Analytic continuation of a parametric polytope and wall-crossing

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Abstract. We define a set theoretic "analytic continuation" of a polytope defined by inequalities. For the regular values of the parameter, our construction coincides with the parallel transport of polytopes in a mirage introduced by Varchenko. We determine the set-theoretic variation when crossing a wall in the parameter space, and we relate this variation to Paradan's wall-crossing formulas for integrals and discrete sums. As another application, we refine the theorem of Brion on generating functions of polytopes and their cones at vertices. We describe the relation of this work with the equivariant index of a line bundle over a toric variety and Morelli constructible support function.

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1 Introduction

Consider a polytope $\mathfrak{q}(b)$ in $\mathbb{R}^d$ defined by a system of N linear inequalities:

$$\mathfrak{q}(b) := \{y \in \mathbb{R}^d;\ \langle \mu_i, y\rangle \leq b_i,\quad 1 \leq i \leq N.\} \tag{1.1}$$

In this article, we study the variation of the polytope $\mathfrak{q}(b)$ when the parameter $b = (b_i)$ varies in $\mathbb{R}^N$, but the linear forms μ_i are fixed (the parametric arrangement of hyperplanes $\langle \mu_i, y\rangle = b_i$ so obtained is called a mirage in [20]).

Our main construction is the following. Starting with a parameter b^0 which is regular (this is defined below), we construct a function $\mathcal{X}(x_1, x_2, \ldots, x_N)$ on $\mathbb{R}^N$ which is a linear combination of characteristic functions of various semi-open coordinate quadrants in $\mathbb{R}^N$. Define

$$A(b)(y) = \mathcal{X}(b_1 - \langle \mu_1, y\rangle, \ldots, b_N - \langle \mu_N, y\rangle).$$

The crucial feature of the function $\mathcal{X}$ is that, for b near b^0, $A(b)(y)$ is the characteristic function of the polytope $\mathfrak{q}(b)$, but $A(b)$ enjoys analyticity properties with respect to the parameter b when b moves in $\mathbb{R}^N$, that we will explain below. So we say that $A(b)$ is the "analytical continuation "of the polytope $\mathfrak{q}(b)$ (with initial value b^0).

Before stating these properties, let us give two examples. We denote by p_i the characteristic function of the closed coordinate half-space, $p_i = [x_i \geq 0]$, and we set $q_i = 1 - p_i = [x_i < 0]$. First, let $\mathfrak{q}$ be the d-dimensional simplex defined by the $d+1$ inequalities $y_i \geq 0$, $\sum_{i=1}^d y_i \leq 1$. In this case we have, (see Example 3.1),

$$\mathcal{X}(x) = p_1 \cdots p_{d+1} + (-1)^d q_1 \cdots q_{d+1}.$$

Thus $\mathcal{X}(x)$ is the sum of the [characteristic function of the] closed positive coordinate quadrant in $\mathbb{R}^{d+1}$ and of $(-1)^d$ times the open negative one. Let $b = (b_1, \ldots, b_{d+1})$. If $b_1 + \cdots + b_{d+1} \geq 0$, then $A(b)(y) =$

$\mathcal{X}(b_1+y_1,\cdots,b_d+y_d,b_{d+1}-(y_1+\cdots+y_d))$ is the characteristic function of the simplex $\{y_i \geq -b_i, \sum_{i=1}^d y_i \leq b_{d+1}\}$, while if $b_1+\cdots+b_{d+1} < 0$, then $A(b)(y)$ is equal to $(-1)^d$ times the characteristic function of the symmetric open simplex $\{y_i < -b_i, \sum_{i=1}^d y_i > b_{d+1}\}$. In particular, in dimension $d = 1$, starting with the closed interval $[0, 1]$, the analytic continuation $A(b)$ is the closed interval $\{-b_1 \leq y \leq b_2\}$ when $b_1 + b_2 \geq 0$, while $A(b)$ is (-1) times the open interval $\{b_2 < y < -b_1\}$ when $b_1 + b_2 < 0$ (Figure 1.1)

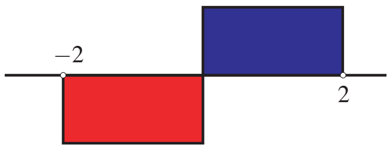

Figure 1.1. In blue for $b = (0, 2)$, $\mathfrak{q}(b) = [0, 2]$, in red for $b = (0, -2)$, $A(b) = (-1)$ times $]-2, 0[$.

For the second example, we start with the tetragon illustrated in Figure 1.2 defined by the 4 inequalities $y_2 + 2 \geq 0$, $y_1 + 1 \geq 0$, $y_1 + y_2 \leq 0$, $y_1 - y_2 \geq 0$. In this case we have (see Example 4.2 and Subsection 4.5)

$$\mathcal{X}(x) = p_1p_2p_3p_4 - p_1q_2q_3p_4 - q_1p_2p_3q_4 + q_1q_2q_3q_4,$$

a signed sum of characteristic functions of 4 semi-open quadrants.

Some values of the analytic continuation $A(b)$ are illustrated in Figures 1.2 and 1.3. For each value of b, it is a signed sum of semi-open polygons. Components with a $+$ sign are colored in blue, components with a -1 sign are colored in red. Semi-openness is indicated by dotted lines.

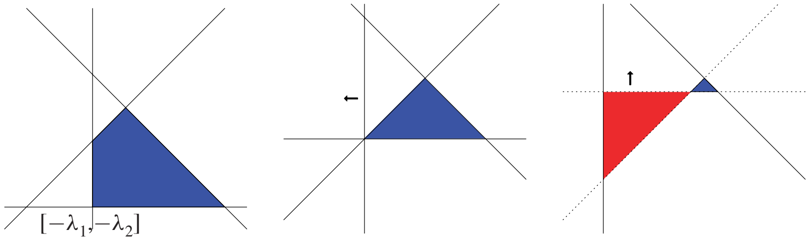

Figure 1.2. Analytic continuation of a tetragon.

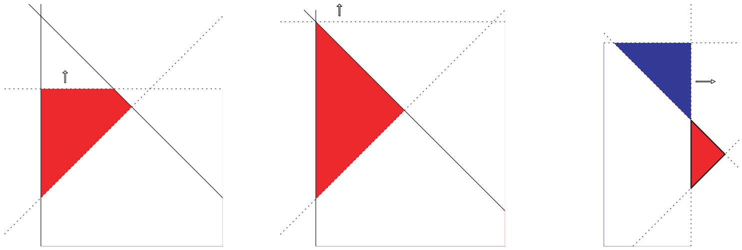

Figure 1.3. More analytic continuation of a tetragon.

Let us describe now some of the properties of $A(b)$.

A point $b = (b_i) \in \mathbb{R}^N$ is called regular (with respect to the sequence of linear forms μ_i) if a subset of k equations among the equations $\{\mu_i = b_i\}$ do not have a common solution if $k > d$. We define a tope τ to be a connected component of the open set of regular points b in $\mathbb{R}^N$. Topes are separated by hyperplanes which we call walls.

Let $b^0 \in \mathbb{R}^N$ be regular. Recall that we assume that $\mathfrak{q}(b^0)$ is compact. In this case, each vertex of the polytope $\mathfrak{q}(b^0)$ belongs to exactly d facets, in other words the polytope $\mathfrak{q}(b^0)$ is simple. Loosely speaking, the shape of the polytope $\mathfrak{q}(b)$ does not change when b remains close to b^0. The facets of $\mathfrak{q}(b)$ remain parallel to those of $\mathfrak{q}(b^0)$, while its vertices depend linearly on b. When b crosses a wall, the shape of $\mathfrak{q}(b)$ changes.

Let $h(y)$ be a polynomial function on $\mathbb{R}^d$. The integral

$$\int_{\mathfrak{q}(b)} h(y)dy,$$

and the discrete sum

$$\sum_{y\in\mathfrak{q}(b)\cap\mathbb{Z}^d} h(y)$$

are classical topics. In particular, if h is the constant function 1, these quantities are respectively the volume of the polytope $\mathfrak{q}(b)$ and the number of integral points in the polytope $\mathfrak{q}(b)$. It is well-known that the function $b \to \int_{\mathfrak{q}(b)} h(y)dy$ is given on each tope by a polynomial function of b. Moreover, if we assume that the linear forms μ_i are rational, the discrete sum $b \to \sum_{y\in\mathfrak{q}(b)\cap\mathbb{Z}^d} h(y)$ is given on each tope by a quasi-polynomial function of b. These results follow for instance from Brion's theorem of decomposing a polytope as a sum of its tangent cones at vertices, [6, 9]. When the parameter b crosses a wall of the tope τ, the integral $b \to \int_{\mathfrak{q}(b)} h(y)dy$ is given by a different polynomial, the discrete sum by a different quasi-polynomial. Their wall-crossing variations have been computed by Paradan, in a more general context of Hamiltonian geometry, using transversally elliptic operators, [18].

The function $\mathcal{X}$ which we construct in this article depends on the tope τ which contains the starting value b^0, and we will study its dependence with respect to τ. Therefore, we write $\mathcal{X}(\tau)(x)$ and $A(\tau, b)(y) = \mathcal{X}(\tau)(b_1 - \langle\mu_1, y\rangle, \ldots, b_N - \langle\mu_N, y\rangle)$ instead of $\mathcal{X}(x)$ and $A(b)(y)$ from now on. The function $y \mapsto A(\tau, b)(y)$ enjoys the following properties.

- When b is in the closure $\overline{\tau}$ of the tope τ, $A(\tau, b)$ coincides with the characteristic function $[\mathfrak{q}(b)]$ of $\mathfrak{q}(b)$.
- The function $A(\tau, b)(y)$ is a linear combination with integral coefficients of characteristic functions of bounded faces of various dimensions of the arrangement of hyperplanes $\langle\mu_i, y\rangle = b_i, \quad 1 \leq i \leq N$.
- The integral
$$\int_{\mathbb{R}^d} A(\tau, b)(y) e^{\langle\xi, y\rangle} dy$$
is an analytic function of $(\xi, b) \in (\mathbb{R}^d)^* \times \mathbb{R}^N$. For $b \in \overline{\tau}$, it coincides with $\int_{\mathfrak{q}(b)} e^{\langle\xi, y\rangle} dy$. If $h(y)$ is a polynomial function, then $b \mapsto \int_{\mathbb{R}^d} A(\tau, b)(y) h(y) dy$ is a polynomial function of $b \in \mathbb{R}^N$ which coincides with $\int_{\mathfrak{q}(b)} h(y) dy$ when $b \in \overline{\tau}$.
- Moreover, if we assume that the μ_i are rational, the discrete sum
$$\sum_{y \in \mathbb{Z}^d} A(\tau, b)(y) h(y)$$
is a quasi-polynomial function of b, (see Definition 5.1 of quasi-polynomial functions). It coincides with $\sum_{y \in \mathfrak{q}(b) \cap \mathbb{Z}^d} h(y)$ for b in the initial tope and even in a neighborhood of its closure (see the precise statement in Corollary 3.1).

For instance, let us look again at the closed interval $[0, b]$. For $b \in \mathbb{N}$, the number of integral points in $[0, b]$ is given by the polynomial function $b + 1$. For a negative integer $b < 0$, the value $b + 1$ is indeed equal to (-1) times the number of integral points in the open interval $b < y < 0$.

The key idea is to define $A(\tau, b)$ as a signed sum of closed affine cones, shifted when b varies, so that their vertices depend linearly on the parameter b. We use decompositions of a polytope $\mathfrak{p}$ as a signed sum of cones, such as the Brianchon-Gram decomposition, (see for instance [8]).

Theorem 1.1 (Brianchon-Gram decomposition). *Let $\mathfrak{p} \subset \mathbb{R}^d$ be a polytope. For each face $\mathfrak{f}$ of $\mathfrak{p}$, let $\mathfrak{t}_{\mathrm{aff}}(\mathfrak{p}, \mathfrak{f}) \subseteq \mathbb{R}^d$ be the affine tangent cone to $\mathfrak{p}$ at the face $\mathfrak{f}$. Then*
$$[\mathfrak{p}] = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} (-1)^{\dim \mathfrak{f}} [\mathfrak{t}_{\mathrm{aff}}(\mathfrak{p}, \mathfrak{f})], \tag{1.2}$$

where $\mathcal{F}(\mathfrak{p})$ is the set of faces of $\mathfrak{p}$.

Here, for a set $E \subset \mathbb{R}^d$, we denote by $[E]$ the function on $\mathbb{R}^d$ which is the characteristic function of the set E.

For regular values of b, our construction of $A(\tau, b)$ coincides with the parallel transport of Varchenko [20], the idea of which is quite simple. For instance, write the Brianchon-Gram formula for the closed interval $0 \leq y \leq b$,

$$[0 \leq y \leq b] = [y \leq b] + [y \geq 0] - [\mathbb{R}].$$

If the vertex b moves to the left, crosses the origin and becomes negative, the right hand side of the Brianchon-Gram formula becomes first, for $b = 0$, the characteristic function of the point 0, then for $b < 0$, the characteristic function of the open interval $b < y < 0$ with a minus sign.

Actually, instead of the Brianchon-Gram decomposition, Varchenko uses the polarized decomposition into semi-closed cones at vertices which he obtains in [20]. However, we go beyond [20] in several ways. First, as we already mentioned, we introduce (and compute) the "precursor" function $\mathcal{X}(\tau)$, a sum of characteristic functions of semi-open quadrants, which gives rise to $A(\tau, b)$ for all b. Moreover, we compute explicitly the wall-crossing variation

$$[\mathfrak{q}(b)] - A(\tau, b)$$

when b belongs to a tope adjacent to the starting tope τ. Actually, we compute the wall crossing variation at the level of the "precursor" function $\mathcal{X}(\tau)$ itself.

Finally, we show that "analytic continuation" of the faces of the polytope $\mathfrak{q}(b^0)$ occurs naturally, when one wants to compute $\sum_{y\in\mathfrak{q}(b_0)\cap\mathbb{Z}^d} e^{\langle\xi,y\rangle}$ for a degenerate value of ξ.

Let us now summarize the results of this article. We need some notations. It is more convenient to work in the framework of partition polytopes. So, let us first recall how one goes from the framework of linear inequalities $\langle\mu_i, y\rangle \leq b_i$ to that of partition polytopes. A partition polytope $\mathfrak{p}(\Phi, \lambda)$ is determined by a sequence $\Phi = (\phi_j)_{1\leq j\leq N}$ of elements of a vector space F (of dimension r) and an element $\lambda \in F$, as follows:

Definition 1.1.

$$\mathfrak{p}(\Phi, \lambda) = \left\{x \in \mathbb{R}^N; \sum_{j=1}^{N} x_j\phi_j = \lambda, \quad x_j \geq 0.\right\}$$

We assume that the cone $\mathfrak{c}(\Phi)$ generated by the ϕ_j's, is salient and that Φ generates F. Thus the set $\mathfrak{p}(\Phi, \lambda)$ is compact whenever $\lambda \in \mathfrak{c}(\Phi)$ (if λ is not in $\mathfrak{c}(\Phi)$, then $\mathfrak{p}(\Phi, \lambda)$ is empty.) The polytope $\mathfrak{p}(\Phi, \lambda)$ is, by definition, the intersection of the affine subspace

$$V(\Phi, \lambda) = \left\{ x \in \mathbb{R}^N; \sum_{j=1}^{N} x_j \phi_j = \lambda \right\}$$

with the standard quadrant

$$Q := \left\{ x \in \mathbb{R}^N; x_j \geq 0 \right\}.$$

A wall in F is a hyperplane generated by $r - 1$ linearly independent elements of Φ. An element $\lambda \in F$ is called Φ-regular, if λ does not lie on any wall. If $\lambda \in \mathfrak{c}(\Phi)$ is regular, the polytope $\mathfrak{p}(\Phi, \lambda)$ is a simple polytope of dimension $d = N - r$ contained in the affine space $V(\Phi, \lambda)$.

Consider the map $M : \mathbb{R}^N \to F$ given by $M(x) = \sum_i x_i \phi_i$. Let $V \subset \mathbb{R}^N$ be the kernel of M.

$$V = \left\{ x \in \mathbb{R}^N; \sum_{j=1}^{N} x_j \phi_j = 0 \right\}.$$

So V has dimension $d = N - r$.

