1$, then the integral behaves like $\log\xi$ and $\varphi_1(\xi)^2/\xi\sim\log\xi$, which is integrable. If $b<1$, the integral tends to a constant as $\xi\to 0$, so $\varphi_1(\xi)^2/\xi\sim\xi^{b-1}$, which is integrable for $b>0$, i.e., $\alpha>n$. Thus we get a solution for φ_1 in L^2 if and only if $n<\alpha<n+2$. However, we must also have (by 3.17) $K_0\varphi_1\in L^2$, which means

$$e^{-\xi/2}\xi^{(b+1)/2}g'(\xi)\in L^2(d\xi/\xi),$$

or

$$e^{-\xi/2}\xi^{(b+1)/2}\int_0^\infty e^{-\xi s}s^a(1+s)^{b-a-1}\,ds\in L^2(d\xi/\xi).$$

Since $b>0$, the integral behaves like ξ^{-b} as $\xi\to 0$, so $\varphi_0(\xi)\sim\xi^{(b+1)/2}\xi^{-b}=\xi^{(1-b)/2}$ and $\varphi_0(\xi)^2/\xi\sim\xi^{-b}$, which is integrable in b only if $b<1$. Thus we have *both* φ_1 and φ_0 in L^2 exactly when $n<\alpha<n+1$. In this case, we can solve (3.17) for infinitely many distinct values of r, which proves $\mathscr{H}^1(\check{\pi}_+,\alpha Z^*,\mathfrak{h})$ is infinite-dimensional.

Equations (3.18) corresponding to $\varepsilon=-1$ are treated similarly. In this case, the substitution (3.19) together with

$$\varphi_1(t_0)=e^{\xi/2}\xi^{(\alpha-n)/2}g(\xi)$$

leads again to an equation resembling (3.20):

$$\xi g''(\xi)+(\xi-b)\,g'(\xi)-ag(\xi)=0,$$

with $b=n-\alpha$ and with $a=(r+1)\,n/2$. Once again, one of the two linearly independent solutions will give a φ_1 with exponential growth as $\xi\to\infty$, so that up to a constant, we must have

$$\varphi_1(\xi)=e^{-\xi/2}\xi^{-b/2}\int_0^\infty e^{-\xi s}s^{a-b-1}(1+s)^{-a-1}\,ds \tag{3.22}$$

(this converges if we take r, hence a, sufficiently large), or a similar expression for a small. When (3.22) is valid, the conditions to be satisfied are that $\varphi_1(\xi)\in L^2(d\xi/\xi)$, and also that

$$e^{-\xi/2}\xi^{(1-b)/2}\int_0^\infty e^{-\xi s}s^{a-b-1}(1+s)^{-a}\,ds\in L^2(d\xi/\xi).$$

Both conditions again depend only on b. If $b>-1$, the integral in (3.22) approaches a positive constant as $\xi\to 0$, so $\varphi_1(\xi)^2/\xi\sim\xi^{-b-1}$, which is integrable in ξ when $b<0$ but not when $b\geqslant 0$. If $-1<b<0$, then for ξ small, we have $\varphi_0(\xi)\sim\xi^{(1-b)/2}\xi^b=\xi^{(1+b)/2}$, so $\varphi_0(\xi)^2/\xi\sim\xi^b$ which is integrable. Thus we have infinitely many independent solutions of (3.18) when $-1<b<0$ or $n<\alpha<n+1$. When $b\leqslant -1$, we again have $\varphi_0(\xi)\sim\xi^{(1+b)/2}$ and $\varphi_0(\xi)^2/\xi\sim\xi^b$, which is not integrable, and (3.18) has no solutions in L^2. Putting everything together, we see thus that

$$\begin{aligned} \pi^1(\alpha Z^*,\mathfrak{h}) &\cong \infty\cdot\pi_+\oplus\infty\cdot\pi_- && \text{if } n<\alpha<n+1,\\ &\cong 0 && \text{otherwise.}\end{aligned}$$

