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MULTIPLICITIES FORMULA FOR GEOMETRIC QUANTIZATION, PART I

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1. Introduction. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ acting on a compact symplectic manifold M by a Hamiltonian action. If $X \in \mathfrak{g}$, we denote by X_M the vector field on M induced by the action of G. We denote by σ the symplectic form on M and by $\mu: M \to \mathfrak{g}^*$ the moment map. To simplify, we will assume in this article that M has a G-invariant spin structure. We will show in the appendix how to remove this assumption.

Let us assume that M is prequantized, and let $\mathscr{L}$ be the Kostant-Souriau line bundle on M. We denote by $R(G)$ the ring of virtual finite-dimensional representations of G. An element of $R(G)$ is thus a difference of two finite-dimensional representations of G. We associate to $(M, \mathscr{L})$ a virtual representation $Q(M, \mathscr{L}) \in R(G)$ of G constructed as follows: Choose a G-invariant Riemannian structure on M. Let $\mathscr{S}^{\pm}$ be the half-spin bundles over M determined by the spin structure and the symplectic orientation of M. Let $\Gamma(M, \mathscr{S}^{\pm} \otimes \mathscr{L})$ be the spaces of smooth sections of $\mathscr{S}^{\pm} \otimes \mathscr{L}$. Consider the twisted Dirac operator

$$D_{\mathscr{L}}^{+}: \Gamma(M, \mathscr{S}^{+} \otimes \mathscr{L}) \to \Gamma(M, \mathscr{S}^{-} \otimes \mathscr{L}).$$

This is an elliptic operator commuting with the action of G. We define a virtual representation $Q(M, \mathscr{L})$ of G by the formula:

$$Q(M, \mathscr{L}) = (-1)^{\dim M/2}([\operatorname{Ker} D_{\mathscr{L}}^{+}] - [\operatorname{Coker} D_{\mathscr{L}}^{+}]).$$

The virtual representation $Q(M, \mathscr{L})$ so obtained is independent of the choice of the Riemannian structure on M. If M and $\mathscr{L}$ have G-invariant complex structure, then $Q(M, \mathscr{L})$ (apart from a shift of parameters) is the direct image of the sheaf $\mathcal{O}(\mathscr{L})$ of holomorphic sections of $\mathscr{L}$ by the map $M \to point$. In the differentiable category, we employ as in Atiyah-Hirzebruch [3] the Dirac operator to define the direct image $Q(M, \mathscr{L}) \in R(G) = K_G(point)$ of $\mathscr{L} \in K_G(M)$. If the group G is trivial, then $Q(M, \mathscr{L}) \in \mathbb{Z}$ is the index of the operator $D_{\mathscr{L}}^{+}$. We call this number the Riemann-Roch number of $(M, \mathscr{L})$.

We are interested in describing the decomposition of $Q(M, \mathscr{L})$ in irreducible representations of G. Let $G = T$ be a torus. Let $P \subset i\mathfrak{t}^*$ be the lattice of weights of T. We have a decomposition

$$Q(M, \mathscr{L}) = \sum_{\xi \in iP} n(\xi, M, \mathscr{L}) e_{i\xi},$$

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where $n(\xi, M, \mathscr{L}) \in \mathbb{Z}$ and is nonzero for a finite number of ξ in the lattice $iP \subset \mathfrak{t}^*$.

The number $n(\xi, M, \mathscr{L})$ is called the multiplicity of the weight $e_{i\xi}$ in $Q(M, \mathscr{L})$. The function $\xi \mapsto n(\xi, M, \mathscr{L})$ defined on the lattice $iP \subset \mathfrak{t}^*$ is the Fourier transform of the character $\operatorname{Tr} Q(M, \mathscr{L})$ of T. It is the quantum analogue of the function over $\mu(M) \subset \mathfrak{t}^*$ equal to the density $\mu_*(\beta_M)$ of the pushforward by μ of the Liouville measure β_M of M. The value of the locally polynomial density $\mu_*(\beta_M)$ on a regular value $\xi \in \mathfrak{t}^*$ of μ is equal to the symplectic volume of the reduced fiber $M_\xi = T\backslash\mu^{-1}(\xi)$ [12]. Let $\xi \in iP \cap \mu(M)$. Assume furthermore that T acts freely on $\mu^{-1}(\xi)$. Then $\mathscr{L}_\xi = T\backslash(\mathscr{L}|_{\mu^{-1}(\xi)})$ is a Kostant-Souriau line bundle on the reduced symplectic manifold M_ξ, and M_ξ inherits a spin structure from M. Guillemin-Sternberg [17] conjectured that the value of $n(\xi, M, \mathscr{L})$ at a regular value $\xi \in iP \cap \mu(M)$ is the Riemann-Roch number $Q(M_\xi, \mathscr{L}_\xi)$. They prove this conjecture in the holomorphic case under some positivity assumptions on $\mathscr{L}$.

Recently, E. Witten [31] suggested a formula relating integrals over $M \times \mathfrak{g}$ of some closed, equivariant cohomology classes $\alpha(X)$ on M and integrals over the reduced fiber M_ξ. Witten's formula has been proved by Jeffrey-Kirwan [19]. We realised [29] that the same idea of Witten can be employed modulo some elaboration to the proof of the formula $n(\xi, M, \mathscr{L}) = Q(M_\xi, \mathscr{L}_\xi)$ for multiplicities. The similarity between Jeffrey-Kirwan-Witten formula and multiplicities formulas was also noticed independently and from a different viewpoint by V. Guillemin [14] and E. Meinrenken [22]. In particular, E. Meinrenken [22] proved the formula $n(\xi, M, \mathscr{L}) = Q(M_\xi, \mathscr{L}_\xi)$, including the case of locally free actions, where the definition of $Q(M_\xi, \mathscr{L}_\xi)$ has to be suitably modified. Note that a simple proof of the formula $n(\xi, M, \mathscr{L}) = Q(M_\xi, \mathscr{L}_\xi)$ is obtained in the case of an S^1-action by Jeffrey-Kirwan [20] using their residue formula.

Our proof relies directly on the universal formula for the character of $Q(M, \mathscr{L})$. Let us explain briefly our technique, which can be applied as well to other index problems. Let us consider an open G-invariant subset U with smooth boundary B of a compact spin manifold M. Let λ be a G-invariant 1-form on M such that the map $\mu_\lambda\colon M \to \mathfrak{g}^*$ defined by $\mu_\lambda(X) = (\lambda, X_M)$ does not vanish at any point of the boundary B. Let $\mathscr{E}$ be a G-equivariant complex vector bundle over M. In [9], the formula of Atiyah-Segal-Singer for the equivariant index $Q(M, \mathscr{E})$ of the twisted Dirac operator $D_\mathscr{E}$ is reformulated in a neighbourhood of the identity of G in terms of the equivariant cohomology of M as follows:

$$\operatorname{Tr} Q(M, \mathscr{E})(\exp X) = (2i\pi)^{-\dim M/2} \int_M \frac{\operatorname{ch}(\mathscr{E})(X)}{J^{1/2}(M)(X)}.$$

Here $\operatorname{ch}(\mathscr{E})(X)$ is the equivariant Chern character of $\mathscr{E}$ and $J^{-1/2}(M)(X)$ is the equivariant $\hat{A}$-genus of M. The class $J(M)(X)$ is invertible for X in a neighbourhood of 0 in $\mathfrak{g}$. Normalisations are as in [13]. Consider for $t \in \mathbb{R}$ the C^∞ functions

$TQ(U, \mathscr{E}, \lambda, t)$ on a neighbourhood of $0 \in \mathfrak{g}$ defined by:

$$TQ(U, \mathscr{E}, \lambda, t)(X) = (2i\pi)^{-\dim M/2} \int_U e^{-itd_{\mathfrak{g}}\lambda(X)} \frac{\mathrm{ch}(\mathscr{E})(X)}{J^{1/2}(M)(X)},$$

where $d_{\mathfrak{g}} = d - \iota(X_M)$ is the equivariant differential.

We conjecture that there exists an (infinite-dimensional) trace-class virtual representation $Q(U, \mathscr{E}, \lambda)$ of G such that we have the identity of generalised functions on a neighbourhood of 0 in $\mathfrak{g}$:

$$\mathrm{Tr}\, Q(U, \mathscr{E}, \lambda)(\exp X) = \lim_{t \mapsto \infty} TQ(U, \mathscr{E}, \lambda, t)(X)$$

(and similar identities near any point $s \in G$).

In this article we prove the following theorem (Theorem 17):

Let T be a torus with Lie algebra $\mathfrak{t}$. Assume that there exists $S \in \mathfrak{t}$ such that $(\lambda, S_M) > 0$ on the boundary B of U. Then there exists a trace-class virtual representation $Q(U, \mathscr{E}, \lambda)$ of T such that

$$\mathrm{Tr}\, Q(U, \mathscr{E}, \lambda)(\exp X) = \lim_{t \mapsto \infty} TQ(U, \mathscr{E}, \lambda, t)(X).$$

Furthermore, the fixed-point formula of Atiyah-Segal-Singer is valid in the generalised sense. Let M^T be the submanifold of fixed points for the action of the torus T. Our hypothesis implies that $M^T \cap B = \varnothing$. Consider the subset $M^T \cap U$. Then we have the fixed-point formula for $g = \exp X$

$$\mathrm{Tr}\, Q(U, \mathscr{E}, \lambda)(g) = \int_{M^T \cap U} \frac{\mathrm{ch}(\mathscr{E})(X)}{J^{1/2}(M^T)} D^{-1}_{0+(S)}(g),$$

where $D(g)$ is a holomorphic function on $T_{\mathbb{C}}$ with value-differential forms on M^T and $D^{-1}_{0+(S)}(g)$ is the boundary value of the holomorphic function $D^{-1}(g)$ on an open subset of $T_{\mathbb{C}}$ determined by S. Thus, the invariant form λ allows us to construct a trace-class virtual representation with character formula given by the Atiyah-Segal-Singer fixed-point formula on U, suitably interpreted as a generalised function.

Let us return to the case of a Hamiltonian manifold M under an action of $G = S^1$. Let us indicate briefly how to write $Q(M, \mathscr{L})$ as a sum of three infinite-dimensional trace-class (virtual) representations related to the geometry of the moment map $\mu: M \to \mathfrak{g}^*$. We consider $0 \in \mathfrak{g}^*$, and we assume G acts freely on $\mu^{-1}(0)$.

Choosing a basis E of $\mathfrak{g}$, we consider the map $f = \mu(E)$ from M to $\mathbb{R}$. Choose r a small, positive real number. Let

$$M_0 = \{x; |f(x)| < r\}, \qquad M_+ = \{x; f(x) > r\}, \qquad M_- = \{x; f(x) < -r\}.$$

Let $(.,.)$ be a G-invariant metric on M, and let $\lambda(\cdot) = \mu(E)(E_M, \cdot)$. Then λ is a G-invariant 1-form on M. The value $\lambda(E_M) = \mu(E)\|E_M\|^2$ is strictly positive on the boundary of M_+ and strictly negative on the boundary of M_-. We can thus construct with the help of the 1-form λ virtual representations $Q_\pm(M, \mathscr{L}) = Q(M_\pm, \mathscr{L}, \lambda)$ with character formula given by Atiyah-Segal-Singer fixed-point formula on $M_\pm$ expanded in the outer directions: it follows that the virtual character $Q_+(M, \mathscr{L})$ is of the form $Q_+(M, \mathscr{L}) = \sum_{n>0} a_n e^{in\theta}$. Similarly, $Q_-(M, \mathscr{L}) = \sum_{n<0} a_n e^{in\theta}$. In particular, neither $Q_+(M, \mathscr{L})$ nor $Q_-(M, \mathscr{L})$ contains the trivial representation of $G = S^1$.

We denote $f^{-1}(0)$ by P. Consider the principal fibration $q: P \to M_{\text{red}} = G\backslash P$ with structure group $G = S^1$. Let $\mathscr{L}_{\text{red}} = G\backslash(\mathscr{L}|_P)$. Consider, for all $n \in \mathbb{Z}$ the characters $\chi_n(\exp \theta E) = e^{in\theta}$ of G, and let $\mathscr{T}_n$ be the associated line bundles on $G\backslash P = M_{\text{red}}$. Then $Q(M_{\text{red}}, \mathscr{L}_{\text{red}} \otimes \mathscr{T}_n)$ is a positive or negative integer. Define the virtual character $Q_0(M, \mathscr{L})$ of G by

$$Q_0(M, \mathscr{L}) = \sum_{n \in \mathbb{Z}} Q(M_{\text{red}}, \mathscr{L}_{\text{red}} \otimes \mathscr{T}_{-n}) e^{in\theta}.$$

In particular, the multiplicity of the trivial representation in $Q_0(M, \mathscr{L})$ is $Q(M_{\text{red}}, \mathscr{L}_{\text{red}})$.

As a corollary of Theorem 17 of this article and of a limit formula à la Witten for integration of equivariant differential forms in a neighbourhood of $f^{-1}(0)$ proved in [28], we obtain the following decomposition of $Q(M, \mathscr{L})$ associated to the partition

$$M = M_0 \cup \overline{M_+} \cup \overline{M_-}$$

of M:

$$Q(M, \mathscr{L}) = Q_0(M, \mathscr{L}) \oplus Q_+(M, \mathscr{L}) \oplus Q_-(M, \mathscr{L}).$$

If a group K commutes with the action of G, then this decomposition is a decomposition of representations of $G \times K$. As a result, the virtual representation $Q(M, \mathscr{L})^G$ of K is the virtual representation $Q(M_{\text{red}}, \mathscr{L}_{\text{red}})$ of K.

To extend the results to the action of a torus, as announced in [29], we can apply successively the decomposition above to analyse the decomposition of $Q(M, \mathscr{L})$ under the action of a torus $T = (S^1)^n$ to obtain $n(\xi, M, \mathscr{L}) = Q(M_\xi, \mathscr{L}_\xi)$ for a regular value ξ. This requires an extension of Theorem 17 to an open subset V of some noncompact manifolds N. The basic example is $T^*G \times M$, where M is a compact $G \times H$ manifold and $V = T^*G \times U$. We will not give the details. Instead, we will give in Part II of this article [30] a proof based directly on a deformation of the Dirac operator itself. This proof, although requiring the machinery of transversally elliptic symbols, requires almost no computation and leads directly to the formula $n(\xi, M, \mathscr{L}) = Q(M_\xi, \mathscr{L}_\xi)$ in the more general case of orbifolds with torus actions. It is parallel to the method we used here. However, we feel worthwhile to publish here a detailed and elementary proof for the case of an S^1-action.

To state the results without spin hypothesis, it is necessary to modify as in [23] the notion of Kostant-Souriau line bundle. This is explained in the appendix and is only a technical modification. However, we believe results are more invariantly stated in terms of quantum bundles as in [27]. Indeed, if τ is a quantum bundle on an even-dimensional, compact oriented manifold M, there is a virtual representation $Q(M, \tau)$ associated to τ. Furthermore, there is a natural map $\tau \mapsto \tau_{\text{red}}$ from quantum bundles on M to quantum bundles on the reduction M_{red}. Our main result on multiplicities is the following.