If E is a subset of $\mathbb{R}^N$, we denote now by $[E]$ the function on $\mathbb{R}^N$ which is the characteristic function of E. Thus, for any subset E of $\mathbb{R}^N$, $[E \cap V] = [E][V]$.

If $\lambda = M(b) = \sum_i b_i \phi_i$, the map

$$x \to x + b \tag{1.3}$$

is an isomorphism between V and the affine space $V(\Phi, \lambda)$.

Let μ_i be the linear form $-x_i$ restricted to V. The bijection $V \to V(\Phi, \lambda)$ maps the polytope $\mathfrak{q}(b) = \{y \in V;\ \langle \mu_i, y \rangle \leq b_i\}$ onto $\mathfrak{p}(\Phi, \lambda)$. Indeed, the point $(y_1 + b_1, \ldots, y_N + b_N)$ is in $\mathfrak{p}(\Phi, \lambda)$ if and only if $-y_i \leq b_i$.

Moreover, b is regular with respect to the sequence of linear forms μ_i on V if and only if $\lambda = M(b)$ is Φ-regular in F. A connected component of the set of Φ-regular elements of F will be called a Φ-tope. Thus a subset $\tau \subset F$ is a Φ-tope if and only if $M^{-1}(\tau) \subset \mathbb{R}^N$ is a connected component of the set of regular parameters, *i.e.* a tope with respect to (μ_i).

It is clearly equivalent to study the variation of the polytope $\mathfrak{q}(b)$ when b varies, or the variation of the partition polytope $\mathfrak{p}(\Phi, \lambda)$, when λ varies.

In this framework, the inequations $x_j \geq 0$ are fixed, while the affine space $V(\Phi, \lambda)$ varies. For example, Figure 1.4 shows the interval $[0, b]$, in blue, now realized as $\{x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = b\}$. The analytic continuation $A(\tau, b)$ for $b < 0$ is colored in red on this figure, where a minus sign is assigned to red.

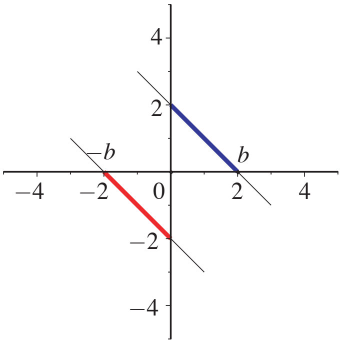

Figure 1.4. $F = \mathbb{R}$, $\Phi = (1, 1)$.

We fix a Φ-tope $\tau \subset F$, and consider $\lambda \in \tau$. Recall the combinatorial description of the faces of the partition polytope $\mathfrak{p}(\Phi, \lambda)$. We denote by $\mathcal{G}(\Phi, \tau)$ (respectively $\mathcal{B}(\Phi, \tau)$) the set of $I \subseteq \{1, \ldots, N\}$ such that $\{\phi_i, i \in I\}$ generates F (respectively is a basis of F) and such that τ is contained in the cone generated by $\{\phi_i, i \in I\}$. The set of faces (respectively vertices) of $\mathfrak{p}(\Phi, \lambda)$ is in one-to-one correspondence with $\mathcal{G}(\Phi, \tau)$ (respectively $\mathcal{B}(\Phi, \tau)$). The face which corresponds to I is

$$\mathfrak{f}_I(\Phi, \lambda) = \left\{ x \in \mathbb{R}^N_{\geq 0}, \sum_{j=1}^{N} x_j \phi_j = \lambda, \quad x_j = 0 \text{ for } j \in I^c \right\}. \tag{1.4}$$

The affine tangent cone to $\mathfrak{p}(\Phi, \lambda)$ at the face $\mathfrak{f}_I(\Phi, \lambda)$ is

$$\mathfrak{t}_I(\Phi, \lambda) = \left\{ x \in \mathbb{R}^N, \sum_{j=1}^{N} x_j \phi_j = \lambda, x_j \geq 0 \text{ for } j \in I^c \right\}. \tag{1.5}$$

If λ is in $\mathfrak{c}(\Phi)$, but is not in the tope τ, then the partition polytope $\mathfrak{p}(\Phi, \lambda)$ is not empty, but its faces are no longer in one-to-one correspondence with $\mathcal{G}(\Phi, \tau)$, (see Figure 1.2). Nevertheless, the cone in (1.5) makes sense **for every** $\lambda \in F$: it remains "the same cone" $\{x \in V; x_j \geq 0, j \in I^c\}$ up to a shift, under the map $V(\Phi, \lambda) \to V$ (see Formula (2.2)).

We introduce now the main character of this story, the function on $\mathbb{R}^N$ previously denoted by $\mathcal{X}(\tau)$.

Definition 1.2. The Geometric Brianchon-Gram function is

$$\mathcal{X}(\Phi, \tau) = \sum_{I \in \mathcal{G}(\Phi,\tau)} (-1)^{|I|-\dim F} \prod_{j \in I^c} [x_j \geq 0].$$

Let us compute this function for the case of $\Phi = (1, 1)$ in $F = \mathbb{R}$. Then

$$\mathcal{X}(\Phi, \tau) = [x_1 \geq 0] + [x_2 \geq 0] - [\mathbb{R}^2]$$

is equal to

$$[x_1 \geq 0, x_2 \geq 0] - [x_1 < 0, x_2 < 0],$$

the characteristic function of the closed positive quadrant minus the characteristic function of the open negative quadrant, (Figure 1.5).

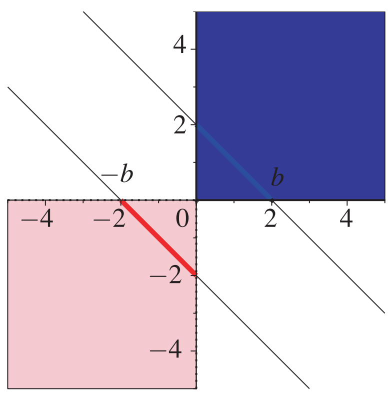

Figure 1.5. The function $\mathcal{X}(\Phi, \tau)$ for $\Phi = (1, 1)$.

If $\lambda \in \tau$, the Brianchon-Gram theorem implies

$$\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)] = [\mathfrak{p}(\Phi, \lambda)], \tag{1.6}$$

the characteristic function of the partition polytope $\mathfrak{p}(\Phi, \lambda)$. However, the function $\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)]$ is defined for any $\lambda \in F$. It is a signed sum of characteristic functions of closed cones intersected with the affine space $V(\Phi, \lambda)$.

For instance, in the case of $\Phi = (1, 1)$, taking the product of $\mathcal{X}(\Phi, \tau)$ with the characteristic function of the affine line $x_1 + x_2 = b$, we clearly recover the analytic continuation pictured in Figure 1.4.

One of our first results (and our main technical tool) (Theorem 3.3) is the fact that the Brianchon-Gram combinatorial function $X(\Phi, \tau)(p, q)$ coincides with the analogous function associated with any Lawrence-Varchenko polarized decomposition of a polytope into semi-closed cones at vertices [15, 20].

From this result, we deduce that the function $\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)]$ is the signed sum of characteristic functions of semi-open polytopes, in particular the support of this function is bounded for any $\lambda \in F$ (Corollary 3.2).

Reverting to the framework of linear inequalities, we define now $A(\tau, b)$ to be the inverse image of $\mathcal{X}(\Phi, \tau)$ under the map $v \to v + b$ from V to $\mathbb{R}^N$. For $b \in \tau$, $A(\tau, b)$ is the characteristic function of the polytope $\mathfrak{q}(b)$. For any value of b, it follows from the definition that $A(\tau, b)$ is the signed sum of the characteristic functions of the tangent cones to the faces of the initial polytope $\mathfrak{q}(b_0)$, with $b_0 \in \tau$, followed"by continuity ". The above qualitative result implies that $A(\tau, b)$ is a signed sum of bounded faces of various dimensions of the mirage $\mu_i = b_i$. It is easy to see that $A(\tau, b)$ enjoys the analyticity properties stated above.

Our main result is a wall crossing formula which we prove in a purely combinatorial context.

As the space $\mathbb{R}^N$ is the disjoint union of the semi-closed quadrants $Q^B_{\mathrm{neg}} := \{x = (x_i); x_i < 0 \text{ for } i \in B,\ x_i \geq 0 \text{ for } i \in B^c\}$, we write $\mathcal{X}(\Phi, \tau)$ in terms of the characteristic functions of these quadrants.

We introduce the following polynomial in the variables p_i and q_i.

Definition 1.3. Let τ be a Φ-tope. The Combinatorial Brianchon-Gram function associated to the pair (Φ, τ) is

$$X(\Phi, \tau)(p, q) = \sum_{I \in \mathcal{G}(\Phi,\tau)} (-1)^{|I|-\dim F} \prod_{i \in I^c} p_i \prod_{i \in I} (p_i + q_i). \tag{1.7}$$

We recover $\mathcal{X}(\Phi, \tau)$ when we substitute $p_i = [x_i \geq 0]$ and $q_i = [x_i < 0]$ in $X(\Phi, \tau)(p, q)$ (so that $p_i + q_i = 1$).

For example, when $\Phi = (1, 1)$, we have

$$X(\Phi, \tau) = p_1(p_2+q_2)+p_2(p_1+q_1)-(p_1+q_1)(p_2+q_2) = p_1p_2-q_1q_2.$$

The polynomial $X(\Phi, \tau)$ enjoys remarkable properties. Let us say that the quadrant Q^B_{neg} is Φ-bounded, if the intersection of its closure $\overline{Q^B_{\mathrm{neg}}}$ with V is reduced to 0. Equivalently, the intersection of Q^B_{neg} with the affine space $V(\Phi, \lambda)$ is bounded for any $\lambda \in F$.

We have

$$X(\Phi, \tau)(p, q) = \sum_B z_B \prod_{i \in B} p_i \prod_{i \in B^c} q_i.$$

where, for any subset $B \subseteq \{1, \ldots, N\}$ such that $z_B \neq 0$, the associated quadrant Q^B_{neg} is Φ-bounded. The coefficients z_B are in $\mathbb{Z}$ and we give an algorithmic formula for them.

As we will observe in the last section, the decomposition in Φ-bounded quadrants of $X(\Phi, \tau)$ is an analogue of the fact that the $\overline{\partial}$ cohomology spaces of a compact complex manifold are finite dimensional.

Our main result is Theorem 4.1, where we compute the function $X(\Phi, \tau_1) - X(\Phi, \tau_2)$, when τ_1 and τ_2 be two adjacent topes (meaning that the intersection of their closures is contained in a wall H and spans this wall).

We will not state the formula for $X(\Phi, \tau_1) - X(\Phi, \tau_2)$ in this introduction, but let us just mention a significant corollary, the wall-crossing formula for the polytope $\mathfrak{p}(\Phi, \lambda)$. Let A be the set of $i \in \{1, \dots, N\}$ such that ϕ_i belongs to the open side of H which contains τ_1 (hence $-\phi_i$ belongs to the side of τ_2). Let

$$\mathfrak{p}_{\text{flip}}(\Phi, A, \lambda) = \{x \in V(\Phi, \lambda); x_i < 0 \text{ if } i \in A, x_i \geq 0 \text{ if } i \notin A\}.$$

Thus $\mathfrak{p}_{\text{flip}}(\Phi, A, \lambda)$ is a semi-closed bounded polytope in $V(\Phi, \lambda)$.

Theorem 1.2. *Let τ_1 and τ_2 be adjacent topes. If $\lambda \in \tau_2$, we have*

$$\mathcal{X}(\Phi, \tau_1)[V(\Phi, \lambda)] = [\mathfrak{p}(\Phi, \lambda)] - (-1)^{|A|}[\mathfrak{p}_{\text{flip}}(\Phi, A, \lambda)]. \tag{1.8}$$

This formula is clearly inspired by the results of Paradan [18]. In turn, we show that it implies the convolution formula of Paradan which involves the number of lattice points of some lower dimensional polytopes associated to $\Phi \cap H$."

The above formula implies that, after crossing a wall, the analytic continuation of the original polytope $\mathfrak{p}(\Phi, \lambda)$ is the signed sum of two polytopes, among which one, but no more than one, may be empty. As illustrated in Section 4.5, we see the new polytope $\mathfrak{p}_{\text{flip}}(\Phi, A, \lambda)$ starting to show his nose when λ crosses the wall. To be precise, the wall H must separate two chambers, not just two topes, (as explained in Remark 4.2) in order for the new polytope $\mathfrak{p}_{\text{flip}}(\Phi, A, \lambda)$ to be not empty.

When F is provided with a lattice Λ, and the ϕ_i's are in Λ, the data (Φ, λ) parameterize a toric variety together with a line bundle. The zonotope

$$\mathfrak{b}(\Phi) := \left\{ \sum_{i=1}^{N} t_i \phi_i ; 0 \leq t_i \leq 1 \right\}$$

plays an important role in the "continuity properties" of our formulae in the discrete case, where, for a tope τ, the "neighborhood" of $\tau \cap \Lambda$ is the fattened tope $(\tau - \mathfrak{b}(\Phi)) \cap \Lambda$. Remark indeed that the semi-closed flipped polytope $\mathfrak{p}_{\text{flip}}(\Phi, A, \lambda)$ of Formula (1.8) may not contain any integral point when λ stays near the wall between τ_1 and τ_2.

In Section 5, where we study discrete sums over partition polytopes, we recover the quasi-polynomiality over fattened topes which was previously obtained in [12, 13, 19], as well as wall crossing formulae. Remarkably, the proofs which we give in the present article are based only on the Brianchon-Gram decomposition of a polytope and some set theoretic computations.

Our original motivation for the present work was to understand Brion's formula when specialized at a degenerate point. Let $\mathfrak{p} \subset V$ be a full-dimensional polytope in a vector space V equipped with a lattice $V_{\mathbb{Z}}$. Consider the discrete sum

$$S(\mathfrak{p})(\xi) = \sum_{x \in V_{\mathbb{Z}} \cap \mathfrak{p}} e^{\langle \xi, x \rangle}.$$

Brion's theorem expresses the analytic function $S(\mathfrak{p})(\xi)$ as the sum

$$S(\mathfrak{p})(\xi) = \sum_{s \in \mathcal{V}(\mathfrak{p})} S(s + \mathfrak{c}_s)(\xi). \tag{1.9}$$

Here s runs over the set of vertices of $\mathfrak{p}$, and $s + \mathfrak{c}_s$ is the tangent cone at $\mathfrak{p}$ at the vertex s. Now the function $S(s + \mathfrak{c}_s)(\xi)$ is a meromorphic function of ξ. Its poles are the points $\xi \in V^*$ such that ξ vanishes on some edge generator of the cone $\mathfrak{c}_s$ (or equivalently, such that ξ takes the same value at the vertex s and some adjacent vertex s' of $\mathfrak{p}$).

It is well known that if ξ is regular with respect to $\mathfrak{p}$, (*i.e.* $\langle \xi, s \rangle \neq \langle \xi, s' \rangle$ for adjacent vertices), Brion's formula is the combinatorial translation of the localization formula in equivariant cohomology [4], in a case where the fixed points are isolated. The case where ξ is not regular corresponds to the case where the variety of fixed points has components of positive dimension. We obtain indeed the combinatorial translation of the localization formula in this degenerate case. The vertices must be replaced by the faces on which ξ is constant which are maximal with respect to this property. For such a face $\mathfrak{f}$, the tangent cone must be replaced by the transverse cone to $\mathfrak{p}$ along $\mathfrak{f}$. However, the formula is "nice" only under some conditions (satisfied for example when the polytope $\mathfrak{p}$ is simple). The formula involves the "analytic continuation" of the face $\mathfrak{f}$ obtained by slicing the polytope $\mathfrak{p}$ by affine subspaces parallel to $\mathfrak{f}$, Figure 6.2

Finally, in the last section, we sketch the relation of this work with the cohomology of line bundles over a toric variety. In the case where the ϕ_i's generate a lattice in F, a Φ-tope τ gives rise to a toric variety M_τ. Then, the value of the function $\mathcal{X}(\Phi, \tau)$ computed at a point $m \in$

$\mathbb{Z}^N \subset \mathbb{R}^N$ is the multiplicity of the character m in the alternate sum of the cohomology groups of the line bundle L_λ on M_τ which corresponds to $\lambda = \sum_i m_i\phi_i$. In other words, the function $\mathcal{X}(\Phi, \tau)$ induces on each affine space $V(\Phi, \lambda)$ the constructible function associated by Morelli [16] to the line bundle L_λ on M_τ.