It remains to deal with the calculation of $\pi^k(\alpha Z^*,\mathfrak{h})$ when $2\leqslant k\leqslant n$. This gets a bit messy but is really quite similar: to illustrate we take the case of $k=2$. If $\omega=\sum_{i<j}\omega_{ij}\otimes(J_i^*\wedge J_j^*)$ and $\delta(\omega)=0$, $\delta^*(\omega)=0$, we obtain the following equations which replace (3.14):

$$\begin{aligned} J_k\cdot\omega_{ij}-J_j\cdot\omega_{ik}+J_i\cdot\omega_{jk}&=0 && \text{if } 0<i<j<k,\\ J_0\cdot\omega_{ij}-2\omega_{ij}+J_j\cdot\omega_{0i}-J_i\cdot\omega_{0j}&=0 && \text{if } 0<i<j,\\ J_1^*\cdot\omega_{01}+J_2^*\cdot\omega_{02}+\cdots+J_n^*\cdot\omega_{0n}&=0, && \\ (J_0^*-1)\cdot\omega_{0i}-\sum_{i<j}J_j^*\cdot\omega_{ij}+\sum_{0<j<i}J_j^*\cdot\omega_{ij}&=0 && \text{if } i>0.\end{aligned}\tag{3.23}$$

Once again, the asymmetry in the equations coming from the non-commutativity of $\mathfrak{h}$ can be remedied by a change of variables, but in place of (3.13), we now let

$$\begin{aligned} K_j&=e^{-t_0}J_j && \text{if } j>0,\\ K_0&=e^{-t_0}(J_0-2). &&\end{aligned}\tag{3.24}$$

(In computing $\mathscr{H}^k$, we would let $K_0=e^{-t_0}(J_0-k)$.) This changes the system (3.23) to the system one would have if the algebra $\mathfrak{h}$ were commutative. As before the variables can be separated, and we have to look for eigenvectors of $K_0K_0^*$ and $K_0^*K_0$ which are in L^2 (as functions of t_0) and have *negative* eigenvalues (for these formally positive operators). One obtains the same equations as before, except for the shift due to the difference between (3.24) and (3.13). In other words, we find

$$\begin{aligned} \pi^k(\alpha Z^*,\mathfrak{h}) &\cong \infty\cdot\pi_+\oplus\infty\cdot\pi_- && \text{if } n+1-k<\alpha<n+2-k,\\ &\cong 0 && \text{otherwise,}\end{aligned}$$

which is the assertion of the theorem. ∎

4. Connections with the Theory of the Discrete Series for Semisimple Lie Groups

Although Theorem 3.8 may seem quite mysterious from the point of view of the representation theory of solvable Lie groups, it has a nice interpretation if we bring semisimple groups into the picture. For purposes of this section, it will be easiest if we alter the notation of Section 3 and instead assume

HYPOTHESIS 4.1. *G is a connected, simply connected, simple Lie group, K a maximal* Ad-*compact subgroup, such that G/K is a hermitian symmetric space of non-compact type. Note that the center Z of G will be infinite, so that K is non-compact; however, K/Z is maximal compact in G/Z. Let $G = KAN$ be an Iwasawa decomposition of G and let $S = AN$.*

By a theorem of C. Moore, S will have open coadjoint orbits on $\mathfrak{s}^*$, the dual of its Lie algebra. Also, $\operatorname{rk} G = \operatorname{rk} K$, so G will have a relative discrete series $\hat{G}_d$ of irreducible unitary representations square-integrable modulo Z. However, since Z is infinite, this discrete series will in fact be "continuous" (think of the universal covering group of $SL(2, \mathbb{R})$).

