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LATTICE POINTS IN SIMPLE POLYTOPES

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1. Introduction

Consider a convex n-dimensional polytope P in $\mathbb{R}^n$ with all vertices in the lattice $\mathbb{Z}^n$. In this article, we give a formula for the number of lattice points in P, in the case where P is simple, that is, if there are exactly n edges through each vertex of P. More generally, for any polynomial function ϕ on $\mathbb{R}^n$, we express

$$\sum_{m\in\mathbb{Z}^n\cap P}\phi(m)$$

in terms of $\int_{P(h)}\phi(x)dx$ where the polytope $P(h)$ is obtained from P by independent parallel motions of all facets. This extends to simple lattice polytopes the Euler-Maclaurin summation formula of Khovanskii and Pukhlikov [8] (valid for lattice polytopes such that the primitive vectors on edges through each vertex of P form a basis of the lattice). As a corollary, we recover results of Pommersheim [9] and Kantor-Khovanskii [6] on the coefficients of the Ehrhart polynomial of P. Our proof is elementary. In a subsequent article, we will show how to adapt it to compute the equivariant Todd class of any complete toric variety with quotient singularities.

The Euler-Maclaurin summation formula for simple lattice polytopes has been obtained independently by Ginzburg-Guillemin-Karshon [4]. They used the dictionary between convex polytopes and projective toric varieties with an ample divisor class, in combination with the Riemann-Roch-Kawasaki formula ([1], [7]) for complex manifolds with quotient singularities. A counting formula for lattice points in lattice simplices has been announced by Cappell and Shaneson [2], as a consequence of their computation of the Todd class of toric varieties with quotient singularities.

2. Euler-Maclaurin formula for polytopes

Let V be a real vector space of dimension n. Let M be a lattice in V. Points of M will be called integral points. The vector space V has a canonical Lebesgue measure dx giving volume 1 to a fundamental domain for M. More precisely, let $e^1, e^2, \ldots, e^n$ be a basis of V such that

$$M = \mathbb{Z}e^1 \oplus \mathbb{Z}e^2 \oplus \cdots \oplus \mathbb{Z}e^n.$$

If $x = x_1e^1 + x_2e^2 + \cdots + x_ne^n$ is a point in V, then $dx = dx_1dx_2\cdots dx_n$.

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We denote by V^* the dual vector space to V. If L is a lattice in a vector space W, we denote by L^* its dual lattice in the dual vector space W^*:

$$L^* = \{y \in W^*, (x, y) \in \mathbb{Z}, \text{ for all}\, x \in L\}.$$

We will denote M^* by N. Then N is a lattice in V^*.

Let P be a convex polytope contained in V with nonempty interior P^0. We denote by $\mathrm{vol}(P)$ the volume of P with respect to the measure dx on V.

We denote by $\mathcal{F}$ the set of closed faces of P. We have

$$\mathcal{F} = \bigcup_{k=0}^{n} \mathcal{F}(k),$$

where $\mathcal{F}(k)$ is the set of faces of dimension k. We have $\mathcal{F}(n) = \{P\}$. By definition, the set $\mathcal{F}(0)$ of extremal points of P is the set of vertices of P. The set $\mathcal{F}(1)$ is the set of edges of P. A face of codimension 1 is called a facet. A facet $F \in \mathcal{F}(n-1)$ is the intersection of P with an affine hyperplane $\{y; (u_F, y) + \lambda_F = 0\}$. We choose the normal vector $u_F \in V^*$ to the facet F such that P is contained in $\{y, (u_F, y) + \lambda_F \geq 0\}$. In other words, we choose the inward-pointing normal vector u_F. This normal vector is determined modulo multiplication by an element of $\mathbb{R}^+$.

If f is a face of P, we define

$$\mathcal{F}^f = \{F \in \mathcal{F}(n-1); f \subset F\}. \tag{2.1}$$

We denote by $\langle f \rangle$ the vector space generated by elements $p - q$ where $p \in f$ and $q \in f$. If f is in $\mathcal{F}(k)$, then $\langle f \rangle$ is of dimension k.

Definition 2.1. Define C_f to be the convex cone generated by elements $p - q$ with $p \in P$ and $q \in f$. We say that C_f is the tangent cone to P at its face f.

The cone C_f is also known as the barrier cone. It contains $\langle f \rangle$ as its largest linear subspace.

Definition 2.2. Define σ_f to be the polar cone to C_f:

$$\sigma_f = \{y \in V^* | (x, y) \geq 0 \text{ for all}\, x \in C_f\}.$$

The cone σ_f is also known as the normal cone.

We have

$$\sigma_f = \sum_{F \in \mathcal{F}^f} \mathbb{R}^+ u_F.$$

If $f = \{P\}$, then $\sigma_f = \{0\}$.

Definition 2.3. A convex polytope P is said to be simple if there are exactly n edges through each vertex.

For example, in $\mathbb{R}^3$, a cube, a pyramid with triangular basis, and a dodecahedron are simple.

Definition 2.4. A convex polytope P is a lattice polytope if all vertices of P are in the lattice M.

Consider a convex lattice polytope P. We can then choose for each facet F the normal vector u_F in the dual lattice N. We normalize u_F in order that u_F be a primitive element of N, that is, if $tu_F \in N$, then t is an integer.

Let us number facets of P as $F_1, F_2, \dots, F_d$. We denote by $u_i \in N$ the normalized normal vector u_{F_i} to F_i. Let $\lambda_i = \lambda_{F_i}$. Thus P is the intersection of d half-spaces:

$$P = \{x \in V, (u_i, x) + \lambda_i \geq 0, 1 \leq i \leq d\}.$$

Consider $h \in \mathbb{R}^d$, $h = (h_1, h_2, \dots, h_d)$. For $h \in \mathbb{R}^d$, define

$$P(h) = \{x \in V, (u_i, x) + \lambda_i + h_i \geq 0, 1 \leq i \leq d\}. \tag{2.2}$$

Then $P(h)$ is a convex polytope. Moreover, for small h, $P(h)$ and P have the same directions of faces. In particular, $P(h)$ is simple if P is simple and h is small enough.

Let C be a closed convex cone in a vector space W with a lattice L. We denote by $\langle C\rangle$ the vector space spanned by C. The dimension of C is defined to be equal to the dimension of the vector space $\langle C\rangle$. We say that C is acute (or pointed) if C does not contain any nonzero linear subspace. A cone C is said to be polyhedral (respectively, rational polyhedral) if C is generated by a finite number of elements of W (respectively, of L). An acute polyhedral cone C of dimension k is said to be simplicial if C has exactly k edges. If P is a convex lattice polytope, then cones C_f associated to faces f of P, and their polar cones σ_f are rational and polyhedral. The cone C_f is acute if and only if f is a vertex of P. The cones σ_f are acute for all $f \in \mathcal{F}$.

A finite collection Σ of rational polyhedral acute cones in V^* is called a fan if

1) for any face τ of an element $\sigma \in \Sigma$, we have $\tau \in \Sigma$;

2) for $\sigma, \tau \in \Sigma$, we have $\sigma \cap \tau \in \Sigma$.

The fan is complete if $\bigcup_{\sigma\in\Sigma} \sigma = V^*$.

We denote by $\Sigma(k)$ the set of cones in the fan Σ of dimension k.

If P is a convex lattice polytope, the collection $\Sigma_P = \{\sigma_f, f \in \mathcal{F}\}$ is a complete fan, called the normal fan of P ([11]). If $f \in \mathcal{F}(n-k)$ is a face of codimension k of P, then σ_f has dimension k. The fan Σ_P depends only on the directions of the faces of P. In particular, the homothetic polytope qP (q a positive integer) has the same fan as P.

Definition 2.5. A fan Σ is said to be simplicial if each cone $\sigma \in \Sigma$ is simplicial.

A lattice polytope is simple if and only if its fan is simplicial.

Let Σ be a simplicial fan. Let d be the cardinal of $\Sigma(1)$. We denote elements of $\Sigma(1)$ as $\ell_1, \ell_2, \dots, \ell_d$. Let u_i, $1 \leq i \leq d$, be the primitive integral vector (with respect to N) on the half-line ℓ_i.

Definition 2.6. Let $\sigma \in \Sigma$. We denote by $\mathcal{E}(\sigma)$ the subset of the set $\{1, 2, \dots, d\}$ consisting of those i such that the half-line ℓ_i is an edge of σ.