THEOREM 1. *Let G be a torus. Let K be a compact Lie group. Let M be a $G \times K$ Hamiltonian manifold. Let $\mathscr{L}$ be a Kostant-Souriau quantum line bundle for $G \times K$. Let $Q(M, \mathscr{L}) \in R(G \times K)$ be the quantized space. Let $\mu: M \to \mathfrak{g}^*$ be the moment map for the G-action. Assume that G acts freely on $\mu^{-1}(0)$. Consider the K-Hamiltonian manifold $M_{\text{red}} = G \backslash \mu^{-1}(0)$ with Kostant-Souriau quantum line bundle $\mathscr{L}_{\text{red}}$. Then the virtual representation $Q(M, \mathscr{L})^G$ is isomorphic to the virtual representation $Q(M_{\text{red}}, \mathscr{L}_{\text{red}})$ of K.*

I am thankful to Michel Duflo for discussion on this problem. The inspiration of this work is our common conjecture [27] on universal formula for characters. Indeed, the character $Q(U, \mathscr{E}, \lambda)$ obtained by a limit procedure is the universal character formula for the manifold U with a cylindrical end attached to it.

2. Quantization on compact manifolds. In this section, we recall some well-known facts due mainly to Atiyah-Bott [1], [2], Atiyah-Segal-Singer [5], [4], [6], and Berline-Vergne [9] on the equivariant index of the Dirac operator.

Let G be a compact Lie group acting on a compact even-dimensional orientable manifold M. For simplicity, we assume first that M has a G-invariant spin structure. We will remove this assumption in the appendix. If $\mathscr{E}$ is a vector bundle over M, we denote by $\Gamma(M, \mathscr{E})$ the space of its smooth sections. Let $K_G(M)$ be the Grothendieck group of G-equivariant complex vector bundles over M.

We denote by $R(G)$ the ring of virtual finite-dimensional representations of G. If $V^{\pm}$ are two finite-dimensional representation spaces of G, then $[V] = [V^+] - [V^-]$ is an element of $R(G)$. We denote $\dim V = \dim V^+ - \dim V^-$. If $G = \{1\}$, we identify $R(G)$ to $\mathbb{Z}$ by the function $\dim V$.

Let o be a G-invariant orientation of M. There is a well-defined quantization map

$$Q^o: K_G(M) \to R(G).$$

This map can be constructed as follows: Choose a G-invariant metric on M. Let $\mathscr{S} = \mathscr{S}^+ \oplus \mathscr{S}^-$ be the spinor bundle (conventions on gradings are as in [7]). Let $\mathscr{E}$ be a G-equivariant complex vector bundle over M. Let $\mathscr{S}^{\pm} \otimes \mathscr{E}$ be the twisted spinor bundle. With the help of a G-invariant connection $\mathbb{A}$ on $\mathscr{E}$, we can construct a twisted Dirac operator $D_{\mathscr{E},\mathbb{A}}$. This gives an elliptic operator

$$D^+_{\mathscr{E},\mathbb{A}}: \Gamma(M, \mathscr{S}^+ \otimes \mathscr{E}) \to \Gamma(M, \mathscr{S}^- \otimes \mathscr{E}),$$

which commutes with the natural action of G. The index space of $D^+_{\mathscr{E},\mathbb{A}}$ is by definition the virtual representation of G in $[\operatorname{Ker} D^+_{\mathscr{E},\mathbb{A}}] - [\operatorname{Coker} D^+_{\mathscr{E},\mathbb{A}}]$. The virtual representation of G so obtained is independent of the choice of the metric on M and of the connection $\mathbb{A}$ on $\mathscr{E}$. It depends only on the bundle $\mathscr{E}$ and on the orientation o of M. We define

$$Q^o(M, \mathscr{E}) = (-1)^{\dim M/2}([\operatorname{Ker} D^+_{\mathscr{E},\mathbb{A}}] - [\operatorname{Coker} D^+_{\mathscr{E},\mathbb{A}}]). \tag{1}$$

In particular, if G is reduced to the identity and $\mathscr{E}$ is a bundle over M, then $Q^0(M, \mathscr{E}) \in \mathbb{Z}$ is a number. This number will be called the Riemann-Roch number of the bundle $\mathscr{E}$ over M.

Indeed, the direct-image map Q^o is the C^∞-version of the direct-image map in algebraic geometry. If M is a complex manifold of complex-dimension d, and $\mathscr{E}$ is a holomorphic vector bundle, then the space $Q^o(M, \mathscr{E})$ coincides up to a sign $\varepsilon(o)$ with the virtual space

$$H(M, \mathscr{E} \otimes \rho^*) = \sum_{k=0}^{d} (-1)^k [H^k(M, \mathscr{O}(\mathscr{E} \otimes \rho^*))]$$

of cohomology of the sheaf of holomorphic sections of $\mathscr{E} \otimes \rho^*$, where ρ is the square root of the line bundle of $(n, 0)$-forms. The line bundle ρ exists, from our assumption on existence of spin structure. In particular, if $\mathscr{E}$ is sufficiently positive, the space $Q^o(M, \mathscr{E})$ is up to sign the space of holomorphic sections of $\mathscr{E} \otimes \rho^*$. Our convention on orientations is such that if $\mathscr{L}$ is a sufficiently positive line bundle and $o_{\mathscr{L}}$ the orientation of M induced by the symplectic form determined by the curvature of $\mathscr{L}$, then $Q^{o_{\mathscr{L}}}(M, \mathscr{L}) = H^0(M, \mathscr{L} \otimes \rho^*)$.

Let $\mathfrak{g}$ be the Lie algebra of G. If $X \in \mathfrak{g}$, we denote by X_M the vector field on M produced by the infinitesimal action of $\mathfrak{g}$:

$$(X_M)_x = \frac{d}{d\varepsilon}(\exp - \varepsilon X) \cdot x|_{\varepsilon=0}.$$

A *G-equivariant differential form* on M is a smooth G-equivariant map, defined on the Lie algebra $\mathfrak{g}$, with values in the space $\mathscr{A}(M)$ of smooth differential forms on M. We denote the space of G-equivariant differential forms by $\mathscr{A}^\infty_G(\mathfrak{g}, M) = C^\infty(\mathfrak{g}, \mathscr{A}(M))^G$. Here X will denote a point on $\mathfrak{g}$ or the function $X \mapsto X$. Thus, we may denote a map $\alpha: \mathfrak{g} \to \mathscr{A}(M)$ by the notation $\alpha(X)$. Similar notations will be common for functions on manifolds, where the notation $f(x)$ will denote (depending on the context) either the value of the function f at the point $x \in M$ or the function f itself.

We will consider equivariant differential forms $\alpha(X)$, which are defined only for X belonging to a G-invariant open subset $W \subset \mathfrak{g}$. We denote by $\mathscr{A}_G^\infty(W, M) = C^\infty(W, \mathscr{A}(M))^G$ the space of these forms. An element of $C^\infty(W, \mathscr{A}(M))$ will also be referred to as a differential form on M depending on $X \in W$. The *equivariant coboundary* $d_\mathfrak{g}$: $\mathscr{A}_G^\infty(W, M) \to \mathscr{A}_G^\infty(W, M)$ is defined for $\alpha \in \mathscr{A}_G^\infty(W, M)$ and $X \in W$ by

$$(d_\mathfrak{g}\alpha)(X) = d(\alpha(X)) - \iota(X_M)(\alpha(X)),$$

where $\iota(X_M)$ is the contraction with the vector field X_M.

A closed equivariant form is by definition a G-equivariant differential form α satisfying $d_\mathfrak{g}\alpha = 0$.

We also write d_X for the operator $d - \iota(X_M)$ acting on forms.

Let G be a compact Lie group acting on a symplectic manifold M by a Hamiltonian action. We denote by (M, σ, μ) such a data, with σ the symplectic form on M and μ: $M \to \mathfrak{g}^*$ the moment map. For $X \in \mathfrak{g}$, let

$$X \mapsto \sigma_\mathfrak{g}(X) = \mu(X) + \sigma \tag{2}$$

be the equivariant symplectic form. It is a closed G-equivariant differential form on M.

Let $\mathscr{E}$ be a G-equivariant vector bundle with G-invariant connection $\mathbb{A}$. Let F be the curvature of $\mathbb{A}$. For $X \in \mathfrak{g}$, let $\mu^\mathbb{A}(X) \in \Gamma(M, \mathrm{End}(\mathscr{E}))$ be the moment of X with respect to the connection $\mathbb{A}$ [7]. Let $X \mapsto F(X) = \mu^\mathbb{A}(X) + F (X \in \mathfrak{g})$ be its equivariant curvature. Then $\mathrm{ch}(\mathscr{E}, \mathbb{A})(X) = \mathrm{Tr}(e^{F(X)})$ is a closed G-equivariant differential form on M called the equivariant Chern character. For $X = 0$, we denote $\mathrm{ch}(\mathscr{E}, \mathbb{A})(0) = \mathrm{Tr}(e^F)$ simply by $\mathrm{ch}(\mathscr{E})$ leaving implicit the choice of connection. The form $\mathrm{ch}(\mathscr{E})$ is up to normalisation factors of 2π the usual Chern character.

Definition 2 (Kostant-Souriau) [21], [25]. The Hamiltonian manifold (M, σ, μ) is said to be quantizable, if there exists a G-equivariant line bundle $\mathscr{L} \to M$ with G-invariant connection $\mathbb{A}$ such that the equivariant curvature of $\mathscr{L}$ is $i\sigma_\mathfrak{g}(X)$. Such a line bundle will be called a Kostant-Souriau line bundle.

Thus, if $(\mathscr{L}, \mathbb{A})$ is a Kostant-Souriau line bundle, we have for $X \in \mathfrak{g}$,

$$\mathrm{ch}(\mathscr{L}, \mathbb{A})(X) = e^{i\sigma_\mathfrak{g}(X)}. \tag{3}$$

If G acts on a set E, we will denote by E^G the set of fixed points of G in E. If $s \in G$, we denote by $E(s)$ the subset of E fixed by s. We denote by $G(s)$ the centraliser of s in G. If $s \in G$, the set $M(s)$ is a submanifold of M. We denote by TM the tangent bundle to M. If N is a closed submanifold of M, we denote by $T_N M = TM|_N/TN$ the normal bundle to N in M. If $S \in \mathfrak{g}$, we denote by $M(S) = \{m \in M; (S_M)_m = 0\}$ the manifold of zeroes of the vector field S_M.

Recall the localisation formula for G-equivariant differential forms with compact support on an oriented G-manifold M. Let $S \in \mathfrak{g}$ and consider the zero set $M(S)$. We choose a $G(S)$-invariant connection on $T_{M(S)}M$, and we denote by $R(T_{M(S)}M)$ its $G(S)$-equivariant curvature. Choose an orientation o on $T_{M(S)}M$. We denote by $\mathrm{Eul}_o(T_{M(S)}M)$ the $G(S)$-equivariant Euler form of $T_{M(S)}M$. We have for $Y \in \mathfrak{g}(S)$

$$\mathrm{Eul}_o(T_{M(S)}M)(Y) = (-2\pi)^{-\mathrm{rank}(T_{M(S)}M)/2} \det{}_o^{1/2} R(T_{M(S)}M)(Y).$$

Let $\mathscr{W}$ be a $G(S)$-invariant neighbourhood of S in $\mathfrak{g}(S)$. Let $\alpha \in \mathscr{A}^{\infty}_{G(S)}(\mathscr{W}, M)$ such that $\alpha(X) \in \mathscr{A}_{cpt}(M)$ for every $X \in \mathscr{W}$. We suppose that α is a closed $G(S)$-equivariant form on M. The form $\mathrm{Eul}_o(T_{M(S)}M)(Y)_x$ is invertible for Y sufficiently near S and x in the compact support of α. Then for $Y \in \mathfrak{g}(S)$ sufficiently close to S, we have [8], [10]; see also [7, Chapter 7]

$$\int_M \alpha(Y) = \int_{M(S)} \frac{\alpha(Y)}{\mathrm{Eul}_o(T_{M(S)}M)(Y)}. \tag{4}$$

Here the orientations of M, $M(S)$ and the orientation o on $T_{M(S)}M$ are chosen in a compatible way.

Assume now that $\alpha(Y)$ depends holomorphically on Y, for Y belonging to an open set W in $\mathfrak{g}(S)_{\mathbb{C}}$. Let U be the open subset of $\mathfrak{g}(S)_{\mathbb{C}}$ consisting of those Y such that $\mathrm{Eul}_o(T_{M(S)}M)(Y)$ is invertible. Let W' be the connected component of $W \cap U$ containing S. By analytic continuation, we then have, for all $Y \in W'$,

$$\int_M \alpha(Y) = \int_{M(S)} \frac{\alpha(Y)}{\mathrm{Eul}_o(T_{M(S)}M)(Y)}. \tag{5}$$

For example, if α is analytic in a small ball $W \subset \mathfrak{g}(S)_{\mathbb{C}}$, we can apply Formula 5 to $Y = zS$, with z a nonzero complex number of small norm.

A bouquet of equivariant-differential forms on M is a family α_s, where, for each $s \in G$, the form α_s is a $G(s)$-closed equivariant-differential form on $M(s)$ satisfying the following conditions of invariance and compatibility [13], [11]; see also [27].

1. Invariance:

$$\alpha_{gsg^{-1}} = g \cdot \alpha_s$$

for all $g \in G$ and $s \in G$.

2. Compatibility: Let $s \in G$; then for all $S \in \mathfrak{g}(s)$ and *sufficiently small*

$$\alpha_{se^S}(Y) = \alpha_s(S + Y)|M(se^S)$$

for all $Y \in \mathfrak{g}(se^S)$.

Remark 2.1. If $S \in \mathfrak{g}(s)$ is sufficiently small, then $M(se^S) = M(s) \cap M(S)$ and $\mathfrak{g}(se^S) = \mathfrak{g}(s) \cap \mathfrak{g}(S)$ so that the right-hand side of the equality (2) has a meaning.

Recall the definition of the bouquet $\mathrm{bch}(\mathscr{E}, \mathbb{A})$ of Chern characters of a G-equivariant vector bundle with G-invariant connection $\mathbb{A}$ [13], [11]; see also [27]. By definition, $\mathrm{bch}(\mathscr{E}, \mathbb{A}) = (\mathrm{ch}_s(\mathscr{E}, \mathbb{A}))_{s \in G}$, where

$$\mathrm{ch}_s(\mathscr{E}, \mathbb{A})(X) = \mathrm{Tr}(s^{\mathscr{E}} e^{F(X)|_{M(s)}}) \quad \text{for } X \in \mathfrak{g}(s). \tag{6}$$

If λ is a G-invariant 1-form, then $\mathbb{A}(t) = \mathbb{A} - it\lambda I$ is also a G-invariant connection for all $t \in \mathbb{R}$. We have

$$\mathrm{ch}_s(\mathscr{E}, \mathbb{A}(t)) = e^{-itd_{\mathfrak{g}(s)}\lambda|_{M(s)}} \wedge \mathrm{ch}_s(\mathscr{E}, \mathbb{A}). \tag{7}$$

Let us give a formula for $\mathrm{Tr}\, Q^o(M, \mathscr{E})$ in the neighbourhood of $s \in G$ in terms of the equivariant cohomology of $M(s)$. As M is oriented and has a G-invariant spin structure, the submanifolds $M(s)$ of M are orientable; see, for example, [7] or [27]. To explain the formula for $\mathrm{Tr}\, Q^o(M, \mathscr{E})$ in the neighbourhood of $s \in G$, we recall the definitions of some other equivariant differential forms.

Let s be an orthogonal transformation of a Euclidean vector space V_1 such that $(1 - s)$ is invertible. In particular, $\det_{V_1}(1 - s) > 0$. Let $\mathfrak{so}(V_1)(s)$ be the space of orthogonal transformations of V_1 commuting with s. Define for $Y \in \mathfrak{so}(V_1)(s)$

$$D_s(V_1)(Y) = \det{}_{V_1}(1 - se^Y). \tag{8}$$

If V_1 is understood, we write $D_s(V_1) = D_s$. The function D_s has an analytic square root on $(\mathfrak{so}(V_1)(s))_{\mathbb{C}}$. We normalise it by $D_s^{1/2}(0) > 0$.

