Pergamon

Topology Vol. 35, No. 1, pp. 243–266, 1996

0040-9383/96 $15.00 + 0.00

0040-9393(95)00007-0

A NOTE ON THE JEFFREY–KIRWAN–WITTEN LOCALISATION FORMULA

MICHÈLE VERGNE

(*Received* 27 *June* 1994)

0. INTRODUCTION

LET M be a compact symplectic manifold provided with a Hamiltonian action of a compact Lie group G with Lie algebra $\mathfrak{g}$. We note by (M, σ, μ) such a data where σ is the symplectic form of M and $\mu: M \to \mathfrak{g}^*$ is the moment map. Let us assume that the action of G on $\mu^{-1}(0)$ is free. We can then consider the symplectic manifold $M_{\mathrm{red}} = G\backslash\mu^{-1}(0)$. It is a symplectic manifold, called the Marsden–Weinstein reduction of M, with symplectic form σ_{red}. It is important to be able to compute the integral $\int_{M_{\mathrm{red}}} v_{\mathrm{red}}$ of a de Rham cohomology class v_{red} on M_{red}. By a theorem of Kirwan [8], any cohomology class v_{red} of M_{red} is obtained from an equivariant cohomology class v on M by restriction and reduction. In [12], Witten proposed a formula relating the integral over M_{red} of v_{red} and an integral over $M \times \mathfrak{g}$ of an equivariant cohomology class given in terms of v and the equivariant symplectic form. Witten's formula has been proven by Kalkman [7], Wu [13] in the case of circle actions and by Jeffrey and Kirwan [6] in the general case. As the localisation formula [1] is an efficient tool to compute integrals over M of equivariant cohomology classes, the formula of Witten can be used to compute $H^*(M_{\mathrm{red}})$ in some cases [7, 6].

Let us explain Witten's statement. Let α be a G-equivariant differential form on M, that is, α is an equivariant map from $\mathfrak{g}$ to the space $\mathscr{A}(M)$ of differential forms on M. Assume that for $X \in \mathfrak{g}$, $\alpha(X) = e^{i\sigma_{\mathfrak{g}}(X)}\beta(X)$ where β is a closed G-equivariant form on M *depending polynomially* on the variable $X \in \mathfrak{g}$ and $\sigma_{\mathfrak{g}}(X) = \mu(X) + \sigma$ is the value at $X \in \mathfrak{g}$ of the equivariant symplectic form of M. Let $\alpha_{\mathrm{red}} = e^{i\sigma_{\mathrm{red}}}\beta_{\mathrm{red}}$ be the de Rham cohomology class of M_{red} determined by α. We denote by $\int_M \alpha$ the C^∞-function on $\mathfrak{g}$ such that its value at $X \in \mathfrak{g}$ is the integral of $\alpha(X)$ over M:

$$\left(\int_M \alpha\right)(X) = \int_M \alpha(X).$$

Consider the Fourier transform $\mathscr{F}(\int_M \alpha)$ of $\int_M \alpha$. This is a tempered distribution on $\mathfrak{g}^*$. Let $d\xi$ be a Euclidean measure on $\mathfrak{g}^*$. Then Witten asserted the following: near 0, the generalised density $\mathscr{F}(\int_M \alpha)$ is a polynomial density $P(\xi)\, d\xi$ and

$$P(0) = (2i\pi)^{\dim G} \operatorname{vol}(G) \int_{M_{\mathrm{red}}} \alpha_{\mathrm{red}}. \tag{1}$$

In this formula dX is the Euclidean measure on $\mathfrak{g}$ dual to $d\xi$, $\operatorname{vol}(G)$ is the volume of G for the Haar measure on G compatible with dX. Moreover, $\mathscr{F}(\int_M \alpha)(\xi)$ near 0 depends only on the equivariant cohomology class of the restriction of α on $\mu^{-1}(0)$ and is described explicitly. In other words, the Fourier transform is local at 0 (or near any regular value of the moment map).

In this note, we start by giving a short proof of the formula for $P(\xi)$ following closely the Jeffrey–Kirwan proof [6] of Witten's formula. Our main observation is the following. Consider the equivariant cohomology complex with C^∞ coefficients $(\mathscr{A}_G^\infty(\mathfrak{g}, M), d_\mathfrak{g})$. Denote by $\mathscr{A}_G^{\mathrm{pol}}(\mathfrak{g}, M)$ the subspace of G-equivariant differential forms depending polynomially on $X \in \mathfrak{g}$. Consider a G-equivariant differential form $\alpha \in \mathscr{A}_G^\infty(\mathfrak{g}, M)$ such that for $X \in \mathfrak{g}$, $\alpha(X) = \mathrm{e}^{\mathrm{i}(\mu, X)}\gamma(X)$ where γ is a G-equivariant form on M *depending polynomially* on the variable $X \in \mathfrak{g}$. The subspace

$$\mathscr{A}_G^\mu(\mathfrak{g}, M) = \{\alpha(X) = \mathrm{e}^{\mathrm{i}(\mu, X)}\gamma(X);\ \gamma \in \mathscr{A}_G^{\mathrm{pol}}(\mathfrak{g}, M)\}$$

of such forms is a subcomplex of $(\mathscr{A}_G^\infty(\mathfrak{g}, M), d_\mathfrak{g})$. Let $\mathscr{H}_G^\mu(\mathfrak{g}, M)$ be the corresponding cohomology space. Let $\alpha \in \mathscr{A}_G^\mu(\mathfrak{g}, M)$ and let $\mathscr{F}(\int_M \alpha)$ be the Fourier transform of $\int_M \alpha$. Then the map $A = \mathscr{F}\int_M : \mathscr{A}_G^\mu(\mathfrak{g}, M) \to \mathscr{M}^{-\infty}(\mathfrak{g}^*)^G$ defines a map from the equivariant cohomology space $\mathscr{H}_G^\mu(\mathfrak{g}, M)$ to the space of G-invariant distributions on $\mathfrak{g}^*$. We remark that the map A is local *in cohomology*: if U is a G-invariant open subset contained in the set of regular values of μ, then A defines a map from $\mathscr{H}_G^\mu(\mathfrak{g}, \mu^{-1}(U))$ to the space of G-invariant C^∞-densities on U. It is then easy to describe the map A using local coordinates on $\mu^{-1}(U)$.

The Jeffrey–Kirwan formula implies Witten's asymptotic estimates, when ε tends to 0 of

$$Z(\varepsilon) = \int_M \int_\mathfrak{g} \alpha(X)\phi_\varepsilon(X)\,dX$$

for $\phi_\varepsilon(X) = \mathrm{e}^{-\varepsilon\|X\|^2/2}$ a Gaussian function on $\mathfrak{g}$ and α a closed element of $\mathscr{A}_G^\mu(\mathfrak{g}, M)$.

For applications to multiplicities formula, we need more generally to give a formula for $\int_M \int_\mathfrak{g} \alpha(X)\phi(X)\,dX$ for any C^∞-function ϕ (with adequate decay properties) on $\mathfrak{g}$ and any G-equivariant closed form α on M with C^∞-coefficients. Thus in the second part of this article (which is independent of the first part) we study more systematically the C^∞-function $(\int_M \alpha)$ considered as a generalised function on $\mathfrak{g}$.

Let M_0 be an open tubular neighbourhood of $\mu^{-1}(0)$ in M. Then G acts freely on M_0. We show that the partition $M = M_0 \cup (M - M_0)$ leads to a decomposition of the C^∞-function $\int_M \alpha$ as a sum of two *generalised functions* Θ_0 and Θ_{out} on $\mathfrak{g}$. These two generalised functions are obtained by a limit formula as in Witten: let us consider the G-invariant function $\frac{1}{2}\|\mu\|^2$ and its Hamiltonian vector field H. Let us choose a G-invariant metric $(\cdot,\cdot)$ on M and consider the G-invariant 1-form λ^M on M given by

$$\lambda^M(\cdot) = (H, \cdot).$$

For any $t \in \mathbb{R}$ and $X \in \mathfrak{g}$, let

$$\Theta(M, t)(X) = \int_M \mathrm{e}^{-\mathrm{i}t d_X \lambda^M}\alpha(X)$$

where $d_X = d - \iota(X_M)$ is the equivariant differential. As α is a closed form, $\Theta(M, t)(X)$ is independent of t. Let us break the integral formula for $\Theta(M, t)$ in two parts

$$\Theta(M_0, t)(X) = \int_{M_0} \mathrm{e}^{-\mathrm{i}t d_X \lambda^M}\alpha(X) \tag{2}$$

and

$$\Theta(M - M_0, t)(X) = \int_{M - M_0} \mathrm{e}^{-\mathrm{i}t d_X \lambda^M}\alpha(X). \tag{3}$$

We prove the following theorem (Theorem 19).

THEOREM 1. *Let α be a closed G-equivariant form on M. The limits Θ_0 and Θ_{out} when $t \to \infty$ of $\Theta(M_0, t)$ and $\Theta(M - M_0, t)$ exist in the space of generalised functions on $\mathfrak{g}$. We have*

$$\int_M \alpha = \Theta_0 + \Theta_{out}.$$

The generalised function Θ_0 is of support 0 and we describe it explicitly. Let $W: C^\infty(\mathfrak{g})^G \to H^*(M_{red})$ be the Chern–Weil homomorphism associated to the principal fibration $\mu^{-1}(0) \to M_{red}$. If α_{red} is the form on M_{red} obtained from α and if ϕ is a G-invariant test function on $\mathfrak{g}$, then

$$\int_{\mathfrak{g}} \Theta_0(X)\phi(X)\,dX = (2i\pi)^{\dim G}\operatorname{vol}(G)\int_{M_{red}} \alpha_{red} W(\phi).$$

Let us stress that this description of Θ_0 follows easily from the determination in [9] of the equivariant cohomology with generalised coefficients of a space with free G-action. However, we will give here a self-contained proof. This formula for Θ_0 implies, for example, the Jeffrey–Kirwan formula for $\mathscr{F}(\int_M \alpha)$ when $\alpha \in \mathscr{H}^\mu_G(\mathfrak{g}, M)$, giving a second proof of the Jeffrey–Kirwan–Witten formula.

We give also an integral formula for the generalised function Θ_{out} as an integral over $M - M_0$ with a boundary term added. In short $\Theta_{out}(X)$ is the integral of an equivariant cohomology class over the noncompact manifold $M - M_0$ with a cylindrical end attached to it. It would be interesting to give a more explicit description of Θ_{out}. Such a description is suggested by Witten as an integral over the critical set of the function $\|\mu\|^2$. An explicit description of this kind is given in case of the integrals $Z(\varepsilon)$ considered by Witten when furthermore G is a circle S^1 acting on M with isolated fixed points in [12].

For some of our purposes, this rough determination of Θ_{out} will be sufficient: we present in [11] an application of the decomposition of the function $\int_M \alpha$ as a sum of two generalised functions to a proof of the Guillemin–Sternberg conjecture [4] on multiplicities when G is a torus.

1. JEFFREY-KIRWAN LOCALISATION FORMULA

1.1. Local Fourier transforms

Let G be a Lie group acting on a manifold M. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the dual vector space.

In this article the letter X denotes either a point $X \in \mathfrak{g}$ or the map $X \mapsto X$ from a subset of $\mathfrak{g}$ to $\mathfrak{g}$. The similar ambiguity is allowed for the letter ξ which denotes either a point of $\mathfrak{g}^*$ or, more often, the map $\xi \mapsto \xi$ from a subset of $\mathfrak{g}^*$ to $\mathfrak{g}^*$. In particular, (ξ, X) is either a scalar (the value at $X \in \mathfrak{g}$ of the linear form $\xi \in \mathfrak{g}^*$), or a function on $\mathfrak{g}^*$ depending linearly on $X \in \mathfrak{g}$, or, more often, a map from $\mathfrak{g}$ to the space of functions on $\mathfrak{g}^*$.

Let $n = \dim \mathfrak{g}$. Let $E^1, E^2, \ldots, E^n$ be a basis of $\mathfrak{g}$. We write $X \in \mathfrak{g}$ as $X = \sum_i x_i E^i$. Let $E_1, E_2, \ldots, E_n$ be the dual basis of $\mathfrak{g}^*$. We write $\xi \in \mathfrak{g}^*$ as $\xi = \sum_i \xi^i E_i$. We denote by dX the density $dx_1\,dx_2 \cdots dx_n$ and by $d\xi = d\xi^1\,d\xi^2 \cdots d\xi^n$. We say that dX and $d\xi$ are dual densities.

If ϕ is a (tempered generalised) function on $\mathfrak{g}$, its Fourier transform $\mathscr{F}(\phi)$ is the (generalised) density on $\mathfrak{g}^*$ such that

$$\int_{\mathfrak{g}^*} e^{i(\xi, X)}\mathscr{F}(\phi)(\xi) = \phi(X).$$

Let $S(\mathfrak{g}^*)$ be the symmetric algebra of $\mathfrak{g}^*$. We identify an element $P \in S(\mathfrak{g}^*)$ either to a polynomial function $X \mapsto P(X)$ on $\mathfrak{g}$ or to a differential operator with constant coefficient $P(\partial_\xi)$ on $\mathfrak{g}^*$. The identification is such that $P(\partial_\xi)(e^{(\xi,X)}) = P(X)e^{(\xi,X)}$. Similarly $S(\mathfrak{g})$ is identified to the space of polynomial functions on $\mathfrak{g}^*$.