The elements $\{u_i, i \in \mathcal{E}(\sigma)\}$ are linearly independent. Let

$$U(\sigma) = \bigoplus_{i\in\mathcal{E}(\sigma)} \mathbb{Z}u_i \tag{2.3}$$

and

$$T(\sigma) = \langle\sigma\rangle / U(\sigma).$$

Let k be the dimension of σ. Then $U(\sigma) = \mathbb{Z}^k$ is a lattice in $\langle\sigma\rangle = \mathbb{R}^k$, and $T(\sigma) = \mathbb{R}^k/\mathbb{Z}^k$ is a k-dimensional torus.

Consider the lattice $N(\sigma) = N \cap \langle\sigma\rangle$ of $\langle\sigma\rangle$. Then $U(\sigma)$ is a sublattice of $N(\sigma)$, which is usually different from $N(\sigma)$. Define

$$G(\sigma) = N(\sigma)/U(\sigma). \tag{2.4}$$

Then $G(\sigma)$ is a finite subgroup of $T(\sigma)$. The order of $G(\sigma)$ is called the multiplicity of σ in toric geometry.

If τ is a face of σ, we have $U(\tau) = \langle\tau\rangle \cap U(\sigma)$ since $\mathcal{E}(\tau) \subset \mathcal{E}(\sigma)$ and the elements $u_i, i \in \mathcal{E}(\sigma)$, are linearly independent. Thus we have a natural inclusion $T(\tau) \subset T(\sigma)$. This induces a natural inclusion of the finite group $G(\tau)$ in $G(\sigma)$. By the definition of u_i as a primitive vector, the group $G(\sigma)$ is trivial if $\sigma \in \Sigma(1)$.

Definition 2.7. Let Σ be a simplicial fan. We denote by T_Σ the set obtained from the disjoint union of the tori $T(\sigma)$ ($\sigma \in \Sigma$) by identifying the subsets $T(\sigma \cap \tau)$ of $T(\sigma)$ and $T(\tau)$ for all $(\sigma, \tau) \in \Sigma \times \Sigma$.

We write $T_\Sigma = \bigcup_{\sigma\in\Sigma} T(\sigma)$. In T_Σ, we have $T(\sigma) \cap T(\tau) = T(\sigma \cap \tau)$. A visual way to represent the set T_Σ associated to a rational fan Σ is the following. We denote by $Q(\sigma)$ the subset

$$Q(\sigma) = \sum_{i\in\mathcal{E}(\sigma)} [0, 1[u_i$$

of $\langle\sigma\rangle$. It is clear that the map $Q(\sigma) \mapsto T(\sigma)$ (restriction of the quotient map $\langle\sigma\rangle \to \langle\sigma\rangle/U(\sigma)$) is an isomorphism. Furthermore, $Q(\sigma) \cap Q(\tau) = Q(\sigma \cap \tau)$. Consider the subset Q_Σ of V^* defined by

$$Q_\Sigma = \bigcup_{\sigma\in\Sigma} Q(\sigma). \tag{2.5}$$

Then the set Q_Σ is isomorphic to the set T_Σ.

Consider the finite subgroup $G(\sigma) \subset T(\sigma)$.

Definition 2.8. The subset Γ_Σ of T_Σ is defined to be

$$\Gamma_\Sigma = \bigcup_{\sigma\in\Sigma} G(\sigma).$$

Thus we can think of Γ_Σ as the union of all finite groups $G(\sigma)$ ($\sigma \in \Sigma$) with equivalence relations given by $G(\sigma \cap \tau) = G(\sigma) \cap G(\tau) \subset T_\Sigma$. In particular, the neutral elements of all the groups $G(\sigma)$ are identified to a unique element of Γ_Σ, denoted by 1. In the identification of T_Σ with the subset Q_Σ of V^*, the subset Γ_Σ of T_Σ is identified with $Q_\Sigma \cap N$.

Example 2.9. Let a, b, c be pairwise coprime integers, and let $P(a, b, c)$ be the simplex in $\mathbb{R}^3$ with vertices

$$O = (0, 0, 0), \qquad A = (a, 0, 0), \qquad B = (0, b, 0), \qquad C = (0, 0, c).$$

The rational fan Σ_P associated to P has edges

$$\ell_1 = \mathbb{R}^+e_1, \quad \ell_2 = \mathbb{R}^+e_2, \quad \ell_3 = \mathbb{R}^+e_3, \quad \ell_0 = \mathbb{R}^+(-bce_1 - cae_2 - abe_3),$$

where (e_1, e_2, e_3) is the canonical basis of $(\mathbb{R}^3)^*$. Let us list the nontrivial abelian groups $G(\sigma)$ for $\sigma \in \Sigma_P$. Denote by $G(j, \dots, k)$ the group associated to a cone in Σ_P generated by $(\ell_j, \dots, \ell_k)$. We have

$$G(0, 2, 3) = \mathbb{Z}/bc\mathbb{Z}, \quad G(0, 3, 1) = \mathbb{Z}/ca\mathbb{Z}, \quad G(0, 1, 2) = \mathbb{Z}/ab\mathbb{Z}$$

and

$$G(0,1) = \mathbb{Z}/a\mathbb{Z}, \qquad G(0,2) = \mathbb{Z}/b\mathbb{Z}, \qquad G(0,3) = \mathbb{Z}/c\mathbb{Z}.$$

Our set Γ_Σ is equal to

$$\Gamma = (\mathbb{Z}/bc\mathbb{Z}) \cup (\mathbb{Z}/ac\mathbb{Z}) \cup (\mathbb{Z}/ab\mathbb{Z}),$$

where we identify the common subsets $\mathbb{Z}/a\mathbb{Z}$, $\mathbb{Z}/b\mathbb{Z}$, $\mathbb{Z}/c\mathbb{Z}$.

Definition 2.10. A simple lattice polytope P is called a Delzant polytope if each cone $\sigma \in \Sigma_P$ is spanned by a part of a basis of N, i.e., if $G(\sigma) = \{1\}$ for each element $\sigma \in \Sigma_P$.

Equivalently, P is a Delzant polytope if the set Γ_Σ constructed from the complete fan Σ_P of P is reduced to $\{1\}$. This is a very strong hypothesis. As shown by the example above, many lattice simplices are not Delzant. As another example, consider the lattice simplex $P(a)$ in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,a)$, where $a \geq 2$ is an integer. Then $P(a)$ is not Delzant, and the only lattice points in $P(a)$ are its vertices. It follows that $P(a)$ is not a union of Delzant polytopes.

Remark 2.11. In the dictionary (that we do not use here) between convex polytopes and toric varieties, a projective toric variety with quotient singularities is associated to a simple lattice polytope P. This toric variety is nonsingular if and only if the polytope P is a Delzant polytope (see [3]).

We now define for $k \in \{1, 2, \ldots, d\}$ functions a^k on T_Σ associated to the d elements ℓ_k of $\Sigma(1)$. The torus $T(\sigma) = \langle\sigma\rangle/U(\sigma)$ comes equipped with a basis of its lattice of characters: for each $k \in \mathcal{E}(\sigma)$, we define $\chi^k_\sigma(g) = e^{2i\pi y^k_\sigma}$ if $y = \sum_{j\in\mathcal{E}(\sigma)} y^j_\sigma u_j$ is an element of $\langle\sigma\rangle$ representing g.

The following lemma is obvious.

Lemma 2.12. *For any $k \in \{1, 2, \ldots, d\}$, there exists a unique function $a^k : T_\Sigma \to \mathbb{C}^*$ such that*

1) if $k \notin \mathcal{E}(\sigma)$, then $a^k(g) = 1$ for all $g \in T(\sigma) \subset T_\Sigma$;

2) if $k \in \mathcal{E}(\sigma)$, then $a^k(g) = \chi^k_\sigma(g)$ if $g \in T(\sigma) \subset T_\Sigma$.

Observe that there exists a unique continuous function ξ^k on V^* which is linear on each cone of Σ, and such that $\xi^k(u_k) = 1$ and $\xi^k(u_j) = 0$ for all $j \neq k$. Let us identify T_Σ with the subset Q_Σ of V^*. Then if $g \in T_\Sigma$ is represented by the element $y \in Q_\Sigma$, we have $a^k(g) = e^{2i\pi\xi^k(y)}$.

We can characterize the subset $G(\sigma)$ of Γ_Σ as follows.