If V_1 is 2-dimensional with (e_1, e_2) as orthonormal basis, and $s \cdot (e_1 + ie_2) = e^{i\theta}(e_1 + ie_2)$ with θ not in $2\pi\mathbb{Z}$, then

$$D_s^{1/2}(0) = 2|\sin(\theta/2)|. \tag{9}$$

Let ∇ be a G-equivariant Euclidean connection on TM. Then ∇ determines Euclidean connections ∇_0 on $TM(s)$ and ∇_1 on $T_{M(s)}M$. Let $R_0(X)$, $R_1(X)$ be the equivariant curvatures of ∇_0 and ∇_1. Then, we define the $G(s)$-equivariant form $J(M(s), \nabla)$ on $M(s)$ by

$$J(M(s), \nabla)(X) = \det\left(\frac{e^{R_0(X)/2} - e^{-R_0(X)/2}}{R_0(X)}\right) \tag{10}$$

for $X \in \mathfrak{g}(s)$. When ∇ is understood, we write the equivariant differential form $J(M(s), \nabla)$ simply as $J(M(s))$.

For X in a small neighbourhood of zero in the complexification of $\mathfrak{g}(s)$, $J^{1/2}(M(s), \nabla)(X)$ is analytic and invertible.

Let us still denote by s the transformation of $T_{M(s)}M$ determined by s. Then, at each point $x \in M(s)$, the transformation s is an orthogonal transformation of $(T_{M(s)}M)_x$ and does not have any eigenvalue equal to 1. We define for $X \in \mathfrak{g}(s)$:

$$(11) \qquad D_s(T_{M(s)}M, \nabla)(X) = \det(1 - se^{R_1(X)}).$$

When ∇ is understood, we write the equivariant differential form $D_s(T_{M(s)}M, \nabla)$ simply as $D_s(T_{M(s)}M)$.

We denote by $L(s, \mathscr{E}, \mathbb{A})$ the $G(s)$-equivariantly closed form on $M(s)$ defined on a sufficiently small neighbourhood of zero in $\mathfrak{g}(s)_{\mathbb{C}}$ by

$$(12) \quad L(s, \mathscr{E}, \mathbb{A})(X) = (2\pi)^{-\dim M(s)/2} \operatorname{ch}_s(\mathscr{E}, \mathbb{A})(X) J^{-1/2}(M(s))(X) D_s^{-1/2}(T_{M(s)}M)(X).$$

Let o be an orientation of M and o' an orientation of $M(s)$. The action of s on the spin bundle $\mathscr{S}$ determines a sign $\varepsilon(s, o, o')$ which is a locally constant function on $M(s)$. If S is small, then $M(e^S) = M(S)$. The element S determines an orientation o_S of the normal bundle $T_{M(S)}M$ and $\varepsilon(e^S, o, o') = 1$ in the case where the orientations o, o', o_{-S} are compatible. The convention for o_S is as in [13].

THEOREM 3. *Let $\mathscr{E}$ be a G-equivariant vector bundle over an even-dimensional compact spin manifold M. Choose a G-invariant connection $\mathbb{A}$ on $\mathscr{E}$. Then, for each $s \in G$, there exists a neighbourhood $U_s(0)$ of 0 in $\mathfrak{g}(s)$ such that we have*

$$\operatorname{Tr} Q^o(M, \mathscr{E})(s \exp X)$$
$$= i^{-\dim M/2} \int_{M(s), o'} (2\pi)^{-\dim M(s)/2} \frac{\varepsilon(s, o, o') \operatorname{ch}_s(\mathscr{E}, \mathbb{A})(X)}{J^{1/2}(M(s))(X) D_s^{1/2}(T_{M(s)}M)(X)}$$

for every $X \in U_s(0)$.

We also write this formula

$$(13) \qquad \operatorname{Tr} Q^o(M, \mathscr{E})(s \exp X) = i^{-\dim M/2} \int_{M(s), o'} \varepsilon(s, o, o') L(s, \mathscr{E}, \mathbb{A})(X)$$

with $L(s, \mathscr{E}, \mathbb{A})$ given by (12). The formula (13) determines $\operatorname{Tr} Q^o(M, \mathscr{E})$ in a neighbourhood of $s \in G$.

Remark 2.2. Let us denote by $\mathscr{L}_G(M)$ the space of bouquets of equivariant differential forms. We have defined in [13] (see also [27]) a direct image (or bouquet-integral) map

$$\int_b^M : \mathscr{L}_G(M) \to C^\infty(G)^G.$$

We can restate the formula of Theorem 3 for the equivariant index of the twisted Dirac operator in function of the bouquet integration as follows

$$\operatorname{Tr} Q^o(M, \mathscr{E}) = i^{-\dim M/2} \int_b^M \operatorname{bch}(\mathscr{E}, \mathbb{A}).$$

Let $g = se^S$ with $S \in \mathfrak{g}(s)$ small. Let $Y \in \mathfrak{g}(s) \cap \mathfrak{g}(S)$. Then the two integral expressions for $\operatorname{Tr} Q^o(M, \mathscr{E})(se^{(S+Y)}) = \operatorname{Tr} Q^o(M, \mathscr{E})(se^S e^Y)$, either as an integral formula over $M(s)$ (formula for the s-part) or over $M(s) \cap M(S)$ (formula for the se^S-part), agree, as follows from the bouquet condition on $\operatorname{ch}_s(\mathscr{E}, \mathbb{A})$ and the localisation formula. For example, the two formulas for $\operatorname{Tr} Q^o(M, \mathscr{E})(e^{(S+Y)}) = \operatorname{Tr} Q^o(M, \mathscr{E})(e^S e^Y)$ coincide: for $s = e^S$ with S small, we have $M(s) = M(S)$ $G(s) = G(S)$ and the following relation between the $G(S)$-equivariant forms over $M(S)$:

$$(14) \qquad (2\pi)^{-\dim M/2} \frac{\operatorname{ch}(\mathscr{E}, \mathbb{A})(S + Y)|_{M(S)}}{J^{1/2}(M)|_{M(S)}(S + Y) \operatorname{Eul}_{o_{-S}}(T_{M(S)}M)(S + Y)}$$

$$= (2\pi)^{-\dim M(S)/2} \frac{\operatorname{ch}_s(\mathscr{E}, \mathbb{A})(Y)}{J^{1/2}(M(S))(Y) D_s^{1/2}(T_{M(S)}M)(Y)}$$

for all $Y \in \mathfrak{g}(S)$ sufficiently small.

Taking $X = 0$ in Formula (13), we obtain the Atiyah-Segal-Singer formula for the equivariant index of the Dirac operator.

$$(15) \quad \operatorname{Tr} Q^o(M, \mathscr{E})(s) = i^{-\dim M/2} \int_{M(s), o'} (2\pi)^{-\dim M(s)/2} \frac{\varepsilon(s, o, o') \operatorname{ch}_s(\mathscr{E}, \mathbb{A})(0)}{J^{1/2}(M(s))(0) D_s^{1/2}(T_{M(s)}M)(0)}.$$

When $G = \{1\}$, we identify the index $Q^o(M, \mathscr{E})$ with the natural number $\operatorname{Tr} Q^o(M, \mathscr{E})(1) = \dim Q^o(M, \mathscr{E})$. We have

$$(16) \qquad Q^o(M, \mathscr{E}) = (2i\pi)^{-\dim M/2} \int_M \operatorname{ch}(\mathscr{E}) J^{-1/2}(M).$$

We have denoted $J^{-1/2}(M)(0)$ simply by $J^{-1/2}(M)$. The form $J^{-1/2}(M)$ is a characteristic form on M which coincides up to normalisation factors of 2π with the $\hat{A}$-genus.

When (M, σ, μ) is a Hamiltonian manifold and $\mathscr{L}$ a Kostant-Souriau line bundle on M, the virtual representation $Q^o(M, \mathscr{L})$ is then the quantized space of the manifold (M, σ, μ). Here the orientation o will always be the symplectic orientation, and we will sometimes omit o in our notation. If the group G is connected, the bouquet of Chern characters $\operatorname{bch}(\mathscr{L}, \mathbb{A})$ is entirely determined by (σ, μ); thus, we could write $Q^o(M, \mathscr{L}) = Q(M, \sigma, \mu)$. The (virtual) representation $Q(M, \sigma, \mu)$ has indeed some deep relations with the original symplectic space (M, σ). For example, if the $\hat{A}$-genus of the manifold M is equal to 1 (as is the case when M is a regular coadjoint orbit of G), then the dimension of $Q(M, \sigma, \mu)$ is equal to the symplectic volume of M. In particular, this volume is an integer. One of the aims of this article is to describe the decomposition of $Q(M, \sigma, \mu)$ in function of the moment map μ.

Of course, Atiyah-Singer-Segal's pointwise formula (15) for $\operatorname{Tr} Q^o(M, \mathscr{E})$ determines it. However, as the dependence of s on the set $M(s)$ is quite chaotic, it is difficult to employ directly this formula for the geometric study of multiplicities. To compute, for example, the multiplicity of the trivial representation of G, we have to compute $\int_G \operatorname{Tr} Q^o(M, \mathscr{E})(s)\, ds$. The equivariant-index formula has a better behaviour: the dependence in X on the integral formula for $\operatorname{Tr} Q^o(M, \mathscr{E})(se^X)$ is C^∞ in X. However, there are still some difficulties, as it seems not possible to give a unique global integral formula valid on G and with C^∞-dependence on s (the formula given above is only valid in a neighbourhood of each point $s \in G$).

Consider the particular case of a manifold M with a trivial action of a torus T.

Let T be a torus. Let $\mathfrak{t}$ be the Lie algebra of T. Let $P \subset i\mathfrak{t}^*$ be the set of differentials of unitary characters of T. We will call an element of P a weight of T. If $\xi \in P$, we denote by $e_\xi \in \hat{T}$ the corresponding character of T. For $X \in \mathfrak{t}$, we have $e_\xi(\exp X) = e^{(\xi, X)}$. Then $R(T)$ is the free $\mathbb{Z}$-module with basis e_ξ, $\xi \in P$. Let $\mathscr{E}$ be a T-equivariant vector bundle over M. The vector bundle $\mathscr{E}$ is a sum of its subbundles $\mathscr{E}_\xi$ such that T acts on $\mathscr{E}_\xi$ by e_ξ. Let $\mathbb{A}_\xi$ be the connection induced by the T-invariant connection $\mathbb{A}$ on $\mathscr{E}_\xi$. Thus, the function $s \mapsto \operatorname{ch}_s(\mathscr{E}, \mathbb{A})(0) = \operatorname{Tr}(se^F) = \sum_\xi e_\xi(s) \operatorname{ch}(\mathscr{E}_\xi, \mathbb{A}_\xi)(0)$ is a regular function on T with value differential forms on M. We denote it by $\operatorname{ch}(\mathscr{E})(s)$, leaving implicit the choice of $\mathbb{A}$. The equivariant index $Q^o(M, \mathscr{E})$ of $\mathscr{E}$ is the element of $R(T)$ such that for $s \in T$

$$(17) \qquad \operatorname{Tr} Q^o(M, \mathscr{E})(s) = (2i\pi)^{-\dim M/2} \int_M \operatorname{ch}(\mathscr{E})(s) J^{-1/2}(M).$$

In particular, the set of weights appearing in the virtual representation $Q^o(M, \mathscr{E})$ is contained in the set of weights ξ such that $\mathscr{E}_\xi$ is nonzero.

Let G be a compact, connected Lie group. Atiyah-Segal-Singer's formula (15) gives us a formula similar to Hermann Weyl's formula for the character of $Q^o(M, \mathscr{E})$. Let T be the maximal torus of G.

We have a decomposition

$$\operatorname{Tr} Q^o(M, \mathscr{E})(s) = \sum_{\xi \in P} n(\xi, M, \mathscr{E}) e_\xi(s)$$

where $n(\xi, M, \mathscr{E}) \in \mathbb{Z}$ and is nonzero only for a finite number of ξ. The number $n(\xi, M, \mathscr{E})$ is called the multiplicity of the weight e_ξ in $Q^o(M, \mathscr{E})$.

Let M^T be the set of fixed points of the action of T in M. Let o' be an orientation of M^T. Let $\mathcal{N}$ be the normal bundle of M^T in M. The formula

$$(18)\quad \operatorname{Tr} Q^o(M,\mathscr{E})(s) = i^{-\dim M/2}\int_{M^T,o'}(2\pi)^{-(\dim M^T)/2}\frac{\varepsilon(s,o,o')\operatorname{ch}_s(\mathscr{E},\mathbb{A})(0)}{J^{1/2}(M^T)(0)D_s^{1/2}(\mathcal{N})(0)}$$

is valid for the dense set of $s\in T$ such that $M(s)=M^T$.

Let $\mathscr{F}$ be the set of connected components of M^T. For $a\in\mathscr{F}$, we denote by M_a the corresponding connected component of M^T. We denote by $\mathscr{E}_a$ the restriction of the vector bundle $\mathscr{E}$ on M_a. Let $\mathcal{N}_a$ be the normal bundle $T_{M_a}M$.

Definition 4. We say that s is a-regular if $\det_{\mathcal{N}_a}(1-s)\neq 0$.

We denote by $T_{\text{reg},a}$ the set of a-regular elements of T. We fix an orientation o_a on M_a. We consider the part of Formula (18) for $\operatorname{Tr} Q^o(M,\mathscr{E})$ coming from integration on the connected component M_a of M^T.

Definition 5. Let $a\in\mathscr{F}$. Let $\Theta_a^o(M,\mathscr{E})$ be the function on $T_{a,\text{reg}}$ given by

$$(19)\quad \Theta_a^0(M,\mathscr{E})(s) = i^{-\dim M/2}\int_{M_a,o_a}(2\pi)^{-(\dim M_a)/2}\frac{\varepsilon(s,o,o_a)\operatorname{ch}_s(\mathscr{E}_a,\mathbb{A})(0)}{J^{1/2}(M_a)(0)D_s^{1/2}(\mathcal{N}_a)(0)}.$$

Remark 2.3. $\Theta_a^o(M,\mathscr{E})(s)$ does not depend of the choice of o_a.

LEMMA 6. *The function $\Theta_a^o(M,\mathscr{E})$ is the restriction to $T_{\text{reg},a}$ of a rational function on T.*

Proof. We need to analyse the behaviour in s of the terms of the above integral formula for $\Theta_a^o(M,\mathscr{E})$. Recall that we have chosen G-invariant connections ∇ on TM and $\mathbb{A}$ on $\mathscr{E}$. The function $s\to\operatorname{ch}_s(\mathscr{E}_a,\mathbb{A})(0)=\operatorname{ch}(\mathscr{E}_a)(s)$ is a regular function on T with value-differential forms on M_a.