If $X \in \mathfrak{g}$, we denote by X_M the vector field on M produced by the infinitesimal action of $\mathfrak{g}$:

$$(X_M)_x = \frac{d}{d\varepsilon}(\exp -\varepsilon X)\cdot x|_{\varepsilon=0}.$$

A *G-equivariant differential form* on M is a smooth G-equivariant map, defined on the Lie algebra $\mathfrak{g}$, with values in the space $\mathscr{A}(M)$ of smooth differential forms on M. We denote the algebra of G-equivariant differential forms on M by $\mathscr{A}_G^\infty(\mathfrak{g}, M) = C^\infty(\mathfrak{g}, \mathscr{A}(M))^G$. Thus, if $\alpha \in \mathscr{A}_G^\infty(\mathfrak{g}, M)$, the value $\alpha(X)$ of $X \in \mathfrak{g}$ is a differential form on M. Allowing the preceding ambiguity for the notation X, we will sometimes denote the map $\alpha: \mathfrak{g} \to \mathscr{A}(M)$ by $\alpha(X)$. In particular a C^∞-function $\mu(X)$ on M depending smoothly on $X \in \mathfrak{g}$ and in such a way that $\mu(g\cdot X)(g\cdot m) = \mu(X)(m)$ for all $X \in \mathfrak{g}$, $m \in M$, $g \in G$ is an element of $\mathscr{A}_G^\infty(\mathfrak{g}, M)$.

For $\alpha \in \mathscr{A}(M)$ we write $\alpha = \sum \alpha_{[i]}$ for the decomposition of α in homogeneous forms of exterior degree i.

The *equivariant coboundary* $d_\mathfrak{g}: \mathscr{A}_G^\infty(\mathfrak{g}, M) \to \mathscr{A}_G^\infty(\mathfrak{g}, M)$ is defined for $\alpha \in \mathscr{A}_G^\infty(\mathfrak{g}, M)$ and $X \in \mathfrak{g}$ by

$$(d_\mathfrak{g}\alpha)(X) = d(\alpha(X)) - \iota(X_M)(\alpha(X))$$

where $\iota(X_M)$ is the contraction with the vector field X_M. We also write d_X for the operator $d - \iota(X_M)$ acting on forms. A closed equivariant form is by definition a G-equivariant differential form satisfying $d_\mathfrak{g}\alpha = 0$. We denote by $\mathscr{H}_G^\infty(\mathfrak{g}, M)$ the space $\operatorname{Ker} d_\mathfrak{g}/\operatorname{Im} d_\mathfrak{g}$.

We denote by $\mathscr{A}_G^{\mathrm{pol}}(\mathfrak{g}, M) = (S(\mathfrak{g}^*) \otimes \mathscr{A}(M))^G$ the complex of G-equivariant forms $\alpha(X)$ depending polynomially on $X \in \mathfrak{g}$.

If M is a compact oriented manifold and $\alpha \in \mathscr{A}_G^\infty(\mathfrak{g}, M)$ an equivariant differential form, $X \mapsto \int_M \alpha(X)$ is an invariant C^∞-function on $\mathfrak{g}$ (the integral of an inhomogeneous form is by definition the integral of the term of maximum exterior degree). We denote by $\int_M: \mathscr{A}_G^\infty(\mathfrak{g}, M) \to C^\infty(\mathfrak{g})^G$ the map so obtained. We also denote by $\int_M: \mathscr{H}_G^\infty(\mathfrak{g}, M) \to C^\infty(\mathfrak{g})^G$ the map derived from $\int_M$ in cohomology.

Consider $\mathfrak{g}^*$ as a G-manifold via the adjoint action. Then the map $X \mapsto (\xi, X)$ is an element of $\mathscr{A}_G^\infty(\mathfrak{g}, \mathfrak{g}^*)$. Let $U \subset \mathfrak{g}^*$ be a G-invariant open subset of $\mathfrak{g}^*$. Let $\beta \in \mathscr{A}_G^\infty(\mathfrak{g}, U)$ and let $\alpha \in \mathscr{A}_G^\infty(\mathfrak{g}, U)$ be defined by $\alpha(X) = e^{i(\xi,X)}\beta(X)$ for $X \in \mathfrak{g}$. Then $(d_\mathfrak{g}\alpha)(X) = e^{i(\xi,X)}(i(d\xi, X) + (d_\mathfrak{g}\beta)(X))$ with $(d\xi, X) = \sum_i d\xi^i x_i$. Thus if $\beta \in \mathscr{A}_G^{\mathrm{pol}}(\mathfrak{g}, U)$, then $(d_\mathfrak{g}\alpha)(X) = e^{i(\xi,X)}\gamma(X)$ with γ depending also polynomially on $X \in \mathfrak{g}$.

Definition 2. The subcomplex $(\mathscr{A}_G^{\mathscr{F}}(\mathfrak{g}, U), d_\mathfrak{g})$ is defined to be

$$\mathscr{A}_G^{\mathscr{F}}(\mathfrak{g}, U) = \{\alpha(X) = e^{i(\xi,X)}\beta(X);\ \beta \in \mathscr{A}_G^{\mathrm{pol}}(\mathfrak{g}, U)\}.$$

Its cohomology is denoted by $\mathscr{H}_G^{\mathscr{F}}(\mathfrak{g}, U)$.

To motivate the next definition, assume first that $\mathscr{A}_G^{\mathscr{F}}(\mathfrak{g}, \mathfrak{g}^*)$ is compactly supported on $\mathfrak{g}^*$. We choose an orientation on $\mathfrak{g}^*$. Then the integral $\int_{\mathfrak{g}^*} \alpha(X)_{[n]}$ of $\alpha(X)$ over $\mathfrak{g}^*$ is well defined and is a rapidly decreasing C^∞-function on $\mathfrak{g}$. The Fourier transform $\mathscr{F}(\int_{\mathfrak{g}^*}\alpha)$ is a C^∞-density on $\mathfrak{g}^*$. It is readily computed: let us write $\alpha(X)_{[n]} = e^{i(\xi,X)}\sum_a P_a(X)\alpha_a(\xi)\,d\xi$ where $P_a \in S(\mathfrak{g}^*)$ and $\alpha_a(\xi) \in C^\infty(\mathfrak{g}^*)$. Then

$$\mathscr{F}\left(\int_{\mathfrak{g}^*}\alpha\right) = \left(\sum_a P_a(i\partial_\xi)\cdot\alpha_a(\xi)\right)d\xi.$$

Definition 3. Let $\alpha \in \mathscr{A}_G^{\mathscr{F}}(\mathfrak{g}, U)$ be a G-equivariant form on U. Let $d\xi = d\xi^1 \wedge d\xi^2 \wedge \cdots \wedge d\xi^n$. We define $V(\alpha) \in \mathscr{A}^n(U)^G$ by

$$V(\alpha) = \left(\sum_a P_a(\mathrm{i}\partial_\xi) \cdot \alpha_a(\xi) \right) d\xi$$

if $\alpha(X)_{[n]} = \mathrm{e}^{\mathrm{i}(\xi, X)} \sum_a P_a(X) \alpha_a(\xi)\, d\xi$ with $P_a \in S(\mathfrak{g}^*)$ and $\alpha_a(\xi) \in C^\infty(U)$.

In abstract sense, V is equal to the composition of the integration $\int_{\mathfrak{g}^*}$ over $\mathfrak{g}^*$ and of the Fourier transform $\mathscr{F}$. However neither $\int_{\mathfrak{g}^*}$ nor $\mathscr{F}$ are generally defined.

LEMMA 4. *Let $\beta \in \mathscr{A}_G^{\mathscr{F}}(\mathfrak{g}, U)$. Then $V(d_\mathfrak{g}\beta) = 0$.*

Proof. It is sufficient to prove this for β of exterior degree $n-1$. If $\beta(X) = \mathrm{e}^{\mathrm{i}(\xi, X)} \sum_{k=1}^n \beta_k(X, \xi)\, d\xi^1 \wedge d\xi^2 \wedge \widehat{d\xi^k} \wedge \cdots \wedge d\xi^n$, then

$$(d_\mathfrak{g}\beta(X))_{[n]} = \left(\sum_k (-1)^{k+1} (\mathrm{i}X_k) \beta_k(X, \xi) + \sum_k (-1)^{k+1} \partial_{\xi_k} \beta_k(X, \xi) \right) \mathrm{e}^{\mathrm{i}(\xi, X)}\, d\xi.$$

To compute V we must replace X_k by $\mathrm{i}\partial_{\xi_k}$ and we obtain $V(d_\mathfrak{g}\beta) = 0$. □

By the preceding lemma, we can define the map

$$V : \mathscr{H}_G^{\mathscr{F}}(\mathfrak{g}, U) \to \mathscr{A}^n(U)^G$$

in cohomology. We will call V the local Fourier transform.

Let M be a G-manifold. Let $\mu : M \to \mathfrak{g}^*$ be a G-invariant map. Then $m \mapsto (\mu(m), X)$ is a function on M depending on $X \in \mathfrak{g}$ that we denote by (μ, X). Then $X \mapsto \mathrm{e}^{\mathrm{i}(\mu, X)}$ is an element of $\mathscr{A}_G^\infty(\mathfrak{g}, M)$. If $\beta \in \mathscr{A}_G^{\mathrm{pol}}(\mathfrak{g}, M)$ then $\alpha(X) = \mathrm{e}^{\mathrm{i}(\mu, X)}\beta(X)$ is in $\mathscr{A}_G^\infty(\mathfrak{g}, M)$. The subspace of such forms α is stable under $d_\mathfrak{g}$.

Definition 5. The subcomplex $(\mathscr{A}_G^\mu(\mathfrak{g}, M), d_\mathfrak{g})$ is defined to be

$$\mathscr{A}_G^\mu(\mathfrak{g}, M) = \{\alpha(X) = \mathrm{e}^{\mathrm{i}(\mu, X)}\beta(X);\ \beta \in \mathscr{A}_G^{\mathrm{pol}}(\mathfrak{g}, M)\}.$$

Its cohomology is denoted by $\mathscr{H}_G^\mu(\mathfrak{g}, M)$.

The space $\mathscr{H}_G^\mu(\mathfrak{g}, M)$ is a module over $\mathscr{H}_G^{\mathrm{pol}}(\mathfrak{g}, M)$.

Let $\mu : M \to \mathfrak{g}^*$ be a proper map. Let U be a G-invariant open subset of $\mathfrak{g}^*$. Assume that U is contained in the subset of regular values of μ. Then μ is a fibration over U with compact fibres. Let $N = \mu^{-1}(U)$. Assume the fibration $\mu : N \to U$ has oriented fibres and that the action of G preserves the family of orientations o of the fibres. Let us denote by $\mu_* : \mathscr{A}(N) \to \mathscr{A}(U)$ the integral over the fibres (we leave implicit the choice of o). If $\alpha(X) = \mathrm{e}^{\mathrm{i}(\mu, X)}\beta(X)$ with $\beta \in \mathscr{A}_G^{\mathrm{pol}}(\mathfrak{g}, N)$, then

$$\mu_*(\alpha(X)) = \mathrm{e}^{\mathrm{i}(\xi, X)} \mu_*(\beta(X))$$

belongs to $\mathscr{A}_G^{\mathscr{F}}(\mathfrak{g}, U)$. The integral over the fibre gives a map of complexes

$$\mu_* : (\mathscr{A}_G^\mu(\mathfrak{g}, N), d_\mathfrak{g}) \to (\mathscr{A}_G^{\mathscr{F}}(\mathfrak{g}, U), d_\mathfrak{g})$$

and a map

$$V\mu_* : \mathscr{H}_G^\mu(\mathfrak{g}, N) \to \mathscr{A}^n(U)^G$$

that we will call also the local Fourier transform.

We assume now M compact and oriented. Let us relate $V\mu_*$ and $\mathscr{F}\int_M$. Let $\alpha(X) = e^{i(\mu, X)}\beta(X)$ with $\beta(X) = \sum_a P_a(X)\omega_a$. Then

$$\left(\int_M \alpha\right)(X) = \sum_a P_a(X) \int_M e^{i(\mu, X)}\omega_a.$$

The manifold M being compact, the push-forward $\mu_*((\omega_a)_{[\dim M]})$ by μ_* of the C^∞-density $(\omega_a)_{[\dim M]}$ is a compactly supported Radon measure on $\mathfrak{g}^*$ and we identify it with a distribution on $\mathfrak{g}^*$. Writing $\int_M = \int_{\mathfrak{g}^*}\mu_*$, we see that

$$\left(\int_M \alpha\right)(X) = \sum_a P_a(X) \int_{\mathfrak{g}^*} e^{i(\xi, X)}\mu_*((\omega_a)_{[\dim M]}).$$

Thus the Fourier transform of $\int_M \alpha$ is the distribution

$$\mathscr{F}\left(\int_M \alpha\right) = \sum_a P_a(i\partial_\xi)\cdot(\mu_*(\omega_a)_{[\dim M]}). \tag{4}$$

Near a regular value of μ, the distribution $\mu_*(\omega_a)_{[\dim M]}$ is a smooth density $\alpha_a(\xi)\,d\xi$ and $\mathscr{F}(\int_M \alpha)$ is equal to $(\sum_a P_a(i\partial_\xi)\cdot\alpha_a(\xi))\,d\xi$. Thus we obtain the following theorem.