Lemma 2.13. *Let $\sigma \in \Sigma$. We have*

$$G(\sigma) = \{\gamma \in \Gamma_\Sigma, a^k(\gamma) = 1 \text{ for all } k \notin \mathcal{E}(\sigma)\}.$$

Now we turn to the definition of Todd operators. Consider the analytic function

$$\operatorname{Todd}(z) = \frac{z}{1 - \exp(-z)} = 1 + \frac{1}{2}z + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k},$$

where B_k are the Bernoulli numbers.

Let a be a complex number. Consider more generally the function

$$\operatorname{Todd}(a, z) = \frac{z}{1 - a\exp(-z)}.$$

This function is analytic in a neighborhood of 0. Consider its Taylor expansion

$$\mathrm{Todd}(a,z)=\sum_{k=0}^{\infty}c(a,k)z^k$$

for z small.

Let h be a real variable. For any $a\in\mathbb{C}$, consider the operator

$$\mathrm{Todd}(a,\partial/\partial h)=\sum_{k=0}^{\infty}c(a,k)(\partial/\partial h)^k.$$

We have

$$\mathrm{Todd}(1,\partial/\partial h)=1+\frac{1}{2}\partial/\partial h+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}(\partial/\partial h)^{2k}, \tag{2.6}$$

while for $a\neq 1$,

$$\mathrm{Todd}(a,\partial/\partial h)=(1-a)^{-1}\partial/\partial h+\sum_{k=2}^{\infty}c(a,k)(\partial/\partial h)^k. \tag{2.7}$$

We denote $\mathrm{Todd}(1,\partial/\partial h)$ simply by $\mathrm{Todd}(\partial/\partial h)$. If ϕ is a polynomial function of h, then $\mathrm{Todd}(a,\partial/\partial h)\phi(h)$ is well defined, as $(\partial/\partial h)^k\phi=0$ for large k.

Definition 2.14. Let Σ be a complete simplicial fan. For $g\in T_\Sigma$, define

$$\mathrm{Todd}(g,\partial/\partial h)=\prod_{k=1}^{d}\mathrm{Todd}(a^k(g),\partial/\partial h_k).$$

Define

$$\mathrm{Todd}(\Sigma,\partial/\partial h)=\sum_{\gamma\in\Gamma_\Sigma}\mathrm{Todd}(\gamma,\partial/\partial h).$$

Recall a version of the Euler-Maclaurin formula. If $\phi(x)$ is a polynomial function on $\mathbb{R}$ and $s\leq t$ are integers, then

$$\sum_{k=s}^{t}\phi(k)=\mathrm{Todd}(\partial/\partial h_1)\mathrm{Todd}(\partial/\partial h_2)\left(\int_{s-h_1}^{t+h_2}\phi(x)dx\right)\Big|_{h_1=h_2=0}. \tag{2.8}$$

We will generalize this formula to simple lattice polytopes.

Let P be a simple lattice polytope with d facets. Let $|M\cap P|$ be the number of lattice points in P, and $|M\cap P^0|$ the number of lattice points in the interior P^0 of P. Let $P(h)$ be the deformed polytope obtained from P after d independent parallel motions of its facets (formula (2.2)). The main theorem of this article is

Theorem 2.15. *Let V be a vector space with a lattice M. Let P be a simple lattice polytope in V, and let Σ be its associated fan. Then, for small h, the volume* $\mathrm{vol}(P(h))$ *of the deformed polytope $P(h)$ is a polynomial function of h, and we have*

$$|M\cap P|=\mathrm{Todd}(\Sigma,\partial/\partial h)\,\mathrm{vol}(P(h))|_{h=0},$$

while

$$|M\cap P^0|=\mathrm{Todd}(\Sigma,-\partial/\partial h)\,\mathrm{vol}(P(h))|_{h=0}.$$

More generally, if ϕ is a polynomial function on V, then

$$I(\phi)(h)=\int_{P(h)}\phi(x)dx$$

is a polynomial function of h for small h, and

$$\sum_{m\in M\cap P}\phi(m)=\mathrm{Todd}(\Sigma,\partial/\partial h)I(\phi)(h)|_{h=0},$$

while

$$\sum_{m\in M\cap P^0}\phi(m)=\mathrm{Todd}(\Sigma,-\partial/\partial h)I(\phi)(h)|_{h=0}.$$

Remark 2.16. If, moreover, P is a Delzant polytope, then the corresponding set Γ_Σ is reduced to $\{1\}$, $\mathrm{Todd}(\Sigma,\partial/\partial h)$ is the usual Todd operator considered by Khovanskii and Pukhlikov [8] and Theorem 2.15 is due to them in this case.

We will prove Theorem 2.15 in the next section.

3. Integral formulas

As an example of our method, let us first prove identity (2.8). It will be convenient to extend the action of Todd operators to exponential functions $h\mapsto e^{hz}$, for z a small complex number. Indeed, for z small, the series

$$\mathrm{Todd}(\partial/\partial h)e^{hz}=e^{hz}\left(1+\frac{1}{2}z+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^{2k}\right)$$

is convergent and equal to $\mathrm{Todd}(z)e^{hz}$.

Let $[s,t]$ be an interval. Then we have

$$\int_s^t e^{zx}dx=\frac{e^{tz}}{z}-\frac{e^{sz}}{z}. \tag{3.1}$$

Assume t and s are integers; then

$$\sum_{k=s}^{t}e^{kz}=e^{sz}(1+e^z+\cdots+e^{(t-s)z})=e^{sz}\frac{1-e^{(t-s+1)z}}{1-e^z},$$

that is,

$$\sum_{k=s}^{t}e^{kz}=\frac{e^{tz}}{1-e^{-z}}+\frac{e^{sz}}{1-e^z}. \tag{3.2}$$

On the other hand,

$$\int_{s-h_1}^{t+h_2}e^{zx}dx=e^{h_2z}\frac{e^{tz}}{z}-e^{-h_1z}\frac{e^{sz}}{z}. \tag{3.3}$$

Therefore

$$\begin{aligned}\mathrm{Todd}(\partial/\partial h_1)\mathrm{Todd}(\partial/\partial h_2)&\left(\int_{s-h_1}^{t+h_2}e^{zx}dx\right)\Big|_{h_1=h_2=0}\\&=\mathrm{Todd}(z)\frac{e^{tz}}{z}-\mathrm{Todd}(-z)\frac{e^{sz}}{z}.\end{aligned}$$

Comparing with formula (3.2), we obtain

$$\mathrm{Todd}(\partial/\partial h_1)\mathrm{Todd}(\partial/\partial h_2)\left(\int_{s-h_1}^{t+h_2}e^{zx}dx\right)\Big|_{h_1=h_2=0}=\sum_{k=s}^{t}e^{kz}.$$

If we take the Taylor expansion at the origin of this identity in z, we obtain formula (2.8).

Our proof of Theorem 2.15 for an n-dimensional lattice polytope P will be based on the same approach. Let $y \in V_{\mathbb{C}}^*$, and let $P \subset V$ be a polytope (not necessarily a lattice polytope). Define

$$E(P)(y) = \int_P e^{(x,y)} dx.$$

Then the volume of P is the value of $E(P)$ at $y = 0$.

If, moreover, P is a lattice polytope, define

$$D(P)(y) = \sum_{m \in M \cap P} e^{(m,y)}$$

and

$$D(P^0)(y) = \sum_{m \in M \cap P^0} e^{(m,y)}.$$

Then the number $|M \cap P|$ of lattice points in P is the value of $D(P)$ at $y = 0$. Although $E(P)(y)$, $D(P)(y)$ and $D(P^0)(y)$ are analytic functions of y, "simple" expressions (similar to formulae (3.1) and (3.2)) for $E(P)(y)$, $D(P)(y)$ and $D(P^0)(y)$ will be given only when P is simple and y is "generic". On this formula for $E(P(h))(y)$, it will be easy to analyze the action of the Todd operator $\mathrm{Todd}(\Sigma, \partial/\partial h)$ and to compare it with $D(P)(y)$.

Recall that C_f denotes the tangent cone to P at its face f. Choose $v_0 \in f$. Set

$$C_P^+(f) = v_0 + C_f.$$

As C_f is invariant by translation by vectors in $\langle f \rangle$, the affine cone $C_P^+(f)$ does not depend of the choice of $v_0 \in f$. We call it the inward pointing *affine* cone tangent to P at f. Thus $C_P^+(f)$ contains P and $P = \bigcap_{f \in \mathcal{F}} C_P^+(f)$.

Let

$$C_P^-(f) = v_0 - C_f$$

be the outward pointing affine cone at f.

If E is a subset of V, we denote by χ_E its characteristic function.