Let us describe the function $s\mapsto D_s^{1/2}(\mathcal{N}_a)(0)$. For $x\in M_a$, the representation of T on the vector space $(\mathcal{N}_a)_x\otimes_{\mathbb{R}}\mathbb{C}$ breaks up into nonzero weights. These weights and their multiplicities are independent of the point $x\in M_a$. We denote by Δ_a the set of weights α appearing in the action of T in $(\mathcal{N}_a)_x\otimes_{\mathbb{R}}\mathbb{C}$ for some $x\in M_a$. They are all nonzero weights. For $\alpha\in\Delta_a$, let $\mathcal{N}_a(\alpha)$ be the subbundle of $\mathcal{N}_a\otimes_{\mathbb{R}}\mathbb{C}$, where T acts by a multiple of the weight e_α. Let $n_a(\alpha)$ be the rank of $\mathcal{N}_a(\alpha)$. The set $T_{a,\text{reg}}$ is the set of elements $t\in T$ such that $(1-e_\alpha(t))\neq 0$ for all $\alpha\in\Delta_a$. Let $R_{a,\alpha}$ be the curvature of the connection determined by ∇ on $\mathcal{N}_a(\alpha)$.

Weights of T on $\mathcal{N}_a$ appear in pairs $\pm\alpha_k$. If $S_a\in\mathfrak{t}$ is such that $i\alpha(S_a)\neq 0$ for all $\alpha\in\Delta_a$, then we define

$$(20)\quad \Delta_a^+(S_a)=\{\alpha\in\Delta_a;\ i\alpha(S_a)<0\}.$$

Definition 7. A subset Δ_a^+ of Δ_a such that there exists $S_a\in\mathfrak{t}$ with $\Delta_a^+=\Delta_a^+(S_a)$ is called a positive system.

Then $\Delta_a = \Delta_a^+ \cup (-\Delta_a^+)$. For a positive system, we denote

$$C(\Delta_a^+) = \{S \in \mathfrak{t};\, i\alpha(S) < 0, \quad \text{for all } \alpha \in \Delta_a^+\}.$$

We fix a positive system Δ_a^+ and define

$$\mathcal{N}_a^+ = \oplus_{\alpha \in \Delta_a^+} \mathcal{N}_a(\alpha). \tag{21}$$

We denote by R_a^+ the curvature of the connection determined by ∇ on $\mathcal{N}_a^+$.

The fact that the representation of T on M lifts to the spin bundle implies that there exists a T-equivariant line bundle $\mathcal{L}_a^+$ over M_a such that

$$(\mathcal{L}_a^+)^2 = \Lambda^{\max}\mathcal{N}_a^+.$$

Let

$$\rho_a^+ = \frac{1}{2}\sum_{\alpha \in \Delta_a^+} n_a(\alpha)\alpha. \tag{22}$$

Then ρ_a^+ is a weight of T.

Let $r_a^+ = (1/2)\sum_{\alpha \in \Delta_a^+} \mathrm{Tr}_{\mathcal{N}_a(\alpha)} R_{a,\alpha}$. Then r_a^+ is a 2-form on M_a. We have

$$\mathrm{ch}(\mathcal{L}_a^+)(s) = e_{\rho_a^+}(s)e^{r_a^+}.$$

We have (see Formula (9))

$$D_s^{1/2}(\mathcal{N}_a)(0) = i^{n_a^+} e_{-\rho_a^+}(s)e^{-r_a^+}\varepsilon(s, \Delta_a^+)\det{}_{\mathcal{N}_a^+}(1 - se^{R_a^+}),$$

where $\varepsilon(s, \Delta_a^+)$ is a sign. The system Δ_a^+ determines an orientation on $\mathcal{N}_a$: we choose as oriented basis $e_1, e_2, e_3, e_4, \ldots$ a basis such that $e_1 - ie_2, e_3 - ie_4, \ldots$ is a basis of $\mathcal{N}_a^+$. Define

$$\varepsilon(o, o_a, \Delta_a^+) = \pm 1 \tag{23}$$

according to the cases where the orientations on $\mathcal{N}_a$ given by Δ_a^+ and o/o_a coincide or not. We have $\varepsilon(s, o, o_a)\varepsilon(s, \Delta_a^+) = \varepsilon(o, o_a, \Delta_a^+)$. Thus,

$$\varepsilon(s, o, o_a)D_s^{1/2}(\mathcal{N}_a)(0) = \varepsilon(o, o_a, \Delta_a^+)i^{n_a^+}e_{-\rho_a^+}(s)e^{-r_a^+}\det{}_{\mathcal{N}_a^+}(1 - se^{R_a^+})$$

and

(24)
$$\Theta_a^o(M, \mathcal{E})(s) = (2i\pi)^{-\dim M_a}\int_{M_a, o_a} \varepsilon(o, o_a, \Delta_a^+)\frac{\mathrm{ch}(\mathcal{E}_a \otimes \mathcal{L}_a^+)(s)}{J^{1/2}(M_a)}\det{}^{-1}_{\mathcal{N}_a^+}(1 - se^{R_a^+}).$$

Thus, we see on this formula that $\Theta_a^o(M, \mathscr{E})(s)$ is a rational function of s on $T_{\mathrm{reg},a}$. We have used a positive system to give a rational expression. However, $\Theta_a^o(M, \mathscr{E})(s)$ is independent of the choice of positive system. ■

Let us say that $s \in T$ is M-regular if s is a-regular (Definition 4) for all $a \in \mathscr{F}$.

LEMMA 8. *For an arbitrary choice of positive systems Δ_a^+ of Δ_a, we have the equality*

$$\mathrm{Tr}\, Q^o(M, \mathscr{E})(s) = \sum_{a \in \mathscr{F}} \Theta_a^o(M, \mathscr{E})(s)$$

for any M-regular element $s \in T$.

Proof. The character of the finite-dimensional virtual representation $Q^o(M, \mathscr{E})$ is an analytic function on T. The preceding formula holds for all $s \in T$ such that $M(s) = M^T$. By analyticity, it holds for all M-regular elements of T. ■

A choice of positive system Δ_a^+ determines a natural extension of the function $\Theta_a^o(M, \mathscr{E})$ defined on $T_{a,\mathrm{reg}}$ as a generalised function on T.

Definition 9. We denote by $R^{-\infty}(T)$ the set of generalised characters of T. An element $\theta \in R^{-\infty}(T)$ is a sum $\sum n_\xi e_\xi$ of characters of T with coefficients n_ξ in $\mathbb{Z}$ and such that the coefficients n_ξ have, at most, polynomial growth. The support of θ is the set of $\xi \in P$ such that $n_\xi \neq 0$. For $\theta \in R^{-\infty}(T)$, we denote by $\mathrm{Tr}\,\theta$ the generalised function

$$\mathrm{Tr}\,\theta(t) = \sum_\xi n_\xi e_\xi(t).$$

Let Δ_a^+ be a choice of positive systems for Δ_a. Let $\mathscr{N}_a^+$ be the T-equivariant vector bundle given by Formula (21). Let $S(\mathscr{N}_a^+) = \oplus_0^\infty S^m(\mathscr{N}_a^+)$ be the series of complex, finite-dimensional vector bundles obtained from the symmetric powers of $\mathscr{N}_a^+$. If $\mathscr{H}$ is a T-equivariant bundle on M_a, then the equivariant index $Q^{o_a}(M_a, \mathscr{H})$ is an element of $R(T)$.

Definition 10. Define the series of characters of T:

$$A_a^o(M, \mathscr{E}, \Delta_a^+) = \varepsilon(o, o_a, \Delta_a^+) \sum_{k=0}^{\infty} Q^{o_a}(M_a, \mathscr{E}_a \otimes S^k(\mathscr{N}_a^+) \otimes \mathscr{L}_a^+).$$

It is easy to see that $A_a^o(M, \mathscr{E}, \Delta_a^+)$ is in $R^{-\infty}(T)$.

PROPOSITION 11. *For $s \in T_{a,\mathrm{reg}}$, we have*

$$\Theta_a^o(M, \mathscr{E})(s) = \mathrm{Tr}(A_a^o(M, \mathscr{E}, \Delta_a^+))(s).$$

Proof. Using Formula (24), this formula is a consequence of the formula $\det_{\mathscr{N}_a^+}(1 - se^{R_a^+})\,\mathrm{Tr}_{S(\mathscr{N}_a^+)}(se^{R_a^+}) = 1$ and of the index formula for a manifold with trivial T-action (Formula (17)). ■

Let $T_{\mathbb{C}} = (\mathbb{C}^*)^{\dim T}$ be the complexification of T. Elements of $R(T)$ extend on holomorphic functions on $T_{\mathbb{C}}$. Let

$$T_{\mathbb{C}}(\Delta_a^+) = \{g = \exp(X + iY); X \in \mathfrak{t}, Y \in C(\Delta_a^+)\}.$$

Remark that for $g \in T_{\mathbb{C}}(\Delta_a^+)$, then $\det_{\mathcal{N}_a^+}^{-1}(1 - ge^{R_a^+})$ is not 0, as $(1 - e_\alpha(g)) \neq 0$ for all $\alpha \in \Delta_a$.

LEMMA 12. *The generalised function* $\operatorname{Tr} A_a^o(M, \mathscr{E}, \Delta_a^+)$ *is the boundary value of a holomorphic function* Ψ_a^o *on* $T_{\mathbb{C}}(\Delta_a^+)$. *We have for* $g \in T_{\mathbb{C}}(\Delta_a^+)$:

$$\Psi_a^o(g) = (2i\pi)^{-\dim M_a} \int_{M_a, o_a} \varepsilon(o, o_a, \Delta_a^+) \frac{\operatorname{ch}(\mathscr{E}_a \otimes \mathscr{L}_a^+)(g)}{J^{1/2}(M_a)} \det{}_{\mathcal{N}_a^+}^{-1}(1 - ge^{R_a^+}).$$

Proof. For each k, the function $g \mapsto \operatorname{Tr} Q^{o_a}(M_a, \mathscr{E}_a \otimes S^k(\mathcal{N}_a^+) \otimes \mathscr{L}_a^+)(g)$ extends holomorphically to $T_{\mathbb{C}}$. Furthermore, the series

$$\sum_{k=0}^{\infty} \operatorname{Tr} Q^{o_a}(M_a, \mathscr{E}_a \otimes S^k(\mathcal{N}_a^+) \otimes \mathscr{L}_a^+)(g)$$

defines a holomorphic function on $T_{\mathbb{C}}(\Delta_a^+)$. Indeed, writing $g = \exp(X + iY)$ with $Y \in C(\Delta_a^+)$ and decomposing $\mathcal{N}_a^+$ in line bundles, this follows from the fact that for any $\alpha \in \Delta_a^+$,

$$\sum_{k=0}^{\infty} e^{k\alpha(X+iY)}$$

defines a holomorphic function on $T_{\mathbb{C}}(\Delta_a^+)$ as $i\alpha(Y) < 0$ for $Y \in C(\Delta_a^+)$.

Consider the case where (M, σ, μ) is a quantizable symplectic manifold. Then $\mu: M \to \mathfrak{t}^*$ takes constant values μ_a on each connected component M_a of M^T. Furthermore, it follows from the definition of the Kostant-Souriau line bundle with connection $(\mathscr{L}, \mathbb{A})$ that T acts by $e_{i\mu_a}$ on $\mathscr{L}_a$. In particular, $i\mu_a$ is a weight. By definition of $A^o(M, \mathscr{L}, \mathbb{A})$ and the remark following Formula (17), we have the following lemma.

LEMMA 13. *The support of* $A^o(M, \mathscr{L}, \mathbb{A})$ *is contained in the set*

$$i\mu_a + \rho_a^+ + \{\textstyle\sum n_\alpha \alpha; n_\alpha \geqslant 0, \alpha \in \Delta_a^+\}.$$

Let $a \mapsto \Delta_a^+$ be an arbitrary choice of positive systems for Δ_a^+, when a varies in $\mathscr{F}$. Then, over the open subset of M-regular elements, we have, from Lemma 8 and Proposition 11, the equality

$$\operatorname{Tr} Q^o(M, \mathscr{E})(s) = \sum_a \operatorname{Tr} A_a^o(M, \mathscr{E}, \Delta_a^+)(s). \tag{25}$$

However, in general, the equality above does not hold over T.

Let $S \in \mathfrak{t}$ be an element such that $\alpha(S) \neq 0$ for all $\alpha \in \bigcup_{a \in \mathscr{F}} \Delta_a$. The element S determines a consistent choice of positive systems $\Delta_a^+(S)$ of Δ_a when a varies in $\mathscr{F}$.

PROPOSITION 14 (Guillemin-Lerman-Sternberg [15], Guillemin-Prato [16]). *Let $\mathscr{F}$ be the set of connected components of M^T. Let $S \in \mathfrak{t}$ be an element such that $\alpha(S) \neq 0$ for all $\alpha \in \bigcup_{a \in \mathscr{F}} \Delta_a$. We have the identity of generalised functions over T:*

$$\operatorname{Tr} Q^o(M, \mathscr{E})(s) = \sum_{a \in \mathscr{F}} \operatorname{Tr} A_a^o(M, \mathscr{E}, \Delta_a^+(S))(s).$$

Proof. Let us give a proof using the integral expression (13) for $\operatorname{Tr} Q^o(M, \mathscr{E})(s \exp X)$, as this proof will generalise easily to a proof of Theorem 17. Consider the function $s \mapsto \operatorname{Tr} Q^o(M, \mathscr{E})(s)$. It extends holomorphically to $T_{\mathbb{C}}$. We can choose S such that $M(S) = M^T$. Let t be a small positive number. We have $\operatorname{Tr} Q^o(M, \mathscr{E})(s) = \lim_{t \to 0} \operatorname{Tr} Q^o(M, \mathscr{E})(s \exp itS)$.

The differential form $L(s, \mathscr{E}, \mathbb{A})(X)$ extends holomorphically in a neighbourhood of 0 in $\mathfrak{t}_{\mathbb{C}}$. It is clear that

$$\operatorname{Tr} Q^o(M, \mathscr{E})(s \exp itS) = i^{-\dim M/2} \int_{M(s), o'} \varepsilon(s, o, o') L(s, \mathscr{E}, \mathbb{A})(itS)$$

for every t sufficiently small.

We now use the localisation formula (5) (applied to $M = M(s)$ and $Y = itS$), and we obtain, for any small positive t,

$$i^{-\dim M/2} \int_{M(s), o'} \varepsilon(s, o, o') L(s, \mathscr{E}, \mathbb{A})(itS)$$

$$= i^{-\dim M/2} \sum_{a \in \mathscr{F}} \int_{M_a, o_a} \varepsilon(s, o, o') \frac{L(s, \mathscr{E}, \mathbb{A})(itS)}{\operatorname{Eul}_{o'/o_a}(T_{M_a} M(s))(itS)}.$$

We can check by calculations similar to those of Lemma 6 that on M_a

(26)

$$i^{-\dim M/2} \varepsilon(s, o, o') \frac{L(s, \mathscr{E}, \mathbb{A})(itS)|_{M_a}}{\operatorname{Eul}_{o'/o_a}(T_{M_a} M(s))(itS)}$$

$$= (2i\pi)^{-\dim M_a} \varepsilon(o, o_a, \Delta_a^+) \frac{\operatorname{ch}(\mathscr{E}_a \otimes \mathscr{L}_a^+)(s \exp itS)}{J^{1/2}(M_a)} \det{}^{-1}_{\mathscr{N}_a^+}(1 - s \exp itS e^{R_a^+}).$$

The element $s \exp itS$ is in $T_{\mathbb{C}}(\Delta_a^+)$ for any $t > 0$. Thus, we obtain that for any

$t > 0$,

$$\operatorname{Tr} Q^o(M, \mathscr{E})(s \exp itS) = \sum_{a \in \mathscr{F}} \Psi_a^o(s \exp itS),$$

and we conclude by Lemma 12. ■

3. Quantization of manifolds with boundaries. Consider a compact G-manifold M (oriented and with G-invariant spin structure), and let U be a G-invariant open subset of M. If $(\mathscr{E}, \mathbb{A})$ is a G-equivariant vector bundle over M with connection $\mathbb{A}$, we would like to give a meaning to the quantized space $Q^o(U, \mathscr{E}, \mathbb{A})$. Consider the character formula (13) for $Q^o(M, \mathscr{E})$. Let $s \in G$ and $Y \in U_s(0)$ a small neighbourhood of 0 in $\mathfrak{g}(s)$; then

$$\operatorname{Tr} Q^o(M, \mathscr{E}, \mathbb{A})(s \exp Y) = i^{-\dim M/2} \int_{M(s), o'} \varepsilon(s, o, o') L(s, \mathscr{E}, \mathbb{A})(Y).$$

It would be naive to try to define a character $\operatorname{Tr} Q^o(U, \mathscr{E}, \mathbb{A})$ by a truncated formula:

$$\operatorname{Tr} Q^o(U, \mathscr{E}, \mathbb{A})(s \exp Y) = i^{-\dim M/2} \int_{U(s), o'} \varepsilon(s, o, o') L(s, \mathscr{E}, \mathbb{A})(Y).$$

This will never define a character on G nor even a global function Θ on G, because the localisation formula (which relies on Stokes's theorem) does not hold for manifolds with boundaries.