THEOREM 6. *Let M be a compact oriented G-manifold and $\mu: M \to \mathfrak{g}^*$ be a G-invariant map. Let U be a G-invariant subset of $\mathfrak{g}^*$ contained in the set of regular values of μ. Let $\alpha \in \mathscr{A}_G^\mu(\mathfrak{g}, M)$. Then over U we have the equality:*

$$\mathscr{F}\left(\int_M \alpha\right) = V(\mu_*\alpha).$$

In particular, if α is closed, then $\mathscr{F}(\int_M \alpha)$ over U depends only on the cohomology class of α in $\mathscr{H}_G^\mu(\mathfrak{g}, \mu^{-1}(U))$.

Thus for $\alpha \in \mathscr{H}_G^\mu(\mathfrak{g}, M)$, in order to determine $\mathscr{F}(\int_M \alpha)$ near a regular value f of μ we need only to determine the class of α in $\mathscr{H}_G^\mu(\mathfrak{g}, \mu^{-1}(U))$ where U is a G-invariant tubular neighbourhood of the orbit $\mathcal{O}$ of f. In this sense the Fourier transform is local over $\mathscr{H}_G^\mu(\mathfrak{g}, M)$.

Remark 1.1. Let $\alpha \in \mathscr{H}_G^\mu(\mathfrak{g}, M)$. Assume that G is connected. Let T be a maximal torus of G. By the localisation formula [1], the integral $\int_M \alpha$ of α over M depends only on the restriction of α to the submanifold M^T of fixed points of T. In the equality

$$\mathscr{F}\left(\int_M \alpha\right) = V(\mu_*\alpha)$$

near an orbit $\mathcal{O}$, the first member depends only on $\alpha|_{M^T}$ while the second member depends only on $\alpha|_{\mu^{-1}(U)}$. This equality between these two localisations formulas has already been fruitfully employed in [6, 7, 13] to compute $H^*(M_{\text{red}})$ if (M, σ, μ) is a Hamiltonian manifold.

In the next section, we determine explicitly the map $V\mu_*$ near $0 \in \mathfrak{g}^*$ when the action of G on $\mu^{-1}(0)$ is infinitesimally free.

1.2. Local Fourier transforms and free actions

Let P be a compact manifold with a free left action of a compact Lie group G. Let $q: P \to G\backslash P$ be the quotient map. Recall (see for example [3]) that $H_G^\infty(\mathfrak{g}, P)$ is isomorphic to the de Rham cohomology $H^*(G\backslash P)$ by the pull-back q^*. Let ω be a connection form on $P \to G\backslash P$. Let $\Omega \in \mathscr{A}(P) \otimes \mathfrak{g}$ be the curvature of ω. If ϕ is a polynomial function on $\mathfrak{g}$, then $\phi(\Omega)$ is a differential form on P. If ϕ is an invariant polynomial function on $\mathfrak{g}$, then $\phi(\Omega)$ is a basic form which determines a closed de Rham cohomology class on $G\backslash P$. More generally, if $X \mapsto \alpha(X)$ is a G-equivariant differential form on P, then $\alpha(\Omega)$ is a form on P. If α is a closed G-equivariant differential form, the horizontal component $h(\alpha(\Omega))$ of $\alpha(\Omega)$ defines a closed de Rham form on $G\backslash P$. Then define

$$\alpha_{\mathrm{red}} = h(\alpha(\Omega)).$$

The cohomology class of the differential form α_{red} depends only on the cohomology class of α in $\mathscr{H}_G^\infty(\mathfrak{g}, P)$ and not on the choice of connection ω. Furthermore the map $\alpha \mapsto \alpha_{\mathrm{red}}$ is the inverse of q^* in cohomology.

Choose a G-invariant Euclidean norm $\|\cdot\|$ on $\mathfrak{g}$. Let U be a G-invariant open ball centred at 0 in $\mathfrak{g}^*$. Consider the manifold

$$N = P \times U.$$

We denote $G\backslash P$ by N_{red} (the motivation for this notation will become clear). We denote by $\mu: N \to U$ the second projection. If $\alpha \in \mathscr{A}_G^\mu(\mathfrak{g}, N)$, the restriction of α to P is a G-equivariant differential form on $P = \mu^{-1}(0)$, thus determines a form α_{red} on N_{red}.

We assume that P has a G-invariant orientation o^P that we will leave implicit most of the time. Choose a basis $E^1, E^2, \ldots, E^n$ of $\mathfrak{g}$. Let us write the connection form

$$\omega = \sum_k \omega_k E^k. \tag{5}$$

Let

$$\Omega = \sum_k \Omega_k E^k \tag{6}$$

be the curvature of ω. If $\xi = \sum_k \xi^k E_k \in \mathfrak{g}^*$, then $(\Omega, \xi) = \sum_k \Omega_k \xi^k$ is a form on P.

Let

$$v_\omega = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n. \tag{7}$$

Then v_ω is a vertical form on P of degree $n = \dim G$.

The basis E^i of $\mathfrak{g}$ determines a volume form $dX = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \in \Lambda^n \mathfrak{g}^*$. Our convention on dual orientations is as follows. We choose as dual positive element $d\xi \in \Lambda^n \mathfrak{g}$ the element $d\xi$ such that

$$E_1 \wedge E^1 \wedge E_2 \wedge E^2 \wedge \cdots \wedge E_n \wedge E^n = dX \wedge d\xi \tag{8}$$

that is $d\xi = (-1)^{n(n-1)/2} d\xi^1 \wedge d\xi^2 \wedge \cdots \wedge d\xi^n$. The next theorem determines the application $V\mu_*: \mathscr{H}_G^\mu(\mathfrak{g}, N) \to \mathscr{A}^n(U)^G$.

THEOREM 7. *Let P be a compact G-oriented manifold with a free action of G. Let U be an open ball centred at 0 in $\mathfrak{g}^*$. Let $N = P \times U$ and let μ be the second projection $P \times U \to U$. Let $N_{\mathrm{red}} = G\backslash P$. Let ω be a connection form on P with curvature Ω. Let $\alpha \in \mathscr{A}_G^\mu(\mathfrak{g}, N)$ be a closed equivariant differential form. Let α_{red} be the element of $H^*(N_{\mathrm{red}})$ determined by $\alpha|_P$. Then*

$$V(\mu_*\alpha) = \mathrm{i}^n \left(\int_P \alpha_{\mathrm{red}} e^{-\mathrm{i}(\Omega, \xi)} v_\omega \right) d\xi.$$

In this formula the elements v_ω and $d\xi$ are determined by an oriented basis of $\mathfrak{g}$ by formulas (7) *and* (8).

As Ω is a 2-form, Theorem 7 shows in particular that $V(\mu_*\alpha)$ is a polynomial density.

Proof of Theorem 7. If v is a form of $G\backslash P$ or P we still denote by v its pull-backs to P and $P\times\mathfrak{g}^*$. The connection form ω gives us the 1-form (ω,ξ) on $P\times\mathfrak{g}^*$

$$(\omega,\xi)=\sum_i \xi^i\omega_i. \tag{9}$$

We denote this 1-form by λ:

$$\lambda=(\omega,\xi). \tag{10}$$

Consider the differential form $e^{-id_{\mathfrak{g}}\lambda}$ on $P\times\mathfrak{g}^*$. By definition of ω, $\iota(X_P)\omega=X$. Thus for $(x,\xi)\in P\times\mathfrak{g}^*$, we have $(d_{\mathfrak{g}}\lambda)_{x,\xi}(X)=-(\xi,X)+((d\omega)_x,\xi)-(\omega_x,d\xi)$. It follows that

$$e^{-i(d_{\mathfrak{g}}\lambda)(X)}=e^{i(\xi,X)}e^{-i(d\omega,\xi)+i(\omega,d\xi)} \tag{11}$$

gives an element of $\mathscr{A}^{\mu}_G(\mathfrak{g},N)$. As the element $e^{-id_{\mathfrak{g}}\lambda}$ is invertible, we have

$$\mathscr{A}^{\mu}_G(\mathfrak{g},N)=e^{-id_{\mathfrak{g}}\lambda}\,\mathscr{A}^{\mathrm{pol}}_G(\mathfrak{g},N).$$

The form $e^{-id_{\mathfrak{g}}\lambda}$ is obviously closed.

Remark 1.2. We have

$$e^{-id_{\mathfrak{g}}\lambda}=1+d_{\mathfrak{g}}\left(\lambda\left(\frac{e^{-id_{\mathfrak{g}}\lambda}-1}{d_{\mathfrak{g}}\lambda}\right)\right)$$

so that $e^{-id_{\mathfrak{g}}\lambda}$ is congruent to 1 in $\mathscr{H}^{\infty}_G(\mathfrak{g},N)$ (but not in $\mathscr{H}^{\mu}_G(\mathfrak{g},N)$).

Let $\alpha\in\mathscr{A}^{\mu}_G(\mathfrak{g},N)$ be a closed equivariant differential form. We may write $\alpha=e^{-id_{\mathfrak{g}}\lambda}\beta$ with β a closed element of $\mathscr{A}^{\mathrm{pol}}_G(\mathfrak{g},N)$. By the Poincaré lemma, as U is contractible, the equivariant cohomology space $\mathscr{H}^{\mathrm{pol}}_G(\mathfrak{g},P\times U)$ is isomorphic to $\mathscr{H}^{\mathrm{pol}}_G(\mathfrak{g},P)$ by the restriction map, thus to $H^*(N_{\mathrm{red}})=H^*(G\backslash P)$ as G acts freely on P. As $\lambda=0$ on P, we see that, if $\alpha=e^{-id_{\mathfrak{g}}\lambda}\beta$, then $\alpha_{\mathrm{red}}=\beta_{\mathrm{red}}$ and $\alpha\in\mathscr{A}^{\mu}_G(\mathfrak{g},N)$ is $d_{\mathfrak{g}}$-equivalent in $\mathscr{A}^{\mu}_G(\mathfrak{g},N)$ to $\alpha_{\mathrm{red}}e^{-id_{\mathfrak{g}}\lambda}$.

Remark 1.3. It is easy to see that $\mathscr{H}^{\mu}_G(\mathfrak{g},N)$ is a free module over $H^*(N_{\mathrm{red}})$ with generator $e^{-id_{\mathfrak{g}}\lambda}$.

We only need to prove Theorem 7 for such an element $\alpha=\alpha_{\mathrm{red}}e^{-id_{\mathfrak{g}}\lambda}$.
We have

$$\alpha(X)=\alpha_{\mathrm{red}}e^{i(\xi,X)}e^{i(d\omega,\xi)+i(\omega,d\xi)}.$$

Let us remark for later use that

$$\alpha(X)=e^{i(\xi,X)}v \tag{12}$$

where $v=\alpha_{\mathrm{red}}e^{-i(d\omega,\xi)+i(\omega,d\xi)}\in\mathscr{A}(N)$ is *independent* of X.

The form α_{red} is a form on $G\backslash P$. It is independent of $(\xi,d\xi)$. Let us write $e^{i(\omega,d\xi)}=\sum_J i^{|J|}\varepsilon_J\omega_J\,d\xi_J$ where J are multi-indexes and ε_J signs. We thus have

$$\mu_*(\alpha(X))=e^{i(\xi,X)}\sum_J i^{|J|}\varepsilon_J\left(\int_P\alpha_{\mathrm{red}}e^{-i(d\omega,\xi)}\omega_J\right)d\xi_J. \tag{13}$$

To compute $V(\mu_*\alpha)$ we must take the component of maximal degree in $d\xi$ of $\mu_*\alpha$. With our conventions of orientations, we have

$$(\mu_*\alpha(X))_{[n]}=i^n e^{i(\xi,X)}\left(\int_P\alpha_{\mathrm{red}}e^{-i(d\omega,\xi)}v_\omega\right)d\xi$$

where $d\xi$ is the element dual (formula (8)) to the element dX determined by the oriented basis of $\mathfrak{g}$. Let $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ be the curvature of ω. As $\omega_i \wedge v_\omega = 0$ we have

$$\alpha_{\text{red}} e^{-i(d\omega, \xi)} v_\omega = \alpha_{\text{red}} e^{-i(\Omega, \xi)} v_\omega.$$

Thus

$$(\mu_* \alpha(X))_{[n]} = i^n e^{i(\xi, X)} \left(\int_P \alpha_{\text{red}} e^{-i(\Omega, \xi)} v_\omega \right) d\xi. \tag{14}$$

By definition of V, we have $V(\mu_* \alpha) = i^n (\int_P \alpha_{\text{red}} e^{-i(\Omega, \xi)} v_\omega) d\xi$ and we obtain Theorem 7. □

Remark 1.4. If the action of G on P is only infinitesimally free, it is easy to see that every element $\alpha \in \mathscr{H}_G^\infty(\mathfrak{g}, P)$ is congruent to a basic form α_{red} (i.e. a form which is independent of $X \in \mathfrak{g}$, horizontal and G-invariant.) We can choose a connection form ω on P and Theorem 7 is valid.

We may reformulate Theorem 7 more intrinsically using integration over $N_{\text{red}} = G \backslash P$ instead of integration over P. First of all, if G is abelian then $e^{-i(\Omega, \xi)}$ is a form on N_{red} and we obtain the following.

LEMMA 8. *Let G be a torus, then with the same notations as in Theorem 7*

$$V(\mu_* \alpha) = (2i\pi)^n \left(\int_{N_{\text{red}}} \alpha_{\text{red}} e^{-i(\Omega, \xi)} \right) d\xi.$$

In this formula the orientation on N_{red} is the orientation o^P / v_ω and the normalisation for the density dX is such that $\text{vol}(G) = (2\pi)^n$ (we choose this normalisation for dX only in the case of the torus).