Proposition 3.1. *Let P be a convex polytope with non empty interior P^0. Then we have the identities*

(1)

$$\chi_P = \sum_{f \in \mathcal{F}} (-1)^{\dim f} \chi_{C_P^+(f)},$$

(2)

$$(-1)^n \chi_{P^0} = \sum_{f \in \mathcal{F}} (-1)^{\dim f} \chi_{C_P^-(f)},$$

(3)

$$\chi_{\{0\}} = \sum_{f \in \mathcal{F}} (-1)^{\dim f} \chi_{C_f}.$$

Proof. A version of these identities can be found in [5]. We give another proof, based on the Euler identities

$$\sum_{f\in\mathcal{F}}(-1)^{\dim f}=1 \tag{3.4}$$

and, for any point m in the boundary of P,

$$\sum_{f\in\mathcal{F},m\in f}(-1)^{\dim f}=0. \tag{3.5}$$

Let m be an arbitrary point of V. We have to prove the relations

(1) If $m\in P$, then

$$\sum_{f\in\mathcal{F},m\in C_P^+(f)}(-1)^{\dim f}=1.$$

(2) If $m\notin P$, then

$$\sum_{f\in\mathcal{F},m\in C_P^+(f)}(-1)^{\dim f}=0.$$

(3) If $m\in P^0$, then

$$\sum_{f\in\mathcal{F},m\in C_P^-(f)}(-1)^{\dim f}=(-1)^n.$$

(4) If $m\notin P^0$, then

$$\sum_{f\in\mathcal{F},m\in C_P^-(f)}(-1)^{\dim f}=0.$$

(5)

$$\sum_{f\in\mathcal{F},0\in C_f}(-1)^{\dim f}=1.$$

(6) If $m\neq 0$, then

$$\sum_{f\in\mathcal{F},m\in C_f}(-1)^{\dim f}=0.$$

First observe that $m\in P$ if and only if $m\in C_P^+(f)$ for all $f\in\mathcal{F}$. So assertion (1) is just the Euler identity (3.4), and the same holds for (5). For (3), if $m\in P^0$, then the unique face f such that $m\in C_P^-(f)$ is $f=P$.

Let us prove (4). Let $m\notin P^0$. Consider the convex hull H of P and m. Let $\mathcal{F}(H)$ be the set of faces of H. Let $\mathcal{F}_m(H)$ be the set of faces of H containing m. Write $\mathcal{F}(H)$ as the disjoint union $\mathcal{F}_m(H)\cup\mathcal{F}_n(H)$. Using relations (3.4), (3.5) for the polytope H and its boundary point m, we obtain $\sum_{g\in\mathcal{F}_n(H)}(-1)^{\dim g}=1$. It is easy to see that the faces g in $\mathcal{F}_n(H)$ are faces of P and that these are all the faces f of P such that $m\notin C_P^-(f)$. Thus we have $\sum_{f\in\mathcal{F},m\notin C_P^-(f)}(-1)^{\dim f}=1$. Subtracting the Euler identity for P, we obtain (4).

Let us prove (2). Let $m\notin P$. Consider the convex hull H of P and m. Let R be the closure of $H\setminus P$. The set R is not convex in general; however, it can be contracted to m. Therefore, the Euler identities holds for R. Let $\mathcal{F}(R)$ be the set of faces of R. Let $\mathcal{F}_m(R)$ be the set of faces of R containing m. Write $\mathcal{F}(R)$ as the disjoint union $\mathcal{F}_m(R) \cup \mathcal{F}_n(R)$. Using relations (3.4), (3.5) for the polytope R and its vertex m, we obtain $\sum_{g\in\mathcal{F}_n(R)}(-1)^{\dim g} = 1$. It is easy to see that the faces g in $\mathcal{F}_n(R)$ are faces of P and that these are all the faces f of P such that $m \notin C_P^+(f)$. Thus we have $\sum_{f\in\mathcal{F}, m\notin C_P^+(f)}(-1)^{\dim f} = 1$. Subtracting the Euler identity for P, we obtain relation (2).

Finally, let us prove (6). We may assume that 0 is an interior point of P. Let $m \neq 0$. Choose a small positive number t. Recall that $C_{tP}^+(tf)$ denotes the inward pointing affine cone for the face tf of tP, where t is a small positive number. Then there exists t sufficiently small such that $m \in C_f$ if and only if $m \in C_{tP}^+(tf)$. Thus the last relation is deduced from relation (2) by considering the polytope tP for t sufficiently small. □

To a point m of V, we associate its δ-measure $\delta(m)$, defined as follows. For any continuous function ϕ on V, we have $(\delta(m), \phi) = \phi(m)$. If S is a discrete subset of V, we denote by $\delta(S) = \sum_{s\in S}\delta(s)$ its δ-measure.

The following proposition follows immediately from Proposition 3.1.

Proposition 3.2. *Let P be a convex lattice polytope. We have the equalities*

(1)

$$\delta(M \cap P) = \sum_{f\in\mathcal{F}}(-1)^{\dim f}\delta(M \cap C_P^+(f)),$$

(2)

$$(-1)^n\delta(M \cap P^0) = \sum_{f\in\mathcal{F}}(-1)^{\dim f}\delta(M \cap C_P^-(f)),$$

(3)

$$\delta(\{0\}) = \sum_{f\in\mathcal{F}}(-1)^{\dim f}\delta(M \cap C_f).$$

We will consider Fourier transforms of the measures $\delta(M\cap C_f)$. They make sense in the framework of generalized functions. We will use the function notation $\Theta(y)$ for a generalized function Θ on V^*, although the value of Θ at a particular point y may not have a meaning. We denote by $\int_{V^*}\Theta(y)\phi(y)dy$ the value of Θ on a test density $\phi(y)dy$. We will say that Θ is smooth on an open subset U of V^* if there exists a smooth function $\theta(y)$ on U such that $\int_{V^*}\Theta(y)\phi(y)dy = \int_{V^*}\theta(y)\phi(y)dy$ for all test functions ϕ with compact support contained in U. Then the value of Θ at $y \in U$ is defined to be $\theta(y)$. If there exist two smooth functions f, g on V^*, with g not identically 0, such that the equation $g(y)\Theta(y) = f(y)$ holds in the space of generalized functions on V^*, then Θ is smooth on the open set $U = \{y, g(y) \neq 0\}$ and $\Theta(y) = f(y)/g(y)$ on U.

Consider for example $V = \mathbb{R}$. Consider the discrete measure $\delta(\mathbb{Z}) = \sum_{n\in\mathbb{Z}}\delta(n)$. We denote its Fourier transform by $\Theta(y) = \sum_{k\in\mathbb{Z}} e^{iky}$. This means that the generalized function $\Theta(y)$ is the limit in the space of generalized functions of the smooth functions $\sum_{|k|\leq K} e^{iky}$. We have thus for a smooth test function ϕ on $\mathbb{R}$

$$\int_{\mathbb{R}}\Theta(y)\phi(y)dy = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} e^{iky}\phi(y)dy.$$

Clearly $(1 - e^{iy})(\sum_{k\in\mathbb{Z}} e^{iky}) = 0$ so that $\Theta(y)$ is supported on $2\pi\mathbb{Z}$. In fact, Poisson summation formula is

$$(2\pi)^{-1}\int_{\mathbb{R}} \Theta(y)\phi(y)dy = \sum_{k\in\mathbb{Z}} \phi(2\pi k).$$

Let

$$\mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}.$$

Let a be a complex number of modulus 1. Consider the discrete measure $h(a) = \sum_{n\in\mathbb{Z}^+} a^n\delta(n)$ and its Fourier transform $\Theta_a^+(y) = \sum_{n\in\mathbb{Z}^+} a^n e^{iny}$. We have the equality

$$(1 - ae^{iy})\Theta_a^+(y) = 1. \tag{3.6}$$

Thus the generalized function $\Theta_a^+(y)$ is smooth outside the set $i\mathrm{Log}a + 2\pi\mathbb{Z}$, and for $y \notin i\mathrm{Log}a + 2\pi\mathbb{Z}$,

$$\Theta_a^+(y) = \frac{1}{1 - ae^{iy}}. \tag{3.7}$$

In particular, $\Theta_a^+(y)$ is a rational function of e^{iy}. We will generalize this formula to higher dimensions.

We call a meromorphic function $\Phi(y)$ on $V_{\mathbb{C}}^*$ a rational function of e^y if for some basis $(e_1, \dots, e_n)$ of N, writing $y = \sum_{i=1}^n y_i^* e_i$, $\Phi(y_1^*, \dots, y_n^*)$ is a rational function of $e^{y_1^*}, \dots, e^{y_n^*}$. This does not depend on the choice of the basis of N.