Because of the Iwasawa decomposition $G = SK$, S acts simply transitively on the hermitian symmetric space G/K. In this way, the open coadjoint orbits of S (which as manifolds may be identified with S or G/K) acquire a canonical invariant complex structure, which may be interpreted as being given by a totally complex polarization. (For details of all of this, see [24, Sects. 1.6 and 5, and Appendix].) Let f_0 be the specific element of $\mathfrak{s}^*$ constructed in [24], and let $\mathfrak{h}$ be the associated positive totally complex polarization for it. We wish to study the representations $\pi^k(\alpha f_0, \mathfrak{h})$ of S for $\alpha \in \mathbb{R}\backslash\{0\}$. In fact, Theorem 3.8 deals with precisely this situation, when G/K is the unit ball in $\mathbb{C}^{n+1}$ and G is the universal covering group of $SU(n+1, 1)$.

We note now that via the identification of $S \cdot f_0$ with G/K (as homogeneous complex manifolds), the line bundle $\mathscr{L}_\chi$ associated to $\chi = e^{if}$, $f = \alpha f_0$, may be identified with the homogenous line bundle on G/K induced by a character ψ_α of K. (It was to make this true without an annoying integrality restriction on α that we took G and K to be simply connected.) It is now clear that $\pi^k(\alpha f_0, \mathfrak{h})$ is just the restriction to S of the natural unitary representation of G on L^2 harmonic $(0, k)$-forms with values in this bundle. In other words, the problem of computing $\pi^k(\alpha f_0, \mathfrak{h})$ is equivalent to the problem of computing the kth harmonically induced representation of G from ψ_α, but restricted to S. Thus if $\mathfrak{p}$ is the parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ defining the anti-holomorphic tangent vectors on G/K,

$$\pi_S^k(\alpha f_0, \mathfrak{h}) \cong \pi_G^k(\psi_\alpha, \mathfrak{p})|_S. \tag{4.2}$$

This relationship can be exploited in two ways. When $\pi_S^k(\alpha f_0, \mathfrak{h})$ is computable, (4.2) gives necessary and sufficient conditions for non-vanishing of $\pi_G^k(\psi_\alpha, \mathfrak{p})$. When $k=0$, this was used in [24] to recapture the results of Harish-Chandra [11] on the holomorphic discrete series. Our Theorem 3.8 does essentially the same for certain representations of $\widetilde{SU(n+1, 1)}$.

However, since harmonic induction for semisimple groups is already well understood by Schmid's theorem (Theorem 3.2 above), whereas the calculations of (3.8) become unmanageable even for $G = \widetilde{Sp(2, \mathbb{R})}$ (corresponding to $G/K =$ the Siegel upper half-space of complex dimension 3), it is more effective to use the correspondence in the other direction. Let g_0 be the elliptic element of $\mathfrak{g}^*$ corresponding to f_0, so that $i\alpha g_0$ is the differential of ψ_α. Since all of K stabilizes g_0, (3.2) does not apply directly, but we may use Remark 3.3(b). This yields

THEOREM 4.3. *Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$, the Lie algebra of K, and choose positive systems of roots $P_c \subseteq P_0$ for the roots Δ_c and Δ of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}$, respectively, so that $\mathfrak{p}$ contains none of the root spaces for $P_n = P_0 \backslash P_c$. Let ρ_0 be half the sum of the roots in P_0. Then if $i\alpha g_0 + \rho_0$ is singular, $\pi_G^k(\psi_\alpha, \mathfrak{p}) = 0$ for all k. Otherwise, $\pi_G^k(\psi_\alpha, \mathfrak{p})$ will be non-zero for exactly one value of k, namely, $\#\{\beta \in P_n \mid \langle i\alpha g_0 + \rho_0, \beta\rangle > 0\}$, and will be the representation in $\hat{G}_d$ with Harish-Chandra parameter $i\alpha g_0 + \rho_0$.*