More generally, if G is not abelian, we write $(V\mu_* \alpha)/d\xi = L(\alpha)(\xi)$ where $L(\alpha)(\xi) \in S(\mathfrak{g})^G$ is a polynomial function of ξ. We denote by (P, Q) the duality between $S(\mathfrak{g})$ and $S(\mathfrak{g}^*)$ given by

$$(P, Q) = P(\partial_\xi) Q(\xi)|_{\xi = 0}$$

for $P \in S(\mathfrak{g}^*)$ and $Q \in S(\mathfrak{g})$. Then $L(\alpha)$ is determined by the duality between $S(\mathfrak{g})^G$ and $S(\mathfrak{g}^*)^G$. Consider the principal fibration $P \to G \backslash P$. If $\phi \in S(\mathfrak{g}^*)^G$, then $\phi(-i\Omega)$ is a closed form on N_{red} (its de Rham cohomology class is independent of ω). Using the same notations as Theorem 7, we have the more invariant formulation of Theorem 7:

THEOREM 9. *For $\phi \in S(\mathfrak{g}^*)^G$,*

$$(\phi, (V\mu_* \alpha)/d\xi) = i^n \text{vol}(G) \int_{N_{\text{red}}} \alpha_{\text{red}} \phi(-i\Omega).$$

Proof. By Theorem 7 and by definition of the duality, we obtain

$$(\phi, (V\mu_* \alpha)/d\xi) = i^n \int_P \alpha_{\text{red}} \phi(-i\Omega) v_\omega.$$

The forms $\phi(-i\Omega)$ and β_{red} are forms on $G \backslash P$ so that the integration of the factor v_ω gives the term $\text{vol}(G)$ and we obtain Theorem 9. □

1.3. Jeffrey–Kirwan localisation theorem

In this section (M, σ, μ) is a compact symplectic manifold with Hamiltonian action of a compact Lie group G. We assume that 0 is a regular value of μ. We note $P = \mu^{-1}(0)$.

Let $\sigma_{\mathfrak{g}}$ be the equivariant symplectic form. It is the closed G-equivariant differential form on M defined for $X \in \mathfrak{g}$ by $\sigma_{\mathfrak{g}}(X) = \mu(X) + \sigma$. Thus $e^{i\sigma_{\mathfrak{g}}(X)} = e^{i(\mu,X)}e^{i\sigma}$ is a closed element in our complex $\mathscr{A}_G^{\mu}(\mathfrak{g}, M)$. As it is an invertible element, we have

$$\mathscr{A}^{\mu}(\mathfrak{g}, M) = \{e^{i\sigma_{\mathfrak{g}}(X)}\beta(X);\ \beta \in \mathscr{A}_G^{\text{pol}}(\mathfrak{g}, M)\}.$$

We first consider the particularly important closed element $e^{i\sigma_{\mathfrak{g}}(X)} = e^{i(\mu,X)}e^{i\sigma}$ of $\mathscr{A}_G^{\mu}(\mathfrak{g}, M)$. Let $\dim M = 2d$. Let $\beta_M = (d!)^{-1}(2\pi)^{-d}\sigma^d$ be the Liouville form on M. Near the regular value 0 the push-forward $\mu_*(\beta_M)$ of the Liouville measure of M is a C^∞-density on $\mathfrak{g}^*$.

The manifold P is a compact manifold. Furthermore, the fact that 0 is a regular value of μ is equivalent to the fact that the action of G on $P = \mu^{-1}(0)$ is locally free. The orbifold $M_{\text{red}} = G\backslash P$ is the Marsden–Weinstein reduction of M.

As 0 is a regular value, there exists a G-invariant open ball $U \subset \mathfrak{g}^*$ such that $\mu^{-1}(U)$ is diffeomorphic to $P \times U$ by a G-invariant diffeomorphism. Let $N = \mu^{-1}(U) = P \times U$. We apply the results of the preceding section. In our case the manifold $N_{\text{red}} = G\backslash P$ is the reduced manifold M_{red}. By definition of V, $\mu_*(\beta_M)$ is the density $i^{-d}(2\pi)^{-d}V(\mu_*(e^{i\sigma_{\mathfrak{g}}}))$. Let ω be a connection form on P and let Ω be the curvature of ω. The restriction of $\sigma_{\mathfrak{g}}(X) = \mu(X) + \sigma$ to $P = \mu^{-1}(0)$ is simply $\sigma|_P$. By definition, it is the pull-back of the symplectic form σ_{red} of the Marsden–Weinstein reduction M_{red} of M at 0. The dimension of M_{red} is $2d_0 = 2(d - n)$ ($n = \dim G$). We obtain from Theorem 7 $\mu_*(\beta_M) = i^{-d_0}(2\pi)^{-d}$ $(\int_P e^{i(\sigma_{\text{red}} - (\xi,\Omega))}v_\omega)\,d\xi$. Checking out useful exterior degrees, we have:

Proposition 10. *Near 0, the push-forward of the Liouville form $\mu_*(\beta_M)$ is given by*

$$\mu_*(\beta_M) = (2\pi)^{-d}(d_0!)^{-1}\left(\int_P (\sigma_{\text{red}} - (\xi,\Omega))^{d_0} v_\omega\right)d\xi.$$

If G is a torus,

$$\mu_*(\beta_M) = (2\pi)^{-d_0}(d_0!)^{-1}\left(\int_{M_{\text{red}}} (\sigma_{\text{red}} - (\xi,\Omega))^{d_0}\right)d\xi. \tag{15}$$

The formula above for a torus G is the Duistermaat–Heckman formula [2]. For a general compact Lie group G, this is due to Jeffrey and Kirwan [6]. Jeffrey and Kirwan deduce this formula from the normal form theorem [10, 5] which asserts that if U is sufficiently small there exists a symplectic diffeomorphism of $(\mu^{-1}(U), \sigma)$ to $U \times P$ equipped with the symplectic form $\sigma^P = \sigma_{\text{red}} - (\xi, \Omega)$.

It follows from Theorem 14 in the next section that $\mu_*(\beta_M)$ is an analytic density on each connected component of the set of regular values of μ. This fact follows also obviously from the localisation formula [1]. In particular, $\mu_*(\beta_M)$ will be a polynomial density on the connected component of 0 in the open subset of regular values of μ. In the case of a torus action it is a polynomial density on each connected component of the open subset of regular values of μ. This is obvious from the previous result as in the case of a torus action we can translate μ to $\mu - \xi_0$ and displace ourselves at 0. Furthermore if G is a torus, the preceding formula determines entirely the push-forward of the Liouville measure of M if we assume that no connected subgroup of T acts trivially on M. Indeed in this case it is easy to see that the push-forward of the Liouville measure can be written as $f(\xi)\,d\xi$ where $f(\xi)$ is a continuous function on the closed convex set with nonzero interior $\mu(M) \subset \mathfrak{t}^*$. If G is nonabelian, the knowledge of $\mu_*\beta_M$ on regular values does not determine $\mu_*\beta_M$. For example, an orbit $\mathcal{O} \subset \mathfrak{g}^*$ of the coadjoint representation is an Hamiltonian space with moment map μ the canonical injection $\mathcal{O} \to \mathfrak{g}^*$. The set of regular values of μ is $\mathfrak{g}^* - \mathcal{O}$ and $\mu_* \beta_{\mathcal{O}}$ is 0 outside $\mathcal{O}$, but is not 0 as a distribution.

Consider now a general element $\alpha \in \mathscr{A}_G^{\mu}(\mathfrak{g}, M)$. From Theorems 6 and 7, we obtain:

THEOREM 11 (Jeffrey and Kirwan [6]). *Let α be a closed element in $\mathscr{A}_G^{\mu}(\mathfrak{g}, M)$. Let α_{red} be the cohomology class of M_{red} determined by $\alpha|_P$. Near $0 \in \mathfrak{g}$ the Fourier transform of the integral $\int_M \alpha$ of α over M is given by*

$$\mathscr{F}\left(\int_M \alpha\right) = \left(\mathrm{i}^n \int_P \alpha_{\text{red}} \mathrm{e}^{-\mathrm{i}(\xi, \Omega)} v_\omega\right) d\xi.$$

In particular for $\xi = 0$, the Jeffrey–Kirwan result gives the particularly beautiful following formula. For α a closed element of $\mathscr{A}_G^{\mu}(\mathfrak{g}, M)$:

$$\mathscr{F}\left(\int_M \alpha\right)(0) = \mathrm{i}^n (\operatorname{vol} G) \int_{M_{\text{red}}} \alpha_{\text{red}}.$$

Consider the function $\|\mu\|^2$ on M.

LEMMA 12. *Let R be the largest number u such that all $f \in \mathfrak{g}^*$ such that $\|f\|^2 < u$ are regular values of μ. Then R is also the smallest critical value of the function $\|\mu\|^2$.*

Proof. Indeed x is a critical point of $\|\mu\|^2$ if and only if x is a zero of the vector field $\mu(x)_M$. Let us consider $\gamma \in \mathfrak{g}$ a nonzero element and let $M(\gamma)$ be the manifold of zeroes of the vector field γ_M on M. Let $M(\gamma)^a$ be a connected component of $M(\gamma)$. Then $\mu(M(\gamma)^a)$ is contained in an affine plane orthogonal to γ. Thus, identifying $\mathfrak{g}$ with $\mathfrak{g}^*$, the nearest point to 0 in this plane is proportional to γ. Changing γ in a proportional vector, we thus see that R is also the smallest value of $\mu(x)$ for those x such that there exists $\gamma \neq 0$ such that $x \in M(\gamma)$ and $\mu(x) = \gamma$. □

It follows from the localisation formula [1] that $\mathscr{F}(\int_M \alpha)$ is an analytic density on each connected component of the set of regular values of μ. This follows also from Theorem 14 of the next section. By analyticity and Lemma 12 above, the Jeffrey–Kirwan formula (Theorem 11) remains valid for $\|\xi\| < R$.

Witten [12] studied the asymptotic behaviour when $\varepsilon \mapsto 0$ of

$$Z(\varepsilon) = \int_M \int_{\mathfrak{g}} \mathrm{e}^{\mathrm{i}\sigma_{\mathfrak{g}}(X)} \beta(X) \mathrm{e}^{-\varepsilon \|X\|^2/2}\, dX$$

where $\beta(X)$ is a G-equivariant closed form on M with polynomial coefficients. Let $\alpha = \mathrm{e}^{\mathrm{i}\sigma_{\mathfrak{g}}}\beta$. Then the value of $Z(\varepsilon)$ at $\varepsilon = 0$ is $(2\pi)^n(\mathscr{F} \int_M \alpha)(0)$.

THEOREM 13 (Witten [12]). *Let (M, σ, μ) be a compact symplectic manifold with a Hamiltonian action of a compact group G. Assume that the action of G on $\mu^{-1}(0)$ is free. Let Ω be the curvature of the fibration $\mu^{-1}(0) \to M_{\text{red}} = G\backslash\mu^{-1}(0)$. Let R be the smallest critical value of $\|\mu\|^2$. Let r be a positive number such that $r < R$. Then for any G-equivariantly closed form β on M with polynomial coefficients, there exists a constant C such that*

$$Z(\varepsilon) = (2\mathrm{i}\pi)^{\dim G} \operatorname{vol}(G)\left(\int_{M_{\text{red}}} \mathrm{e}^{\mathrm{i}\sigma_{\text{red}}} \beta_{\text{red}} \mathrm{e}^{-\varepsilon\|\Omega\|^2/2}\right) + N(\varepsilon)$$

with $|N(\varepsilon)| \leq C\mathrm{e}^{-r/2\varepsilon}$ for any $\varepsilon > 0$.

Proof. Let $\alpha(X) = e^{i\sigma_g(X)}\beta(X)$ and let $w(X) = \int_M \alpha(X)$. Recall from formula (4) that the Fourier transform $\mathscr{F}(w)$ of w is a derivative of compactly supported Radon measures on $\mathfrak{g}^*$. Furthermore for $\|\xi\| < R \in \mathfrak{g}$, $\mathscr{F}(w)$ is a polynomial density given (Theorem 11) by

$$\mathscr{F}(w) = i^n\left(\int_P e^{-i(\Omega,\xi)}e^{i\sigma_{red}}\beta_{red}v_\omega\right)d\xi.$$

We have

$$Z(\varepsilon) = \int_{\mathfrak{g}} w(X)e^{-\varepsilon\|X\|^2/2}\,dX$$

and by Fourier transforms

$$Z(\varepsilon) = \int_{\mathfrak{g}^*} \mathscr{F}(w)(\xi)\varepsilon^{-n/2}(2\pi)^{n/2}e^{-\|\xi\|^2/2\varepsilon}.$$

Thus by partition of unity, we see that modulo a rest $N(\varepsilon)$ less than $Ce^{-r/2\varepsilon}$,

$$Z(\varepsilon) = \int_{\mathfrak{g}^*} i^n\left(\int_P e^{-i(\Omega,\xi)}e^{i\sigma_{red}}\beta_{red}v_\omega\right)\varepsilon^{-n/2}(2\pi)^{n/2}e^{-\|\xi\|^2/2\varepsilon}\,d\xi + N(\varepsilon).$$

By the inversion formula

$$\int_{\mathfrak{g}^*} i^n e^{-i(\Omega,\xi)}\varepsilon^{-n/2}(2\pi)^{n/2}e^{-\|\xi\|^2/2\varepsilon}\,d\xi = (2i\pi)^n e^{-\varepsilon\|\Omega\|^2/2}$$

and one obtains Witten's estimate. □

1.4. Induction formula

In this section, we prove an induction formula for the map $V\mu_*$. This section will not be used in the remainder of this article.