Definition 3.3. Let C be a rational polyhedral convex cone in V. Denote by $\Theta(C)(y)$ the Fourier transform of $\delta(M \cap C)$:

$$\Theta(C)(y) = \sum_{m\in M\cap C} e^{i(m,y)}.$$

Proposition 3.4. *Let C be a rational polyhedral convex cone in V; let W be the largest linear subspace contained in C.*

(1) *The generalized function $\Theta(C)$ is supported on a discrete union of translates of $W^\perp$.*

(2) *If C is acute (i.e. $W = 0$), then there exists a meromorphic function ϕ on $V_{\mathbb{C}}^*$ such that for y outside a union of a discrete set of affine hyperplanes*

$$\Theta(C)(y) = \phi(iy).$$

The order at 0 of the function ϕ is at least $-n$. Moreover, $\phi(y)$ is a rational function of e^y.

Proof. Observe that W is a rational subspace of V. Moreover, for any $m_0 \in M\cap W$, $(1 - e^{i(m_0,y)})\Theta(C)(y) = 0$ as $M \cap C$ is invariant under translation by elements of $M \cap W$. Hence the support of Θ is contained in the set

$$\{y \in V^* \mid (y, m_0) \in 2\pi\mathbb{Z} \text{ for all } m_0 \in M \cap W\}$$

and this set is a discrete union of translates of $W^\perp$.

It is enough to prove (2) when C is simplicial. Indeed, we can always subdivide C by simplicial cones C_j, and then $\Theta(C)$ is a sum (with signs) of the generalized functions $\Theta(C_j)$. Now consider a simplicial cone $C \subset V$, the intersection of n distinct hyperplanes $(w_k, x) \geq 0$ with w_k in the lattice N of V^*. Let $w^k \in V$ be the dual basis of w_k. An element $x \in V$ is written as $x = \sum \eta_k w^k$, with $\eta_k = (w_k, x)$. Thus

$$C = \bigoplus_k \mathbb{R}^+ w^k.$$

Let $L = \bigoplus_{k=1}^n \mathbb{Z} w^k$ be the lattice of V spanned by w^k. We have

$$L \cap C = \bigoplus_k \mathbb{Z}^+ w^k.$$

Let χ be a multiplicative character of L. Consider the discrete measure

$$h(C, \chi, L) = \sum_{m \in L \cap C} \chi(m) \delta(m)$$

and its Fourier transform

$$\Theta(C, \chi, L)(y) = \sum_{m \in L \cap C} \chi(m) e^{i(m,y)}.$$

Formula (3.6) gives

$$\prod_{k=1}^n (1 - \chi(w^k) e^{i(w^k,y)}) \Theta(C, \chi, L)(y) = 1.$$

This equation implies that $\Theta(C, \chi, L)$ is smooth outside the zeroes of the analytic function $g(y) = \prod_{k=1}^n (1 - \chi(w^k) e^{i(w^k,y)})$. This zero set is a union of a discrete set of hyperplanes. Thus we obtain

Lemma 3.5. *For y outside a union of a discrete set of hyperplanes,*

$$\Theta(C, \chi, L)(y) = \frac{1}{\prod_{k=1}^n (1 - \chi(w^k) e^{i(w^k,y)})}.$$

With the notation as above, a basis of the dual lattice L^* to L consists of $w_1, \dots, w_n$. Let $T = V^*/L^* = V^*/(\bigoplus_k \mathbb{Z} w_k)$. Then T is an n-dimensional torus. Characters of L are parametrized by T: an element $g \in T$ gives a character χ_g by writing $\chi_g(x) = e^{2i\pi(x,y)}$ if $x \in L$ and if $y \in V^*$ represents $g \in V^*/L^*$.

Consider the finite subgroup $G = N/(\bigoplus_k \mathbb{Z} w_k) \subset T$. Recall that N is the dual lattice to M. Thus, for $x \in L$,

$$\begin{aligned} \sum_{g \in G} \chi_g(x) &= 0, \qquad \text{if } x \notin M, \\ \sum_{g \in G} \chi_g(x) &= |G|, \qquad \text{if } x \in M. \end{aligned}$$

We obtain

$$\delta(M \cap C) = |G|^{-1} \sum_{g \in G} h(C, \chi_g, L).$$

From Lemma 3.5, we obtain

Lemma 3.6. *Let $C \subset V$ be a rational simplicial cone. Then, for y outside a union of a discrete set of hyperplanes, we have*

$$\Theta(C)(y) = |G|^{-1} \sum_{g \in G} \frac{1}{\prod_{k=1}^n (1 - \chi_g(w^k) e^{i(w^k,y)})}.$$

This explicit formula for simplicial cones implies Proposition 3.4. To check that $\phi(y)$ is a rational function of e^y, let us state another formula for $\Theta(C)(y)$. Let $\alpha^1, \dots, \alpha^n$ be the primitive vectors of M on the edges $\mathbb{R}^+w^1, \dots, \mathbb{R}^+w^n$ of C. Then the set

$$S(C) = M \cap \left\{ \sum_{k=1}^{n} t_k \alpha^k \mid 0 \leq t_k < 1 \right\}$$

is finite and

$$\Theta(C)(y) \cdot \prod_{k=1}^{n} (1 - e^{i(\alpha^k, y)}) = \sum_{m \in S(C)} e^{i(m,y)} \ .$$

□

In the sequel, we will say that a property holds for generic $y \in V^*$ if there exists some nonzero analytic function g such that the property holds for all y with $g(y) \neq 0$.

Let P be a lattice polytope. Consider the Fourier transform of identities (1), (2), and (3) of Proposition 3.2. By definition, the Fourier transform of $\delta(M \cap P)$ is the function $y \mapsto D(P)(iy)$. We choose an integral element $v_0 \in M$ on each face f of P. Then $M \cap C_P^+(f) = v_0 + (M \cap C_f)$, while $M \cap C_P^-(f) = v_0 - (M \cap C_f)$. We obtain the following proposition.

Proposition 3.7. *We have the equality of generalized functions:*

(1)

$$D(P)(iy) = \sum_{f \in \mathcal{F}} (-1)^{\dim f} e^{i(v_0, y)} \Theta(C_f)(y),$$

(2)

$$(-1)^n D(P^0)(iy) = \sum_{f \in \mathcal{F}} (-1)^{\dim f} e^{i(v_0, y)} \Theta(C_f)(-y),$$

(3)

$$1 = \sum_{f \in \mathcal{F}} (-1)^{\dim f} \Theta(C_f)(y).$$

For each vertex s, consider the acute cone C_s. Proposition 3.4 shows that there exists a meromorphic function $\phi(C_s)$ on $V_{\mathbb{C}}^*$ (in fact, a rational function of e^y) such that $\Theta(C_s)(y) = \phi(C_s)(iy)$ for y generic.

Proposition 3.8. *We have the equalities of meromorphic functions on $V_{\mathbb{C}}^*$:*

(1)

$$D(P)(y) = \sum_{s \in \mathcal{F}(0)} e^{(s,y)} \phi(C_s)(y),$$

(2)

$$D(P^0)(y) = (-1)^n \sum_{s \in \mathcal{F}(0)} e^{(s,y)} \phi(C_s)(-y).$$

Furthermore

(3)
$$\sum_{s\in\mathcal{F}(0)} \phi(C_s)(y) = 1.$$

Proof. Consider formulae (1) and (2) of Proposition 3.7. When $f \in \mathcal{F}$ is a face of strictly positive dimension, the cone C_f contains the nonzero linear vector space $\langle f\rangle$, and $\Theta(C_f)$ is supported on a union of affine spaces of dimension strictly less than n. Thus we obtain the identities above for $y \in iV^*$ generic. The last identity is obtained from relation (3). □

Consider a simple lattice polytope P with associated fan Σ. In this case, we have explicit expressions for the meromorphic functions $\phi(C_s)$. Indeed C_s is the intersection of the n half-spaces $(u_F, y) \geq 0, F \in \mathcal{F}^s$. The elements u_F belong to N. Let $\sigma_s \in \Sigma$ be the polar cone to C_s. Recall the definitions of $U(\sigma)$ and of $G(\sigma)$ by formulae (2.3) and (2.4). The lattice $U(\sigma_s)$ is the lattice with $\mathbb{Z}$-basis $(u_F, F \in \mathcal{F}^s)$ and the group $G(\sigma_s)$ is the group $N/U(\sigma_s)$. Let $(m_s^F, F \in \mathcal{F}^s)$ be the dual basis to $(u_F, F \in \mathcal{F}^s)$. Applying Lemma 3.6, we obtain
$$\phi(C_s)(y) = \frac{1}{|G(\sigma_s)|}\sum_{g\in G(\sigma_s)} \frac{1}{\prod_{F\in\mathcal{F}^s}(1-\chi_g(m_s^F)e^{(m_s^F,y)})}.$$
This leads to the following explicit formulae for $D(P)$ and $D(P^0)$ as a sum of meromorphic functions attached to each vertex s of P. These formulae are the generalization of formula (3.2) in the 1-dimensional case.