Consider a G-invariant 1-form λ on M. As suggested by Witten's localisation procedure, we introduce for every $t \in \mathbb{R}$ the connection $\mathbb{A}(t) = \mathbb{A} - it\lambda I$ on $\mathscr{E}$. By Formula (7) we have $L(s, \mathscr{E}, \mathbb{A}(t))(Y) = e^{-itd_Y\lambda} L(s, \mathscr{E}, \mathbb{A})(Y)$.

We consider

$$\operatorname{Tr} Q^o(M, \mathscr{E}, \mathbb{A}(t))(s \exp Y) = i^{-\dim M/2} \int_{M(s), o'} \varepsilon(s, o, o') e^{-itd_Y\lambda} L(s, \mathscr{E}, \mathbb{A})(Y).$$

The integral is independent of t as seen from the fact that $e^{-itd_Y\lambda}$ is congruent to 1 in equivariant cohomology. However, for every $s \in G$, the truncated integral

$$TQ^o(U, \mathscr{E}, \mathbb{A}, \lambda, s, t)(Y) = i^{-\dim M/2} \int_{U(s), o'} \varepsilon(s, o, o') e^{-itd_Y\lambda} L(s, \mathscr{E}, \mathbb{A})(Y)$$

is a C^∞-function on $U_s(0)$ depending on t.

We assume that $U = \{x \in M; f(x) > 0\}$, where f is a G-invariant function from M to $\mathbb{R}$ such that zero is a regular value of f.

LEMMA 15. *Assume that the map $\mu_\lambda\colon M \to \mathfrak{g}^*$ given by $\mu_\lambda(X) = \lambda(X_M)$ does not vanish at any point of the boundary B of U. Then for each $s \in G$, the limit $\Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)$ when $t \mapsto \infty$ of $TQ^o(U, \mathscr{E}, \mathbb{A}, \lambda, s, t)$ exists in the space of generalised functions on $U_s(0)$.*

Proof. For $s \in G$, the boundary of $U(s)$ is smooth and given by $B(s)$. Indeed, we have $B = f^{-1}(0)$. Then the differential df vanishes on $(1 - s)T_xM$ for $x \in M(s)$. It follows that zero is a regular value for the restriction of f to $M(s)$.

Let α be any closed $G(s)$-equivariant form on $M(s)$. We write $\mathfrak{g}(s) = \mathfrak{z}$. For $Y \in \mathfrak{z}$, write $\Theta(s, t)(Y) = \int_{U(s)} e^{-itd_Y\lambda}\alpha(Y)$. Then we have

$$\frac{d}{dt} e^{-itd_{\mathfrak{z}}\lambda}\alpha = -id_{\mathfrak{z}}(\lambda e^{-itd_{\mathfrak{z}}\lambda}\alpha).$$

We then obtain

$$e^{-itd_{\mathfrak{z}}\lambda}\alpha = \alpha - id_{\mathfrak{z}}\left(\int_0^t \lambda e^{-iud_{\mathfrak{z}}\lambda}\alpha\, du\right).$$

Integration over $U(s)$ and using Stokes's formula leads to

$$\Theta(s, t)(Y) = \int_{U(s)} e^{-itd_Y\lambda}\alpha(Y) = \int_{U(s)} \alpha(Y) - i\int_{B(s)}\left(\int_0^t \lambda e^{-iud_Y\lambda}\alpha(Y)\, du\right).$$

Let us see that when t tends to ∞, $\Psi(B, s, t)(Y) = -i\int_{B(s)}\int_0^t \lambda e^{-iud_Y\lambda}\alpha(Y)\, du$ has a limit in the sense of generalised functions given by

$$\Psi(B, s)(Y) = -i\int_{B(s)}\int_0^\infty \lambda e^{-iud_Y\lambda}\alpha(Y)\, du.$$

Consider a test function ϕ on $\mathfrak{z}$. Let, for $f \in \mathfrak{z}^*$, $(\widehat{\alpha\phi})(f) = \int_{\mathfrak{z}} e^{if(Y)}\alpha(Y)\phi(Y)\, dY$. Then $(\widehat{\alpha\phi})(f)$ is a differential form on M depending on $f \in \mathfrak{z}^*$. When $f \mapsto \infty$, this differential form converges uniformly to 0 on M. We have $d_Y\lambda = -\mu_\lambda(Y) + d\lambda$ by definition of μ_λ so that $e^{-iud_Y\lambda} = e^{iu\mu_\lambda(Y)}e^{-iud\lambda}$. The restriction of $\mu_\lambda\colon M \to \mathfrak{g}^*$ to $M(s)$ is valued in $\mathfrak{z}^*$. Thus,

$$(27)\qquad \int_{\mathfrak{z}} \Psi(B, s, t)(Y)\phi(Y)\, dY = -i\int_{B(s)\times[0,t]} \lambda e^{-iud\lambda}(\widehat{\alpha\phi})(u\mu_\lambda(m))\, du.$$

In this integral expression, we see that $\Psi(B, s, t)$ has a limit. Indeed, for $m \in B(s)$, the differential form $(\widehat{\alpha\phi})(u\mu_\lambda(m))$ is rapidly decreasing in u (as $\mu_\lambda(m)$ is never zero on $B(s)$) while $e^{-iud\lambda}$ is polynomial in u.

Applying this calculation to $TQ^o(U, \mathscr{E}, \mathbb{A}, \lambda, s, t)$, we see that $TQ^o(U, \mathscr{E}, \mathbb{A}, \lambda, s, t)$ has limit when $t \to \infty$ the generalised function $\Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)$ given for $Y \in U_s(0)$ by:

$$\Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)(Y) = i^{-\dim M/2}\left(\int_{U(s)} \varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(Y) - i\int_{B(s)}\left(\int_0^\infty \lambda e^{-iud_Y\lambda}\varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(Y)\, du\right)\right). \tag{28}$$

■

We conjecture that under the same hypothesis as Lemma 15, there exists a virtual trace class representation $Q^o(U, \mathscr{E}, \mathbb{A}, \lambda)$ of G such that for every $s \in G$, we have for $Y \in U_s(0)$:

$$\operatorname{Tr} Q^o(U, \mathscr{E}, \mathbb{A}, \lambda)(s \exp Y) = \lim_{t\to\infty} TQ^o(U, \mathscr{E}, \mathbb{A}, \lambda, s, t)(Y) = \Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)(Y).$$

Remark that it is not even clear that there exists a G-invariant generalised function Θ on G such that, for $Y \in U_s(0)$,

$$\Theta(s \exp Y) = \Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)(Y).$$

Remark 3.1. It is possible to understand Formula (28) for $\Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)$ in the framework of bouquet integrals. Let us consider the manifold $\tilde{M} = M \times \mathbb{R}$ where G acts trivially on $\mathbb{R}$. We embed M in $M \times \mathbb{R}$ by $m \mapsto (m, 0)$. We write (m, u) an element of $\tilde{M}$. We consider the differential form $\tilde{\lambda} = u\lambda$ as a differential form on $\tilde{M}$. Let us consider $C \subset \tilde{M}$ the cylinder with base B:

$$C = B \times \mathbb{R}^+.$$

The boundary of C in $M \times \mathbb{R}$ is equal to the boundary of U, both being the manifold B. If R is a tubular neighbourhood of B in M, we can identify C to the open subset $R - \bar{U}$ of M. This gives an orientation o_{out} to C.

Define

$$Z = U \cup (C, o_{\text{out}}). \tag{29}$$

Then Z is an oriented cycle in $\tilde{M}$. It can be also identified to the manifold U with the cylindrical end C attached to it. Consider on $\tilde{M}$ the pullback $\tilde{\mathscr{E}}$ of the vector bundle $\mathscr{E}$ with connection $\tilde{\mathbb{A}} = \mathbb{A} - i\tilde{\lambda}$. Then Formula (28) for $\Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)(Y)$ is nothing but the s-part of the universal formula for $i^{-\dim M/2} \int_b^{U\cup C} \operatorname{bch}(\tilde{\mathscr{E}}, \tilde{\mathbb{A}})$. The conjecture above is then: There exists a virtual trace-class representation

$Q^o(U \cup C, \tilde{\mathscr{E}}, \tilde{\mathbb{A}}) = Q^o(U, \mathscr{E}, \mathbb{A}, \lambda)$ such that

$$\text{(30)} \qquad \operatorname{Tr} Q^o(U, \mathscr{E}, \mathbb{A}, \lambda) = i^{-\dim M/2} \int_b^{U \cup C} \operatorname{bch}(\tilde{\mathscr{E}}, \tilde{\mathbb{A}}).$$

This conjecture is consistent with the hope (see [27]) that for good noncompact manifolds and good bouquets, then the bouquet integration produces global (generalised) functions on G.

We will prove this conjecture together with an explicit formula for $Q^o(U, \mathscr{E}, \mathbb{A}, \lambda)$ in a simple case. The following localisation formula for the manifold U with boundary B is due to Kalkman [18].

LEMMA 16. *Let α be a closed G-equivariant differential form on M. Assume there exists a central element $S \in \mathfrak{g}$ such that (λ, S_M) does not vanish on B. Then for every $Y \in \mathfrak{g}$ sufficiently close to S*

$$\int_U \alpha(Y) - \int_B \frac{\lambda}{(d_{\mathfrak{g}}\lambda)(Y)} \alpha(Y) = \int_{U(S)} \frac{\alpha(Y)}{\operatorname{Eul}_o(T_{M(S)}M)(Y)}.$$

Proof. Our hypothesis implies $M(S) \cap B = \varnothing$. Recall (see, for example, [13]) that we may write $\alpha(Y) = \beta(Y) + d_{\mathfrak{g}}\nu(Y)$, where $\beta(Y)$ is supported in a small neighbourhood of $M(S)$. Using a G-invariant partition of unity, we may write also $\nu(Y) = \nu_0(Y) + \nu_1(Y)$, where $\nu_0(Y)$ is supported on a neighbourhood of B and $\nu_1(Y)$ is identically 0 on B. Applying the localisation formula (4) to the compactly supported closed form $\alpha_1(Y) = \beta(Y) + (d_{\mathfrak{g}}\nu_1)(Y)$ on U, we obtain

$$\int_U \alpha_1(Y) = \int_{U(S)} \frac{\alpha_1(Y)}{\operatorname{Eul}_o(T_{M(S)}M)(Y)}.$$

As α_1 is in the same cohomology class as α,

$$\int_{U(s)} \frac{\alpha_1(Y)}{\operatorname{Eul}_o(T_{M(S)}M)(Y)} = \int_{U(S)} \frac{\alpha(Y)}{\operatorname{Eul}_o(T_{M(S)}M)(Y)}.$$

Let $\alpha_0(Y) = (d_{\mathfrak{g}}\nu_0)(Y)$. It remains to show that

$$\int_U \alpha_0(Y) = \int_B \frac{\lambda}{(d_{\mathfrak{g}}\lambda)(Y)} \alpha(Y).$$

We have $\alpha_0(Y) = d_Y(\lambda(d_Y\lambda)^{-1}\alpha_0(Y))$ as $\alpha_0(Y)$ is compactly supported near B and $d_Y\lambda$ invertible on B. Let $n = \dim M$. This implies that the term of maximal degree of $\alpha_0(Y)$ is exact and equal to $d(\lambda(d_Y\lambda)^{-1}\alpha_0(Y))_{[n-1]}$. By Stokes's theorem, we obtain $\int_U \alpha_0(Y) = \int_B \lambda(d_Y\lambda)^{-1}\alpha_0(Y)$. But $\alpha = \alpha_0$ on B, and we obtain our result. ∎

Assume that $\alpha(Y)$ depends holomorphically on Y for Y belonging to an open subset W of $\mathfrak{g}_{\mathbb{C}}$. Let U be the open subset consisting of the elements $Y \in \mathfrak{g}_{\mathbb{C}}$ such that $d_{\mathfrak{g}}\lambda(Y)|_B$ and $\mathrm{Eul}_o(T_{M(S)}M)(Y)$ are invertible. Let W' be the connected component of S in $W \cap U$. Then for all $Y \in W'$, we have

$$(31) \qquad \int_U \alpha(Y) - \int_B \frac{\lambda}{(d_{\mathfrak{g}}\lambda)(Y)} \alpha(Y) = \int_{U(S)} \frac{\alpha(Y)}{\mathrm{Eul}_o(T_{M(S)}M)(Y)}.$$

We consider now a compact manifold M with an action of a torus T. Let U be a T-invariant open subset of M with smooth boundary B. Assume that there exists $S \in \mathfrak{t}$ such that $\mu_\lambda(S) > 0$ on B. The element $S \in \mathfrak{t}$ can be assumed sufficiently generic so that $M(S) = M^T$. We have $M^T \cap B = \emptyset$. Let $\mathscr{F}$ be the set of connected components of M^T. Let $\mathscr{F}(U)$ be the subset of connected components of M^T, which are contained in U. The element $S \in \mathfrak{t}$ determines positive systems $\Delta_a^+(S)$ defined by Formula (20). Recall Definition 10 for $A_a^o(M, \mathscr{E}, \Delta_a^+(S))$ and Formula (28) for $\Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)(X)$. The main theorem of this section is as follows.

THEOREM 17. *Let T be a torus. Let U be a T-invariant open subset of M with boundary B. Let λ be a T-invariant 1-form on M such that there exists $S \in \mathfrak{t}$ with $(\lambda, S_M) > 0$ on B. Define*

$$Q^o(U, \mathscr{E}, \mathbb{A}, \lambda) = \sum_{a \in \mathscr{F}(U)} A_a^o(M, \mathscr{E}, \Delta_a^+(S)).$$

Then for each $s \in T$, there exists small neighbourhood $U_s(0)$ of $0 \in \mathfrak{t}$, such that in $C^{-\infty}(U_s(0))$:

$$\mathrm{Tr}\, Q^o(U, \mathscr{E}, \mathbb{A}, \lambda)(s \exp X) = \Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)(X).$$

Remark 3.2. If $M = U$, then B is empty, and we obtain the formula of Guillemin-Lerman-Sternberg, Guillemin-Prato (Proposition 14).