Let $\mathcal{O} \subset \mathfrak{g}^*$ be an orbit of the coadjoint representation. Let $f \in \mathcal{O}$. Let $G_0 = G(f)$ and $\mathfrak{g}_0 = \mathfrak{g}(f)$. Let $n = \dim \mathfrak{g}$ and $n_0 = \dim \mathfrak{g}_0$. Let

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{r} \tag{16}$$

be a G_0-invariant decomposition of $\mathfrak{g}$. Let $\dim \mathfrak{r} = 2r = \dim \mathcal{O}$.

Using decomposition (16), we consider $\mathfrak{g}_0^* \subset \mathfrak{g}^*$. Thus $\mathfrak{g}_0^*$ is a G_0-invariant supplementary subspace to the tangent space $\mathfrak{r}^* = \mathfrak{g}_0^\perp = \mathfrak{g}\cdot f$ to the orbit $\mathcal{O}$ at f.

Let $\xi \in \mathfrak{g}_0^*$. We denote by $B_\xi \in \Lambda^2\mathfrak{r}^*$ the alternate bilinear map $B_\xi(R_1, R_2) = -(\xi, [R_1, R_2])$. Let $dR \in \Lambda^{2r}\mathfrak{r}^*$ be a volume form on $\mathfrak{r}$ and let $D_0(\xi)$ be the G_0-invariant polynomial function (depending of dR) on $\mathfrak{g}_0^*$ such that

$$D_0(\xi)\,dR = (r!)^{-1}B_\xi^r.$$

As B_f is nondegenerate on $\mathfrak{r} = \mathfrak{g}/\mathfrak{g}(f)$, the value of $D_0(\xi)$ at f is nonzero.

Consider U_0 a G_0-invariant small ball around 0 in $\mathfrak{g}_0^*$. Then

$$\mathscr{W} = \{g\cdot(f+\xi_0);\ \xi_0 \in U_0\}$$

is a tubular neighbourhood of $\mathcal{O}$ isomorphic to $G \times_{G_0} U_0$ by $(g, \xi_0) \mapsto g\cdot(f+\xi_0)$. Any G_0-invariant function $L(\xi_0)$ on U_0 extends to a G-invariant function $\tilde{L}$ on $\mathscr{W}$ by $\tilde{L}(g\cdot(f+\xi_0)) = L(\xi_0)$. If L is polynomial, the extension $\tilde{L}$ is a G-invariant algebraic function (it is rational on a $|W|$-cover, where $|W|$ is the order of the Weyl group of $\mathfrak{g}$). In particular, if L is polynomial, $\tilde{L}$ is analytic. The function $\xi_0 \mapsto D_0(f+\xi_0)$ does not vanish for $\xi_0 \in U_0$. It admits a G-invariant analytic extension to $\mathscr{W}$.

Consider a G_0-oriented manifold P_0 where G_0 acts infinitesimally freely. Let $N_0 = P_0 \times U_0$. Let $\mu_0 : N_0 \to U_0$ be the second projection. The element $f \in \mathfrak{g}_0^*$ is G_0-invariant, thus the map $f + \mu_0$ is a G_0-invariant map from N_0 to $\mathfrak{g}_0^*$. We consider the induced manifold $N = G \times_{G_0} N_0$. We denote by $[g, n_0]$ the image of the element (g, n_0) in the quotient manifold $N = (G \times N_0)/G_0$. The manifold N is fibred over G/G_0. Its fibre above the base point of G/G_0 is N_0. Thus we consider N_0 as a G_0-invariant submanifold of N. The map $\mu(g, n_0) = g \cdot (f + \mu_0(n_0))$ is a G-invariant fibration from N to $\mathscr{W} = G \times_{G_0} U_0$ with typical fibre P_0.

Consider the restriction map

$$r_0 : \mathscr{A}_G^\infty(\mathfrak{g}, N) \to \mathscr{A}_{G_0}^\infty(\mathfrak{g}_0, N_0)$$

given by $r_0(\alpha)(Y) = \alpha(Y)|_{N_0}$ for $\alpha \in \mathscr{A}_G^\mu(\mathfrak{g}, N)$ and $Y \in \mathfrak{g}_0$. We have $r_0(\mu)(Y) = (f, Y) + \mu_0(Y)$. If $\alpha \in \mathscr{A}_G^\mu(\mathfrak{g}, N)$, then $\beta(Y) = r_0(\alpha)(Y)$ is in $\mathscr{A}_{G_0}^{f+\mu_0}(\mathfrak{g}_0, N_0)$. The element $e^{-i(f,Y)} r_0(\alpha)$ is in $\mathscr{A}_{G_0}^{\mu_0}(\mathfrak{g}_0, N_0)$.

Consider the maps

$$V\mu_* : \mathscr{H}_G^\mu(\mathfrak{g}, N) \to \mathscr{A}^n(\mathscr{W})^G$$

and

$$V_0(\mu_0)_* : \mathscr{H}_{G_0}^{\mu_0}(\mathfrak{g}_0, N_0) \to \mathscr{A}^{n_0}(U_0)^{G_0}.$$

An element $h \in \mathscr{A}^n(\mathscr{W})^G$ is a G-invariant map from $\mathscr{W}$ to $\Lambda^n \mathfrak{g}$. It is thus determined by its restriction $\xi_0 \mapsto h(f + \xi_0)$ to $f + U_0$. An element $\phi \in \mathscr{A}^{n_0}(U_0)^{G_0}$ is a G_0-invariant map from U_0 to $\Lambda^{n_0}\mathfrak{g}_0$. Let $dR \in \Lambda^{2r}\mathfrak{r}^*$ and $dR^* \in \Lambda^{2r}\mathfrak{r}$ be the dual element. Thus if $\phi \in \mathscr{A}^{n_0}(U_0)^{G_0}$, and $\xi_0 \in U_0$ then $D_0(f + \xi_0)^{-1}\phi(\xi_0) \wedge dR^*$ is an element of $\Lambda^n\mathfrak{g}$. Note that it is independent of the choice of dR.

THEOREM 14. *Let $N_0 = P_0 \times U_0$ where G_0 acts infinitesimally freely on P_0. Let $N = G \times_{G_0} N_0$. Let $\alpha \in \mathscr{H}_G^\mu(\mathfrak{g}, N)$. Let $\beta = e^{-if} r_0(\alpha) \in \mathscr{H}_{G_0}^{\mu_0}(\mathfrak{g}_0, N_0)$. Then*

$$V(\mu_*\alpha)(f + \xi_0) = i^r D_0(f + \xi_0)^{-1}(V_0(\mu_0)_*\beta)(\xi_0) \wedge dR^*.$$

Proof. Decomposition (16) determines a connection form θ for the fibration $G \mapsto G/G_0$. Let $\Theta \in \Lambda^2\mathfrak{r}^* \otimes \mathfrak{g}_0$ be the curvature of θ at $e \in G$. Then $\Theta(R_1, R_2)(\xi_0) = -(\xi_0, [R_1, R_2])$ for $R_i \in \mathfrak{r}$ and $\xi_0 \in \mathfrak{g}_0^*$.

Let Z_0 be a G_0-manifold. Let Z be the induced manifold $G \times_{G_0} Z_0$. We constructed in [2] a homomorphism of differential algebras $W_\theta : \mathscr{A}_{G_0}^\infty(\mathfrak{g}_0, Z_0) \to \mathscr{A}_G^\infty(\mathfrak{g}, Z)$ which gives the inverse in cohomology to $r_0 : \mathscr{A}_G^\infty(\mathfrak{g}, Z) \to \mathscr{A}_{G_0}^\infty(\mathfrak{g}_0, Z_0)$. The formula for $W_\theta(\beta)$ is given as follows: we identify a neighbourhood of $z_0 \in Z_0$ in Z to a neighbourhood of $(0, z_0)$ in $\mathfrak{r} \times Z_0$ by the map $(R, z_0) \to [\exp R, z_0]$. Thus the tangent space to Z at $z_0 \in Z_0$ is identified to $\mathfrak{r} \oplus T_{z_0}Z_0$.

Let $X \in \mathfrak{g}$. We write $X = Y + R$ with $Y \in \mathfrak{g}_0$ and $R \in \mathfrak{r}$. By definition

$$(W_\theta\beta)_{z_0}(X) = \beta(Y + \Theta)_{z_0} \in \Lambda\mathfrak{r}^* \otimes \Lambda T_{z_0}^* Z_0. \tag{17}$$

By G-invariance, this formula determines $W_\theta\beta$ everywhere.

PROPOSITION 15. *The map $\alpha \mapsto r_0(\alpha)$ induces an isomorphism from $\mathscr{H}_G^\mu(\mathfrak{g}, N)$ to $\mathscr{H}_{G_0}^{f+\mu_0}(\mathfrak{g}_0, N_0)$.*

Proof of Proposition 15. Let us see first that $W_\theta(e^{if+\mu_0})(X) = e^{i(\mu, X)}v$ where $v \in \mathscr{A}(N)$ is independent of $X \in \mathfrak{g}$. Indeed at $n_0 \in N_0$ and for $X = Y + R$, we have $(f + \mu_0(n_0), X) = (f + \mu_0(n_0), Y)$ as $f + \mu_0(n_0) \in \mathfrak{g}_0^*$. Thus $W_\theta(e^{if+\mu_0})_{n_0}(X) = e^{i(\mu, X)}v$ with

$\nu = e^{i(f+\mu_0(n_0),\Theta)}$ independent of X. By G-invariance the result follows. Thus W_θ sends $\mathscr{A}_{G_0}^{f+\mu_0}(\mathfrak{g}_0, N_0)$ to $\mathscr{A}_G^{\mu}(\mathfrak{g}_0, N_0)$. The explicit formula for $\alpha - W_\theta(r_0\alpha)$ given in [3] gives the proposition. □

It follows that any closed element of $\mathscr{A}_G^{\mu}(\mathfrak{g}, N)$ is congruent to an element $\alpha = W_\theta(e^{if}\beta)$ with β a closed element of $\mathscr{A}_{G_0}^{\mu_0}(\mathfrak{g}_0, N_0)$. It remains to prove Theorem 14 for

$$\alpha = W_\theta(e^{if}\beta).$$

If $m_0: Z_1 \to Z_2$ is a fibration of G_0-manifolds, and m is the induced fibration $G \times_{G_0} Z_1 \to G \times_{G_0} Z_2$, the map W_θ satisfies $m_* W_\theta = W_\theta (m_0)_*$. Thus we obtain

$$\mu_*\alpha = \mu_* W_\theta(e^{if}\beta) = W_\theta((\mu_0)_*(e^{if}\beta)) = W_\theta(e^{if}(\mu_0)_*\beta).$$

Let $Y \in \mathfrak{g}_0$. In the proof of Theorem 7 we have seen (formula (12)) that we can suppose that $\beta \in \mathscr{A}_{G_0}^{\mu_0}(\mathfrak{g}_0, N_0)$ is such that $\beta(Y) = e^{i(\mu_0, Y)}\nu$ with ν independent of Y. Thus $(\mu_0)_*\beta(Y) = e^{i(\xi_0, Y)}(\mu_0)_*\nu$ where $(\mu_0)_*\nu \in \mathscr{A}(U_0)$ is a form independent of $Y \in \mathfrak{g}_0$. The highest exterior degree term $((\mu_0)_*\nu)_{[n_0]}$ of $(\mu_0)_*\nu$ is by definition equal to $V_0(\mu_0)_*\beta$. As before, we see that $\mu_*\alpha = W_\theta(e^{if}(\mu_0)_*\beta) = W_\theta(e^{i(f+\xi_0, Y)}(\mu_0)_*\nu)$ is such that $\mu_*\alpha(X) = e^{i(\xi, X)}\kappa$ where $\kappa \in \mathscr{A}(\mathscr{W})$ is independent of X. In particular there will be no differentiation in computing $V(\mu_*\alpha) = V(W_\theta(e^{if}(\mu_0)_*\beta))$ and we can restrict ourselves to the slice $f + U_0$ of $\mathscr{W}$. Consider the map $\mathfrak{r} \times U_0 \mapsto \mathscr{W}$ given by $(R, \xi_0) \mapsto \exp R \cdot (f + \xi_0)$. In the local coordinates $\mathfrak{r} \times U_0$ then by definition of W_θ (formula (17)),

$$W_\theta(e^{if}(\mu_0)_*\beta)_{(0,\xi_0)}(X) = e^{i(f+\xi_0, Y+\Theta)} \wedge (\mu_0)_*\nu.$$

The highest exterior degree term of $W_\theta(e^{if}(\mu_0)_*\beta)_{(0,\xi_0)}(X)$ is

$$i^r e^{i(f+\xi_0, Y)} D_0(f + \xi_0)\, dR \wedge ((\mu_0)_*\nu_{[n_0]}).$$

We rewrite $W_\theta(e^{if}(\mu_0)_*\beta)_{(0,\xi_0)}(X)$ in the coordinates $\xi = \exp R(f + \xi_0)$. The image of $(0, \xi_0)$ under this map is the point $f + \xi_0$. The Jacobian of this change of coordinates at the point $(0, \xi_0)$ is $D_0(f + \xi_0)^{-2}$ and we obtain

$$(W_\theta(e^{if}(\mu_0)_*\beta)_{[n]})_{f+\xi_0}(X) = i^r e^{i(f+\xi_0, Y)} D_0(f + \xi_0)^{-1}\, dR^* \wedge ((\mu_0)_*\nu_{[n_0]}).$$

This formula implies Theorem 14. □

Let us come back to the situation where (M, σ, μ) is a Hamiltonian manifold. Let f be a regular value of μ. Let $\mathcal{O}$ be the orbit of f. Let $G_0 = G(f)$. Let $P_0 = \mu^{-1}(f)$. Then G_0 acts by an infinitesimally free action on P_0. If $U_0 \subset \mathfrak{g}_0^*$ is a sufficiently small ball, the manifold $N_0 = \mu^{-1}(f + U_0)$ is a submanifold of M diffeomorphic to $P_0 \times U_0$. We denote by μ_0 the projection of N_0 on U_0. The manifold $N = \mu^{-1}(\mathscr{W})$ is diffeomorphic to the G-manifold $G \times_{G_0} N_0$. Applying Theorems 7 and 14 we conclude:

COROLLARY 16. *Let (M, σ, μ) be a symplectic manifold with a Hamiltonian action of G. Let $\alpha \in \mathscr{H}_G^{\mu}(\mathfrak{g}, M)$. Then the Fourier transform $\mathscr{F}(\int_M \alpha)$ of the function $\int_M \alpha(X)$ is an analytic density on each connected component of the set of regular values. It is a polynomial density on the connected component of* 0.