Proposition 3.9. *For $y \in V_{\mathbb{C}}^*$ generic, we have*
$$\sum_{m\in M\cap P} e^{(m,y)} = \sum_{s\in\mathcal{F}(0)} \frac{e^{(s,y)}}{|G(\sigma_s)|}\sum_{g\in G(\sigma_s)} \frac{1}{\prod_{F\in\mathcal{F}^s}(1-\chi_g(m_s^F)e^{(m_s^F,y)})},$$
while
$$\sum_{m\in M\cap P^0} e^{(m,y)} == (-1)^n\sum_{s\in\mathcal{F}(0)} \frac{e^{(s,y)}}{|G(\sigma_s)|}\sum_{g\in G(\sigma_s)} \frac{1}{\prod_{F\in\mathcal{F}^s}(1-\chi_g(m_s^F)e^{-(m_s^F,y)})}.$$

Let P be a simple polytope in V. We now analyze the continuous version
$$E(P)(y) = \int_P e^{(x,y)}dx$$
of $D(P)$. We do not assume that P is a lattice polytope, as we will have to consider deformed polytopes $P(h)$. Let s be a vertex of P. We choose inward pointing normal vectors $(u_F, F \in \mathcal{F}^s)$ and dual elements $(m_s^F, F \in \mathcal{F}^s)$. Then the volume of the parallelepiped constructed on $(m_s^F, F \in \mathcal{F}^s)$ is equal to $|\det(m_s^F)|_{F\in\mathcal{F}^s}$. The following formula expresses the analytic function $E(P)$ as a sum over all vertices of meromorphic functions attached to each vertex s of P. This is the n-dimensional analogue of formula (3.1).

Proposition 3.10. *Let P be a simple polytope. Let $y \in V_{\mathbb{C}}^*$ be such that $(m_s^F, y) \neq 0$ for all vertices s and all $F \in \mathcal{F}^s$. Then*
$$\int_P e^{(x,y)}dx = (-1)^n \sum_{s\in\mathcal{F}(0)} e^{(s,y)}(|\det(m_s^F)|_{F\in\mathcal{F}^s})\frac{1}{\prod_{F\in\mathcal{F}^s}(m_s^F,y)}.$$

Proof. It is possible to give a direct argument for this proposition using Proposition 3.1 and explicit formulas for Fourier transforms of characteristic functions of simplicial cones. However, we can also deduce the value of $E(P)$ from the value of $D(P)$ by a limit argument using Riemann sums to evaluate an integral. Indeed, it is sufficient to prove this formula for lattice polytopes (choosing lattice M with smaller and smaller fundamental domain). We have

$$E(P)(y) = \lim_{q\to\infty} q^{-n} \sum_{m\in(M/q)\cap P} e^{(m,y)}$$

when q becomes a large integer.

We replace M by M/q in the formula of Proposition 3.9. We obtain

$$q^{-n} \sum_{m\in(M/q)\cap P} e^{(m,y)} = \sum_{s\in\mathcal{F}(0)} \frac{e^{(s,y)}}{|G(\sigma_s)|} \sum_{g\in G(\sigma_s)} \frac{1}{\prod_{F\in\mathcal{F}^s} q(1-\chi_g(m_s^F)e^{(m_s^F,y/q)})}.$$

We see that only the trivial term $g = 1$ in each group $G(\sigma_s)$ will contribute to the limit at $q = \infty$, and we obtain our proposition, as we observe that $|G(\sigma_s)|^{-1}$ is the absolute value of $\det(m_s^F)_{F\in\mathcal{F}^s}$. □

In particular, we have $\mathrm{vol}(P) = \lim_{t\to 0} E(P)(ty)$, and we obtain that for any generic y,

$$\mathrm{vol}(P) = \frac{(-1)^n}{n!} \sum_{s\in\mathcal{F}(0)} (|\det(m_s^F)|_{F\in\mathcal{F}^s}) \frac{(s,y)^n}{\prod_{F\in\mathcal{F}^s}(m_s^F,y)}. \tag{3.8}$$

Let P be a simple lattice polytope with facets $F_1, F_2, \ldots, F_d$. Let $h=(h_1,\ldots,h_d)$ be a small parameter of deformation.

Lemma 3.11. *Let ϕ be a polynomial function on V. For $h \in \mathbb{R}^d$ small, the function $I(\phi)(h) = \int_{P(h)} \phi(x)dx$ is polynomial in h.*

Proof. Consider

$$E(y)(h) = E(P(h))(y) = \int_{P(h)} e^{(x,y)}dx.$$

We compute $E(y)(h)$ using Proposition 3.10. Let s be a vertex of P. Let σ_s be the polar cone to C_s. We have

$$\sigma_s = \sum_{j,F_j\in\mathcal{F}^s} \mathbb{R}^+ u_j.$$

The subset of $\{1, 2, \ldots, d\}$ consisting of those j with $F_j \in \mathcal{F}^s$ is the set $\mathcal{E}(\sigma_s)$ of Definition 2.6. We denote by $(m_s^j, j \in \mathcal{E}(\sigma_s))$ the dual basis to $(u_j, j \in \mathcal{E}(\sigma_s))$. When h is small, the point $s(h)$ given by

$$s(h) = s - \sum_{j\in\mathcal{E}(\sigma_s)} h_j m_s^j$$

is a vertex of $P(h)$. Thus, for generic y,

$$E(y)(h) = \sum_{s\in\mathcal{F}(0)} E(s,y)(h), \tag{3.9}$$

where

$$E(s,y)(h) = \frac{(-1)^n}{|G(\sigma_s)|} \frac{e^{(s-\sum_{j\in\mathcal{E}(\sigma_s)} h_j m_s^j, y)}}{\prod_{j\in\mathcal{E}(\sigma_s)}(m_s^j, y)}.$$

The function $E(y)(h)$ is analytic in y. Considering the Taylor expansion of $t \mapsto E(ty)(h)$ at $t=0$, we obtain for every k and generic y,

$$\text{(3.10)} \qquad \frac{1}{k!}\int_{P(h)} (x,y)^k dx = \frac{(-1)^n}{(n+k)!} \sum_{s\in\mathcal{F}(0)} \frac{((s,y) - \sum_{j\in\mathcal{E}(\sigma_s)} h_j (m_s^j, y))^{n+k}}{|G(\sigma_s)| \prod_{j\in\mathcal{E}(\sigma_s)}(m_s^j, y)}.$$

The polynomial behaviour in h of $\int_{P(h)}(x,y)^k dx$ is apparent from this formula. As this result holds for any generic y and any k, we obtain our lemma. □

Remark that the Todd operator $\mathrm{Todd}(a, \partial/\partial h)$ is well defined on functions $h \mapsto e^{hz}$ provided z is sufficiently small. We rewrite also formula (3.9) as

$$\text{(3.11)} \qquad E(y)(h) = \sum_{s\in\mathcal{F}(0)} E(s,y)(h)$$

with

$$E(s,y)(h) = (-1)^n \left(\prod_{j\in\mathcal{E}(\sigma_s)} e^{-h_j (m_s^j, y)} \right) \frac{e^{(s,y)}}{|G(\sigma_s)|} \prod_{j\in\mathcal{E}(\sigma_s)} \frac{1}{(m_s^j, y)}.$$

This shows that $\mathrm{Todd}(\Sigma, \partial/\partial h)$ is well defined on

$$E(y)(h) = \int_{P(h)} e^{(x,y)} dx$$

provided that y is sufficiently small.