The continuity result (Corollary 3.1) implies that the function $\lambda \to \dim H^0(M_\tau, \lambda)$ is a quasi-polynomial on the fattened tope $(\tau - \mathfrak{b}(\Phi)) \cap \Lambda$. We give some examples of computations in the last section.

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List of notations

$A(b)$, $A(\tau, b)$	a function on $\mathbb{R}^d$, the analytic continuation of a polytope
$[E]$	characteristic function of a set $E \subseteq R^d$ or $E \subseteq \mathbb{R}^N$
F	r-dimensional real vector space; $\lambda \in F$
Φ	a sequence of N non zero vectors ϕ_i in F
e_i	canonical basis of $\mathbb{R}^N$
x_i	coordinates functions on $\mathbb{R}^N$
M	the map $\mathbb{R}^N \to F$; $M(e_i) = \phi_i$
$V(\Phi)$, V	$\{x \in \mathbb{R}^N; \sum_i x_i\phi_i = 0\}$
d	$N - r$; the dimension of V
$V(\Phi, \lambda)$	$\{x \in \mathbb{R}^N; \sum_i x_i\phi_i = \lambda\}$
$\mathfrak{p}(\Phi, \lambda)$	$\{x \in \mathbb{R}^N; x_i \geq 0; \sum_i x_i\phi_i = \lambda\}$
Partition polytope	a polytope $\mathfrak{p}(\Phi, \lambda)$
Λ	lattice in F; $\lambda \in \Lambda$
$k(\Phi)$	the function $\lambda \to$ cardinal $\mathfrak{p}(\Phi, \lambda)$
Partition function	the function $k(\Phi)$
Q	the standard quadrant $\{x \in \mathbb{R}^N; x_i \geq 0\}$
I, J, K, A, B	subsets of $\{1, 2, \ldots, N\}$
I^c	complementary subsets to I in $\{1,2,...,N\}$
Φ_I	$(\phi_i, i \in I)$
$\mathfrak{c}(\Phi)$, $\mathfrak{c}(\Phi_I)$	cone generated by Φ, Φ_I
$\mathfrak{a}(K)$	the cone in $\mathbb{R}^N$ defined as $\{x \in \mathbb{R}^N; x_j \geq 0$ for $j \in K^c\}$
$\mathfrak{a}_0(K)$	the cone $V \cap \mathfrak{a}(K)$
$\mathfrak{t}_K(\Phi, \lambda)$	$\mathfrak{a}(K) \cap V(\Phi, \lambda)$
Φ-basic subset I	a subset I such that ϕ_i, $i \in I$, is a basis of F
Φ-generating subset I	a subset I such that ϕ_i, $i \in I$, generates F
$\mathcal{B}(\Phi)$	the set of Φ-basic subsets
$\mathcal{G}(\Phi)$	the set of Φ-generating subsets
$\rho_{\Phi,I}$	$\rho_{\Phi,I} : \mathbb{R}^N \to V(\Phi)$ with kernel $\oplus_{i \in I}\mathbb{R}e_i$
g_j^I	$\rho_{\Phi,I}(e_j)$, $j \in I^c$
wall H	hyperplane in F generated by $r - 1$ vectors in Φ
regular λ	λ does not belong to any wall H
tope τ	$\tau \subset F$, a connected component of the set of regular elements
$\mathcal{B}(\Phi, \tau)$	the set of Φ-basic subsets I such that $\tau \subset \mathfrak{c}(\Phi_I)$

$\mathcal{G}(\Phi, \tau)$	the set of Φ-generating subsets I such that $\tau \subset \mathfrak{c}(\Phi_I)$
arrangement$\mathcal{H}(\lambda)$	the collection of the hyperplanes $x_i = 0$ in $V(\Phi, \lambda)$
vertex s	of the arrangement$\mathcal{H}(\lambda)$; s belongs to d hyperplanes of $\mathcal{H}(\lambda)$
$s_I(\Phi, \lambda)$	the vertex of $\mathcal{H}(\lambda)$ such that $s_j = 0$ for $j \in I^c$
$\mathfrak{f}_I(\Phi, \lambda), \mathfrak{f}_I$	the face of $\mathfrak{p}(\Phi, \lambda)$ indexed by I; defined by $\mathfrak{p}(\Phi, \lambda) \cap \{x_j = 0, j \in I^c\}$
$\mathfrak{t}_{\text{aff}}(\mathfrak{p}, \mathfrak{f})$	tangent affine cone to a polytope $\mathfrak{p}$ at the face $\mathfrak{f}$
$\mathcal{X}(\Phi, \tau)$	$\sum_{I \in \mathcal{G}(\Phi,\tau)} (-1)^{\lvert I\rvert - \dim F} \prod_{j \in I^c} [x_j \geq 0]$
$X(\Phi, \tau)(p, q)$	$\sum_{I \in \mathcal{G}(\Phi,\tau)} (-1)^{\lvert I\rvert - \dim F} \prod_{j \in I^c} p_j \prod_{i \in I} (p_i + q_i)$
w_B	$\prod_{j \in B^c} p_j \prod_{i \in B} q_i$
W	space of polynomials with basis w_B
Geom	substituting $p_i = [x_i \geq 0], q_j = [x_j < 0]$ in w_B
$\mathfrak{b}(\Phi)$	the zonotope generated by Φ; $\left\{\sum_{i=1}^{N} t_i \phi_i; 0 \leq t_i \leq 1\right\}$
Q_{neg}^B	$\{x = (x_i), x_i < 0 \text{ for } i \in B \,; x_i \geq 0 \text{ for } i \in B^c\}$
Φ_{flip}^B	the sequence $(\sigma_i \phi_i, 1 \leq i \leq N)$, where $\sigma_i = -1$ if $i \in B$; $\sigma_i = 1$ if $i \notin B$
$\widetilde{\mathfrak{c}}(\Phi_{\text{flip}}^B)$	$\left\{\sum_i x_i \phi_i, x \in Q_{\text{neg}}^B\right\}$
$\widetilde{\mathfrak{c}}_{\mathbb{Z}}(\Phi_{\text{flip}}^B)$	$\left\{\sum_i x_i \phi_i, x \in Q_{\text{neg}}^B \cap \mathbb{Z}^N\right\}$
β	linear form on $\mathbb{R}^N$
$K_\beta^{c,+}$	$\{j \in K^c; \langle \beta, g_j^K \rangle > 0\}$
$K_\beta^{c,-}$	$\{j \in K^c; \langle \beta, g_j^K \rangle < 0\}$
$\mathfrak{a}(K, \beta)$	$\{x \in \mathbb{R}^N; x_i \geq 0 \,, i \in K_\beta^{c+}; x_i < 0 \,, i \in K_\beta^{c-}\}$
$\mathfrak{a}_0(K, \beta)$	the cone $V \cap \mathfrak{a}(K, \beta)$
$Y(\Phi, \tau, \beta)$	$\sum_{K \in \mathcal{B}(\Phi,\tau)} (-1)^{\lvert K_\beta^{c-}\rvert} \prod_{i \in K_\beta^{c+}} p_i \prod_{i \in K_\beta^{c-}} q_i \prod_{i \in K} (p_i + q_i)$
$\mathfrak{p}(\Phi, A, \lambda)$	$\{x \in V(\Phi, \lambda), \;\; x_i > 0 \text{ for } i \in A, x_i \geq 0 \text{ for } i \in A^c\}$
$\mathfrak{p}_{\text{flip}}(\Phi, A, \lambda)$	$\{x \in V(\Phi, \lambda) \;\; x_i < 0 \text{ for } i \in A, x_i \geq 0 \text{ for } i \in A^c\}$

2 Definition of the analytic continuation

2.1 Some cones related to a partition polytope

In this article, there will be plenty of cones. A cone will always be an affine polyhedral convex cone. A cone will be called flat if it contains an affine line, otherwise, it will be called salient.

Let F be a real vector space of dimension r, and let $\Phi = (\phi_1, \dots, \phi_N)$ be a sequence of N non zero elements of F. We assume that Φ generates F as a vector space.

The standard basis of $\mathbb{R}^N$ is denoted by e_i with dual basis the linear forms x_i. We denote by $M : \mathbb{R}^N \to F$ the surjective map which sends the vector e_i to the vector ϕ_i. The kernel of M is a subspace of dimension $d = N - r$ which will be denoted by $V(\Phi)$ or simply V when Φ is understood.

$$V(\Phi) := \{x \in \mathbb{R}^N; \sum_i x_i \phi_i = 0\}.$$

We denote by Q the standard quadrant

$$Q := \{x \in \mathbb{R}^N; x_i \geq 0\}.$$

The cone $\mathfrak{c}(\Phi)$ generated by Φ is the image of Q by M. Assume that the cone $\mathfrak{c}(\Phi)$ is salient. In other words, there exists a linear form $a \in F^*$ such that $\langle a, \phi_i \rangle > 0$ for all $1 \leq i \leq N$. This is also equivalent to the fact that $V \cap Q = 0$.

If I is a subset of $\{1, 2, \dots, N\}$, we denote by I^c the complementary subset to I in $\{1, 2, \dots, N\}$.

Definition 2.1. If I is a subset of $\{1, 2, \dots, N\}$, let

$$\mathfrak{a}(I) = \{x \in \mathbb{R}^N; x_j \geq 0 \text{ for } j \in I^c\}$$

and let

$$\mathfrak{a}_0(I) = V \cap \mathfrak{a}(I) = \{x \in V;\ x_j \geq 0 \text{ for } j \in I^c\}$$

be the intersection of V with the cone $\mathfrak{a}(I)$.

Thus $\mathfrak{a}(I)$ is the product of the positive quadrant in the variables I^c, with a vector space of dimension $|I|$. The cone $\mathfrak{a}(I)$ is called an angle by Varchenko. It is never salient, except if $I = \emptyset$. With this notation, the positive quadrant Q is $\mathfrak{a}(\emptyset)$.

We now analyze the cone $\mathfrak{a}_0(I) \subseteq V$. A subset $I \subseteq \{1, 2, \dots, N\}$ such that $\{\phi_i, i \in I\}$ is a basis of F will be called Φ-basic. We denote by $\mathcal{B}(\Phi)$ the set of Φ-basic subsets. A subset $I \subseteq \{1, 2, \dots, N\}$ such that $\{\phi_i, i \in I\}$ generates F will be called Φ-generating. We denote by $\mathcal{G}(\Phi)$ the set of Φ-generating subsets.

Let I be Φ-basic. Then the cardinal of I^c is $d = N - r$ and the restrictions to V of the linear forms x_j, with $j \in I^c$, form a basis of V^*. Hence $\mathfrak{a}_0(I)$ is a cone of dimension d in V with d generators, in other words a simplicial cone of full dimension in the vector space V. Let us describe the edges of the simplicial cone $\mathfrak{a}_0(I)$. We have

$$\mathbb{R}^N = V(\Phi) \oplus (\oplus_{i \in I} \mathbb{R} e_i),$$

and we denote by $\rho_{\Phi,I}$ the corresponding linear projection $\mathbb{R}^N \to V$. For $j \in I^c$, we write $\phi_j = \sum_{i \in I} u_{i,j} \phi_i$.

Lemma 2.1. *Let I be Φ-basic. For $j \in I^c$, let*

$$g_j^I = \rho_{\Phi,I}(e_j) = e_j - \sum_{i \in I} u_{i,j} e_i.$$

Then the d vectors g_j^I are the generators of the edges of the simplicial cone $\mathfrak{a}_0(I)$.

Now, let I be a generating subset. Then the restrictions to V of the linear forms x_j, $j \in I^c$, are linearly independent elements of V^*. The cone $\mathfrak{a}_0(I)$ is again the product of a simplicial cone of dimension $|I^c|$ by a vector space of dimension $|I| - r$ More precisely, if K is any Φ-basic subset contained in I, the cone $\mathfrak{a}_0(I)$ is the product of the cone generated by $\rho_{\Phi,K}(e_j)$, $j \in I^c$, by the vector space generated by $\rho_{\Phi,K}(e_i)$ with $i \in I \setminus K$.

2.2 Vertices and faces of a partition polytope

Recall that, for $\lambda \in F$, we denote by $V(\Phi, \lambda) \subset \mathbb{R}^N$ the affine subspace

$$\{x \in \mathbb{R}^N; \sum_i x_i \phi_i = \lambda\}.$$

The intersections of the coordinates hyperplanes $\{x_i = 0\}$ with $V(\Phi, \lambda)$ form an arrangement $\mathcal{H}(\lambda)$ of N affine hyperplanes of $V(\Phi, \lambda)$.

By definition, a vertex of this arrangement is a point $s \in V(\Phi, \lambda)$ such that s belongs to at least d independent hyperplanes. The arrangement $\mathcal{H}(\lambda)$ is called regular if no vertex belongs to more than d hyperplanes. A Φ-wall H is a hyperplane of F spanned by $r - 1$ linearly independent elements of Φ. Thus $\mathcal{H}(\lambda)$ is regular if and only if λ does not belong to any Φ-wall, that is, if λ is regular.

By definition, a face of the arrangement $\mathcal{H}(\lambda)$ is the set of elements $x \in V(\Phi, \lambda)$ which satisfy a subset of the set of relations $\{x_i \geq 0, x_j \leq 0, x_k = 0\}$.

Recall that the partition polytope $\mathfrak{p}(\Phi, \lambda)$ is the intersection of the affine space $V(\Phi, \lambda)$ with the positive quadrant Q. Thus it is a bounded face of the arrangement of hyperplanes $\mathcal{H}(\lambda)$.

If $I \subset \{1, \ldots, N\}$ is Φ-basic, then λ has a unique decomposition $\lambda = \sum_{i \in I} x_i \phi_i$. If λ is regular, $x_i \neq 0$ for all i.

Definition 2.2. Let I be a Φ-basic subset, let $\lambda = \sum_{i \in I} x_i \phi_i$. Then $s_I(\Phi, \lambda)$ is the vertex of the arrangement $\mathcal{H}(\lambda)$ defined by $s_I(\Phi, \lambda) = (s_i)$ where $s_i = x_i$ if $i \in I$, and $s_j = 0$ if $j \in I^c$.

Observe that $s_I(\Phi, \lambda)$ depends linearly on λ.

If λ is regular, the vertices of the arrangement $\mathcal{H}(\lambda)$ are in one to one correspondence $I \mapsto s_I(\Phi, \lambda)$ with the set $\mathcal{B}(\Phi)$ of Φ-basic subsets of $\{1, \ldots, N\}$.

Definition 2.3. For I a subset of $\{1, 2, \ldots, N\}$, define

$$\mathfrak{t}_I(\Phi, \lambda) = \mathfrak{a}(I) \cap V(\Phi, \lambda).$$

If I is a Φ-basic subset, then the cone $\mathfrak{t}_I(\Phi, \lambda)$ is the shift $s_I(\Phi, \lambda) + \mathfrak{a}_0(I)$ of the **fixed simplicial cone** $\mathfrak{a}_0(I)$ by the vertex $s_I(\Phi, \lambda)$ which depends linearly of λ. We will use the formula

$$\mathfrak{t}_I(\Phi, \lambda) = s_I(\Phi, \lambda) + \mathfrak{a}_0(I). \tag{2.1}$$

Similarly, if I is a Φ-generating subset, choose a Φ-basic subset K contained in I, then the cone $\mathfrak{t}_I(\Phi, \lambda)$ is the shift of the **fixed cone** $\mathfrak{a}_0(I)$ by the vertex $s_K(\Phi, \lambda)$ which depends linearly of λ.

$$\mathfrak{t}_I(\Phi, \lambda) = \mathfrak{a}(I) \cap V(\Phi, \lambda) = s_K(\Phi, \lambda) + \mathfrak{a}_0(I). \tag{2.2}$$

So, one can say that the set $\mathfrak{t}_I(\Phi, \lambda)$ varies analytically with λ, whenever I is a generating subset. At least it "keeps the same shape". This is not the case when I is not generating, for example when $I = \emptyset$. Indeed $\mathfrak{t}_\emptyset(\Phi, \lambda)$ is the partition polytope $\mathfrak{p}(\Phi, \lambda)$, and it certainly does not vary "analytically".