Proof. We merely indicate how this follows from (3.2) and (3.3)(b). By Borel–Weil–Bott, if $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{g}_\mathbb{C}$ containing $\mathfrak{t}_\mathbb{C}$ and the root spaces for $-P_0$, $\pi_K^k(\alpha g_0, \mathfrak{b} \cap \mathfrak{k}_\mathbb{C})$ will be 0 in degrees $k > 0$ and will be just ψ_α in degree 0. So by "harmonic induction in stages," $\pi_G^k(\alpha g_0, \mathfrak{b}) \cong \pi_G^k(\psi_\alpha, \mathfrak{p})$. By Schmid's theorem, $\pi_G^k(\alpha g_0, \mathfrak{b})$ will be zero for all k if $i\alpha g_0 + \rho_0$ is singular, and will be the appropriate discrete series representation in degree $k = q(\alpha g_0 - i\rho_0, \mathfrak{p})$ otherwise. Since $\mathfrak{p}$ is positive for g_0, this is easily seen to be the indicated number of roots. ∎

COROLLARY 4.4. *With notation as in* 4.1 *and* 4.3, *$\pi_S^k(\alpha f_0, \mathfrak{h})$ vanishes if $i\alpha g_0 + \rho_0$ is singular. Otherwise, $\pi_S^k(\alpha f_0, \mathfrak{h})$ is non-zero for exactly one value of k, namely, $\#\{\beta \in P_n \mid \langle i\alpha g_0 + \rho_0, \beta\rangle > 0\}$.*

EXAMPLE 4.5. Let $G = \widetilde{SU(n+1, 1)}$, $n \geqslant 1$, and let S be the group of (3.8). In the notation used there, $f_0 = Z^*$. We have $K = \widetilde{U(n+1)} \cong SU(n+1) \times \mathbb{R}$. In this case P_0 has $n+1$ simple roots $\beta_1, \ldots, \beta_{n+1}$, with all but the last one in P_c, and

$$P_n = \{\beta_{n+1}, \beta_{n+1} + \beta_n, \ldots, \beta_{n+1} + \beta_n + \cdots + \beta_1\}.$$

We may normalize the inner product so that $\langle\beta_j, \beta_j\rangle = 2$, $\langle\beta_j, \beta_{j+1}\rangle = -1$, $\langle\rho_0, \beta_j\rangle = 1$. Then ig_0 restricted to $\mathfrak{t}$ is perpendicular to $\beta_1, \ldots, \beta_n$, and $\langle ig_0, \beta_{n+1}\rangle = -1$, so that

$$\left\langle i\alpha g_0 + \rho_0, \sum_{j=0}^{k} \beta_{n+1-j}\right\rangle = \langle i\alpha g_0, \beta_{n+1}\rangle + k + 1 = k + 1 - \alpha.$$

Thus $\pi_S^k(\alpha f_0, \mathfrak{h})$ vanishes for all k if $\alpha \in \{1, 2, \ldots, n+1\}$, and otherwise is non-zero in exactly the degree given by (3.8)(b).

We have now explained parts (a) and (b) of Theorem 3.8, and shown how they generalize to other hermitian symmetric spaces, but it remains to explain the mysterious aspect of (3.8)(c), namely, the appearance of harmonically induced representations which are not only non-irreducible, but equivalent to the regular representation of S! The underlying principle here was explained to us by Nolan Wallach, and will be discussed and generalized in another paper. Since the methods required for the proof of the following theorem involve a deep understanding of semisimple groups, and have almost nothing in common with the techniques of this paper, we shall illustrate the statement but say very little about how it is proved.