Theorem 3.12. *Let P be a simple lattice polytope, and let Σ be the associated fan. If $y \in V_{\mathbb{C}}^*$ is small, then*

$$\mathrm{Todd}(\Sigma, \partial/\partial h) \left(\int_{P(h)} e^{(x,y)} dx \right) \Big|_{h=0} = \sum_{m\in M\cap P} e^{(m,y)},$$

while

$$\mathrm{Todd}(\Sigma, -\partial/\partial h) \left(\int_{P(h)} e^{(x,y)} dx \right) \Big|_{h=0} = \sum_{m\in M\cap P^0} e^{(m,y)}.$$

Consider the Taylor expansion of both members of the first equality above at $y = 0$. We obtain

$$\mathrm{Todd}(\Sigma, \partial/\partial h) \left(\int_{P(h)} (x,y)^k dx \right) \Big|_{h=0} = \sum_{m\in M\cap P} (m,y)^k$$

for all $y \in V_{\mathbb{C}}^*$ and $k \in \mathbb{N}$. Thus Theorem 3.12 implies Theorem 2.15.

Proof. Consider formula (3.11) for the function $E(y)(h)$. For s a vertex of P, the function $E(s,y)(h)$ depends only of the variables h_j such that $j \in \mathcal{E}(\sigma_s)$. Let k be such that $k \notin \mathcal{E}(\sigma_s)$. From formula (2.7), we see that $\mathrm{Todd}(a^k, \partial/\partial h_k)E(s,y)(h) = 0$ if $a^k \neq 1$, while if $a^k = 1$, we have $\mathrm{Todd}(1, \partial/\partial h_k)E(s,y)(h) = E(s,y)(h)$. By

Lemma 2.13, if $\gamma \in \Gamma_\Sigma$ is not in $G(\sigma_s)$, then there is $k \notin \mathcal{E}(\sigma_s)$ such that $a^k(\gamma)$ is not 1. Thus $\mathrm{Todd}(\gamma, \partial/\partial h)E(s,y)(h) = 0$ if $\gamma \notin G(\sigma_s)$. We obtain

$$\mathrm{Todd}(\Sigma, \partial/\partial h)E(s,y)(h) = \sum_{\gamma \in G(\sigma_s)} \mathrm{Todd}(\gamma, \partial/\partial h)E(s,y)(h),$$

and for $\gamma \in G(\sigma_s)$,

$$\mathrm{Todd}(\gamma, \partial/\partial h)E(s,y)(h) = \left(\prod_{j \in \mathcal{E}(\sigma_s)} \mathrm{Todd}(a^j(\gamma), \partial/\partial h_j) \right) E(s,h).$$

We have

$$\mathrm{Todd}(a, \partial/\partial h)e^{uh}|_{h=0} = \mathrm{Todd}(a,u) = \frac{u}{1 - ae^{-u}}.$$

We obtain, for $\gamma \in G(\sigma_s)$,

$$\begin{aligned}
&\mathrm{Todd}(\gamma, \partial/\partial h)E(s,y)(h)|_{h=0} \\
&\quad = (-1)^n \prod_{j \in \mathcal{E}(\sigma_s)} \frac{-(m_s^j, y)}{(1 - a^j(\gamma)e^{(m_s^j,y)})} \frac{e^{(s,y)}}{|G(\sigma_s)|} \prod_{j \in \mathcal{E}(\sigma_s)} \frac{1}{(m_s^j, y)} \\
&\quad = \frac{e^{(s,y)}}{|G(\sigma_s)| \prod_{j \in \mathcal{E}(\sigma_s)} (1 - a^j(\gamma)e^{(m_s^j,y)})}.
\end{aligned}$$

By the definition of a^j, $a^j(\gamma) = \chi_\gamma(m_s^j)$. Comparing with the first formula of Proposition 3.9, we obtain the first formula of our theorem. By a similar proof, we obtain the second formula. □

4. The coefficients of the Ehrhart polynomial

Let P be a convex lattice polytope in V with nonempty interior P^0. Consider, for q a positive integer, the polytope qP. Let ϕ be a function on V. Let

$$i(\phi, P)(q) = \sum_{m \in M \cap (qP)} \phi(m)$$

and

$$i(\phi, P^0)(q) = \sum_{m \in M \cap (qP^0)} \phi(m).$$

As a consequence of Proposition 3.8, let us prove the following generalization of a well-known theorem of Ehrhart.

Proposition 4.1. *If ϕ is a homogeneous polynomial function of degree k, then the functions $q \mapsto i(\phi, P)(q)$ and $q \mapsto i(\phi, P^0)(q)$ are polynomial of degree $n+k$. Moreover, we have*

$$i(\phi, P^0)(q) = (-1)^{n+k} i(\phi, P)(-q)$$

and

$$i(\phi, P)(0) = \phi(0).$$

Proof. Observe that the vertices of qP are the qs (s a vertex of P) and that the tangent cone at qs to qP is C_s. Therefore, using Proposition 3.8, we obtain for generic y:

$$\sum_{m\in M\cap(qP)} e^{(m,y)} = \sum_{s\in\mathcal{F}(0)} e^{(qs,y)}\phi(C_s)(y)$$

and

$$\sum_{m\in M\cap(qP^0)} e^{(m,y)} = (-1)^n \sum_{s\in\mathcal{F}(0)} e^{(qs,y)}\phi(C_s)(-y).$$

Now replace y by ty for small nonzero t and consider the expansion into Laurent series in t. As $\phi(C_s)(y)$ is of order at least $-n$, we have $\phi(C_s)(ty) = \sum_{j\geq -n} t^j a_j^s(y)$, where $a_j^s(y)$ are homogeneous rational functions of degree j. We thus see that

$$\frac{1}{k!}\sum_{m\in M\cap(qP)} (m,y)^k = \sum_{s\in\mathcal{F}(0)}\sum_{j=0}^{k+n}\frac{1}{j!}q^j(s,y)^j a_{k-j}^s(y)$$

and that

$$\frac{1}{k!}\sum_{m\in M\cap(qP^0)} (m,y)^k = (-1)^n\sum_{s\in\mathcal{F}(0)}\sum_{j=0}^{k+n}\frac{1}{j!}q^j(s,y)^j(-1)^{k-j} a_{k-j}^s(y).$$

We thus obtain the polynomial behaviour in q of $i(\phi,P)(q)$ and of $i(\phi,P^0)(q)$ for the polynomial function $\phi(x)=(x,y)^k$ and the first identity as well. As this result holds for all y and k, it holds for all polynomial functions on V. We also obtain, for $q=0$ and for $\phi(x)=(x,y)^k$, that

$$\sum \frac{t^k}{k!} i(\phi,P)(0)$$

is equal to the Laurent series expansion of $\sum_{s\in\mathcal{F}(0)}\phi(C_s)(ty)$. Thus we obtain the second identity from formula (3) of Proposition 3.8 as we have identically $\sum_{s\in\mathcal{F}(0)}\phi(C_s)=1$. □

For $\phi=1$, the polynomial $i(\phi,P)(q)$ is called the Ehrhart polynomial. We denote it simply by $i(P)$. We write

$$i(P)(q) = |M\cap(qP)| = \sum_{k=0}^{n} q^k a_k(P).$$

It follows from Proposition 4.1 that the term $a_0(P)$ is equal to 1.

Let us give, for example, the values of $a_k(P)$ for the simplex $P(a,b,c)$ considered in Example 2.9. Let p and $q\geq 1$ be two coprime integers. Let $s(p,q)$ be the Dedekind sum, defined by

(4.1) $$s(p,q) = \sum_{i=1}^{q}\left(\left(\frac{i}{q}\right)\right)\left(\left(\frac{pi}{q}\right)\right),$$

where $((x))=0$ if x is integral and $((x)) = x-[x]-\frac{1}{2}$ otherwise.

We have

$$a_3(P)=abc/6, \qquad a_2(P)=(ab+bc+ca+1)/4, \qquad a_0(P)=1,$$

while $a_1(P)$ is equal to

$$\frac{1}{12}\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}+\frac{1}{abc}\right)+(a+b+c)/4-s(bc,a)-s(ca,b)-s(ab,c)+3/4.$$

This formula, originally due to Mordell, has been generalized recently by Pommersheim [9] and Kantor-Khovanskii [6]: more generally, they computed the coefficient $a_{n-2}(P)$ of the Ehrhart polynomial. Let us show how to deduce their results from Theorem 2.15, which gives (in principle) an explicit formula for the Ehrhart polynomial of any simple polytope.