More precisely, we know that if f is a regular value and $\mathfrak{g}_0 = \mathfrak{g}(f)$, then $\mathscr{F}(\int_M \alpha) = L(\xi)\, d\xi$ where on the transverse subspace $f + U_0$ to the orbit $\mathcal{O}$,

$$L(f + \xi_0) = i^r D_0(f + \xi_0)^{-1} L_0(\xi_0)$$

is the quotient of two G_0-invariant polynomials. The polynomial L_0 is computed in function of the G_0-Hamiltonian manifold $\mu^{-1}(f + U_0)$. This result can also be proven directly using Harish–Chandra relations between the Fourier transform on $\mathfrak{g}$ and $\mathfrak{g}_0$. However the above proof is a local proof.

2. ON WITTEN'S LOCALISATION FORMULA

2.1. *An integral formula for free actions*

Let G be a Lie group acting on a manifold M.

If M is a compact oriented manifold and $\alpha \in \mathcal{A}_G^\infty(\mathfrak{g}, M)$ an equivariant differential form, $X \mapsto \int_M \alpha(X)$ is an invariant C^∞-function on $\mathfrak{g}$. It determines *a fortiori* a generalised function on $\mathfrak{g}$ denoted $\int_M \alpha$. If $\phi\, dX$ is a test density on $\mathfrak{g}$, the formula

$$\int_{\mathfrak{g}} \left(\int_M \alpha \right)(X)\phi(X)\, dX = \int_M \int_{\mathfrak{g}} \alpha(X)\phi(X)\, dX$$

defines the generalised function $\int_M \alpha$.

We can define a generalised function $(\int_M \alpha)(X)$ when M is a noncompact manifold by the same formula as above provided the differential form $\int_{\mathfrak{g}} \alpha(X)\phi(X)\, dX$ is integrable over M. We formalise this notion as follows. If $\mathcal{V} \to P$ is a vector bundle over a compact manifold P, we say that an equivariant differential form $\alpha \in \mathcal{A}_G^\infty(\mathfrak{g}, \mathcal{V})$ is rapidly decreasing in $\mathfrak{g}$-mean if for any test function ϕ on $\mathfrak{g}$, $\int_{\mathfrak{g}} \alpha(X)\phi(X)\, dX$ is a differential form on $\mathcal{V}$ rapidly decreasing over the fibres of $\mathcal{V} \to P$. Assume the total space $\mathcal{V}$ is oriented. Then the generalised function $(\int_{\mathcal{V}} \alpha)(X)$ is well defined: if ϕ is a test function on $\mathfrak{g}$,

$$\int_{\mathfrak{g}} \left(\int_{\mathcal{V}} \alpha \right)(X)\phi(X)\, dX = \int_{\mathcal{V}} \left(\int_{\mathfrak{g}} \alpha(X)\phi(X)\, dX \right).$$

Let P be a manifold where G acts freely. We employ the notations of Section 1.2. Consider the manifold

$$N = P \times \mathfrak{g}^*.$$

Let us first describe a particular closed G-equivariant differential form on $N = P \times \mathfrak{g}^*$ which is rapidly decreasing in $\mathfrak{g}$-mean over the fibre $\mathfrak{g}^*$.

Let ω be a connection form for $P \to G\backslash P$. Let $\lambda = (\omega, \xi)$ (see formula (10)).

LEMMA 17. *The differential form* $e^{-\mathrm{i} d_{\mathfrak{g}} \lambda}$ *on* N *is rapidly decreasing in* $\mathfrak{g}$*-mean.*

Proof. We have (see formula (11))

$$e^{-\mathrm{i}(d_{\mathfrak{g}}\lambda)(X)} = e^{\mathrm{i}(\xi, X)} e^{-\mathrm{i}(d\omega, \xi) + \mathrm{i}(\omega, d\xi)}$$

and for a test function ϕ on $\mathfrak{g}$,

$$\int_{\mathfrak{g}} e^{-\mathrm{i}(d_{\mathfrak{g}}\lambda)(X)} \phi(X)\, dX = \hat{\phi}(\xi) e^{-\mathrm{i}(d\omega, \xi) + \mathrm{i}(\omega, d\xi)}$$

where $\hat{\phi}(\xi) = \int_{\mathfrak{g}} e^{\mathrm{i}(\xi, X)} \phi(X)\, dX$ is the Fourier transform of the test function ϕ. The form $e^{-\mathrm{i}(d\omega, \xi) + \mathrm{i}(\omega, d\xi)}$ is polynomial in ξ. As the Fourier transform $\hat{\phi}(\xi)$ of the test function ϕ is rapidly decreasing in ξ we obtain the lemma. □

Let $\alpha \in \mathscr{A}_G^\infty(\mathfrak{g}, P)$. Then $e^{-id_\mathfrak{g}\lambda}\alpha$ is rapidly decreasing in $\mathfrak{g}$-mean over N: in local coordinates m_i on P, we have $\alpha(X) = \sum_I \alpha_I(X,m)\,dm_I$ where $\alpha_I(X,m)$ depends smoothly on X, m. By the same calculation as before $\int_\mathfrak{g} e^{-i(d_\mathfrak{g}\lambda)(X)}\alpha(X)\phi(X)\,dX$ is rapidly decreasing in ξ for any test function ϕ on $\mathfrak{g}$. The generalised function $(\int_N e^{-id_\mathfrak{g}\lambda}\alpha)(X)$ is well defined.

THEOREM 18. *Assume that G acts freely on P. Let $\alpha \in \mathscr{A}_G^\infty(\mathfrak{g}, P)$ be a closed G-equivariant differential form on P. Then, if ϕ is a test function on $\mathfrak{g}$,*

$$\int_\mathfrak{g}\left(\int_N e^{-id_\mathfrak{g}\lambda}\alpha\right)(X)\phi(X)\,dX = (2i\pi)^{\dim G}\int_P \alpha_{\mathrm{red}}\phi(\Omega)\wedge v_\omega.$$

In this formula if the orientation of P is o^P, the orientation of N is $o^P \wedge d\xi$, where v_ω and $d\xi$ are determined by formulas (7) and (8). If G acts only infinitesimally freely on P, we obtain the same theorem.

If ϕ is a G-invariant test function, then $\phi(\Omega)$ is a form on N_{red} and we obtain the more invariant formulation of Theorem 18:

$$\int_\mathfrak{g}\left(\int_N e^{-id_\mathfrak{g}\lambda}\alpha\right)(X)\phi(X)\,dX = (2i\pi)^{\dim G}\operatorname{vol}(G)\int_{N_{\mathrm{red}}} \alpha_{\mathrm{red}}\phi(\Omega).$$

In this formula the volume of G is computed using the Haar measure on G compatible with dX. The orientation of N_{red} is o^P/v_ω.

Proof of Theorem 18. Let $\beta \in C^\infty_{\mathrm{cpt}}(\mathfrak{g}, \mathscr{A}(P))$ be a smooth map with compact support from $\mathfrak{g}$ to the space of differential forms $\mathscr{A}(P)$ on a compact manifold P. Define for $\xi \in \mathfrak{g}^*$ the Fourier transform $\hat\beta(\xi) = \int_\mathfrak{g} e^{i(\xi, X)}\beta(X)\,dX$. It is a differential form on P depending on ξ. When ξ tends to ∞, the form $\hat\beta(\xi)$ converges uniformly to 0 on P.

Let $u \in \mathscr{A}(P) \otimes \mathfrak{g}$ be an even form without constant term. For $\beta \in C^\infty(\mathfrak{g}, \mathscr{A}(P))$, we can define $\beta(u) \in \mathscr{A}(P)$ via the Taylor expansion of β at 0. We still have the Fourier inversion formula for $\beta \in C^\infty_{\mathrm{cpt}}(\mathfrak{g}, \mathscr{A}(P))$:

$$(2\pi)^{-n}\int_{\mathfrak{g}^*} e^{-i(u,\xi)}\hat\beta(\xi)\,d\xi = (2\pi)^{-n}\int_{\mathfrak{g}^*}\left(\int_\mathfrak{g} e^{i(X-u,\xi)}\beta(X)\,dX\right)d\xi = \beta(u). \tag{18}$$

Let ϕ be a test function on $\mathfrak{g}$. We have to compute $\int_N\int_\mathfrak{g}\alpha(X)e^{-id_X\lambda}\phi(X)\,dX$. This integral depends only of the equivariant cohomology class of α in $\mathscr{H}_G^\infty(\mathfrak{g}, P)$. Indeed if $\alpha = d_\mathfrak{g}\beta$, then $\alpha(X)e^{-id_X\lambda} = d_X(\beta(X)e^{-id_X\lambda})$. The term of maximal exterior degree of $\alpha(X)e^{-id_X\lambda}$ is equal to $d((\beta(X)e^{-id_X\lambda})_{[\dim N-1]})$. Thus

$$\left(\int_\mathfrak{g}\alpha(X)e^{-id_X\lambda}\phi(X)\,dX\right)_{[\dim N]} = d\left(\int_\mathfrak{g}(\beta(X)e^{-id_X\lambda})_{[\dim N-1]}\phi(X)\,dX\right).$$

The same calculation as in Lemma 17 shows that the form on N given by $\nu = \int_\mathfrak{g}\beta(X)e^{-id_X\lambda}\phi(X)\,dX$ is rapidly decreasing in ξ, so that $\int_N d\nu = 0$.

We choose as representative of the cohomology class of α the form α_{red} which is independent of $X \in \mathfrak{g}$. Let us choose an orientation on $\mathfrak{g}$ and let $E^1, E^2, \ldots, E^n$ be an oriented basis of $\mathfrak{g}$. This determines the form v_ω (formula (7)). We denote by $\int_{N/P}$ the integral over the fibre $\mathfrak{g}^*$ of the fibration $N \to P$. Then

$$\int_N\int_\mathfrak{g}\alpha(X)e^{-id_X\lambda}\phi(X)\,dX = \int_P\alpha_{\mathrm{red}}\int_{N/P}\int_\mathfrak{g} e^{-id_X\lambda}\phi(X)\,dX.$$

Consider $e^{-id_X\lambda} = e^{i(\xi,X)}e^{-i(d\omega,\xi)+i(\omega,d\xi)}$. Its term of maximal degree in $d\xi$ is equal to $c\,d\xi_1\wedge d\xi_2\wedge\cdots\wedge d\xi_n\wedge v_\omega = c\,d\xi\wedge v_\omega$ where $c = i^n\varepsilon$ and ε is a sign.

Then

$$\int_{N/P}\int_{\mathfrak g} e^{-id_X\lambda}\phi(X)\,dX = c\int_{N/P} e^{-i(d\omega,\xi)}\left(\int_{\mathfrak g} e^{i(\xi,X)}\phi(X)\,dX\right)d\xi v_\omega.$$

Let $\Omega = d\omega + \frac{1}{2}[\omega,\omega]$ be the curvature of ω. As $\omega_i\wedge v_\omega = 0$, for all i, we have $e^{-i(d\omega,\xi)}v_\omega = e^{-i(\Omega,\xi)}v_\omega$. We obtain

$$\int_{N/P}\int_{\mathfrak g} e^{-id_X\lambda}\phi(X)\,dX = c\int_{N/P} e^{-i(\Omega,\xi)}\left(\int_{\mathfrak g} e^{i(\xi,X)}\phi(X)\,dX\right)d\xi v_\omega$$

and Fourier inversion formula gives

$$\int_{N/P}\int_{\mathfrak g} e^{-i(\Omega,\xi)}e^{i(\xi,X)}\phi(X)\,dX\,d\xi = (2\pi)^n\phi(\Omega).$$

We obtain Theorem 18. □

Remark 2.1. It is in fact more natural to use the equivariant cohomology space $\mathscr{H}_G^{-\infty}(\mathfrak g, P)$ with generalised coefficients [9]. Let $\gamma_\omega \in \mathscr{A}_G^{-\infty}(\mathfrak g, P)$ defined by

$$\gamma_\omega(X) = v_\omega\wedge\delta(X-\Omega)$$

where δ is the δ-function at 0 on $\mathfrak g$: i.e.

$$\int_{\mathfrak g}\gamma_\omega(X)\phi(X)\,dX = v_\omega\wedge\phi(\Omega).$$

Then $\gamma_\omega(X)$ is a closed equivariant differential form on P. It is proved in [9, Proposition 79] that γ_ω is a generator of $\mathscr{H}_G^{-\infty}(\mathfrak g, P)$ over $H^*(M_{\mathrm{red}})$ and that

$$\int_{N/P} e^{-id_{\mathfrak g}\lambda} = \varepsilon(2i\pi)^n\gamma_\omega$$

where ε is a sign. Theorem 18 follows.