We now analyze the faces of the partition polytope $\mathfrak{p}(\Phi, \lambda)$ and the corresponding tangent cones.

If τ is a Φ-tope, we denote by $\mathcal{B}(\Phi, \tau) \subseteq \mathcal{B}(\Phi)$ the set of basic subsets I such that τ is contained in the cone $\mathfrak{c}(\phi_I)$ generated by the ϕ_i, $i \in I$. In other words, the equation $\lambda = \sum_{i \in I} x_i \phi_i$ can be solved with positive x_i. Equivalently, the corresponding vertex $s_I(\Phi, \lambda)$ belongs to the polytope $\mathfrak{p}(\Phi, \lambda)$. Thus when λ is regular, there is a one-to-one correspondence between the elements $I \in \mathcal{B}(\Phi, \tau)$ and the vertices of the polytope $\mathfrak{p}(\Phi, \lambda)$.

When λ belongs to the closure of a tope τ, every vertex of $\mathfrak{p}(\Phi, \lambda)$ is still of the form $s_I(\Phi, \lambda)$ with $I \in \mathcal{B}(\Phi, \tau)$, but two Φ-basic subsets can give rise to the same vertex.

Let $I \in \mathcal{B}(\Phi, \tau)$. Assume that λ is regular, so that all coordinates s_i of $s_I(\Phi, \lambda)$ with $i \in I$ are positive. Then it is clear that the tangent cone to $\mathfrak{p}(\Phi, \lambda)$ at the vertex $s_I(\Phi, \lambda)$ is the cone determined by the inequations $x_i \geq 0$ for $i \in I^c$, while the sign of the coordinates x_i with $i \in I$ are arbitrary, In other words, it is the simplicial affine cone $\mathfrak{t}_I(\Phi, \lambda)$

We denote by $\mathcal{G}(\Phi, \tau) \subseteq \mathcal{G}(\Phi)$ the set of generating subsets I such that τ is contained in the cone $\mathfrak{c}(\phi_I)$ generated by the ϕ_i, $i \in I$.

If $I \in \mathcal{G}(\Phi, \tau)$, the intersection of $\mathfrak{p}(\Phi, \lambda)$ with $\{x_j = 0,\ j \in I^c\}$ is a face $\mathfrak{f}_I(\Phi, \lambda)$ of dimension $|I|-r$ of the polytope $\mathfrak{p}(\Phi, \lambda)$. The vertices of this face are the points $s_K(\Phi, \lambda)$ corresponding to all the Φ-basic subsets K contained in I. The affine tangent cone $\mathfrak{t}_{\text{aff}}(\mathfrak{p}(\Phi, \lambda),\ f_K(\Phi, \lambda))$ to the polytope $\mathfrak{p}(\Phi, \lambda)$ along the face $f_K(\Phi, \lambda)$ is

$$\mathfrak{t}_{\text{aff}}(\mathfrak{p}(\Phi, \lambda),\ f_K(\Phi, \lambda)) = \mathfrak{t}_I(\Phi, \lambda) = \mathfrak{a}(I) \cap V(\Phi, \lambda).$$

2.3 The Brianchon-Gram function

Summarizing, for $\lambda \in \tau$, there is a one-to one correspondence between the set of faces of the polytope $\mathfrak{p}(\Phi, \lambda)$ and the set $\mathcal{G}(\Phi, \tau)$. The Brianchon-Gram theorem implies, for $\lambda \in \tau$,

$$[\mathfrak{p}(\Phi, \lambda)] = \left(\sum_{I \in \mathcal{G}(\Phi, \tau)} (-1)^{|I| - \dim F} [\mathfrak{a}(I)] \right) [V(\Phi, \lambda)].$$

When λ varies, the right hand side is obtained by intersecting a number of fixed cones in $\mathbb{R}^N$ with the varying affine space $V(\Phi, \lambda)$. It is natural to introduce the function on $\mathbb{R}^N$

$$\begin{aligned} \mathcal{X}(\Phi, \tau) &= \sum_{I \in \mathcal{G}(\Phi, \tau)} (-1)^{|I| - \dim F} [\mathfrak{a}(I)] \\ &= \sum_{I \in \mathcal{G}(\Phi, \tau)} (-1)^{|I| - \dim F} \prod_{j \in I^c} [x_j \geq 0]. \end{aligned} \tag{2.3}$$

that is, the Geometric Brianchon-Gram function which we mentioned in the introduction.

For $\lambda \in \tau$, we have

$$\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)] = [\mathfrak{p}(\Phi, \lambda)], \tag{2.4}$$

the characteristic function of the partition polytope $\mathfrak{p}(\Phi, \lambda) \subset \mathbb{R}^N$.

Let us now consider the function $\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)]$ for any $\lambda \in F$. By Equations (2.1) and (2.2), we have

$$\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)] = \sum_{I \in \mathcal{G}(\Phi,\tau)} (-1)^{|I|-\dim F}[s_K(\Phi, \lambda) + \mathfrak{a}_0(I)].$$

Here, for each $I \in \mathcal{G}(\Phi, \tau)$, we choose $K \subset I$, a basic subset contained in I.

We thus see that $\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)]$ is constructed as follows. Start from the polytope $\mathfrak{p}(\Phi, \lambda_0)$ with $\lambda_0 \in \tau$, write the characteristic function of $\mathfrak{p}(\Phi, \lambda_0)$ as the alternating sum of its tangent cones at faces, and when moving λ in the whole space F, follow these cones by moving their vertex linearly in function of λ. As all the sets I entering in the formula for $\mathcal{X}(\Phi, \tau)$ are generating, the individual pieces $\mathfrak{a}(I) \cap V(\Phi, \lambda) = s_K(\Phi, \lambda) + \mathfrak{a}_0(I)$ keep the same shape.

It is clear that the support of the function $\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)]$ is a union of faces of various dimensions of the arrangement $\mathcal{H}(\lambda)$. We will show that it is is a union of bounded faces of this arrangement, for any $\lambda \in F$ (Corollary 3.2).

Remark 2.1. Chambers rather than topes are relevant to wall crossing. However, we preferred to use topes, because topes are naturally related to the whole set of vertices of the arrangement $\mathcal{H}(\lambda)$. A chamber is a connected component of the complement in F of the union of all the **cones** spanned by $(r-1)$-elements of $\Phi \cup (-\Phi)$. Chambers are bigger than topes, the closure of a chamber is a union of closures of topes. See Figure 2.1. But if τ_1 and τ_2 are contained in the same chamber, we have $\mathcal{G}(\Phi, \tau_1) = \mathcal{G}(\Phi, \tau_2)$, hence $X(\Phi, \tau_1) = X(\Phi, \tau_2)$.

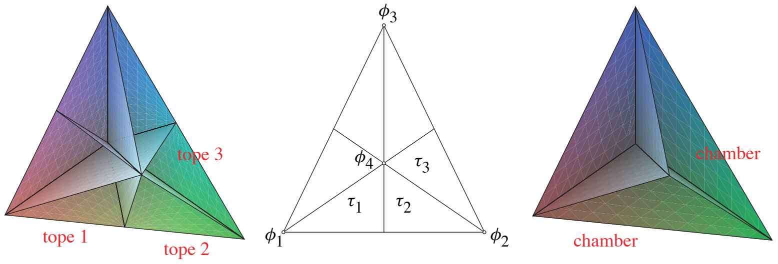


Figure 2.1. Left, topes for $\Phi = (\phi_1, \phi_2, \phi_3, \phi_1 + \phi_2 + \phi_3)$. Right, chambers.

3 Signed sums of quadrants

3.1 Continuity properties of the Brianchon-Gram function

Recall that we defined in the introduction the following polynomial in the variables p_i and q_i.

Definition 3.1. Let τ be a Φ-tope. The Combinatorial Brianchon-Gram function associated to the pair (Φ, τ) is

$$X(\Phi, \tau)(p, q) = \sum_{I \in \mathcal{G}(\Phi,\tau)} (-1)^{|I|-\dim F} \prod_{j \in I^c} p_j \prod_{i \in I} (p_i + q_i). \tag{3.1}$$

If the tope τ is not contained in $\mathfrak{c}(\Phi)$, the set $\mathcal{G}(\Phi, \tau)$ is empty and $X(\Phi, \tau) = 0$. Otherwise, if $\tau \subset \mathfrak{c}(\Phi)$, the sum defining $X(\Phi, \tau)$ is indexed by all the faces of the polytope $\mathfrak{p}(\Phi, \lambda_0)$ (for any choice of $\lambda_0 \in \tau$).

We recover $\mathcal{X}(\Phi, \tau)$ when we substitute $[x_i \geq 0]$ for p_i and $[x_i < 0]$ for q_i in $X(\Phi, \tau)(p, q)$ (so that $p_i + q_i = 1$).

The Combinatorial Brianchon-Gram function is a particular element of the space W below.

Definition 3.2. Let W be the subspace of $\mathbb{Q}[p_1, ..., p_N, q_1, ..., q_N]$ which consists of linear combinations of the monomials

$$w_B = \prod_{j \in B^c} p_j \prod_{i \in B} q_i$$

where B runs over the subsets of $\{1, \ldots, N\}$.

Thus we have

$$X(\Phi, \tau) = \sum_B z(\Phi, \tau, B) w_B, \tag{3.2}$$

with coefficients $z(\Phi, \tau, B) \in \mathbb{Z}$.
Remark: For the subset $B = \{1, 2, \ldots, N\}$, the coefficient $z(\Phi, \tau, B)$ is $(-1)^d = (-1)^{N-r}$.

Example 3.1 (The standard knapsack). Let $F = \mathbb{R}$, $\phi_i = 1$ for $i = 1, \ldots, N$, and $\tau = \mathbb{R}_{>0}$. From the usual inclusion-exclusion relations, we get

$$X(\Phi, \tau) = p_1 \cdots p_N - (-1)^N q_1 \cdots q_N. \tag{3.3}$$

An element in W gives a function on $\mathbb{R}^N$ by the following substitution.

Definition 3.3. We denote by Geom the map from W to the space of functions on $\mathbb{R}^N$ defined by substituting $[x_i \geq 0]$ for p_i and $[x_j < 0]$ for q_j.

Thus if w is an element of W, we may denote by w_{geom} the corresponding function on $\mathbb{R}^N$ so obtained. In particular, we may denote $\mathcal{X}(\Phi, \tau)$ by $X_{\text{geom}}(\Phi, \tau)$.

Later, we will use other substitutions.

We prove now some "continuity" properties of the Combinatorial Brianchon-Gram function when λ reaches the closure of the tope τ. Actually, these properties are shared by any element of the space W which satisfies the hypothesis of Proposition 3.1 below. We first introduce some definitions and prove an easy lemma.

Definition 3.4. The zonotope $\mathfrak{b}(\Phi)$ is the subset of F defined by

$$\mathfrak{b}(\Phi) = \left\{ \sum_{i=1}^{N} t_i \phi_i ; 0 \leq t_i \leq 1 \right\}.$$

When τ is a tope contained in $\mathfrak{c}(\Phi)$, the domain $\tau - \mathfrak{b}(\Phi) := \{x - y, x \in \tau, y \in \mathfrak{b}(\Phi)\}$ will play a crucial role in "continuity properties" of our functions. Remark that $\tau - \mathfrak{b}(\Phi)$ is a fattening of τ which contains the closure of the tope τ. Usually the set of integral points in $\tau - \mathfrak{b}(\Phi)$ is larger than the set of integral points in $\overline{\tau}$.

Definition 3.5. For $B \subseteq \{1, \ldots, N\}$,

- $Q^B_{\text{neg}} \subset \mathbb{R}^N$ is the semi-closed quadrant

$$Q^B_{\text{neg}} = \{x = (x_i), x_i < 0 \text{ for } i \in B, x_i \geq 0 \text{ for } i \in B^c\},$$

- Φ^B_{flip} is the sequence $[\sigma_i \phi_i, 1 \leq i \leq N]$, where $\sigma_i = -1$ if $i \in B$ and $\sigma_i = 1$ if $i \notin B$.
- $\tilde{\mathfrak{c}}(\Phi^B_{\text{flip}}) \subset F$ is the semi-closed cone

$$\tilde{\mathfrak{c}}(\Phi^B_{\text{flip}}) = \left\{ \sum_i x_i \phi_i, x \in Q^B_{\text{neg}} \right\}$$

  and

$$\tilde{\mathfrak{c}}_{\mathbb{Z}}(\Phi^B_{\text{flip}}) = \left\{ \sum_i x_i \phi_i, x \in Q^B_{\text{neg}} \cap \mathbb{Z}^N \right\}.$$

With this notation, the standard quadrant is

$$Q = Q^{\emptyset}_{\text{neg}}.$$

Remark that the closure of the semi-closed cone $\tilde{\mathfrak{c}}(\Phi^B_{\text{flip}})$ is the closed cone $\mathfrak{c}(\Phi^B_{\text{flip}})$.

We recall the following lemma.

Lemma 3.1. *The following conditions are equivalent:*

(i) *The cone* $\mathfrak{c}(\Phi^B_{\text{flip}})$ *is salient*
(ii) $\overline{Q}^B_{\text{neg}} \cap V = \{0\}$
(iii) *For any* $\lambda \in F$, $V(\Phi, \lambda) \cap \overline{Q}^B_{\text{neg}}$ *is bounded.*

Lemma 3.2. *Let* $\tau \subset \mathfrak{c}(\Phi)$ *be a tope and* $\overline{\tau}$ *its closure. Let* B *be a subset of* $\{1, 2, \ldots, N\}$. *Assume that the semi-open cone* $\widetilde{\mathfrak{c}}(\Phi^B_{\text{flip}})$ *and the tope* τ *are disjoint. Then*

(i) τ *is disjoint from the closed cone* $\mathfrak{c}(\Phi^B_{\text{flip}})$.
(ii) *The closure* $\overline{\tau}$ *of* τ *is disjoint from the semi-open cone* $\widetilde{\mathfrak{c}}(\Phi^B_{\text{flip}})$.
(iii) $\tau - \mathfrak{b}(\Phi)$ *is disjoint from* $\widetilde{\mathfrak{c}}_{\mathbb{Z}}(\Phi^B_{\text{flip}})$.

Proof. (i) Assume that the semi-open cone $\widetilde{\mathfrak{c}}(\Phi^B_{\text{flip}})$ and the tope τ are disjoint. As τ is open, it is disjoint from the closure $\mathfrak{c}(\Phi^B_{\text{flip}})$ of $\widetilde{\mathfrak{c}}(\Phi^B_{\text{flip}})$.

(ii) Choose z small in τ. As $\tau \subset \mathfrak{c}(\Phi)$, we can write $z = \sum_{a \in A} \epsilon_a \phi_a$ with A a subset of $\{1, 2, \ldots, N\}$ and $\epsilon_a > 0$. As τ is a cone, we may assume the ϵ_a very small. Let $\lambda \in \overline{\tau}$. Then $\lambda + z \in \tau$. Now, if λ belongs also to $\widetilde{\mathfrak{c}}(\Phi^B_{\text{flip}})$, we may write $\lambda = \sum_{i=1}^N x_i \phi_i$ with $x_i < 0$ if $i \in B$ and $x_i \geq 0$ if $i \in B^c$ and we see that $\lambda + z$ is still in $\widetilde{\mathfrak{c}}(\Phi^B_{\text{flip}})$ if ϵ_a are sufficiently small. This contradicts the fact that $\widetilde{\mathfrak{c}}(\Phi^B_{\text{flip}}) \cap \tau$ is empty. So (ii) is proven.

Let us prove (iii). Assume that there exist $(n_i) \in \mathbb{Z}^N$, with $n_i < 0$ for $i \in B$ and $n_i \geq 0$ for $i \notin B$, such that $\sum_i n_i \phi_i \in \tau - \mathfrak{b}(\Phi)$. Thus there exist (t_i) with $0 \leq t_i \leq 1$, for $i = 1, \ldots, N$, and $\lambda \in \tau$, such that $\sum_i (n_i + t_i)\phi_i = \lambda$. As n_i are integers, we have $n_i \leq -1$ hence $n_i + t_i \leq 0$ for $i \in B$. We have also $n_i + t_i \geq 0$ for $i \notin B$. It follows that $\lambda \in \tau \cap \mathfrak{c}(\Phi^B_{\text{flip}})$. This contradicts (ii). $\square$

The following proposition states continuity properties on closures and beyond.