Proof. Let $s \in T$. Let $\Theta(s) = \Theta^o(U, \mathscr{E}, \mathbb{A}, \lambda, s)$. Then by Formula (28),

$$\Theta(s)(X) = i^{-\dim M/2}(F(U, s)(X) + \Psi(B, s)(X))$$

with

$$F(U, s)(X) = \int_{U(s)} \varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(X)$$

and

$$\Psi(B, s)(X) = -i \int_{B(s)} \left(\int_0^\infty \lambda e^{-iud_X\lambda} \varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(X)\, du \right).$$

The form $L(s, \mathscr{E}, \mathbb{A})(X)$ is analytic on a neighbourhood of zero in $\mathfrak{t}_{\mathbb{C}}$. Let $t \geqslant 0$ be a small positive number. Define

$$F_t(U, s)(X) = \int_{U(s)} \varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(X + itS).$$

Then

$$\lim_{t \mapsto 0} F_t(U, s)(X) = F(U, s)(X).$$

Define $d_t\lambda(X + itS) = -\mu_\lambda(X + itS) + d\lambda$. Define

$$\Psi_t(B, s)(X) = -i \int_{B(s)} \int_0^\infty \lambda e^{-iu(d_t\lambda)(X+itS)}\varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(X + itS)\, du.$$

We have

$$\lim_{t \to 0} \Psi_t(B, s)(X) = \Psi(B, s)(X)$$

in the space of generalised functions.

Let $t > 0$. As $(\mu_\lambda, S) > 0$ on B, the function $u \mapsto e^{-ut(\mu_\lambda, S)}$ is rapidly decreasing on $B(s) \times \mathbb{R}^+$. Now

$$\begin{aligned}\Psi_t(B, s)(X) &= -i \int_{B(s)} \int_0^\infty \lambda e^{-iu(d_t\lambda)(X+itS)}\varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(X + itS)\, du \\ &= -i \int_{B(s)} \int_0^\infty \lambda e^{-ut(\mu_\lambda, S)} e^{i\mu_\lambda(X)} e^{-iud\lambda}\varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(X + itS)\, du\end{aligned}$$

is an analytic function of $X \in U_s(0)$. We have

$$\Psi_t(B, s)(X) = -\int_{B(s)} \frac{\lambda}{(d_t\lambda)(X + itS)}\varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(X + itS).$$

We write $\Delta_a^+(S) = \Delta_a^+$. As $S \in C(\Delta_a^+)$, the element $g = s \exp(X + itS)$ is in $T_{\mathbb{C}}(\Delta_a^+)$. We can define also (see Lemma 12):

$$\operatorname{Tr} A_a^o(M, \mathscr{E}, \Delta_a^+)(s \exp(X + itS))$$

and

$$\lim_{t \to 0} \operatorname{Tr} A_a^o(M, \mathscr{E}, \Delta_a^+)(s \exp(X + itS)) = \operatorname{Tr} A_a^o(M, \mathscr{E}, \Delta_a^+)(s \exp X)$$

in the space of generalised functions.

To prove the theorem, we must see that if $P_t(X) = F_t(U, s)(X) + \Psi_t(B, s)(X)$:

$$i^{-\dim M/2} P_t(X) = \sum_{a \in \mathscr{F}(U)} \operatorname{Tr} A_a^o(M, \mathscr{E}, \Delta_a^+)(s \exp(X + itS))$$

for any $t > 0$.

Define $\alpha(Z) = \varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(Z)$ for Z in a small neighbourhood of zero in $\mathfrak{t}_{\mathbb{C}}$. Then $\alpha(Z)$ depends holomorphically on Z. For $t > 0$,

$$P_t(X) = \int_{U(s)} \alpha(X + itS) - \int_{B(s)} \frac{\lambda}{(d_t\lambda)(X + itS)} \alpha(X + itS).$$

Recall from our assumption on S that $U(s)(S) = U^T = \bigcup_{a \in \mathscr{F}(U)} M_a$. Consider the localisation formula (31) (with U replaced by $U(s)$). We can apply it to $Y = X + itS$. As X is of small norm, W' contains $Y = X + zS$ with $|z| = 1$. Furthermore, as X is real, it contains elements $Y = X + itS$ with $t \neq 0$. We obtain

$$P_t(X) = \sum_{a \in \mathscr{F}(U)} \int_{M_a, o_a} \frac{\varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})(X + itS)}{\operatorname{Eul}_{o'/o_a}(T_{M_a}M(s))(X + itS)}.$$

Comparing with the formula in Lemma 12 for $\operatorname{Tr} A_a^o(M, \mathscr{E}, \Delta_a^+)(g)$, with

$$g = s \exp(X + itS),$$

it remains to be seen that on M_a

$$i^{-\dim M/2} \frac{\varepsilon(s, o, o')L(s, \mathscr{E}, \mathbb{A})|_{M_a}(X + itS)}{\operatorname{Eul}_{o'/o_a}(T_{M_a}M(s))(X + itS)}$$

$$= (2i\pi)^{-\dim M_a} \varepsilon(o, o_a, \Delta_a^+) \frac{\operatorname{ch}(\mathscr{E}_a \otimes \mathscr{L}_a^+)(g)}{J^{1/2}(M_a)} \det{}^{-1}_{\mathscr{N}_a^+}(1 - g e^{R_a^+}). \quad \blacksquare$$

4. Geometry of the moment map and decomposition of the quantized representation for an S^1-action. Let $G = \{e^{i\theta}; \theta \in \mathbb{R}\}$. Let $\mathfrak{g} = \operatorname{Lie}(S^1) = \mathbb{R}E$ with $E \in \mathfrak{g}$ such that $\exp \theta E = e^{i\theta}$. Let (M, σ, μ) be a G-Hamiltonian manifold with symplectic form σ and moment map μ. Let $f: M \to \mathbb{R}$ be the G-invariant map

$$f(m) = \mu(E)(m).$$

We assume that (M, σ, μ) is prequantized, and let $\mathscr{L}$ be a Kostant-Souriau line bundle (Definition 2) with its connection $\mathbb{A}$. As before we assume for simplicity that M carries a G-invariant spin structure. We fix the orientation o given by the symplectic structure and write $Q(M, \mathscr{L})$ instead of $Q^o(M, \mathscr{L})$. Our aim is to understand the decomposition of $Q(M, \mathscr{L})$ in irreducible representations of G in function of the geometry of the map f. Let ξ be a regular value of f. Let $P_\xi = f^{-1}(\xi)$. We assume that G acts freely on $f^{-1}(\xi)$. We then can consider the manifold $M_{\mathrm{red}}(\xi) = G\backslash P_\xi$. It is a symplectic manifold with symplectic form σ_ξ. Then, as we will show in Lemma 26, the manifold $M_{\mathrm{red}}(\xi)$ carries also a spin structure. Consider the line bundle $\mathscr{L}$. Then $\mathscr{L}_{\mathrm{red}}(\xi) = (G\backslash \mathscr{L}|_{P_\xi})$ is a line bundle on $M_{\mathrm{red}}(\xi)$, which is a Kostant-Souriau line bundle for σ_ξ. In this section, we show the following theorem.

THEOREM 18. *Let $G = S^1$. Let $\xi \in \mathfrak{g}^*$ such that $i\xi$ is a weight of G. Assume that G acts freely on $f^{-1}(\xi)$. Then the multiplicity $n(i\xi, M, \mathscr{L})$ of $e_{i\xi}$ in $Q(M, \mathscr{L})$ is equal to $Q(M_{\mathrm{red}}(\xi), \mathscr{L}_{\mathrm{red}}(\xi))$.*

By changing the moment map μ to $\mu - \xi$, we can suppose that $\xi = 0$. We will then denote $M_{\mathrm{red}}(0)$ simply by M_{red}, and $\mathscr{L}_{\mathrm{red}}(0)$ simply by $\mathscr{L}_{\mathrm{red}}$.

To study the multiplicity of the trivial representation in $Q(M, \mathscr{L})$, we will decompose the virtual character $Q(M, \mathscr{L})$ as the sum of *three infinite-dimensional virtual characters* Θ_0, Θ_+ and Θ_-.

We denote $f^{-1}(0)$ by P. Consider the principal fibration $q: P \to M_{\mathrm{red}} = G\backslash P$ with structure group $G = S^1$. Let $n \in \mathbb{Z}$. Consider the character $\chi_n(\exp \theta E) = e^{in\theta}$ of G, and let $\mathscr{T}_n$ be the associated line bundle on $G\backslash P = M_{\mathrm{red}}$. Then $Q(M_{\mathrm{red}}, \mathscr{L}_{\mathrm{red}} \oplus \mathscr{T}_n)$ is a relative integer.

Definition 19. Define the virtual character $Q_0(M, \mathscr{L})$ of G by

$$Q_0(M, \mathscr{L}) = \sum_{n \in \mathbb{Z}} Q(M_{\mathrm{red}}, \mathscr{L}_{\mathrm{red}} \otimes \mathscr{T}_{-n}) e^{in\theta}.$$

The multiplicity of the trivial representation in $Q_0(M, \mathscr{L})$ is $Q(M_{\mathrm{red}}, \mathscr{L}_{\mathrm{red}})$.

It follows from Atiyah-Singer's formula (16) that the function $n \mapsto Q(M_{\mathrm{red}}, \mathscr{L}_{\mathrm{red}} \otimes \mathscr{T}_{-n})$ is polynomial in n so that $Q_0(M, \mathscr{L})$ is indeed a trace-class virtual representation. Furthermore, $\mathrm{Tr}\, Q_0(M, \mathscr{L})$ defines a generalised function on G supported at the identity of G.

Let $\mathscr{F}$ be the set of connected components of M^T. Let $a \in \mathscr{F}$ and let M_a be a connected component of M^T. Then f is constant on M_a. Let $\mathscr{F}^+$ (respectively, $\mathscr{F}^-$) be the set of $a \in \mathscr{F}$ such that $f(M_a) > 0$ (respectively, $f(M_a) < 0$). Then $\mathscr{F} = \mathscr{F}^+ \cup \mathscr{F}^-$. We then choose for each component $a \in \mathscr{F}$ the following outer order:

$$\Delta_a^{\mathrm{out}} = \{\alpha \in \Delta_a, i\alpha(E) f(M_a) < 0\}.$$

More precisely, for $a \in \mathscr{F}^+$ we choose $\Delta_a^+ = \Delta_a^+(E) = \{\alpha \in \Delta_a, i\alpha(E) < 0\}$, while we choose for each component $a \in \mathscr{F}^-$ the order $\Delta_a^+(-E)$.

Recall the definition of the element $A_a(M, \mathscr{L}, \Delta_a^+)$ in $R^{-\infty}(G)$ (Definition 10).

Definition 20. Define $Q_{\pm}(M, \mathscr{L}) \in R^{-\infty}(G)$ by

$$Q_+(M, \mathscr{L}) = \sum_{a \in \mathscr{F}+} A_a(M, \mathscr{L}, \Delta_a^{\text{out}}).$$

$$Q_-(M, \mathscr{L}) = \sum_{a \in \mathscr{F}-} A_a(M, \mathscr{L}, \Delta_a^{\text{out}}).$$

By definition of $\mathscr{F}_+$, the constant value μ_a of $\mu(E)$ on M_a is a positive integer. By definition of $\Delta_a^+(E)$, the number $p_a = -i\rho_a^+(E)$ is a positive integer. The following lemma follows from Lemma 13.

LEMMA 21. *Let $m > 0$ be the minimum of all numbers $\mu_a + p_a$ for $a \in \mathscr{F}^+$. The virtual character $Q_+(M, \mathscr{L})$ is of the form $Q_+(M, \mathscr{L}) = \sum a_n e^{in\theta}$ with $n \geqslant m$. In particular, $Q_+(M, \mathscr{L})$ does not contain the trivial representation of $G = S^1$.*

Similarly, the virtual character $Q_-(M, \mathscr{L})$ is of the form $Q_-(M, \mathscr{L}) = \sum_{n<0} a_n e^{in\theta}$ and $Q_-(M, \mathscr{L})$ does not contain the trivial representation of $S^1 = G$.

The main theorem of this section is the following.

THEOREM 22. *We have the decomposition*

$$Q(M, \mathscr{L}) = Q_0(M, \mathscr{L}) \oplus Q_+(M, \mathscr{L}) \oplus Q_-(M, \mathscr{L}).$$

It is clear that this theorem implies Theorem 18 for the case of an action of S^1. Furthermore, this theorem should allow us to compare multiplicities when we cross a singular value of f.

Remark 4.1. This writing of $Q(M, \mathscr{L})$ is in agreement with Formula (25) for $Q(M, \mathscr{L})$. Indeed, we know from Lemma 8 that $\mathrm{Tr}(Q(M, \mathscr{L}) - \mathrm{Tr}(Q_+(M, \mathscr{L}) \oplus Q_-(M, \mathscr{L})))$ is a generalised function supported on singular elements. Theorem 22 is an explicit description of this generalised function in function of the fiber $f^{-1}(0)$.

Proof. Choosing a basis E of $\mathfrak{g}$, we consider the map $f = \mu(E)$ from $M \to \mathbb{R}$. Let r be a small positive number. Let

$$M_0 = \{x; |f(x)| < r\}, \qquad M_+ = \{x; f(x) > r\}, \qquad M_- = \{x; f(x) < -r\}.$$

We can choose r such that S^1 acts freely on $\overline{M_0}$. Thus, the vector field E_M does not vanish on $\overline{M_0}$. As in Witten, consider the function $w = (1/2)f^2$. Then w gives rise to the Hamiltonian vector field $H_w = \mu(E)E_M = fE_M$. Let $(.,.)$ be a G-invariant metric on M. Let

$$\lambda(\cdot) = (H_w, \cdot) = f(E_M, \cdot) \tag{32}$$

be the G-invariant 1-form determined by H_w and the choice of G-invariant metric $(.,.)$ on M.

Remark that $\lambda(E_M) = \mu(E)\|E_M\|^2$ is strictly positive on the boundary of M_+ and strictly negative on the boundary of M_-.

We associated to M_0, M_+, and M_- three generalised functions on G with the help of λ by truncating formulas for the s-part of the character of $Q(M, \mathscr{L}, \mathbb{A}(t))$ with $\mathbb{A}(t) = \mathbb{A} - it\lambda$ on $\mathscr{L}$. Clearly, as $Q(M, \mathscr{L}, \mathbb{A}(t))$ is independent of t for all $s \in G$ and all $t \in \mathbb{R}$, and $X \in \mathfrak{g}$ small, we have

$$\operatorname{Tr} Q(M, \mathscr{L})(s \exp X) = TQ(M_0, \mathscr{L}, \mathbb{A}, \lambda, s, t)(X) + TQ(M_+, \mathscr{L}, \mathbb{A}, \lambda, s, t)(X) + TQ(M_-, \mathscr{L}, \mathbb{A}, \lambda, s, t)(X).$$

Furthermore, Lemma 15 and Theorem 17 imply that

$$\lim_{t\to\infty} TQ(M_+, \mathscr{L}, \mathbb{A}, \lambda, s, t)(X) = \operatorname{Tr} Q_+(M, \mathscr{L}, \mathbb{A})(s \exp X)$$

and

$$\lim_{t\to\infty} TQ(M_-, \mathscr{L}, \mathbb{A}, \lambda, s, t)(X) = \operatorname{Tr} Q_-(M, \mathscr{L}, \mathbb{A})(s \exp X).$$

Thus, it remains to see the next lemma.