2.2. Witten localisation formula

Let (M,σ,μ) be a compact symplectic manifold with a Hamiltonian action of a compact Lie group G. Let us assume that 0 is a regular value of μ. We assume to simplify that G acts freely on $P = \mu^{-1}(0)$. Let ω be a connection form on P with curvature Ω. Let $M_{\mathrm{red}} = G\backslash P$ be the Marsden–Weinstein reduction of M. Let α be a closed G-equivariant differential form on M. We denote by α_{red} the de Rham cohomology class $(\alpha|_P)_{\mathrm{red}}$ on M_{red} determined by $\alpha|_P$. In particular $(\sigma_{\mathfrak g})_{\mathrm{red}}$ is the symplectic form σ_{red} of M_{red}.

Following Witten, we introduce the function $\frac{1}{2}\|\mu\|^2$ and its Hamiltonian vector field H. This is a G-invariant vector field on M. Let us choose a G-invariant metric $(\cdot,\cdot)$ on M. Let

$$\lambda^M(\cdot) = (H,\cdot).$$

Then λ^M is a G-invariant 1-form on M.

Let R be the smallest critical value of the function $\|\mu\|^2$. Let $r < R$ and let

$$M_0 = \{x\in M;\ \|\mu(x)\|^2 < r\},\qquad M_{\mathrm{out}} = \{x\in M;\ \|\mu(x)\|^2 > r\}. \tag{19}$$

The manifold M is oriented by its symplectic form.

Let $\alpha(X)$ be a closed G-equivariant differential form on M. Let us consider

$$\Theta(M,t)(X) = \int_M e^{-itd_X\lambda^M}\alpha(X).$$

As α is a closed form and $e^{-itd_\mathfrak{g}\lambda^M}$ congruent to 1 in cohomology, $\Theta(M,t)(X)$ is independent of t. Let us break the integral formula for $\Theta(M,t)$ in two parts:

$$\Theta(M_0,t)(X) = \int_{M_0} e^{-itd_X\lambda^M}\alpha(X) \tag{20}$$

and

$$\Theta(M_{\text{out}},t)(X) = \int_{M_{\text{out}}} e^{-itd_X\lambda^M}\alpha(X). \tag{21}$$

The functions $\Theta(M_0,t)(X)$ and $\Theta(M_{\text{out}},t)(X)$ are C^∞-functions on $\mathfrak{g}$.

THEOREM 19. *For every $t\in\mathbb{R}$ and $X\in\mathfrak{g}$, we have*

$$\left(\int_M \alpha\right)(X) = \Theta(M_0,t)(X) + \Theta(M_{\text{out}},t)(X).$$

Furthermore, the limits Θ_0 and Θ_{out} when $t\to\infty$ of $\Theta(M_0,t)$ and $\Theta(M_{\text{out}},t)$ exist in the space of generalised functions on $\mathfrak{g}$. If ϕ is a test function on $\mathfrak{g}$, we have

$$\int_\mathfrak{g} \Theta_0(X)\phi(X)\,dX = (2i\pi)^{\dim G}\int_P \alpha_{\text{red}}\phi(\Omega)v_\omega.$$

Remark 2.2. If ϕ is G-invariant, we obtain

$$\int_\mathfrak{g} \Theta_0(X)\phi(X)\,dX = i^{\dim G}(2\pi)^{2\dim G}\int_{M_{\text{red}}} \alpha_{\text{red}}\phi(\Omega).$$

Remark that Θ_0 is a generalised function with support $0\in\mathfrak{g}$. Its Fourier transform is a polynomial on $\mathfrak{g}^*$.

Proof of Theorem 19. The fact that for every $t\in\mathbb{R}$, we have $(\int_M\alpha)(X) = \Theta(M_0,t)(X) + \Theta(M_{\text{out}},t)(X)$ has already been mentioned. Thus we need only to prove that the limit Θ_0 when $t\to\infty$ of $\Theta(M_0,t)$ exists in the space of generalised functions on $\mathfrak{g}$.

We choose an orthornormal basis E^i of $\mathfrak{g}$. We write $\mu = \sum_i \mu(E^i)E_i$. We have $\frac{1}{2}d\|\mu\|^2 = \sum_i \mu(E^i)\,d\mu(E^i) = \sum_i \mu(E^i)\iota((E^i)_M)\sigma$ so that

$$H = \sum_i \mu(E^i)E^i_M.$$

Let

$$\lambda^M(\cdot) = (H,\cdot).$$

Then $\lambda^M = \sum_i \mu(E^i)\omega_i^M$ where $\omega_i^M(\cdot) = ((E^i)_M,\cdot)$. We write $\omega^M = \sum_i \omega_i^M E^i$. Then

$$\lambda^M = (\omega^M,\mu).$$

On M_0 the action of G is infinitesimally free, as follows from Lemma 12. Thus we may choose our metric $(\cdot,\cdot)$ such that $((E^i)_M,(E^j)_M) = \delta_i^j$ on M_0. Thus on M_0, $\omega^M(X_M) = X$, for $X\in\mathfrak{g}$ so that ω^M is a connection form on M_0. Furthermore on M_0, we have $\lambda^M(X_M) = \mu(X)$.

Let $f_{\lambda^M} : M \to \mathfrak{g}^*$ be the map determined by $f_{\lambda^M}(X) = \lambda^M(X_M)$, then f_{λ^M} coincides with μ on M_0. On M, we have

$$\langle f_{\lambda^M}, \mu \rangle = \sum_i \mu(E^i)((E^i)_M, H) = (H, H) \geq 0. \tag{22}$$

On M_0, we have

$$d_X \lambda^M = -\mathrm{i}(\mu, X) + d\lambda^M = -\mathrm{i}(\mu, X) + (\mu, d\omega^M) - (\omega^M, d\mu)$$

and we study

$$\int_{M_0} \int_{\mathfrak{g}} \mathrm{e}^{\mathrm{i}t(\mu, X) - \mathrm{i}t(\mu, d\omega^M) + \mathrm{i}t(\omega^M, d\mu)} \alpha(X) \phi(X)\, dX. \tag{23}$$

Let $\varepsilon > 0$ be a small number. Let $M_\varepsilon = \{x \in M;\ \|\mu(x)\| < \varepsilon\}$ and let $m \mapsto \chi(m)$ be a cut-off function on M identically 1 on $M_{\varepsilon/2}$ and identically 0 outside M_ε.

LEMMA 20. *We have*

$$\lim_{t \to \infty} \int_{M_0} (1 - \chi(m)) \left(\int_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}t d_X \lambda^M} \alpha(X) \phi(X)\, dX \right) = 0.$$

Proof of Lemma 20. Let $\beta(X) = \phi(X)\alpha(X)$. Then $\beta \in C^\infty_{\mathrm{cpt}}(\mathfrak{g}, \mathscr{A}(M))$. On M_0

$$\int_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}t d_X \lambda^M} \alpha(X)\phi(X)\, dX = \int_{\mathfrak{g}} \mathrm{e}^{\mathrm{i}t(\mu, X)} \mathrm{e}^{-\mathrm{i}t d\lambda^M} \alpha(X)\phi(X)\, dX = \mathrm{e}^{-\mathrm{i}t d\lambda^M} \hat{\beta}(t\mu).$$

On the support of $1 - \chi$, the function μ satisfies $\|\mu(m)\| \geq \frac{1}{2}\varepsilon > 0$. Thus the differential form $\hat{\beta}(t\mu(m))$ tends rapidly to 0 when $t \mapsto \infty$. The differential form $\mathrm{e}^{-\mathrm{i}t d\lambda^M}$ is polynomial in t so that we obtain our lemma. □

Thus

$$\lim_{t \to \infty} \int_{M_0} \left(\int_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}t d_X \lambda^M} \alpha(X)\phi(X)\, dX \right) = \lim_{t \to \infty} \int_{M_\varepsilon} \chi(m) \left(\int_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}t d_X \lambda^M} \alpha(X)\phi(X)\, dX \right).$$

Let

$$N = P \times \mathfrak{g}^*.$$

We write any element of $N = P \times \mathfrak{g}^*$ as (x, ξ). Let $\omega = \omega^M|_P$. Then ω is a connection form on P. Let

$$\lambda = (\omega, \xi)$$

be the 1-form on $N = P \times \mathfrak{g}^*$ determined by the connection form ω (formula (10)). Choosing ε sufficiently small, we can identify in a G-invariant way M_ε to an open set of $N = P \times \mathfrak{g}^*$, the map μ becoming the second projection $(x, \xi) \mapsto \xi$. This isomorphism is the identity on P. As χ has compact support contained in M_ε, we consider the integral $\int_{M_\varepsilon} \chi(m) (\int_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}d_X \lambda^M} \alpha(X) \phi(X)\, dX)$ as an integral over N. We still write ω^M for the 1-form on N corresponding to ω^M. We have $\omega^M|_P = \omega$. Thus

$$\begin{aligned} &\lim_{t \to \infty} \int_{M_0} \left(\int_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}t d_X \lambda^M} \alpha(X)\phi(X)\, dX \right) \\ &= \lim_{t \to \infty} \int_N \chi(m) \left(\int_{\mathfrak{g}} \mathrm{e}^{\mathrm{i}t(\xi, X)} \mathrm{e}^{\mathrm{i}t(\omega^M, d\xi) - \mathrm{i}t(\xi, d\omega^M)} \alpha(X)\phi(X)\, dX \right). \end{aligned} \tag{24}$$

The differential form $e^{it(\omega^M, d\xi) - it(\xi, d\omega^M)}$ can be written $\sum_k P_k(t\xi, t\,d\xi)\mu_k$ where $P_k(\xi, d\xi)$ is a polynomial in the forms $\xi, d\xi$ while μ_k is a differential form on N independent of t. If $v_k(X) = \chi\mu_k \wedge \alpha(X)\phi(X)$, we need to study the limit when $t \mapsto \infty$ of

$$\int_N \left(\int_{\mathfrak{g}} e^{it(\xi, X)} P_k(t\xi, t\,d\xi) v_k(X)\,dX \right).$$

If $v \in C^\infty_{\text{cpt}}(\mathfrak{g}, \mathscr{A}(N))$ we write $v_0(X) = (v(X)|_P)$. Then $X \mapsto v_0(X)$ is a compactly supported C^∞-function on $\mathfrak{g}$, with values in $\mathscr{A}(P)$. Its Fourier transform $\xi \mapsto \hat{v}_0(\xi)$ is a differential form on P depending smoothly on ξ. We can consider $\hat{v}_0(\xi)$ as a differential form on $N = P \times \mathfrak{g}^*$.

Lemma 21. *Let $G(\xi, d\xi)$ be a polynomial. For any $v \in C^\infty_{\text{cpt}}(\mathfrak{g}, \mathscr{A}_{\text{cpt}}(N))$ we have*

$$\lim_{t\to\infty} \int_N \left(\int_{\mathfrak{g}} e^{it(\xi, X)} G(t\xi, t\,d\xi) v(X)\,dX \right) = \int_N G(\xi, d\xi)\hat{v}_0(\xi).$$

Proof of Lemma 21. For $t > 0$, let us consider the map h_t on $N = P \times \mathfrak{g}^*$ to N given by $h_t(m, \xi) \mapsto (m, t^{-1}\xi)$ for $m \in P$ and $\xi \in \mathfrak{g}^*$. Change of coordinates shows that

$$\int_N \left(\int_{\mathfrak{g}} e^{it(\xi, X)} G(t\xi, t\,d\xi) v(X)\,dX \right) = \int_N G(\xi, d\xi) \left(\int_{\mathfrak{g}} e^{i(\xi, X)} h_t^*(v(X))\,dX \right).$$

We write the differential form $v(X) = v(X, \xi, d\xi, m_i, dm_i)$ for a local system of coordinates m_i on P. Then $h_t^*(v(X)) = v(X, \xi/t, d\xi/t, m_i, dm_i)$. For a smooth compactly supported function $\phi(X, x)$ of several variables we denote by $(F_1\phi)(\xi, x) = \int_{\mathfrak{g}} e^{i(\xi, X)}\phi(X, x)\,dX$ the Fourier transform of ϕ with respect to the first variable X. Then for any integer K, there exists a constant C_K such that $|F_1\phi(\xi, x)| \leq C_K(1 + \|\xi\|^2)^{-K}$ for all x, ξ. We have

$$\int_{\mathfrak{g}} e^{i(\xi, X)} h_t^*(v(X))\,dX = (F_1 v)(\xi, \xi/t, d\xi/t, m_i, dm_i).$$

The function $\xi \mapsto (F_1 v)(\xi, \xi/t, d\xi/t, m_i, dm_i)$ is rapidly decreasing when ξ tends to ∞. Furthermore for any K, there exists a constant C_K independent of t such that the function $\xi \mapsto (F_1 v)(\xi, \xi/t, d\xi/t, m_i, dm_i)$ is bounded by $C_K(1 + \|\xi\|^2)^{-K}$. The function $(F_1 v)(\xi, \xi/t, d\xi/t, m_i, dm_i)$ tends to $(F_1 v)(\xi, 0, 0, m_i, dm_i) = \hat{v}_0(\xi)$ when $t \mapsto \infty$. Thus by dominated convergence

$$\int_N G(\xi, d\xi) \int_{\mathfrak{g}} e^{i(\xi, X)} h_t^*(v(X))\,dX = \int_N G(\xi, d\xi)(F_1 v)(\xi, \xi/t, d\xi/t, m_i, dm_i)$$

tends to $\int_N G(\xi, d\xi)\hat{v}_0(\xi)$. □

Applying Lemma 21 to the study of (24) we obtain, as $\chi|_P = 1$, $\omega^M|_P = \omega$,

$$\lim_{t\to\infty} \int_N \chi(m) \left(\int_{\mathfrak{g}} e^{it(\xi, X)} e^{it(\omega^M, d\xi) - it(\xi, d\omega^M)} \alpha(X)\phi(X)\,dX \right) = \int_N e^{i(\omega, d\xi) - i(\xi, d\omega)} \phi\hat{\alpha}_0(\xi).$$