Proposition 3.1. *Let* $\tau \subset \mathfrak{c}(\Phi)$ *be a tope and let* $Z = \sum_B z_B w_B \in W$ *be such that*

$$\sum_B z_B [Q^B_{\text{neg}}]\,[V(\Phi, \lambda)] = [\mathfrak{p}(\Phi, \lambda)] \text{ for every } \lambda \in \tau. \tag{3.4}$$

Then

(i) $z_{\emptyset} = 1$.
(ii) *The equation*

$$\sum_B z_B[Q^B_{\mathrm{neg}}]\,[V(\Phi, \lambda)] = [\mathfrak{p}(\Phi, \lambda)]$$

still holds for every $\lambda \in \overline{\tau}$.
(iii) *For* $\lambda \in \tau - \mathfrak{b}(\Phi)$*, we have*

$$\sum_B z_B[Q^B_{\mathrm{neg}}][V(\Phi, \lambda) \cap \mathbb{Z}^N] = [\mathfrak{p}(\Phi, \lambda) \cap \mathbb{Z}^N].$$

Proof. Let $\lambda \in \tau$ and $x \in \mathfrak{p}(\Phi, \lambda)$. Then $x \in Q = Q^{\emptyset}_{\mathrm{neg}}$. As the quadrants Q^B_{neg} are pairwise disjoint, $x \notin Q^B_{\mathrm{neg}}$ for $B \neq \emptyset$ hence (3.4) implies (i).

Next, let $B \neq \emptyset$. Let $\lambda \in \tau$. Assume there is an $x \in Q^B_{\mathrm{neg}} \cap V(\Phi, \lambda)$. Then $x \notin \mathfrak{p}(\Phi, \lambda)$, thus (3.4) implies that $z_B = 0$. Hence, if $z_B \neq 0$, the semi-open cone $\widetilde{\mathfrak{c}}(\Phi^B_{\mathrm{flip}})$ and the tope τ are disjoint. We can then apply Lemma 3.2. As $\overline{\tau}$ is disjoint from $\widetilde{\mathfrak{c}}(\Phi^B_{\mathrm{flip}})$, we see that $V(\Phi, \lambda)$ does not intersect any of the Q^B_{neg} with $z_B \neq 0$ and Q^B_{neg} different of Q. This implies (ii). In the same way, we obtain (iii). $\square$

Example 3.2 (See Figure 3.1). Let $N = 3$, $\dim F = 2$, $\Phi = (\phi_1, \phi_2, \phi_3 = \phi_1 + \phi_2)$. If τ_1 is the open cone generated by (ϕ_1, ϕ_3), we have $X(\Phi, \tau_1) = (p_1 + q_1)(p_2 + q_2)p_3 + (p_1 + q_1)p_2 - (p_1 + q_1)(p_2 + q_2)(p_3 + q_3) = p_1p_2p_3 - p_1q_2q_3 + q_1p_2p_3 - q_1q_2q_3$. We can check that $X(\phi, \tau_1)$ satisfies the properties (ii) and (iii) of Proposition 3.1 on Figure 3.1.

Corollary 3.1 (Continuity on the closure of a tope τ). *Let* $\Phi = (\phi_j)_{1 \le j \le N}$ *be a sequence of non zero elements of a vector space* F*, generating* F*, and spanning a salient cone and let* $\tau \subset \mathfrak{c}(\Phi)$ *be a tope relative to* Φ .

(i) *If* λ *belongs to the closure* $\overline{\tau}$ *of the tope* τ*, then we have the equality of characteristic functions of sets*

$$\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)] = [\mathfrak{p}(\Phi, \lambda)].$$

(ii) *If* $\lambda \in \tau - \mathfrak{b}(\Phi)$*, we still have the equality of characteristic functions of sets of lattice points*

$$\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)]\,[\mathbb{Z}^N] = [\mathfrak{p}(\Phi, \lambda) \cap \mathbb{Z}^N].$$

Remark 3.1. There are other elements $Z = \sum z_B w_B$ which satisfy (3.4). The simplest one is $Z = w_{\emptyset} = p_1 \cdots p_N$. However it does not enjoy the analytic properties of $\mathcal{X}(\Phi, \tau)$, see Theorem 5.1.

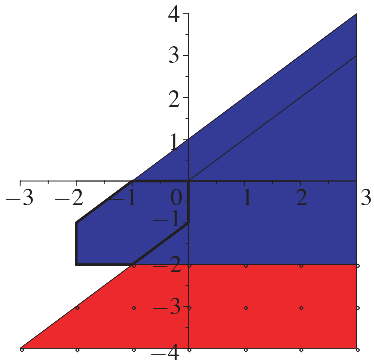


Figure 3.1. (Example 3.2). In blue, $-\mathfrak{b}(\Phi)$ is the closed hexagon, τ and $\tau - \mathfrak{b}(\Phi)$ are open sets. For $B = (2, 3)$, $\widetilde{\mathfrak{c}}_Z(\Phi^B_{\text{neg}})$ is the set of lattice points in the red zone, $m\phi_1 - n\phi_3$ with $m \geq 1$ and $n \geq 2$.

3.2 Polarized sums

We now introduce another function on $\mathbb{R}^N$ related to the polarized decomposition of a polytope as a signed sums of polarized semi-closed cones at the vertices.

In addition to the data of the previous section, we use here a linear form β on $\mathbb{R}^N$, regular with respect to Φ in the following sense.

Let I be a basic subset for Φ, and recall the description of the cone $\mathfrak{a}_0(I)$ given in Lemma 2.1 with generators $g_j^I = \rho_{\Phi,I}(e_j)$:

$$\mathfrak{a}_0(I) = \sum_{j \in I^c} \mathbb{R}_{\geq 0} g_j^I.$$

We assume that β is such that its restriction to V does not vanish on any edge g_j^I of the simplicial cones $\mathfrak{a}_0(I)$ when I varies in $\mathcal{B}(\Phi)$. That is $\langle \beta, \rho_{\Phi,I}(e_j)\rangle \neq 0$ for all $I \in \mathcal{B}(\Phi)$ and $j \in I^c$.

We associate to β the "polarized" cone

$$\mathfrak{a}_0(I, \beta) = \sum_{j; \langle \beta, g_j^I\rangle > 0} \mathbb{R}_{\geq 0} g_j^I + \sum_{j; \langle \beta, g_j^I\rangle < 0} \mathbb{R}_{<0} g_j^I.$$

This is the cone obtained by reversing the direction of some of the generators of the simplicial cone $\mathfrak{a}_0(I)$, in order that β takes positive value on all of them. Note however the delicate condition on signs.

Now all the cones $\mathfrak{a}_0(I, \beta)$ are contained in the half space of V determined by $\beta \geq 0$.

For each $K \in \mathcal{B}(\Phi)$, we denote by K_β^{c+} (respectively K_β^{c-}) the set of $j \in K^c$ such that $\langle \beta, \rho_{\Phi,K} e_j\rangle > 0$ (respectively $\langle \beta, \rho_{\Phi,K} e_j\rangle < 0$).

Definition 3.6. If K is a subset of $\{1, 2, \ldots, N\}$, we denote by

$$\mathfrak{a}(K,\beta) = \{x \in \mathbb{R}^N; x_i \geq 0 \text{ for } i \in K_\beta^{c+}, x_i < 0 \text{ for } i \in K_\beta^{c-}\}.$$

Thus the set $\mathfrak{a}(K,\beta)$ is the product of three terms: the closed quadrant in the variables in K_β^{c+}, , the opposite of the open quadrant in the variables in K_β^{c-}, and a vector space in the variable in K.

If $K \in \mathcal{B}(\Phi)$, the cone $\mathfrak{a}(K,\beta) \cap V(\Phi,\lambda)$ is the translate by the vertex $s_K(\Phi,\lambda)$ of the semi-open cone $\mathfrak{a}_0(K,\beta)$ of dimension d. In particular $\mathfrak{a}(K,\beta) \cap V(\Phi,\lambda)$ is contained in the half space $s_K(\Phi,\lambda) + \beta \geq 0 \cap V(\Phi,\lambda)$ of $V(\Phi,\lambda)$.

If $\lambda \in \tau$ and $K \in \mathcal{B}(\Phi,\tau)$, the cone $\mathfrak{a}(K,\beta) \cap V(\Phi,\lambda)$ is obtained by reversing some of the generators of the tangent cone to the polytope $\mathfrak{p}(\Phi,\lambda)$ at the vertex $s_K(\Phi,\lambda)$ so that β takes positive values on them. We say that it is the polarized tangent cone.

Recall the Lawrence-Varchenko polarized decomposition of $\mathfrak{p}(\Phi,\lambda)$, (actually, we will give a proof below).

$$[\mathfrak{p}(\Phi,\lambda)] = \sum_{K \in \mathcal{B}(\Phi,\tau)} (-1)^{|K_\beta^{c-}|}[\mathfrak{a}(K,\beta) \cap V(\Phi,\lambda)].$$

Again this equality is obtained by intersecting a number of fixed cones in $\mathbb{R}^N$ with the varying affine subspace $V(\Phi,\lambda)$. Therefore it is natural to define the following function.

Definition 3.7. The Combinatorial Lawrence-Varchenko function is the following element of W:

$$Y(\Phi,\tau,\beta) = \sum_{K \in \mathcal{B}(\Phi,\tau)} (-1)^{|K_\beta^{c-}|} \prod_{i \in K_\beta^{c+}} p_i \prod_{i \in K_\beta^{c-}} q_i \prod_{i \in K}(p_i + q_i). \quad (3.5)$$

If the tope τ is not contained in $\mathfrak{c}(\Phi)$, then $Y(\Phi,\tau,\beta) = 0$. Otherwise, if $\tau \subset \mathfrak{c}(\Phi)$, the sum defining $Y(\Phi,\tau,\beta)$ is indexed by all the vertices of the polytope $\mathfrak{p}(\Phi,\lambda_0)$ (for any choice of $\lambda_0 \in \tau$).

If we replace p_i by the characteristic function of $x_i \geq 0$ and q_i by the characteristic function of $x_i < 0$ ($p_i + q_i = 1$), we obtain a function $Y_{\text{geom}}(\Phi,\tau,\beta)$ on $\mathbb{R}^N$.

By construction, if $\lambda \in \tau$, the product $[V(\Phi,\lambda)]Y_{\text{geom}}(\Phi,\tau,\beta)$ is the signed sum of polarized semi-closed cones at the vertices of the (simple) partition polytope $\mathfrak{p}(\Phi,\lambda)$. Lawrence-Varchenko's theorem can be restated as $[V(\Phi,\lambda)]Y_{\text{geom}}(\Phi,\tau,\beta) = [\mathfrak{p}(\Phi,\lambda)]$ while Brianchon-Gram's theorem is $[V(\Phi,\lambda)]X_{\text{geom}}(\Phi,\tau) = [\mathfrak{p}(\Phi,\lambda)]$ for any $\lambda \in \tau$. Indeed by construction, $X_{\text{geom}}(\Phi,\tau)$ is our function $\mathcal{X}(\Phi,\tau)$, our main object of study.

The Lawrence-Varchenko decomposition of a simple polytope can be derived from the Brianchon-Gram one, by grouping some faces with a common vertex, [15]. It is remarkable that their combinatorial precursors actually coincide as elements of the space W, as we show in the next theorem.

Theorem 3.3. *Let $\Phi = (\phi_j)_{1\leq j\leq N}$ be a sequence of non zero elements of a vector space F, generating F, and spanning a salient cone, and let $\tau \subset \mathfrak{c}(\Phi)$ be a Φ-tope. Let $X(\Phi,\tau)$ be the Combinatorial Brianchon-Gram function. For any linear form β which is regular with respect to Φ, let $Y(\Phi,\tau,\beta)$ be the Combinatorial Lawrence-Varchenko function. Then*

$$Y(\Phi,\tau,\beta) = X(\Phi,\tau).$$

Proof. In the sum $X(\Phi,\tau)$, for a given $K \in \mathcal{B}(\Phi,\tau)$, we group together the $I \in \mathcal{G}(\Phi,\tau)$ such that $K \subseteq I$ and $\langle \beta, \rho_{\Phi,K} e_i\rangle < 0$, for every $i \in I \setminus K$. We denote the set of these I by $\mathcal{G}(\Phi,\tau)^K_\beta$. □

Lemma 3.3. *Let $I \in \mathcal{G}(\Phi,\tau)$ and $K \in \mathcal{B}(\Phi,\tau)$ such that $K \subseteq I$. Then $I \in \mathcal{G}(\Phi,\tau)^K_\beta$ if and only if for any $\lambda \in \tau$, on the face $\mathfrak{f}_I(\Phi,\lambda)$ of $\mathfrak{p}(\Phi,\lambda)$ which is indexed by I, the linear form β reaches its maximum at the vertex s_K indexed by K.*

Proof. Let $\lambda \in \tau$. Let $x = \sum_{i\in I} x_i e_i \in \mathfrak{f}_I(\Phi,\lambda)$, with $x \neq s_K$. Then $x - s_K$ is the projection $\rho_{\Phi,K}(x) = \sum_{i\in I\setminus K} x_i \rho_{\Phi,K}(e_i)$. Hence

$$\langle \beta, x - s_K\rangle = \sum_{i\in I\setminus K} x_i \langle \beta, \rho_{\Phi,K}(e_i)\rangle.$$

Assume that $\langle \beta, \rho_{\Phi,K} e_i\rangle < 0$, for every $i \in I\setminus K$. As $x_i \geq 0$ and $x_i > 0$ for at least one index $i \in I\setminus K$, we have $\langle \beta, x - s_K\rangle < 0$.

Conversely, let $i \in I\setminus K$. Take an $x = \sum_{k\in K} x_k e_k + x_i e_i \in \mathfrak{f}_I(\Phi,\lambda)$ with $x_i > 0$. Then by assumption we have $\langle \beta, x - s_K\rangle < 0$, hence $x_i\langle \beta, \rho_{\Phi,K} e_i\rangle < 0$.

It follows that, when K runs over $\mathcal{B}(\Phi,\tau)$ (the set of vertices of $\mathfrak{p}(\Phi,\lambda)$), the subsets $\mathcal{G}(\Phi,\tau)^K_\beta$ form a partition of $\mathcal{G}(\Phi,\tau)$ (the set of faces). Therefore, in order to prove Theorem 3.3, there remains to show that for every $K \in \mathcal{B}(\Phi,\tau)$, we have

$$\sum_{I\in\mathcal{G}(\Phi,\tau)^K_\beta} (-1)^{|I|-\dim F} \prod_{i\in I^c} p_i \prod_{i\in I}(p_i + q_i)$$
$$= (-1)^{|K^{c-}_\beta|} \prod_{i\in K^{c+}_\beta} p_i \prod_{i\in K^{c-}_\beta} q_i \prod_{i\in K}(p_i+q_i). \quad (3.6)$$

We factor out $\prod_{i\in K}(p_i+q_i)$. We need to prove

$$\sum_{I\in\mathcal{G}(\Phi,\tau)_\beta^K}(-1)^{|I|-\dim F}\prod_{i\in I^c}p_i\prod_{i\in I\setminus K}(p_i+q_i)$$
$$=(-1)^{|K_\beta^{c-}|}\prod_{i\in K_\beta^{c+}}p_i\prod_{i\in K_\beta^{c-}}q_i. \quad (3.7)$$

We make several observations. First, $\mathcal{G}(\Phi,\tau)_\beta^K$ is precisely the set of $I\subseteq\{1,\ldots,N\}$ such that $K\subseteq I$ and $I\setminus K\subseteq K_\beta^{c-}$. Moreover, for each $I\in\mathcal{G}(\Phi,\tau)_\beta^K$, the set of indices $I^c\bigsqcup(I\setminus K)$ is exactly the complement K^c and $|I|-\dim F=|I\setminus K|$, as $\dim F=|K|$. Let $B\subseteq K^c$. A given monomial $\prod_{i\in K^c\setminus B}p_i\prod_{i\in B}q_i$ appears on the left hand side of (3.7) with coefficient

$$\sum_{\{I\in\mathcal{G}(\Phi,\tau)_\beta^K,\ B\subseteq I\setminus K\}}(-1)^{|I\setminus K|}.$$

By the usual inclusion-exclusion relations applied to the subsets $I\setminus K$ of K^c, this sum is equal to $(-1)^{|K_\beta^{c-}|}$ if $B=K_\beta^{c-}$ and to 0 otherwise. □

Example 3.4. In Example 3.1 of the standard knapsack with $N=3$, we take $\langle\beta,x\rangle=x_1+\frac{x_2}{2}+\frac{x_3}{3}$. We obtain $Y(\Phi,\tau,\beta)=(p_1+q_1)q_2q_3+p_1(p_2+q_2)p_3-p_1q_2(p_3+q_3)$. It is indeed equal to $X(\Phi,\tau)=p_1p_2p_3+q_1q_2q_3$.