LEMMA 23. *For each $s \in G$ and $X \in U_s(0)$ a small neighbourhood of 0 in $\mathfrak{g}$, we have in $C^{-\infty}(U_s(0))$:*

$$\operatorname{Tr} Q_0(M, \mathscr{L})(s \exp X) = \lim_{t\to\infty} TQ(M_0, \mathscr{L}, \mathbb{A}, \lambda, s, t)(X).$$

Proof. By our hypothesis on the free action, $TQ(M_0, \mathscr{L}, \mathbb{A}, \lambda, s, t)(X)$ is equal to 0 for $s \neq 1$ as $M_0(s) = \varnothing$. The generalised function $\operatorname{Tr} Q_0(M, \mathscr{L})$ is supported at 1, thus $\operatorname{Tr} Q_0(M, \mathscr{L})(s \exp X) = 0$ for $s \neq 1$ and X small. We need only to verify the formula of Lemma 23 for $s = 1$. Let $s = 1$ and let us study the limit in the space of generalised functions on $U_1(0)$ of the truncated integral

$$TQ(M_0, t, 1)(X) = (2i\pi)^{-\dim M/2} \int_{M_0} \frac{\operatorname{ch}(\mathscr{L}, \mathbb{A}(t))(X)}{J^{1/2}(M)(X)}.$$

As $\mathbb{A}(t) = \mathbb{A} - it\lambda$, we have $\operatorname{ch}(\mathscr{L}, \mathbb{A}(t))(X) = e^{-itd_X\lambda} \operatorname{ch}(\mathscr{L}, \mathbb{A})(X)$.

Let ϕ be a test function on $\mathfrak{g}$ with support in $U_1(0)$ so that $J^{1/2}(M)(X)$ is invertible on the support of ϕ. We have

$$(33) \qquad \int_{\mathfrak{g}} TQ(M_0, t, 1)(X)\phi(X)\, dX = \int_{M_0} \left(\int_{\mathfrak{g}} e^{-itd_X\lambda} L(X)\phi(X)\, dX \right)$$

with

$$L(X) = (2i\pi)^{-\dim M/2} \operatorname{ch}(\mathscr{L}, \mathbb{A})(X) J^{-1/2}(M)(X)$$

a closed G-equivariant-differential form on M.

Let us recall the results of [28]. As G acts freely on P, there exists an isomorphism $W: H_G^\infty(\mathfrak{g}, P) \to H^*(M_{\mathrm{red}})$. If ω is a connection form with curvature Ω for the fibration $q: P \to M_{\mathrm{red}}$, then W coincides on $S(\mathfrak{g}^*)$ with the Chern-Weil homomorphism $\phi \mapsto \phi(\Omega)$. The inverse of W is simply given by q^*.

Let $\alpha \in \mathscr{A}_G^\infty(\mathfrak{g}, M)$ be a G-equivariant closed form on M. The restriction $\alpha|_P$ of α to P is a G-equivariant closed form on P. We denote by α_{red} the element $W(\alpha|_P)$. Then α_{red} is a De Rham cohomology class on M_{red}.

We have the following proposition [28, Theorem 19 and Remark].

PROPOSITION 24. *Let $\alpha \in \mathscr{A}_G^\infty(\mathfrak{g}, M)$ be a G-equivariant closed form on M. Let ϕ be a test function on $\mathfrak{g}$. Then*

$$\lim_{t \to \infty} \int_{M_0} \int_{\mathfrak{g}} e^{-itd_X\lambda} \alpha(X) \phi(X)\, dX = i(2\pi)^2 \int_{M_{\mathrm{red}}} \alpha_{\mathrm{red}} \phi(\Omega).$$

Remark 4.2. The form $\phi(\Omega)$ is defined via the Taylor series of ϕ at zero. As Ω is nilpotent, the form $\phi(\Omega)$ involves only finitely many derivatives of ϕ at zero. Thus, the map $\phi \mapsto \int_{M_{\mathrm{red}}} \alpha_{\mathrm{red}} \phi(\Omega)$ is a distribution of support zero.

We then obtained that the limit when t tends to ∞ of the generalised function $TQ(M_0, t, 1)(X)$ exists. We denote it by Θ_0. We have by Proposition 24, for ϕ a test function on $\mathfrak{g}$:

$$\int_{\mathfrak{g}} \Theta_0(X) \phi(X)\, dX = i(2\pi)^2 \int_{M_{\mathrm{red}}} L_{\mathrm{red}} \phi(\Omega),$$

where $L(X) = (2i\pi)^{-\dim M/2} \operatorname{ch}(\mathscr{L}, \mathbb{A})(X) J^{-1/2}(M)(X)$.

LEMMA 25. *We have*

$$L_{\mathrm{red}} = (2i\pi)^{-\dim M/2} e^{i\sigma_{\mathrm{red}}} J^{-1/2}(M_{\mathrm{red}}).$$

Proof. Recall that $\sigma|_P$ is the pullback of the symplectic form σ_{red} on M_{red}. Thus, as $\mu|_P = 0$, we have $\operatorname{ch}(\mathscr{L}, \mathbb{A})|_P = e^{i\sigma|_P}$, and $\operatorname{ch}(\mathscr{L}, \mathbb{A})|_P$ is already in the form $q^*(e^{i\sigma_{\mathrm{red}}})$.

Let $J(M_{\mathrm{red}})$ be the J-genus of the tangent bundle to M_{red}. Let us see that

$$J(M)(X)|_P \cong q^* J(M_{\mathrm{red}})$$

in cohomology. Indeed, $TM|_P = TP \oplus P \times \mathfrak{g}^*$ and $TP = q^*TM_{\text{red}} \oplus V$, where q^*TM_{red} is identified to the horizontal tangent bundle and V to the vertical tangent bundle via the connection form ω. Thus, $TM|_P = q^*TM_{\text{red}} \oplus P \times (\mathfrak{g} \oplus \mathfrak{g}^*)$. The equivariant J-genus of the trivial bundle $P \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ is identically 1. Thus,

$$J(M)|_P \cong q^*(J(M_{\text{red}})),$$

and we obtain our lemma. ■

LEMMA 26. *The manifold M_{red} has a spin structure.*

Proof. Let $\mathscr{S}$ be the spin bundle of M. Consider the horizontal vector $H = \partial_f$ on a neighbourhood of P in M of the form $P \times \{f \in]u, v[\}$. Let $V = E_M$ be the vertical vector field generated by E. Let c_0 be the endomorphism of $\mathscr{S}|_P$ obtained by the Clifford action of $H + iV$ on $\mathscr{S}$. Then $\mathscr{S}_0 = \operatorname{Ker} c_0 \subset \mathscr{S}|_P$ has typical fiber over x the spinor space of $T_{q(x)}M_{\text{red}}$. It is a G-equivariant subbundle of $\mathscr{S}|_P$. The bundle $\mathscr{S}_0/G$ over M_{red} is a spinor bundle for M_{red}. ■

As $\dim M_{\text{red}} = \dim M - 2$, we obtain the following expression of Θ_0:

$$\int_{\mathfrak{g}} \Theta_0(X)\phi(X)\, dX = (2\pi)(2i\pi)^{-\dim M_{\text{red}}/2} \int_{M_{\text{red}}} \phi(\Omega) e^{i\sigma_{\text{red}}} J^{-1/2}(M_{\text{red}}).$$

As $\phi(\Omega) = \sum_k (\Omega^k/k!)((d/dX)^k\phi)|_{X=0}$, we see that Θ_0 is a distribution on $\mathfrak{g}$ with support 0. Let us compute its Fourier expansion. Let $\chi_n(\theta) = e^{in\theta}$. We have $\Theta_0(\exp \theta E) = \sum_n a_n e^{in\theta}$ with

$$a_n = (2\pi)^{-1} \int_{\mathbb{R}} \Theta_0(\exp \theta E)\chi_{-n}(\theta)\, d\theta = (2i\pi)^{-\dim M_{\text{red}}/2} \int_{M_{\text{red}}} e^{i\sigma_{\text{red}}} J^{-1/2}(M_{\text{red}}) e^{-in\Omega}.$$

The associated line bundle $\mathscr{T}_{-n}$ to χ_{-n} has Chern character $e^{-in\Omega}$. Thus, the index $Q(M_{\text{red}}, \mathscr{L}_{\text{red}} \otimes \mathscr{T}_{-n})$ of the twisted Dirac operator $D^+_{\mathscr{L}_{\text{red}} \otimes \mathscr{T}_{-n}}$ on the spin manifold M_{red} is in $\mathbb{Z}$. It is given by the integral formula (16):

$$Q(M_{\text{red}}, \mathscr{L}_{\text{red}} \otimes \mathscr{T}_{-n}) = (2i\pi)^{-\dim M_{\text{red}}/2} \int_{M_{\text{red}}} e^{i\sigma_{\text{red}}} e^{-in\Omega} J^{-1/2}(M_{\text{red}}).$$

We thus obtain Lemma 23 and hence Theorem 22. ■

Assume that a compact group K acts on M commuting with the action of S^1 and such that $(\mathscr{L}, \mathbb{A})$ is a K-equivariant vector bundle and that the action of K preserves $\mathbb{A}$. Then f is a K-invariant function. The manifold $f^{-1}(0)$ carries a K-action, and M_{red} is a K-Hamiltonian manifold with Kostant-Souriau line bundle $\mathscr{L}_{\text{red}}$. All terms of Theorem 22 are virtual representations of $G \times K$, and it is clear that the same theorem holds as (virtual) representations of $G \times K$.

THEOREM 27. *We have the decomposition*

$$Q(M, \mathscr{L}) = Q_0(M, \mathscr{L}) \oplus Q_+(M, \mathscr{L}) \oplus Q_-(M, \mathscr{L})$$

as virtual representations of $G \times K$.

The space $Q(M, \mathscr{L})^G$ is a virtual representation of K. We have the next theorem.

THEOREM 28. *Let* $G = S^1$. *Let* K *be a compact Lie group. Let* M *be a* $G \times K$ *Hamiltonian manifold with a* $(G \times K)$*-invariant spin structure. Let* $\mathscr{L}$ *be a Kostant-Souriau line bundle for* $G \times K$. *Let* $Q(M, \mathscr{L}) \in R(G \times K)$ *be the quantized space. Let* $f: M \to \mathfrak{g}^*$ *be the moment map for the* G*-action. Assume that* G *acts freely on* $f^{-1}(0)$. *Consider the* K*-Hamiltonian manifold* $M_{\mathrm{red}} = G \backslash f^{-1}(0)$ *with Kostant-Souriau line bundle* $\mathscr{L}_{\mathrm{red}}$. *Then the virtual representation* $Q(M, \mathscr{L})^G$ *is isomorphic to the virtual representation* $Q(M_{\mathrm{red}}, \mathscr{L}_{\mathrm{red}})$ *of* K.

Theorem 27 should be interpreted in the noncompact manifold $\tilde{M} = M \times \mathbb{R}$. We embed M in $M \times \mathbb{R}$ by $x \mapsto (x, 0)$. Let us consider

$$C_\pm = B_\pm \times \mathbb{R}^+,$$

where $B_\pm$ is the boundary of $M_\pm$. We define orientations o_{out} on $C_\pm$ as explained in formula (29). Define

$$Z_\pm = M_\pm \cup (C_\pm, o_{\mathrm{out}})$$

$$Z_0 = (C_-, -o_{\mathrm{out}}) \cup M_0 \cup (C_+, -o_{\mathrm{out}}).$$

Thus, Z_0, Z_+, Z_- are G-invariant oriented cycles in $\tilde{M}$. Clearly, as a sum of oriented cycles, we have

$$M = [Z_-] + [Z_0] + [Z_+].$$

As shown in the preceding section (Formula (30)), the decomposition $Q(M, \mathscr{L}) = Q_0(M, \mathscr{L}) \oplus Q_+(M, \mathscr{L}) \oplus Q_-(M, \mathscr{L})$ as a sum of 3 virtual characters of G corresponds to the decomposition $M = [Z_-] + [Z_0] + [Z_+]$ in $\tilde{M}$. Indeed, we have

$$\operatorname{Tr} Q_\pm(M, \mathscr{L}) = i^{-\dim Z/2} \int_{Z_\pm}^b \operatorname{bch}(Z_\pm, \tilde{\mathscr{L}}, \mathbb{A}^\lambda).$$

$$\operatorname{Tr} Q_0(M, \mathscr{L}) = i^{-\dim Z/2} \int_{Z_0}^b \operatorname{bch}(Z_0, \tilde{\mathscr{L}}, \mathbb{A}^\lambda).$$

Appendix

5. The universal character formula for quantum bundles. In this appendix, we state the universal character formula without the restrictive assumption of spin structure on M.

Let G be a compact Lie group acting on a smooth oriented manifold M and preserving the orientation of M. Let us recall the definition of a G-equivariant quantum bundle over M.

If V is a real vector space, we denote by $GL^+(V)$ the group of linear transformations of V which are of positive determinant. We denote by $j: ML^+(V) \to GL^+(V)$ the 2-fold connected cover of $GL^+(V)$ and by $\varepsilon \in ML^+(V)$ the nontrivial element above $1 \in GL^+(V)$.

Let W be a Hermitian space. Let $U(W)$ be the group of unitary transformations of W. We denote by $-I$ the transformation $w \to -w$ of W. We embed $\mathbb{Z}/2\mathbb{Z}$ as a central subgroup Z in $ML^+(V) \times U(W)$ obtained by sending $(-1) \in \mathbb{Z}/2\mathbb{Z}$ to $(\varepsilon, -I) \in ML^+(V) \times U(W)$. Let

$$ML^+(V)^W = (ML^+(V) \times U(W))/Z.$$

We denote by $[g, u]$ the class of the element $(g, u) \in ML^+(V) \times U(W)$ in $ML^+(V)^W$. We call the group $ML^+(V)^W$ the metalinear group with coefficients in W. There are canonical morphisms

$$f: ML^+(V)^W \to GL^+(V)$$

and

$$u: ML^+(V)^W \to U(W)/\pm I.$$

If $V_1 \oplus V_2$ is a direct sum decomposition of V, the subgroup $\{g \in ML^+(V_1 \oplus V_2)^W; f(g) \in GL^+(V_1)\}$ is isomorphic to $ML^+(V_1)^W$. We thus embed $ML^+(V_1)^{W_1}$ and $ML^+(V_2)^{W_2}$ as 2 commuting subgroups of $ML^+(V)^{W_1 \otimes W_2}$.

If $P \to M$ is a principal space with group $ML^+(V)^W$, we denote by P^f the vector bundle over M associated to the representation f of $ML^+(V)^W$ in V.

Recall the following definition.

Definition 29. Let M be a manifold of dimension n. Let V be a real vector space of dimension n. Let W be a Hermitian space. A quantum bundle $\tau: P \to M$ over M with fiber W is a principal bundle over M with structure group $ML^+(V)^W$ such that the associated bundle P^f is the tangent bundle TM.

In this definition, it is necessary that the manifold M is orientable for quantum bundles to exist. The space W is called the fiber of the quantum bundle P, although there is no true vector bundle $\mathcal{W}$ with fiber W associated to P. However, there is an associated bundle P^u to P with structure group $U(W)/\pm I$, i.e., a pseudovector bundle with fiber W in the terminology of [27]. In particular, as the adjoint action of $U(W)$ on $\mathrm{End}(W)$ factors through $U(W)/\pm I$, the bundle $\mathrm{End}(\mathscr{W})$ is well defined.

It is clear that we can add two quantum bundles of fibers W_1 and W_2 and obtain a quantum bundle of fiber $W_1 \oplus W_2$. We can tensor a quantum bundle with fiber W with a Hermitian vector bundle with fiber E and obtain a quantum bundle with fiber $W \otimes E$.

Let G be a Lie group acting on M and preserving the orientation of M. A G-equivariant quantum bundle is a quantum bundle with a left action of G such that the associated action of G on the associated bundle P^f is the natural action of G on the tangent bundle TM.