The last integral is equal to $\int_N \int_{\mathfrak{g}} e^{-id_X\lambda}\alpha_0(X)\phi(X)\,dX$. Thus the limit Θ_0 when $t \to \infty$ of $\Theta(M_0, t)$ exists and

$$\int_{\mathfrak{g}} \Theta_0(X)\phi(X)\,dX = \int_N \int_{\mathfrak{g}} e^{-id_X\lambda}\alpha_0(X)\phi(X)\,dX.$$

We now apply Theorem 18 and obtain Theorem 19. □

Let us give some immediate applications of Theorem 19. Let $\alpha = e^{i\sigma_{\mathfrak{g}}}\beta$ with β a form with polynomial coefficients. Let $\phi(X)$ be a rapidly decreasing function on $\mathfrak{g}$. Then the integral $\int_{\mathfrak{g}} e^{i\sigma_{\mathfrak{g}}(X)}\beta(X)\phi(X)\,dX$ is convergent and defines a form on M. We can thus consider $\int_M \alpha$ as a tempered generalised function. The same estimates show that Theorem 19 is valid for $\int_M \alpha$ in the space of tempered generalised functions: for all $t \in \mathbb{R}$,

$$\int_M \alpha = \Theta(M_0, t) + \Theta(M_{\mathrm{out}}, t)$$

the limit of $\Theta_0 = \Theta(M_0, t)$ exists in the sense of tempered generalised functions and

$$\int_{\mathfrak{g}} \Theta_0(X)\phi(X)\,dX = (2i\pi)^n \int_P \alpha_{\mathrm{red}}\phi(\Omega)v_\omega.$$

Let

$$\phi(X) = \int_{\mathfrak{g}^*} e^{-i(\xi, X)}k(\xi)\,d\xi$$

where $k(\xi)$ is a C^∞-function supported on $\|\xi\| < r < R$. The function ϕ is rapidly decreasing on $\mathfrak{g}$. By definition

$$\int_{\mathfrak{g}}\int_M \alpha(X)\phi(X)\,dX = (2\pi)^n \int_{\mathfrak{g}^*} \mathscr{F}\left(\int_M \alpha\right)(\xi)k(\xi)\,d\xi.$$

We have

$$\int_{\mathfrak{g}} \Theta_0(X)\phi(X)\,dX = (2i\pi)^n \int_P \alpha_{\mathrm{red}} \int_{\mathfrak{g}^*} e^{-i(\Omega, \xi)}k(\xi)\,d\xi.$$

Let us show that $\int_{\mathfrak{g}} \Theta(M_{\mathrm{out}}, t)(X)\phi(X)\,dX$ is equal to 0 for all $t \geq 0$. Indeed

$$\int_{\mathfrak{g}} \Theta(M_{\mathrm{out}}, t)(X)\phi(X)\,dX = \int_{M_{\mathrm{out}}}\int_{\mathfrak{g}} e^{-itd_X\lambda^M} e^{i\sigma_{\mathfrak{g}}(X)}\beta(X)\left(\int_{\mathfrak{g}^*} e^{-i(\xi, X)}k(\xi)\,d\xi\right).$$

We have

$$e^{-id_X\lambda^M} e^{i\sigma_{\mathfrak{g}}(X)} = e^{i(\mu + tf_{\lambda^M}, X)}e^{-itd\lambda^M}e^{i\sigma}.$$

By (22), we have

$$\|\mu + tf_{\lambda^M}\|^2 \geq \|\mu\|^2 + t^2\|f_{\lambda^M}\|^2$$

as $\langle \mu, f_{\lambda^M}\rangle$ is positive. By the double Fourier inversion formula and our hypothesis on the support of k, we see that for every polynomial Q on $\mathfrak{g}$,

$$\int_{\mathfrak{g}} e^{i(\mu + tf_{\lambda^M}, X)}Q(X)\phi(X)\,dX = (2\pi)^n(Q(i\partial_\xi)\cdot k)(tf_{\lambda^M} + \mu) = 0$$

on $M - M_0$ as $\| tf_{\lambda^M} + \mu \| > r$ on $M - M_0$. Thus we obtain from Theorem 19 that for all $t \geq 0$,

$$\int_M \int_{\mathfrak{g}} \alpha(X)\phi(X)\,dX = \int_{\mathfrak{g}} \Theta(M_0,t)(X)\phi(X)\,dX.$$

Taking limits when t tends to $+\infty$, we obtain

$$\int_{\mathfrak{g}^*} \mathscr{F}\left(\int_M \alpha\right)(\xi)k(\xi) = \int_{\mathfrak{g}} \Theta_0(X)\phi(X)\,dX = \mathrm{i}^n \int_{\mathfrak{g}^*}\left(\int_P \alpha_{\mathrm{red}} \mathrm{e}^{-\mathrm{i}(\Omega,\xi)}\right) k(\xi)\,d\xi.$$

This gives another proof of the Jeffrey–Kirwan formula (Theorem 11). Remark that in this proof we obtain immediately that the Jeffrey–Kirwan formula holds on the ball $\| \xi \| < R$, with R equal to the smallest critical value of the function $\| \mu \|^2$ while we had to use some (easy) analyticity arguments in the previous proof.

2.3. The outer term

For further applications to multiplicity formulas, we give a rough analysis of the outer term Θ_{out} in the decomposition $\int_M \alpha = \Theta_0 + \Theta_{\mathrm{out}}$. We consider the generalised function Θ_{out} on $\mathfrak{g}$ given by

$$\Theta_{\mathrm{out}}(X) = \lim_{t\to\infty} \int_{M_{\mathrm{out}}} \alpha(X) \mathrm{e}^{-\mathrm{i}t d_X \lambda^M}.$$

Let us consider the manifold $\tilde{M} = M \times \mathbb{R}$ where G acts trivially on $\mathbb{R}$. We embed M in $M \times \mathbb{R}$ by $m \mapsto (m,0)$. We write (m,t) for an element of $\tilde{M}$. We consider the differential form $\tilde{\lambda}^M = t\lambda^M$ as a differential form on $\tilde{M}$. If α is a form on M we still denote by α its pull-back to $M \times \mathbb{R}$. Let us consider $0 < r < R$ and let

$$P_r = \{ m \in M;\ \| \mu(m) \|^2 = r \}.$$

Let $C \subset \tilde{M}$ be the cyclinder with base P_r:

$$C = P_r \times \mathbb{R}^+.$$

The boundary of C in $M \times \mathbb{R}$ is equal to the boundary of M_{out} both being the manifold P_r. If U is a tubular neighbourhood of P_r in M, we can identify C to the open subset $U - \bar{M}_{\mathrm{out}}$ of M. This gives an orientation o_{out} to C.

Define

$$Z = M_{\mathrm{out}} \cup (C, o_{\mathrm{out}}).$$

Then Z is an oriented cycle in $\tilde{M}$. We can also identify Z to the manifold M_{out} with a cylindrical end C attached to it.

THEOREM 22. *The limit Θ_{out} when $t \to \infty$ of $\Theta(M_{\mathrm{out}},t)$ exists in the space of generalised functions on $\mathfrak{g}$. We have*

$$\Theta_{\mathrm{out}}(X) = \int_{M_{\mathrm{out}} \cup C} \mathrm{e}^{-\mathrm{i}d_X \tilde{\lambda}^M} \alpha(X).$$

Proof. We first give some more explicit expression for Θ_{out}. We have $d_X\tilde{\lambda}^M = dt \wedge \lambda^M + t\,d_X\lambda^M$ and $\mathrm{e}^{-\mathrm{i}d_X \tilde{\lambda}^M} = (1 - \mathrm{i}\,dt \wedge \lambda^M)\mathrm{e}^{-\mathrm{i}t d_X \lambda^M}$. Thus

$$\int_C \mathrm{e}^{-\mathrm{i}d_X \tilde{\lambda}^M} \alpha(X) = -\mathrm{i} \int_{P_r \times \mathbb{R}^+} dt \wedge \lambda^M \mathrm{e}^{-\mathrm{i}t d_X \lambda^M} \alpha(X).$$

As $\tilde{\lambda}^M = 0$ on M,

$$\int_{M_{\text{out}} \cup C} e^{-id_X \tilde{\lambda}^M} \alpha(X) = \int_{M_{\text{out}}} \alpha(X) - i \int_{P_r \times \mathbb{R}^+} dt \wedge \lambda^M e^{-itd_X \lambda^M} \alpha(X).$$

On the other hand, we have

$$\frac{d}{dt} e^{-itd_{\mathfrak{g}} \lambda^M} \alpha = -id_{\mathfrak{g}}(\lambda^M e^{-itd_{\mathfrak{g}} \lambda^M} \alpha).$$

We then obtain

$$e^{-isd_{\mathfrak{g}} \lambda^M} \alpha = \alpha - id_{\mathfrak{g}} \left(\int_0^s \lambda^M e^{-itd_{\mathfrak{g}} \lambda^M} \alpha \, dt \right).$$

Integration over M_{out} and using the Stokes formula leads to

$$\Theta(M_{\text{out}}, s) = \int_{M_{\text{out}}} e^{-isd_X \lambda^M} \alpha(X) = \int_{M_{\text{out}}} \alpha(X) - i \int_{P_r} \left(\int_0^s \lambda^M e^{-itd_X \lambda^M} \alpha(X) \, dt \right).$$

When s tends to ∞, and checking the orientations, we obtain our proposition. □

As $d_X \lambda^M = -\mu(X) + d\lambda^M$ on P_r, we can also explicitly write the integral expression of Θ_{out} on test functions ϕ as follows:

$$\int_{\mathfrak{g}} \Theta_{\text{out}}(X) \phi(X) \, dX = i \int_{P_r \times \mathbb{R}^+} \lambda^M e^{-itd\lambda^M} (\hat{\alpha}\phi)(t\mu(m)) \, dt + \int_{M_{\text{out}}} \int_{\mathfrak{g}} \alpha(X) \phi(X) \, dX.$$

In this integral expression, we see that Θ_{out} is indeed well defined as for $m \in P_r$, $(\hat{\alpha}\phi)(t\mu(m))$ is rapidly decreasing in t (as $\mu(m)$ is never 0 on P_r) while $e^{-itd\lambda^M}$ is polynomial in t.

Remark 2.3. Let $G = S^1$. If $E \in \mathfrak{g}$ is a basis of $\mathfrak{g}$, we denote by

$$M_+ = \{x \in M;\ \mu(E)(x) > r\}, \qquad M_- = \{x \in M;\ \mu(E)(x) < -r\}$$

so that $M_{\text{out}} = M_+ \cup M_-$. It follows from the previous discussions that both

$$\Theta(M_+, t) = \int_{M_+} \alpha(X) e^{-itd_X \lambda^M}$$

and

$$\Theta(M_-, t) = \int_{M_-} \alpha(X) e^{-itd_X \lambda^M}$$

have limits when t tends to ∞.

REFERENCES

1. N. Berline and M. Vergne: Classes caractéristiques équivariantes. Formules de localisation en cohomologie équivariante, *C.R. Acad. Sci. Paris* **295** (1982), 539–541.
2. M. Duflo and M. Vergne: Cohomologie équivariante et descente, *Astérisque* **215** (1993), 5–108.
3. J. J. Duistermaat and G. Heckman: On the variation in the cohomology of the symplectic form of the reduced phase space, *Invent. Math.* **69** (1982), 259–268.
4. V. Guillemin and S. Sternberg: Geometric quantization and multiplicities of group representations, *Invent. Math.* **67** (1982), 515–538.
5. V. Guillemin and S. Sternberg: *Symplectic techniques in physics*, Cambridge Univ. Press, Cambridge (1984).
6. L. C. Jeffrey and F. C. Kirwan: Localization for nonabelian group actions, *Topology* **34** (1995), 291–327.
7. J. Kalkman: Cohomology rings of symplectic quotients, preprint, University Utrecht (1993).

8. F. C. KIRWAN: *Cohomology of quotients in symplectic and algebraic geometry*, Princeton University Press, Princeton, NJ (1985).
9. S. KUMAR and M. VERGNE: Equivariant cohomology with generalised coefficients, *Astérisque* **215** (1993), 109–204.
10. C.-M. MARLE: Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique, *Rend. Sem. Mat. Torino* **43** (1985), 227–251.
11. M. VERGNE: Quantification géométrique et multiplicités, *C. R. Acad. Sci. Paris* **319** (1994), 327–332.
12. E. WITTEN: Two dimensional guage theories revisited, *J. Geom. Phys.* **9** (1992), 303–368.
13. S. WU: An integration formula for the square of moment maps of circle actions, preprint hep-th/921207, to appear in *Lett. Math. Phys.*

E.N.S. et UA 762 du CNRS
DMI, Ecole Normale Supérieure
45 *rue d'Ulm*
75005 *Paris*
France