THEOREM 4.6. *Let G and S satisfy Hypothesis* 4.1, *and let $\pi \in \hat{G}_d$. Then if π is holomorphic or anti-holomorphic, $\pi|_S$ is of finite multiplicity. However, if π is neither holomorphic nor anti-holomorphic, $\pi|_S$ has infinite multiplicity (i.e., contains some $\sigma \in \hat{S}_d$ with infinite multiplicity).*

Remarks 4.7. Let us at least define the terms in the statement and see how it applies to the situation at hand. Any $\pi_\lambda \in \hat{G}_d$ has a Harish-Chandra parameter λ in the weight "lattice" of $\mathfrak{t}$ as above, and λ is regular. (The word "lattice" is slightly misleading since K contains a copy of $\mathbb{R}$, hence we really have a product of a lattice with a line.) Having ordered the roots as in (4.3), we may assume λ is dominant for P_c. Let $P_\lambda = \{\beta \in \Delta \mid \langle\lambda, \beta\rangle > 0\}$. Then P_λ is a new positive system for Δ and, by assumption, $P_\lambda \supseteq P_c$. If $P_\lambda = P_c \cup (-P_n)$, we say π is *holomorphic*. This is equivalent to saying that $\pi_\lambda \cong \pi_G^0((\lambda - \rho_0)/i), \mathfrak{b})$. (Note that $P_c \cup (-P_n)$ is indeed a positive system since if $\gamma, \delta \in P_n$, $\gamma + \delta$ is not a root, and if $\gamma \in P_n$ and $\delta \in \Delta_c$, $\gamma + \delta$ either lies in P_n or is not a root.) Similarly, π_λ is said to be *anti-holomorphic* if $P_\lambda = P_0$. This implies $\pi_\lambda \cong \pi_G^{|P_n|}((\lambda - \rho_0)/i, \mathfrak{b})$; equivalently, π is holomorphic for the opposite complex structure on G/K. Among discrete series representations, the holomorphic and anti-holomorphic representations are unusual in (a) being particularly easy to realize geometrically, since they come from holomorphic induction, and (b) being particularly "small" in having a rather "narrow" set of K-types. One may characterize them algebraically by the property that

$$\gamma, \delta \in P_\lambda \cap \Delta_n \Rightarrow \gamma + \delta \qquad \text{is not a root.}$$

That $\pi|_S$ is irreducible when π is holomorphic or anti-holomorphic follows immediately from (4.2) and the Blattner–Kobayashi Theorem (applied to S). However, showing that $\pi|_S$ has infinite multiplicity in the other cases requires making precise the difference in "size" between holomorphic and non-holomorphic representations. A nice way of viewing (4.6) is in terms of the orbit method, since the restriction of π to S has something to do with the geometry of the projection of the coadjoint orbit of λ/i from $\mathfrak{g}^*$ to $\mathfrak{s}^*$.

We conclude by applying (4.6) to Example 4.5, thereby recovering the last part of Theorem 3.8. If $\alpha > n+1$, then $\pi_S^0(\alpha f_0, \mathfrak{h}) \cong \pi_G^0(\psi_\alpha, \mathfrak{p})|_S$ is the restriction of a holomorphic representation, and so is irreducible. If $\alpha < 1$, then $\pi_S^{n+1}(\alpha f_0, \mathfrak{h}) \cong \pi_G^{|P_n|}(\psi_\alpha, \mathfrak{p})|_S$ is the restriction of an anti-holomorphic representation, and again is irreducible. But if $1 < \alpha < n+1$ and α is not an integer, then with $\lambda = i\alpha g_0 + \rho_0$,

$$P_\lambda = P_c \cup \left\{ \sum_{j=0}^{k} \beta_{n+1-j} : k = [\alpha], [\alpha]+1, ..., n \right\}$$
$$\cup \left\{ -\sum_{j=0}^{k} \beta_{n+1-j} : k = 0, 1, ..., [\alpha]-1 \right\}.$$

The positive system contains two non-compact roots whose sum is a root:

$$-\sum_{j=0}^{[\alpha]-1} \beta_{n+1-j} + \sum_{j=0}^{[\alpha]} \beta_{n+1-j} = \beta_{n+1-[\alpha]},$$

so $\pi_S^{n+1-[\alpha]}(\alpha f_0, \mathfrak{h})$ is the restriction of a discrete series representation of G which is neither holomorphic nor anti-holomorphic, hence has infinite multiplicity by Theorem 4.6.

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