We denote by $F_G^t(M)$ the set of G-equivariant quantum bundles on M (up to isomorphism). We denote by $K_G^t(M)$ the associated Grothendieck group. If $G = \{1\}$, we denote $F_G^t(M)$ simply by $F^t(M)$ and $K_G^t(M)$ by $K^t(M)$.

A G-equivariant quantum bundle with Hermitian connection is a couple $(\tau, \mathbb{A})$ consisting of a G-equivariant quantum bundle τ and of a G-invariant connection $\mathbb{A}$ on the associated bundle τ^u with group $U(W)/\pm I$.

Let τ be a G-equivariant quantum bundle on M. If $\mathbb{A}$ is a G-equivariant connection on P^u, the equivariant curvature $F(X)$ of $\mathbb{A}$ is a differential form on M with values in the bundle $\mathrm{End}(\mathscr{W})$. Thus, we can define the Chern character $\mathrm{ch}(\tau, \mathbb{A})(X)$ as in the case of vector bundles by the formula $\mathrm{ch}(\tau, \mathbb{A})(X) = \mathrm{Tr}(e^{F(X)})$. It is a G-equivariant closed form on the manifold M.

Let (M, σ, μ) be a symplectic manifold with Hamiltonian action of G. We slightly modify the notion of the Kostant-Souriau line bundle.

Definition 30. We say that (M, σ, μ) is prequantizable if there exists a G-equivariant quantum line bundle $(\tau, \mathbb{A})$ over M with equivariant curvature $F(X) = i\sigma_{\mathfrak{g}}(X)$.

If the manifold M is a compact, even-dimensional manifold and has a G-invariant orientation o, there is a well-defined quantization map

$$Q^o\colon K_G^t(M) \to R(G).$$

This map can be constructed as follows: Let V be an oriented even-dimensional Euclidean space. Let $S = S^+ \oplus S^-$ be the spinor space of V. Given a quantum bundle τ over M with fiber W, we can construct with the help of a G-invariant metric on M, a graded G-equivariant Clifford bundle $\mathscr{S}_W^{\pm}$ on M with typical fiber $S^{\pm} \otimes W$. We denote by $\Gamma(M, \mathscr{S}_W^{\pm})$ the space of its smooth sections. With the help of a G-invariant connection $\mathbb{A}$ on P^u, we can construct a twisted Dirac operator $D_{W,\mathbb{A}}^+$. This is an elliptic operator

$$D_{W,\mathbb{A}}^+\colon \Gamma(M, \mathscr{S}_W^+) \to \Gamma(M, \mathscr{S}_W^-),$$

which commutes with the natural action of G. We define

$$Q^o(M,\tau) = (-1)^{\dim M/2}([\operatorname{Ker} D^+_{W,\mathbb{A}}] - [\operatorname{Coker} D^+_{W,\mathbb{A}}]).$$

The virtual representation $Q^o(M,\tau)$ depends only on the quantum bundle τ and on the orientation o of M.

The character formula for Tr $Q^o(M,\tau)$ is a slight modification of the character formula 13 when M is a spin manifold. The crucial difference is that if we do not assume the existence of a G-invariant spin structure, submanifolds $M(s)$ are not necessarily oriented; thus, we need to produce densities on $M(s)$ rather than differential forms. The Chern character of equivariant quantum bundles produces such families as we now recall.

Recall that if s is an elliptic transformation of an oriented vector space V and if $\tilde{s}$ is an element of $ML^+(V)$ above s, then $\tilde{s}$ determines an orientation $o_{\tilde{s}}$ of $V/V(s)$. The convention is as in [27].

If N is a manifold, we introduce the 2-fold cover $N_{or} = \{(m,\xi)\}$ of N, where $m \in N$ and ξ is an orientation of T_mM. We say that α is a *folded differential form* on N if α is a differential form on N_{or} such that $\alpha_\xi = -\alpha_{-\xi}$. The term of maximum exterior degree of a folded differential form α is a density on N. We can then define $\int_N \alpha$.

Let $\tau: P \to M$ be a G-equivariant quantum bundle over M with fiber W and G-invariant Hermitian connection $\mathbb{A}$. Then we can define the equivariant curvature $F(X)$ of $\mathbb{A}$. For $X \in \mathfrak{g}$, it is a differential form on M with values in End($\mathcal{W}$). If $s \in G$, $m \in M(s)$, and p is an element of P above m, we denote by $g(p,s)$ the element $g(p,s) \in ML^+(V)^W$ such that $sp = pg(p,s)$. Let $(\tilde{s}, s^W) \in ML^+(V) \times U(W)$ such that $[\tilde{s}, s^W] = g(p,s)$. Let o' be a local orientation of $M(s)$. We write $\operatorname{sign}(\tilde{s}, o, o') = \pm 1$, depending on whether the orientations $o_{\tilde{s}}, o, o'$ are compatible or not.

Definition 31. The Chern character $\operatorname{bch}(\tau,\mathbb{A}) = (\operatorname{ch}_s(\tau,\mathbb{A}))_{s\in G}$ of the equivariant quantum bundle τ with G-invariant connection $\mathbb{A}$ is the family of folded equivariant-differential forms

$$\operatorname{ch}_{s,o'}(\tau,\mathbb{A})(X) = \operatorname{sign}(\tilde{s}, o, o')\operatorname{Tr}(s^W e^{F(X)|M(s)}).$$

The Chern character $\operatorname{bch}(\tau,\mathbb{A})$ of a G-equivariant quantum bundle with G-invariant connection $\mathbb{A}$ is an admissible bouquet in the sense of [13]. The integration $\int_b^M$ on the space of admissible bouquets is defined in [13].

THEOREM 32. *Let τ be a G-equivariant quantum bundle over an even-dimensional compact manifold M. Choose a G-invariant connection $\mathbb{A}$ on τ. Then*

$$\operatorname{Tr} Q^o(M,\tau,\mathbb{A}) = i^{-\dim M/2}\int_b^M \operatorname{bch}(\tau,\mathbb{A}).$$

The meaning of this formula is as follows. Let $s \in G$. There exists a neighbourhood $U_s(0)$ of 0 in $\mathfrak{g}(s)$ such that we have

(34)

$$\operatorname{Tr} Q^o(M, \tau)(s \exp X) = i^{-\dim M/2} \int_{M(s)} (2\pi)^{-\dim M(s)/2} \frac{\operatorname{ch}_s(\tau, \mathbb{A})(X)}{J^{1/2}(M(s))(X) D_s^{1/2}(T_{M(s)}M)(X)}$$

for $X \in U_s(0)$.

Remark that the term of maximal exterior degree under the integral sign is a density on $M(s)$, so that it can be integrated. If (M, σ, μ) is a prequantizable, compact, symplectic manifold with Kostant-Souriau quantum line bundle τ, the virtual representation $Q(M, \tau, \mathbb{A})$ is the quantized space of the Hamiltonian space (M, σ, μ).

Let (M, σ, μ) be a symplectic manifold with a Hamiltonian action of a compact Lie group G. Let $\mu: M \to \mathfrak{g}^*$ be the moment map. Assume that G acts freely on $\mu^{-1}(0)$. We can then form the reduced manifold M_{red}. In the next subsection, we show that there is a canonical map $\tau \mapsto \tau_{\text{red}}$ from G-equivariant bundles on M to quantum bundles on M_{red}. This is the generalisation of Lemma 26 and was already observed in the symplectic context in [24].

The main result (Theorem 1) on multiplicities can thus be stated without spin hypothesis on M (see [30, Part II] for the proof).

Denote $\mu^{-1}(0)$ by P. A neighbourhood of P in M is diffeomorphic to $P \times \mathfrak{g}^*$. In the next subsection, we show that there is an isomorphism between G-equivariant quantum bundles on $P \times \mathfrak{g}^*$ and quantum bundles on $M_{\text{red}} = G \backslash P$.

6. Reduction of quantum bundles. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let P be a compact manifold that is a principal space for the action of G. Let N be the manifold

$$N = P \times \mathfrak{g}^*$$

with diagonal action of G. Let $N_{\text{red}} = G \backslash P$ be the quotient space. We assume that the manifold N_{red} is oriented. We denote by $q: P \to N_{\text{red}}$ the quotient map.

We denote an element of N by (x, ξ), where $x \in P$ and $\xi \in \mathfrak{g}^*$. We denote by $q^t: P \times \mathfrak{g}^* \to N_{\text{red}}$ the map $q^t(x, \xi) = q(x)$. The fiber of q^t is $G \times \mathfrak{g}^*$. In particular, the manifold N is covered by open subsets of the form $U \times G \times \mathfrak{g}^*$, so that it looks locally as the product of N_{red} by a cotangent space.

If $\mathcal{W}$ is a vector bundle over N_{red}, then $(q^t)^*\mathcal{W}$ is a vector bundle over N. It is provided with an action of G given by $g \cdot ((x, \xi), v) = ((gx, g \cdot \xi), v)$ for $g \in G$, $x \in P$, $\xi \in \mathfrak{g}^*$, $v \in \mathcal{W}_{q(x)}$. Every G-equivariant vector bundle over $P \times \mathfrak{g}^*$ is isomorphic to the pullback of a G-equivariant vector bundle over P. As the action of G on P is free, every G-equivariant vector bundle over P is isomorphic to a bundle $q^*(\mathcal{W})$. Thus, every G-equivariant vector bundle over N is isomorphic to the pullback

$(q^t)^*\mathscr{W}$ of a bundle over N_{red}. We show that the same result is true for quantum bundles over $N = P \times \mathfrak{g}^*$.

To state the correspondence between quantum bundles over N_{red} and N, we need some lemmas on the groups $ML^+(V)$.

Assume $\dim V = 2d$ even. Let o be an orientation of V. Let $f_1, f_2, \ldots, f_{2d-1}, f_{2d}$ be an oriented basis of V. Let

$$S^o f_{2j-1} = f_{2j}, \qquad S^o f_{2j} = -f_{2j-1}.$$

The element $r^o = \exp \pi S^o$ of $ML^+(V)$ covers the element -1 of $SL(V)$ and depends only on the orientation o of V. We have $(r^o)^2 = \varepsilon$, and r^o is central in $ML^+(V)$.

Let V_1 be a real vector space. Let $V = V_1 \oplus V_1^*$. On V, consider the orientation $o = o_{V_1 \oplus V_1^*}$ given as follows. If E_i is a basis of V_1 with dual basis E^i, then an oriented basis is $E_1, E^1, E_2, E^2, \ldots$. Consider the homomorphism $d(g) = (g^t, g^{-1})$ of $GL(V_1)$ into $SL(V)$. Let $ML^+(V)^{\mathbb{C}}$ be the metalinear group with coefficients in $\mathbb{C}$.

Lemma 33. *Let V_1 be a real vector space. Let $V = V_1 \oplus V_1^*$. There exists a homomorphism $\tilde{d}: GL(V_1) \to ML^+(V)^{\mathbb{C}}$ such that $f\tilde{d}(s) = d(s)$.*

We normalise this isomorphism when $\dim V_1$ *is odd such that*

$$\tilde{d}(-1) = [r^o, i^{\dim V_1}].$$

If $\dim V_1$ *is even, we embed V_1 in $V_1 \oplus \mathbb{R}$ and normalise $\tilde{d}$ such that*

$$\tilde{d}(g) = \tilde{d}(g, I).$$

Remark 5.1. If $\dim V_1$ is odd, we have $[r^o, i^{\dim V_1}]^2 = [\varepsilon, -1] = 1$ in $ML^+(V)^{\mathbb{C}}$ so that the image of (-1) is of order 2 as it should be.

Let $n_0 = \dim N_{\text{red}}$, and let $d = \dim G$. The dimension n of N is equal to $n_0 + 2d$. By the choice of a connection on P, the tangent bundle to the fiber of $q^t: N \to N_{\text{red}}$ is isomorphic to $N \times (\mathfrak{g} \oplus \mathfrak{g}^*)$. If $(N_{\text{red}}, o_{\text{red}})$ is oriented, we choose as orientation of N the orientation

$$o^N = o_{\text{red}} \wedge o_{\mathfrak{g} \oplus \mathfrak{g}^*}. \tag{35}$$

Let V_0 be a real vector space of dimension n_0. Let $\tau_0: R_0 \to N_{\text{red}}$ be a quantum bundle on N_{red} with Hermitian fiber W. It is a principal bundle over N_{red} with structure group $ML^+(V_0)^W$. Let $V = V_0 \oplus \mathfrak{g} \oplus \mathfrak{g}^*$. Then V is of dimension $n = n_0 + 2d$. Consider the natural homomorphism $h_0: ML^+(V_0)^W \to ML^+(V)^W$ coming from the direct-sum decomposition $V = V_0 \oplus \mathfrak{g} \oplus \mathfrak{g}^*$. Let $R(\tau_0) = R_0 \times_{ML^+(V_0)^W} ML^+(V)^W$ be the associated bundle to the homomorphism h_0. The group $ML^+(V)^W$ acts on the right on $R(\tau_0)$. Thus, $R(\tau_0)$ is a principal bundle over N_{red} with structure group $ML^+(V)^W$. We denote by $[r, a]$ the class of the element

(r, a) $r \in \tau_0$, $a \in ML^+(V)^W$ in $R(\tau_0)$. Consider the adjoint action $G \to GL(\mathfrak{g})$. Consider the homomorphism $\tilde{d}: G \to ML^+(\mathfrak{g} \oplus \mathfrak{g}^*)^{\mathbb{C}}$ given in Lemma 33. As the subgroups $ML^+(V_0)^W$ and $ML^+(\mathfrak{g} \oplus \mathfrak{g}^*)^{\mathbb{C}}$ commute, the space $R(\tau_0)$ is provided with an action of G given by $g \times [r, a] = [r, \tilde{d}(g)a]$. The bundle $(q^t)^*R(\tau_0)$ is thus a G-equivariant principal bundle over N with structure group $DL^+(V)^W$.

LEMMA 34. *If τ_0 is a quantum bundle on N_{red}, the bundle $(q^t)^*R(\tau_0)$ is a G-equivariant quantum bundle over N.*

Proof. Let $\tau = (q^t)^*R(\tau_0)$. We have to prove that the associated bundle τ^f with fiber $V = V_0 \oplus \mathfrak{g} \oplus \mathfrak{g}^*$ is isomorphic to the tangent bundle to N. It is clear that the associated bundle τ^f is isomorphic to $(q^t)^*TN_{\text{red}} \oplus N \times (\mathfrak{g} \oplus \mathfrak{g}^*)$. Let us choose a connection form for the principal fibration $P \to N_{\text{red}}$. This provides an isomorphism between TN and $(q^t)^*TN_{\text{red}} \oplus N \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ and identifies τ^f to TN as G-equivariant bundles.

PROPOSITION 35. *Every G-equivariant quantum bundle τ on N is of the form $\tau = (p_0^t)^*R(\tau_0)$ for a unique quantum bundle τ_{red} on N_{red}.*

The quantum bundle τ_{red} over the reduced manifold N_{red} such that $\tau \sim (q^t)^*\tau_{\text{red}}$ will be called the reduction of τ. The associated bundle τ^u is isomorphic to $(q^t)^*\tau^u_{\text{red}}$.

It is also natural to consider the situation where a product of compact Lie groups $K \times G$ acts on P, and G acts freely on P. Then there is an action of K on N_{red} and we can similarly prove the following proposition.

PROPOSITION 36. *The map $\tau_0 \to (q^t)^*(R(\tau_0))$ is an isomorphism of $F^t_K(N_{\text{red}})$ and $F^t_{G\times K}(N)$.*

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