Let P be a simple lattice polytope, and let Σ be its fan. Consider the Todd operator $\mathrm{Todd}(\Sigma,\partial/\partial h)$. We write it as the sum of its homogeneous components

$$\mathrm{Todd}(\Sigma,\partial/\partial h)=\sum_{k=0}^{\infty}A_k(\partial/\partial h).$$

Lemma 4.2. *We have*

$$a_{n-k}(P)=A_k(\partial/\partial h)\,\mathrm{vol}(P(h))|_{h=0}.$$

Proof. Let q be a positive integer. We have

$$|M\cap(qP)|=T(\Sigma,\partial/\partial h)\,\mathrm{vol}((qP)(h))|_{h=0}.$$

Formula (3.8) shows that for any generic y,

$$\mathrm{vol}((qP)(h))=\frac{(-1)^n}{n!}\sum_{s\in\mathcal{F}(0)}\frac{((qs,y)-\sum_{j\in\mathcal{E}(\sigma_s)}h_j(m_s^j,y))^n}{|G(\sigma_s)|\prod_{j\in\mathcal{E}(\sigma_s)}(m_s^j,y)}. \tag{4.2}$$

Thus the lemma follows. □

Let us write

$$\mathrm{Todd}(\Sigma,\partial/\partial h)=\mathrm{Todd}(\partial/\partial h)+R(\partial/\partial h)$$

with

$$\mathrm{Todd}(\partial/\partial h)=\prod_{j=1}^{d}\mathrm{Todd}(\partial/\partial h_j)$$

and

$$R(\partial/\partial h)=\sum_{\gamma\in\Gamma_\Sigma,\gamma\neq1}\mathrm{Todd}(\gamma,\partial/\partial h).$$

We write

$$\mathrm{Todd}(\partial/\partial h)=\sum_{k=0}^{\infty}T_k(\partial/\partial h)$$

and

$$R(\partial/\partial h)=\sum_{k=0}^{\infty}R_k(\partial/\partial h),$$

where T_k, R_k are homogeneous polynomials of degree k.

We have thus

$$a_{n-k}(P)=m_{n-k}(P)+r_{d-k}(P)$$

with

$$m_{n-k}(P) = T_k(\partial/\partial h) \cdot \operatorname{vol}(\Delta(h))|_{h=0}$$

and

$$r_{n-k}(P) = R_k(\partial/\partial h) \cdot \operatorname{vol}(\Delta(h))|_{h=0}.$$

Lemma 4.3. *We have*

$$T_0(\partial/\partial h) = I, \qquad T_1(\partial/\partial h) = \frac{1}{2}\sum_{j=1}^{d} \partial/\partial h_j,$$

while

$$R_0(\partial/\partial h) = 0, \qquad R_1(\partial/\partial h) = 0.$$

Proof. The first two equalities follow readily from formula (2.6).

The groups $G(\sigma)$ are trivial for $\sigma \in \Sigma(1)$. Thus there is no element γ of Γ_Σ with $a_k(\gamma) = 1$ for all k but one, and the last equalities follow from formula (2.7). □

More generally, by the same argument, we obtain the following

Lemma 4.4. *Assume $G(\sigma) = \{1\}$ for all cones $\sigma \in \Sigma$ of dimension at most K. Then $R_k(\partial/\partial h) = 0$ and hence $a_{n-k}(P) = m_{n-k}(P)$ for all $k \leq K$.*

Let $f \in \mathcal{F}$ be a face of P. Consider the vector space $\langle f \rangle$ and its lattice $M \cap \langle f \rangle$. We denote by $\operatorname{vol}(f)$ the volume of the face f with respect to the Lebesgue measure on $\langle f \rangle$ determined by this lattice.

Let f be a face of codimension 2. Then f is the intersection of two facets. To simplify notation, we assume that $f = F_1 \cap F_2$. Then σ_f (the polar cone of C_f) is generated by two normal vectors u_1 and u_2 to F_1 and F_2. The elements u_1, u_2 generate a sublattice $U(\sigma_f)$ of $N \cap \langle \sigma_f \rangle$. We have

$$G(\sigma_f) = (N \cap \langle \sigma_f \rangle)/U(\sigma_f).$$

As u_1 is primitive, we can always choose a $\mathbb{Z}$-basis n_1, n_2 of $N \cap \langle \sigma_f \rangle$ such that $u_1 = n_1$ and $u_2 = pn_1 + qn_2$ with $1 \leq p \leq q$. The integers (p, q) are coprime. We have $q = |G(\sigma)|$. Recall formula (4.1) for $s(p, q)$.

Definition 4.5. Let f be a face of codimension 2. Using notation above, define

$$\mu(f) = \frac{1}{4} - \frac{1}{4q} + s(p, q).$$

Proposition 4.6 ([9], [6])**.** *We have*

$$m_n(P) = \operatorname{vol}(P), \qquad r_n(P) = 0,$$

$$m_{n-1}(P) = \frac{1}{2}\sum_{F \in \mathcal{F}(n-1)} \operatorname{vol} F, \qquad r_{n-1}(P) = 0,$$

$$r_{n-2}(P) = \sum_{f \in \mathcal{F}(n-2)} \mu(f) \operatorname{vol}(f).$$

Proof. Let σ be an element of the fan Σ. Define

$$e(\sigma, \partial/\partial h) = \prod_{j\in\mathcal{E}(\sigma)} \partial/\partial h_j.$$

From formula (4.2), we readily obtain

Lemma 4.7. *Let f be a face of P, and $\sigma_f \in \Sigma$ the corresponding cone. Then we have*

$$e(\sigma_f, \partial/\partial h)\operatorname{vol}(P(h))|_{h=0} = |G(\sigma_f)|^{-1}\operatorname{vol}(f).$$

The values of $a_n(P)$, $a_{n-1}(P)$ are well known and easily obtained from Lemmas 4.3 and 4.7. It remains to obtain the value of $r_{n-2}(P)$. Consider the subset Γ_2 of Γ_Σ defined by

$$\Gamma_2 = \bigcup_{\sigma\in\Sigma(2)} G(\sigma).$$

Let $\Gamma_2' = \Gamma_2 - \{1\}$. We have

$$R_2(\partial/\partial h) = \sum_{\gamma\in\Gamma_2'} \operatorname{Todd}(\gamma, \partial/\partial h).$$

For $\sigma \in \Sigma(2)$, let $G(\sigma)' = G(\sigma) - \{1\}$. Then Γ_2' is the disjoint union of the sets $G(\sigma)'$ when σ varies in $\Sigma(2)$. We study

$$R_2(\sigma, \partial/\partial h) = \sum_{\gamma\in G(\sigma)'} \operatorname{Todd}(\gamma, \partial/\partial h)$$

for $\sigma \in \Sigma(2)$. To simplify notation, we write $\sigma = \mathbb{R}^+u_1 + \mathbb{R}^+u_2$, and as before we choose a $\mathbb{Z}$-basis n_1, n_2 of $\langle\sigma\rangle \cap N$ such that $u_1 = n_1, u_2 = pn_1 + qn_2$. Elements of $G(\sigma) = (\langle\sigma\rangle \cap N)/U(\sigma)$ are represented by elements jn_2 with $0 \leq j < q$. If $\gamma = jn_2$, we write $\gamma = -(jp/q)u_1 + (j/q)u_2$. By definition $a^k(\gamma) = 1$ except for $k = 1, 2$. We have $a^1(\gamma) = e^{-2\pi ijp/q}$, while $a^2(\gamma) = e^{2i\pi j/q}$. Thus by formula (2.7)

$$R_2(\sigma, \partial/\partial h) = \left(\sum_{j=1}^{q-1}(1 - e^{-2\pi ijp/q})^{-1}(1 - e^{2i\pi j/q})^{-1}\right)(\partial/\partial h_1)(\partial/\partial h_2).$$

If f is the face of P such that $\sigma = \sigma_f$, we have by Lemma 4.7,

$$(\partial/\partial h_1)(\partial/\partial h_2)\operatorname{vol}(P(h))|_{h=0} = q^{-1}\operatorname{vol}(f).$$

By formula (18a) of [10], we have

$$q^{-1}\sum_{j=1}^{q}(1 - e^{2i\pi jp/q})^{-1}(1 - e^{-2i\pi j/q})^{-1}$$
$$= -s(-p, q) + \frac{q-1}{4q} = s(p, q) - \frac{1}{4q} + \frac{1}{4}.$$

Thus we obtain

$$R_2(\sigma_f)\operatorname{vol}(P(h))_{h=0} = \mu(f)\operatorname{vol}(f).$$

Summing over all faces of codimension 2, we obtain the desired formula for $r_{n-2}(P)$.

□

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