Example 3.5. (continues Example 3.2). The subspace V is generated by the vector $e_1+e_2-e_3$. Hence, the projections $\rho_{(\Phi,K)}(e_i)$ are $\rho_{\Phi,(1,2)}(e_3)=e_3-e_1-e_2$, $\rho_{\Phi,(1,3)}(e_2)=\rho_{\Phi,(2,3)}(e_1)=e_1+e_2-e_3$. We can take $\langle\beta,x\rangle=x_1+x_2+x_3$. Let τ_1 be the cone generated by (ϕ_1,ϕ_3). We obtain

$$Y(\Phi,\tau_1,\beta)=-(p_1+q_1)(p_2+q_2)q_3+(p_1+q_1)p_2(p_3+q_3) \quad (3.8)$$
$$=p_1p_2p_3+p_1q_2p_3-q_1p_2q_3-q_1q_2q_3. \quad (3.9)$$

Comparing with Example 3.2, we check that $Y(\Phi,\tau_1,\beta)=X(\Phi,\tau_1)$.

Corollary 3.2. *For any $\lambda\in F$, $\mathcal{X}(\Phi,\tau)[V(\Phi,\lambda)]$ is a signed sum of bounded polytopes.*

Proof. Choose any β regular, then the function $\mathcal{X}(\Phi,\tau)[V(\Phi,\lambda)]$ is equal to $Y_{\text{geom}}(\Phi,\tau,\beta)[V(\Phi,\lambda)]$. Taking $m(\beta)$ to be the minimum of the values $\langle\beta,s_K(\Phi,\lambda)\rangle$ over the $K\in\mathcal{B}(\Phi,\lambda)$, we see that the support of the function $\mathcal{X}(\Phi,\tau)[V(\Phi,\lambda)]$ is contained in the half space $\{\langle\beta,x\rangle\geq m(\beta)\}$ of $V(\Phi,\lambda)$. As this equality holds for any regular linear form β, this implies that the support of $\mathcal{X}(\Phi,\tau)[V(\Phi,\lambda)]$ is bounded. □

4 Wall-crossing

4.1 Combinatorial wall-crossing

In this section we prove the main theorem of this article: a formula for $X(\Phi, \tau_2) - X(\Phi, \tau_1)$, when τ_1 and τ_2 are adjacent topes.

The computation comes out nicely when we use the polarized expression $Y(\Phi, \tau, \beta)$ as a sum over $\mathcal{B}(\Phi, \tau)$ (Theorem 3.3), because it is easy to analyze how $\mathcal{B}(\Phi, \tau)$ changes as we cross the wall H between the two topes.

We recall that two topes τ_1 and τ_2 are called adjacent if the intersection of their closures spans a wall H. We denote by $\Phi \cap H$ the subsequence of Φ formed by the elements ϕ_i belonging to H.

Lemma 4.1. *Let τ_1 and τ_2 be adjacent Φ-topes such that $\tau_1 \subset \mathfrak{c}(\Phi)$. Let $K \in \mathcal{B}(\Phi, \tau_1)$ such that $K \notin \mathcal{B}(\Phi, \tau_2)$.*

(i) *For all $k \in K$ but one, say k_1, we have $\phi_k \in H$. The vector ϕ_{k_1} is in the open side of H which contains τ_1.*
(ii) *Let τ_{12} be the unique tope of $\Phi \cap H$ such that $\overline{\tau_1} \cap \overline{\tau_2} \subset \overline{\tau_{12}}$. Then τ_{12} is contained in the cone generated by the vectors ϕ_k for $k \in K, k \neq k_1$.*

Proof. Up to renumbering, we can assume that $K = \{1, \ldots, r\}$. Let $x_1, \ldots, x_r$ be the coordinates on F relative to this basic subset. If $K \notin \mathcal{B}(\Phi, \tau_2)$, at least one of these coordinates, say x_1, is < 0 on τ_2. Then the wall H must be the hyperplane $\{x_1 = 0\}$. (i) follows immediately.

The proof of (ii) is also easy. □

Recall Definition 3.5 of flipped systems Φ^A_{flip}.

Lemma 4.2. *Let τ_1 and τ_2 be adjacent Φ-topes such that $\tau_1 \subset \mathfrak{c}(\Phi)$. Let H be their common wall. Let A be the set of $i \in \{1, \ldots, N\}$ such that ϕ_i belongs to the open side of H which contains τ_1, (hence $-\phi_i$ belongs to the side of τ_2). Then $\mathcal{B}(\Phi^A_{\text{flip}}, \tau_2)$ is equal to the symmetric difference $\mathcal{B}(\Phi, \tau_1) \bigtriangleup \mathcal{B}(\Phi, \tau_2)$. More precisely*

$$K \in \mathcal{B}(\Phi, \tau_1), K \notin \mathcal{B}(\Phi, \tau_2) \Leftrightarrow K \in \mathcal{B}(\Phi^A_{\text{flip}}, \tau_2), K \cap A \neq \emptyset,$$

$$K \in \mathcal{B}(\Phi, \tau_2), K \notin \mathcal{B}(\Phi, \tau_1) \Leftrightarrow K \in \mathcal{B}(\Phi^A_{\text{flip}}, \tau_2), K \cap A = \emptyset.$$

Moreover, the cone $\mathfrak{c}(\Phi^A_{\text{flip}})$ is salient , and τ_2 is contained in at least one of the cones $\mathfrak{c}(\Phi)$ or $\mathfrak{c}(\Phi^A_{\text{flip}})$.

Proof. It follows easily from Lemma 4.1 (i) and the definition of Φ^A_{flip}. □

It will be convenient to have a notation.

Definition 4.1. Let τ_1, τ_2 be adjacent Φ-topes. We denote by $A(\Phi, \tau_1, \tau_2)$ the set of $i \in \{1, \dots, N\}$ such that ϕ_i belongs to the open side of the common wall which contains τ_1 .

Theorem 4.1. *Let $\Phi = (\phi_j)_{1 \le j \le N}$ be a sequence of non zero elements of a vector space F, generating F, and spanning a salient cone. Let τ_1 and τ_2 be adjacent Φ-topes such that $\tau_1 \subset \mathfrak{c}(\Phi)$. Let H be their common wall. Let A be the set of $i \in \{1, \dots, N\}$ such that ϕ_i is in the open side of H which contains τ_1. Let Φ^A_{flip} be the sequence $\sigma_i^A \phi_i$, where $\sigma_i^A = -1$ if $i \in A$ and $\sigma_i^A = 1$ if $i \notin A$. Let $X(\Phi, \tau_1)$, $X(\Phi, \tau_2)$ and $X(\Phi^A_{\mathrm{flip}}, \tau_2)$ be the corresponding Combinatorial Brianchon-Gram polynomials $\in \mathbb{Z}[p_i, q_i]$. Let Flip_A be the ring homomorphism from $\mathbb{Z}[p_i, q_i]$ to itself defined by exchanging p_i and q_i for $i \in A$. Then we have the wall-crossing formula*

$$X(\Phi, \tau_1) = X(\Phi, \tau_2) - (-1)^{|A|} \operatorname{Flip}_A X(\Phi^A_{\mathrm{flip}}, \tau_2). \tag{4.1}$$

Remark 4.1. If the tope τ_2 is not contained in $\mathfrak{c}(\Phi)$, (res $\mathfrak{c}(\Phi^A_{\mathrm{flip}})$), then $\mathcal{G}(\Phi, \tau_2)$, (respectively $\mathcal{G}(\Phi^A_{\mathrm{flip}}, \tau_2)$), is empty, hence $X(\Phi, \tau_2) = 0$, (resp $X(\Phi^A_{\mathrm{flip}}, \tau_2) = 0$).

Remark 4.2. By Remark 2.1, the actual wall crossing variations occur only on walls between chambers. This in agreement with this formula: indeed if H is not a wall between chambers, the tope τ_{12} is not contained in $\mathfrak{c}(\Phi \cap H)$ and τ_2 is not contained in $\mathfrak{c}(\Phi^A_{\mathrm{flip}})$.

Remark 4.3. The theorem is trivially true if $\tau_1 \not\subseteq \mathfrak{c}(\Phi)$. Indeed, in this case, we have $X(\phi, \tau_1) = 0$, and $A(\Phi, \tau_1, \tau_2) = \emptyset$, so that the right hand side of (4.1) is $X(\phi, \tau_2) - X(\phi, \tau_2) = 0$.

Proof of Theorem 4.1. Let β be a linear form on $\mathbb{R}^N$ which is regular for Φ. Let β^A be the linear form defined by

$$\langle \beta^A, e_i \rangle = \sigma_i^A \langle \beta, e_i \rangle$$

where $\sigma_i^A = -1$ if $i \in A$ and $\sigma_i^A = 1$ if $i \in A^c$. We have, for every i,

$$\langle \beta^A, \rho_{\Phi^A_{\mathrm{flip}}, K} e_i \rangle = \sigma_i^A \langle \beta, \rho_{\Phi, K} e_i \rangle. \tag{4.2}$$

It follows, in particular, that β^A is regular for Φ^A_{flip}.

First we will prove the following relation between the polarized sums

$$Y(\Phi, \tau_2, \beta) - Y(\Phi, \tau_1, \beta) = (-1)^{|A|} \operatorname{Flip}_A Y(\Phi^A_{\mathrm{flip}}, \tau_2, \beta^A). \tag{4.3}$$

Then we obtain (4.1) by applying Theorem 3.2. We write

$$\begin{aligned} &Y(\Phi,\tau_2,\beta) - Y(\Phi,\tau_1,\beta) \\ &\quad = \sum_{K\in\mathcal{B}(\Phi,\tau_2)} (-1)^{|K_\beta^{c-}|} \prod_{i\in K_\beta^{c+}} p_i \prod_{i\in K_\beta^{c-}} q_i \prod_{i\in K}(p_i+q_i) \\ &\qquad - \sum_{K\in\mathcal{B}(\Phi,\tau_1)} (-1)^{|K_\beta^{c-}|} \prod_{i\in K_\beta^{c+}} p_i \prod_{i\in K_\beta^{c-}} q_i \prod_{i\in K}(p_i+q_i) \end{aligned} \tag{4.4}$$

The terms for which $K \in \mathcal{B}(\Phi,\tau_1)\cap\mathcal{B}(\Phi,\tau_2)$ cancel out. For the other terms, we apply Lemma 4.2. Take a K in $\mathcal{B}(\Phi,\tau_1) \vartriangle \mathcal{B}(\Phi,\tau_2) = \mathcal{B}(\Phi^A_{\mathrm{flip}},\tau_2)$. Using (4.2), we check easily that the unique K-term in $Y(\Phi,\tau_2,\beta) - Y(\Phi,\tau_1,\beta)$ is equal to the K-term in $(-1)^{|A|}\,\mathrm{Flip}_A\, Y(\Phi^A_{\mathrm{flip}},\tau_2,\beta^A)$. □

Example 4.2. $N = 4$ and $\Phi = (\phi_1,\phi_2,\phi_3 = \frac{1}{2}(\phi_2-\phi_1),\phi_4 = \frac{1}{2}(\phi_1+\phi_2))$. The tope τ_1 is the open cone generated by ϕ_2 and ϕ_4. The adjacent tope τ_2 is the open cone generated by ϕ_4 and ϕ_1, see Figure 4.1. Then ϕ_2 and ϕ_3 lie on the τ_1-side of the common wall, so that $A = \{2,3\}$ and $\Phi^A_{\mathrm{flip}} = (\phi_1,-\phi_2,-\phi_3,\phi_4)$. We obtain

$$\begin{aligned} X(\Phi,\tau_1) &= p_1p_2p_3p_4 - p_1q_2q_3p_4 - q_1p_2p_3q_4 + q_1q_2q_3q_4, \\ X(\Phi,\tau_2) &= p_1p_2p_3p_4 + p_1q_2q_3q_4 + q_1p_2p_3p_4 + q_1q_2q_3q_4, \\ X(\Phi^A_{\mathrm{flip}},\tau_2) &= p_1p_2p_3p_4 + p_1p_2p_3q_4 + q_1q_2q_3p_4 + q_1q_2q_3q_4, \\ \mathrm{Flip}_A\, X(\Phi^A_{\mathrm{flip}},\tau_2) &= p_1q_2q_3p_4 + p_1q_2q_3q_4 + q_1p_2p_3p_4 + q_1p_2p_3q_4, \\ &= X(\Phi,\tau_2) - X(\Phi,\tau_1). \end{aligned}$$

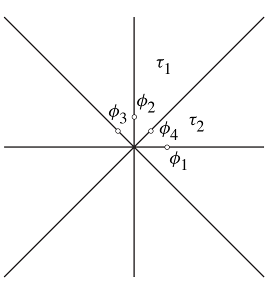


Figure 4.1. Topes τ_1,τ_2 for $\Phi = (\phi_1,\phi_2,\phi_3 = \frac{1}{2}(\phi_2-\phi_1),\phi_4 = \frac{1}{2}(\phi_1+\phi_2))$.

Later, in order to give a formula for the decomposition of $\mathcal{X}(\Phi,\tau)$ in quadrants, we will need to iterate the wall-crossing through a sequence of consecutively adjacent topes. To help understand what we obtain, let us first cross two consecutive walls.

Corollary 4.1. *Let τ_1, τ_2, τ_3 be pairwise different Φ-topes such that τ_2 is adjacent to τ_1 and to τ_3. Let $A_1 = A(\Phi, \tau_1, \tau_2)$, $A_2 = A(\Phi, \tau_2, \tau_3)$ and let $A_{1,2}$ be the symmetric difference $A_1 \triangle A(\Phi_{\text{flip}}^{A_1}, \tau_2, \tau_3)$. Then the three cones $\mathfrak{c}(\Phi_{\text{flip}}^{A_1})$, $\mathfrak{c}(\Phi_{\text{flip}}^{A_2})$ and $\mathfrak{c}(\Phi_{\text{flip}}^{A_{1,2}})$ are salient .*

We have

$$\begin{aligned} X(\Phi, \tau_1) = X(\Phi, \tau_3) &- (-1)^{|A_1|} \operatorname{Flip}_{A_1} X(\Phi_{\text{flip}}^{A_1}, \tau_3) \\ &- (-1)^{|A_2|} \operatorname{Flip}_{A_2} X(\Phi_{\text{flip}}^{A_2}, \tau_3) \\ &+ (-1)^{|A_{1,2}|} \operatorname{Flip}_{A_{1,2}} X(\Phi_{\text{flip}}^{A_{1,2}}, \tau_3). \end{aligned} \tag{4.5}$$

Proof. We apply the wall-crossing theorem to (Φ, τ_1, τ_2). We obtain

$$X(\Phi, \tau_1) = X(\Phi, \tau_2) - (-1)^{|A_1|} \operatorname{Flip}_{A_1} X(\Phi_{\text{flip}}^{A_1}, \tau_2).$$

In the right hand side, we transform each term by crossing the wall from τ_2 into τ_3. First,

$$X(\Phi, \tau_2) = X(\Phi, \tau_3) - (-1)^{|A_2|} \operatorname{Flip}_{A_2} X(\Phi_{\text{flip}}^{A_2}, \tau_3).$$

In order to apply the wall-crossing to $(\Phi_{\text{flip}}^{A_1}, \tau_2, \tau_3)$, we observe that the sign rule implies

$$\begin{aligned} (\Phi_{\text{flip}}^{A_1})^{A(\Phi_{\text{flip}}^{A_1}, \tau_2, \tau_3)} &= \Phi_{\text{flip}}^{A_1 \triangle A(\Phi_{\text{flip}}^{A_1}, \tau_2, \tau_3)}, \\ \operatorname{Flip}_{A_1} \circ \operatorname{Flip}_{A(\Phi_{\text{flip}}^{A_1}, \tau_2, \tau_3)} &= \operatorname{Flip}_{A_1 \triangle A(\Phi_{\text{flip}}^{A_1}, \tau_2, \tau_3)}. \end{aligned}$$

Moreover, $(-1)^{|A_1|}(-1)^{A(\Phi_{\text{flip}}^{A_1}, \tau_2, \tau_3)} = (-1)^{A_1 \triangle A(\Phi_{\text{flip}}^{A_1}, \tau_2, \tau_3)}$. Hence we obtain

$$\begin{aligned} -(-1)^{|A_1|} \operatorname{Flip}_{A_1} X(\Phi_{\text{flip}}^{A_1}, \tau_2) = &-(-1)^{|A_1|} \operatorname{Flip}_{A_1} X(\Phi_{\text{flip}}^{A_1}, \tau_3) \\ &+ (-1)^{|A_{1,2}|} \operatorname{Flip}_{A_{1,2}} X(\Phi_{\text{flip}}^{A_{1,2}}, \tau_3). \end{aligned}$$

This proves (4.5). The cones are salient by the very definition of the flipped systems $\Phi_{\text{flip}}^{A_i}$ and $(\Phi_{\text{flip}}^{A_1})^{A(\Phi_{\text{flip}}^{A_1}, \tau_2, \tau_3)}$. □

We now cross a number of walls to go from a tope τ to another tope ν. A signed subset of $\{1, 2, \ldots, N\}$ is a list $[\epsilon, I]$ where I is a subset of $\{1, 2, \ldots, N\}$ and $\epsilon = \pm 1$ a sign.

Definition 4.2. Let τ and ν be two topes, and let us choose a sequence τ_k, $k = 1, \ldots, \ell$ of topes such that τ_{k+1} is adjacent to τ_k for every $1 \leq k \leq \ell - 1$, and $\tau_1 = \tau$, $\tau_\ell = \nu$.

For every sequence $K = (1 \leq k_1 < \cdots < k_s \leq \ell - 1)$, let $A_K \subseteq \{1, \ldots, N\}$ be the subset defined recursively as follows. If $s = 0$, that is $K = \emptyset$, then $A_\emptyset = \emptyset$. If $K = (k_1, k_2, \ldots, k_s)$ with $s \geq 1$, let $A = A_{(k_1,\ldots,k_{s-1})}$, then

$$A_K = A \bigtriangleup A(\Phi^A_{\text{flip}}, \tau_{k_s}, \tau_{k_s+1}).$$

We define the list $\mathcal{A}(\nu, \tau)$ to be the list of signed subsets $[(-1)^{|K|}, A_K]$ of $\{1, 2, \ldots, N\}$ so obtained.

Remark 4.4. The list $\mathcal{A}(\nu, \tau)$ depends of the choice of path of adjacent topes from τ to ν , but we do not indicate this in the notation.

We obtain the following result, if there are $\ell - 1$ wall crossings to go from τ to ν.

Corollary 4.2. *Let ν be a tope. Then for every $[\epsilon, A] \in \mathcal{A}(\nu, \tau)$, the cone $\mathfrak{c}(\Phi^A_{\text{flip}})$ is salient. Furthermore, we have*

$$X(\Phi, \tau) = \sum_{[\epsilon, A] \in \mathcal{A}(\nu, \tau)} \epsilon \operatorname{Flip}_A X(\Phi^A_{\text{flip}}, \nu).$$

Proof. The recursion rule means that, when we travel through the sequence of topes $\tau_i, i = 1, \ldots, \ell$, and apply Formula (4.1), we choose the flipped term when we cross the wall between τ_{k_i} and τ_{k_i+1}, and the unflipped term for the other walls. For instance, $A_{\{1\}} = A(\Phi, \tau_1, \tau_2)$, and $A_{\{1,2\}} = A_{\{1\}} \bigtriangleup A(\Phi^{A_{\{1\}}}_{\text{flip}}, \tau_2, \tau_3)$, in agreement with the two-step wall-crossing formula. The general case is immediate by induction. $\square$

4.2 Semi-closed partition polytopes

In order to state the geometric consequences of the above combinatorial wall-crossing formulas, we introduce some semi-closed partition polytopes, to which the Brianchon-Gram theorem extends naturally.

Definition 4.3.

$$\mathfrak{p}(\Phi, A, \lambda) = \{x \in V(\Phi, \lambda), \quad x_i > 0 \text{ for } i \in A, x_i \geq 0 \text{ for } i \in A^c.\}$$

When λ is regular, the closure of $\mathfrak{p}(\Phi, A, \lambda)$ is the partition polytope $\mathfrak{p}(\Phi, \lambda)$.

Definition 4.4. For $A \subseteq \{1, \ldots, N\}$ we denote by Geom_A the map from W to the space of functions on $\mathbb{R}^N$ defined by substituting $1 - p_i$ for q_i, then $[x_i \geq 0]$ for p_i if $i \notin A$ and $[x_i > 0]$ for p_i if $i \in A$.

When A is the empty set, the substitution $\mathrm{Geom}_\emptyset$ coincide with the usual substitution Geom defined before. When we consider non empty subsets $A \subseteq \{1, \dots, N\}$, we obtain an extension of the Brianchon-Gram theorem to these semi-closed polytopes.

Proposition 4.1. *Let $A \subseteq \{1, \dots, N\}$. For $\lambda \in \tau$, we have*

$$\mathrm{Geom}_A\, X(\Phi, \tau)\, [V(\Phi, \lambda)] = [\mathfrak{p}(\Phi, A, \lambda)]. \tag{4.6}$$

Proof. When $A = \emptyset$, it is exactly the Brianchon-Gram theorem. We proceed by induction on the cardinality of A. If $A \neq \emptyset$, we can assume that $N \in A$, up to renumbering. Let $A' = A \setminus \{N\}$.

We write $[\mathfrak{p}(\Phi, A, \lambda)] = [\mathfrak{p}(\Phi, A', \lambda)] + \big([\mathfrak{p}(\Phi, A, \lambda)] - [\mathfrak{p}(\Phi, A', \lambda)]\big)$. We have

$$[\mathfrak{p}(\Phi, A', \lambda)] - [\mathfrak{p}(\Phi, A, \lambda)] = [\mathfrak{p}(\Phi, A', \lambda)][x_N = 0].$$

Let us show that we have

$$\begin{aligned} &(\mathrm{Geom}_{A'}\, X(\Phi, \tau) - \mathrm{Geom}_A\, X(\Phi, \tau))[V(\Phi, \lambda)] \\ &= [\mathfrak{p}(\Phi, A', \lambda)][x_N = 0]. \end{aligned} \tag{4.7}$$

We first prove (4.7) in the case where $A = \{N\}$, hence $A' = \emptyset$.

We observe that the right hand side of (4.7) is the characteristic function of the face of $\mathfrak{p}(\Phi, \lambda)$ defined by $x_N = 0$. If we identify the hyperplane $\{x_N = 0\}$ with $\mathbb{R}^{N-1}$, this face is the partition polytope $\mathfrak{p}(\Phi', \lambda)$ corresponding to $\Phi' = (\phi_i)$, $1 \leq i \leq N - 1$ and $\lambda \in F$.

We now look at the left hand side of (4.7). We see that

$$\begin{aligned} &\mathrm{Geom}_\emptyset\, X(\Phi, \tau) - \mathrm{Geom}_{\{N\}}\, X(\Phi, \tau) \\ &\quad = \left(\sum_{\substack{I \in \mathcal{G}(\Phi, \tau), \\ I^c \ni N}} (-1)^{|I| - \dim F} \prod_{i \in I^c, i \neq N} [x_i \geq 0] \right) [x_N = 0], \end{aligned} \tag{4.8}$$

because the terms indexed by the subsets I such that $N \notin I^c$ cancel out in the difference.

If Φ' does not generate Φ, we see that both sides of the equation (4.7) are equal to 0. Indeed as λ is regular, it cannot be contained in the smaller dimensional space generated by Φ', and every generating subset in $\mathcal{G}(\Phi, \tau)$ contains the index N.

Now assume that Φ' generates F. The Φ-tope τ is contained in a unique Φ'-tope τ'. The set $\mathcal{G}(\Phi', \tau')$ consists precisely of the subsets $I' \subseteq \{1, \dots, N - 1\}$ such that $I' \in \mathcal{G}(\Phi, \tau)$.

Therefore the right hand side of (4.8) is the Brianchon-Gram decomposition of the facet $\mathfrak{p}(\Phi', \lambda)$. Thus we have proved (4.7) in the case where $A = \{N\}$.

The general case when $A' \neq \emptyset$ is similar. We have now

$$\begin{aligned}&\mathrm{Geom}_{A'} X(\Phi, \tau) - \mathrm{Geom}_A X(\Phi, \tau)\\ &= \left(\sum_{\substack{I \in \mathcal{G}(\Phi,\tau),\\ I^c \ni N}} (-1)^{|I| - \dim F} \prod_{i \in I^c \cap A'} [x_i > 0] \prod_{\substack{i \in I^c \cap A'^c,\\ i \neq N}} [x_i \geq 0] \right) [x_N = 0].\end{aligned}$$

By the induction hypothesis, the right hand side of this equality is the Brianchon-Gram decomposition of the semi-closed polytope $\mathfrak{p}(\Phi', A', \lambda) = \mathfrak{p}(\Phi, A', \lambda) \cap \{x_N = 0\}$. □

Remark 4.5. The formula is not necessarily true on the boundary of τ, as shown by the trivial example $\Phi = (\phi_1)$, $A = \{1\}$, $\lambda = 0$.

It will be useful to rephrase Proposition 4.1 in the terms which arise in the combinatorial wall-crossing Theorem 4.1.

Definition 4.5. For $A \subseteq \{1, \ldots, N\}$ such that $\mathfrak{c}(\Phi^A_{\mathrm{flip}})$ is salient and $\lambda \in F$, let

$$\begin{aligned}&\mathfrak{p}_{\mathrm{flip}}(\Phi, A, \lambda)\\ &= \{x \in \mathbb{R}^N \; ; \sum_i x_i \phi_i = \lambda, x_i < 0 \text{ for } i \in A, x_i \geq 0 \text{ for } i \notin A\}. \quad (4.9)\end{aligned}$$

Proposition 4.2. *Let $A \subseteq \{1, \ldots, N\}$ be such that the cone $\mathfrak{c}(\Phi^A_{\mathrm{flip}})$ is salient. Let τ be a Φ-tope. Then, for $\lambda \in \tau$ we have*

$$\mathrm{Geom}_\emptyset \, \mathrm{Flip}_A \, X(\Phi^A_{\mathrm{flip}}, \tau)[V(\Phi, \lambda)] = [\mathfrak{p}_{\mathrm{flip}}(\Phi, A, \lambda)]. \quad (4.10)$$

Proof. For any polynomial $Z \in \mathbb{C}[p_i, q_i]$, we have

$$\mathrm{Geom}_\emptyset \, \mathrm{Flip}_A(Z) = \mathrm{Geom}_A(Z) \circ \sigma^A \quad (4.11)$$

where $\sigma^A x = (\sigma^A_i x_i)$ with $\sigma^A_i = -1$ if $i \in A$ and $\sigma^A_i = 1$ if $i \notin A$. Moreover we have

$$[V(\Phi, \lambda)] = [V(\Phi^A_{\mathrm{flip}}, \lambda)] \circ \sigma^A.$$

Thus

$$\begin{aligned}&\mathrm{Geom}_\emptyset \, \mathrm{Flip}_A \, X(\Phi^A_{\mathrm{flip}}, \tau)[V(\Phi, \lambda]\\ &\qquad = \big(\mathrm{Geom}_A \, X(\Phi^A_{\mathrm{flip}}, \tau)[V(\Phi^A_{\mathrm{flip}}, \lambda)]\big) \circ \sigma^A. \quad (4.12)\end{aligned}$$

We apply Proposition 4.1 to the sequence Φ_{flip}^{A} and the Φ_{flip}^{A}-tope τ. We obtain that the right hand side of (4.12) is equal to

$$[\mathfrak{p}(\Phi_{\text{flip}}^{A}, A, \lambda] \circ \sigma^{A}.$$

By definition of $\mathfrak{p}(\Phi_{\text{flip}}^{A}, A, \lambda)$, this is precisely the characteristic function of the set of x such that $\sigma_i^A x_i = -x_i > 0$ for $i \in A$ and $\sigma_i^A x_i = x_i \geq 0$ for $i \notin A$, and $\sigma^A x \in V(\Phi_{\text{flip}}^{A}, \lambda)$, *i.e.* $x \in \mathfrak{p}_{\text{flip}}(\Phi, A, \lambda)$. □

4.3 Decomposition in quadrants

Recall that when τ and ν are two topes, we have defined a list $\mathcal{A}(\nu, \tau)$ of signed subsets $[\epsilon, A]$ of $\{1, 2, \ldots, N\}$.

Theorem 4.3. *Let $z(\Phi, \tau, B)$, $B \subset \{1, \ldots, N\}$ be the collection of coefficients of the Combinatorial Brianchon-Gram polynomial associated to the Φ-tope τ.*

$$X(\Phi, \tau) = \sum_{B} z(\Phi, \tau, B) \prod_{i \notin B} p_i \prod_{i \in B} q_i.$$

(i) *If $B = \emptyset$, then $z(\Phi, \tau, B) = 1$ while if $B = \{1, 2, \ldots, N\}$, then $z(\Phi, \tau, B) = (-1)^d$.*
(ii) *If $z(\Phi, \tau, B) \neq 0$, then the cone $\mathfrak{c}(\Phi_{\text{flip}}^{B})$ is salient .*
(iii) *More precisely, if $z(\Phi, \tau, B)$ is not equal to 0, choose $x = (x_i) \in Q_{\text{neg}}^{B}$ such that $\lambda = \sum_i x_i \phi_i$ is a regular element in F and let ν be the tope containing λ. Then B occurs in the list in $\mathcal{A}(\nu, \tau)$ and we have*

$$z(\Phi, \tau, B) = (-1)^{|B|} \sum_{\{[\epsilon, B] \in \mathcal{A}(\nu, \tau)\}} \epsilon. \tag{4.13}$$

Proof. We already remarked (i).

For the choice of x as in (iii), the coefficient $z_B = z(\Phi, \tau, B)$ is the value of Geom $X(\Phi, \tau)[V(\Phi, \lambda)]$ at such a point x. We now apply Corollary 4.2 and Proposition 4.2 using the tope ν, where λ belongs.

We obtain that the value z_B is the signed sum of the values of $\mathfrak{p}_{\text{flip}}(\Phi, A, \lambda) = [Q_{\text{neg}}^{A}] \cap [V(\Phi, \lambda)]$ at x. By definition, this value is zero if the set of i with $x_i < 0$ is different from B. Otherwise, is equal to 1. □

Example 4.4. Let τ_1 and τ_2 be the adjacent topes and $A = A(\Phi, \tau_1, \tau_2)$. If $\tau_2 \subset \mathfrak{c}(\Phi_{\text{flip}}^{A})$, then $z(\Phi, \tau, A) = -(-1)^{|A|}$. An example of this situation is the standard knapsack, (Example 3.1), where there are exactly two topes, $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. The theorem implies that the Brianchon-Gram polynomial is $p_1 \cdots p_N - (-1)^N q_1 \cdots q_N$, as we found directly.

Example 4.5. If there is no subset K in the sum (4.13), or if there are more than one, then the coefficient $z(\Phi, \tau, B)$ may be 0 although the cone $\mathfrak{c}(\Phi^B_{\text{flip}})$ is salient . Let us take Example 3.2. The direct computation gave $X(\Phi, \tau_1) = p_1p_2p_3 - p_1q_2q_3 + q_1p_2p_3 - q_1q_2q_3$. We see that there are no terms corresponding to $B = \{2\}$ and $B = \{1, 3\}$. For $B = \{2\}$, the tope $\tau_2 \stackrel{\circ}{=} \mathfrak{c}\ (\phi_1, -\phi_2)$ is contained in $\mathfrak{c}(\phi_1, -\phi_2, \phi_3)$, hence we take the sequence (τ_1, τ_2). For $K = (1)$ we have $A_{(1)} = \{2, 3\} \neq B$, so there is no K such that $A_K = B$. For $B = \{1, 3\}$, we need a sequence of three topes, (τ_1, τ_2, τ_3) where now $\tau_2 \stackrel{\circ}{=} \mathfrak{c}\ (\phi_2, \phi_3)$ and $\tau_3 \stackrel{\circ}{=} \mathfrak{c}\ (-\phi_1, \phi_2) \subset \mathfrak{c}(\Phi^B_{\text{flip}})$. The K such that $A_K = \{1, 3\}$ are $K = (2)$ and $K = (1, 2)$, which indeed lead to opposite signs in the sum (4.13).

4.4 Geometric wall-crossing

By "intersecting" Geom $X(\Phi, \tau)$ with $[V(\Phi, \lambda)]$, we translate the results on $X(\Phi, \tau)$ in geometric terms.

Corollary 4.3 (Notations of Theorem 4.1 and Definition 4.5). *Let τ_1 be a Φ-tope. For λ in the adjacent tope τ_2, we have the geometric wall-crossing for.mula*

$$\mathcal{X}(\Phi, \tau_1)[V(\Phi, \lambda)] = [\mathfrak{p}(\Phi, \lambda)] - (-1)^{|A|}[\mathfrak{p}_{\text{flip}}(\Phi, A, \lambda)]. \tag{4.14}$$

Proof. We apply the map Geom on both sides of the combinatorial wall-crossing formula (4.1), then we multiply by the characteristic function $[V(\Phi, \lambda)]$. We obtain, by definition,

$$\mathcal{X}(\Phi,\tau_1)[V(\Phi,\lambda)] = [\mathfrak{p}(\Phi,\lambda] - (-1)^{|A|}\,\text{Geom}\,\text{Flip}_A X(\Phi^A_{\text{flip}},\tau_2)[V(\Phi,\lambda)],$$

hence (4.14) by applying the semi-closed Brianchon-Gram formula, as stated in Proposition 4.2, to the tope τ_2. ☐

Corollary 4.4. *For any $\lambda \in F$, the function*

$$\mathcal{X}(\Phi, \tau)[V(\Phi, \lambda)] = \sum_{\{B, \lambda \in \mathfrak{c}(\Phi^B_{\text{flip}})\}} z(\Phi, \tau, B)[\mathfrak{p}_{\text{flip}}(\Phi, B, \lambda)] \tag{4.15}$$

is a linear combination with integral coefficients of semi-closed partition polytopes.

4.5 An example

We return to Example 4.2, see Figure 4.1, with $\Phi = (\phi_1, \phi_2, \phi_3 = \frac{1}{2}(\phi_2 - \phi_1), \phi_4 = \frac{1}{2}(\phi_1 + \phi_2))$.

For $\lambda = \lambda_1\phi_1 + \lambda_2\phi_2$, we parametrize the 2-dim subspace $V(\Phi, \lambda) \subset \mathbb{R}^4$ by $(y_1, y_2) \mapsto (y_1+\lambda_1, y_2+\lambda_2, y_1-y_2, -(y_1+y_2))$. We start with λ in the tope τ_1 generated by ϕ_2 and ϕ_4, *i.e.* $\lambda_2 > \lambda_1 > 0$. Then $\mathfrak{p}(\lambda)$ corresponds under the parameterization to the tetragon in Figure 4.2, defined by the inequations

$$\begin{aligned} y_1 + \lambda_1 &\geq 0 \\ y_2 + \lambda_2 &\geq 0 \\ y_1 - y_2 &\geq 0 \\ y_1 + y_2 &\leq 0 \end{aligned}$$

We describe its analytic continuation, as the parameter λ visits the topes, one after the other. In the figures, the polytopes which are counted positively are coloured in blue, those wich are counted negatively are coloured in red. Semi-openness is indicated with dashed lines. When λ moves to the right and reaches the wall generated by ϕ_4, the tetragon $\mathfrak{p}(\lambda)$ transforms into a triangle (Figure 4.2). When λ enters the adjacent tope τ_2 generated by (ϕ_1, ϕ_4), *i.e.* $\lambda_1 > \lambda_2 > 0$, the wall-crossing polytope appears (Figure 4.3). It is (with sign -1) the semi-closed triangle

$$\begin{aligned} y_1 + \lambda_1 &\geq 0 \\ y_2 + \lambda_2 &< 0 \\ y_1 - y_2 &< 0 \\ y_1 + y_2 &\leq 0, \quad \text{(this condition is redundant).} \end{aligned}$$

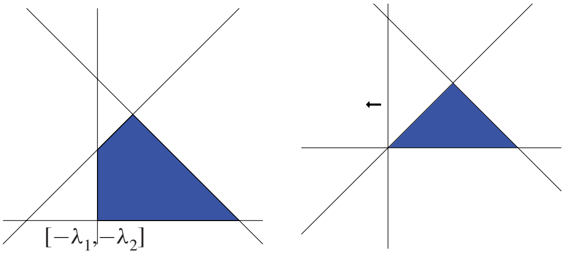


Figure 4.2. $\lambda_2 > \lambda_1 > 0$, (tope (ϕ_2, ϕ_4)), then $\lambda_2 = \lambda_1 > 0$, (wall (ϕ_4)).

Then λ moves downwards towards the wall generated by ϕ_1. The positive closed triangle shrinks while the negative semi-closed one increases. When λ reaches the wall, the closed triangle is reduced to a point (Figure 4.3). When λ enters the tope $(\phi_1, -\phi_3)$, the negative semi-closed triangle deforms into a negative semi-closed quadrileral (Figure 4.4).

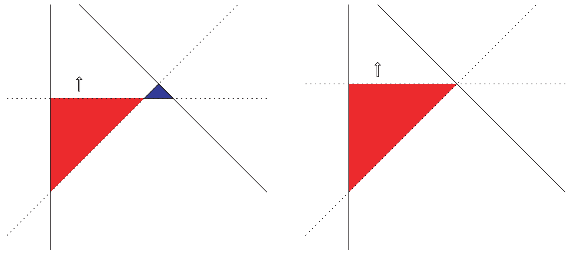


Figure 4.3. $\lambda_1 > \lambda_2 > 0$, (tope (ϕ_4, ϕ_1)), then $\lambda_1 > 0 = \lambda_2$, (wall (ϕ_1)).

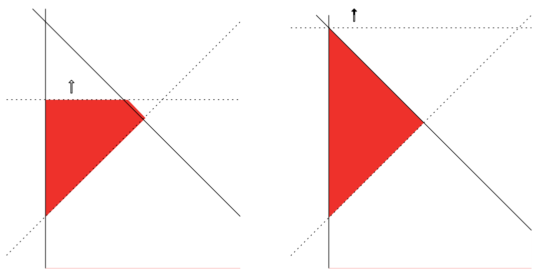


Figure 4.4. $\lambda_1 > -\lambda_2 > 0$, (tope $(\phi_1, -\phi_3)$), then $\lambda_1 = -\lambda_2 > 0$, (wall $(-\phi_3)$).

When λ reaches the wall generated by $-\phi_3$ (Figure 4.5), then enters the tope $(-\phi_3, -\phi_2)$, a new positive open triangle appears. Then λ reaches the wall generated by $-\phi_2$ (Figure 4.5) and enters the tope $(-\phi_2, -\phi_4)$ (Figure 4.6). The analytic continuation is now a positive open tetragon opposite to the (closed) initial one. (This case is pointed out in [20]).

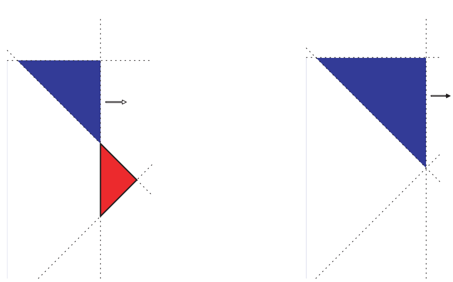


Figure 4.5. $-\lambda_2 > \lambda_1 > 0$, (tope $(-\phi_3, -\phi_2)$), then $-\lambda_2 > 0 = \lambda_1$, (wall $-\phi_2$).

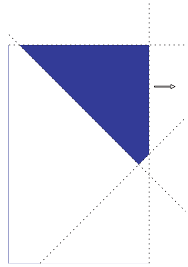


Figure 4.6. $\lambda_1 < \lambda_2 < 0$, (tope $(-\phi_2, -\phi_4)$). The polygon is now the interior of the opposite of the initial tetragon of Figure 4.2, cf. [20].

5 Integrals and discrete sums over a partition polytope

As a consequence of the compactness result of Corollary 4.4, together with the set theoretic relations of Corollary 3.1, we recover properties of sums and integrals over partition polytopes which were previously obtained in [9] and [12, 13, 19]. Moreover, the set-theoretic wall-crossing formula has obvious implications for sums and integrals. In particular, when applied to the number of points of a partition polytope, it implies the wall-crossing formula of [18, Theorem 5.2]. We will explain in more details this last point in Subsection 5.3.

5.1 Generating functions of polyhedra and Brion's theorem

Let V be a real dimensional vector space. We choose a Lebesgue measure dv on V. Let us recall the notion of valuations and of generating functions of cones (see the survey [3]).

Recall that a *valuation* F is a map from a set of polyhedra $\mathfrak{p} \subset V$ to a vector space $\mathcal{M}$ such that whenever the characteristic functions $[\mathfrak{p}_i]$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i[\mathfrak{p}_i] = 0$, then the elements $F(\mathfrak{p}_i)$ satisfy the same relation $\sum_i r_i F(\mathfrak{p}_i) = 0$. Thus any valuation defined on the set of all polyhedra can be extended to the "analytic continuation", which is a signed sum of polytopes. In particular, the valuation defined on the set of polyhedra by the Euler characteristic (see [3]) is identically equal to 1 on the "analytic continuation" as follows from Brianchon-Gram decomposition and Euler relations.

We now study two other classical instances of valuations.

There exists a unique valuation $\mathfrak{p} \mapsto I(\mathfrak{p})$ which associates to every polyhedron $\mathfrak{p} \subseteq V$ a meromorphic function $I(\mathfrak{p})(\xi)$ on V^*, so that the following properties hold:

(i) If $\mathfrak{p}$ contains a straight line, then $I(\mathfrak{p}) = 0$.

(ii) If $\xi \in V^*$ is such that $\mathrm{e}^{\langle \xi, x\rangle}$ is integrable over $\mathfrak{p}$ for the measure dv, then

$$I(\mathfrak{p})(\xi) = \int_{\mathfrak{p}} \mathrm{e}^{\langle \xi, x\rangle}\, dv.$$

Moreover, for every point $s \in V$, one has

$$I(s + \mathfrak{p})(\xi) = \mathrm{e}^{\langle \xi, s\rangle} I(\mathfrak{p})(\xi).$$

$I(\mathfrak{p})(\xi)$ is called the *continuous generating function* of $\mathfrak{p}$.

Assume that V is equipped with a lattice $V_{\mathbb{Z}}$.

There exists a unique valuation $\mathfrak{p} \mapsto S(\mathfrak{p})$ which associates to every rational polyhedron $\mathfrak{p} \subseteq V$ a meromorphic function $S(\mathfrak{p})(\xi)$ on V^*, so that

(i) if $\mathfrak{p}$ contains a straight line, then $S(\mathfrak{p}) = 0$;
(ii) if $\xi \in V^*$ is such that $\mathrm{e}^{\langle \xi, x\rangle}$ is summable over the set of lattice points of $\mathfrak{p}$, then

$$S(\mathfrak{p})(\xi) = \sum_{x \in \mathfrak{p} \cap V_{\mathbb{Z}}} \mathrm{e}^{\langle \xi, x\rangle}.$$

Moreover, for every point $s \in V_{\mathbb{Z}}$, one has

$$S(s + \mathfrak{p})(\xi) = \mathrm{e}^{\langle \xi, s\rangle} S(\mathfrak{p})(\xi).$$

$S(\mathfrak{p})(\xi)$ is called the (*discrete*) *generating function* of $\mathfrak{p}$.

These valuations are easily constructed, either by algebraic methods (see [3]), or by introducing the Fourier transforms of discrete or continuous measures associated to the polyhedron $\mathfrak{p}$ (see Section 6). Furthermore, there is an important property of the generating functions $S(\mathfrak{p})(\xi)$ and $I(\mathfrak{p})(\xi)$. Introduce the space $\mathcal{M}_\ell(V^*)$ of meromorphic functions on V^* which can be written as the quotient of a function which is holomorphic near $\xi = 0$ by a product of linear forms. The functions $I(\mathfrak{p})(\xi)$ and $S(\mathfrak{p})(\xi)$ belong to the space $\mathcal{M}_\ell(V^*)$. Then a function $f(\xi) \in \mathcal{M}_\ell(V^*)$ has a unique expansion into homogeneous rational functions

$$f(\xi) = \sum_{m \geq m_0} f_{[m]}(\xi),$$

where the summands $f_{[m]}(\xi)$ have degree m as we define now: if P is a homogeneous polynomial on V^* of degree p, and D a product of r linear forms, then $\frac{P}{D}$ is an element in $\mathcal{M}_\ell(V^*)$ homogeneous of degree $m = p - r$.

Let $\mathfrak{p}$ be a polytope with set of faces $\mathcal{F}(\mathfrak{p})$, and affine tangent cones $\mathfrak{t}_{\mathrm{aff}}(\mathfrak{p}, \mathfrak{f})$ at $\mathfrak{f}$. We obtain from the Brianchon-Gram theorem:

$$\int_{\mathfrak{p}} e^{\langle \xi, v\rangle} dv = \sum_{\mathfrak{f}} (-1)^{\dim \mathfrak{f}} I(\mathfrak{t}_{\mathrm{aff}}(\mathfrak{p}, \mathfrak{f}))(\xi).$$

Furthermore, as the cone $\mathfrak{t}_{\mathrm{aff}}(\mathfrak{p}, \mathfrak{f})$ contains a straight line, when the dimension of $\mathfrak{f}$ is strictly greater than 0, this gives the well-known Brion's formula:

$$\int_{\mathfrak{p}} e^{\langle \xi, v\rangle} dv = \sum_{s} I(s + \mathfrak{c}_s)(\xi).$$

Here s runs through the vertices of $\mathfrak{p}$ and $\mathfrak{c}_s$ is the tangent cone at s.

Similarly, when V is a rational vector space with lattice $V_{\mathbb{Z}}$, and $\mathfrak{p}$ a rational polytope, we have

$$\sum_{x \in \mathfrak{p} \cap V_{\mathbb{Z}}} e^{\langle \xi, x\rangle} = \sum_{s} S(s + \mathfrak{c}_s)(\xi). \tag{5.1}$$

These formulae are at the heart of Varchenko's "analytic continuation procedure": we see intuitively that if the vertices of a polytope $\mathfrak{q}(b)$ vary "analytically" with a parameter b, the integrals and discrete sums will also vary "analytically". We will state precise results in the next section.

5.2 Polynomiality and wall-crossing for integrals and sums

Recall the following definition

Definition 5.1. If F is equipped with a lattice Λ, a quasi-polynomial function f on Λ is a function such that there exists a sublattice $\Lambda' \subset \Lambda$ so that, for any $\lambda_0 \in \Lambda$, the function $\lambda' \to f(\lambda_0 + \lambda')$ is given by the restriction to Λ' of a polynomial function f_{λ_0} on F.

Theorem 5.1. *Let $\Phi = (\phi_j)_{1 \leq j \leq N}$ be a sequence of non zero elements of a vector space F, generating F, and spanning a salient cone. Let $\tau \subset F$ be a Φ-tope such that τ is contained in the cone $\mathfrak{c}(\Phi)$ generated by Φ. For $\lambda \in F$, let $V(\Phi, \lambda)$ be the affine subspace of $\mathbb{R}^N$ defined by $\sum_{i=1}^{N} x_i \phi_i = \lambda$. Let*

$$\mathcal{X}(\Phi, \tau) = \sum_{I \in \mathcal{G}(\Phi, \tau)} (-1)^{|I| - \dim F} \prod_{i \in I^c} [x_i \geq 0],$$

where $\mathcal{G}(\Phi, \tau)$ is the set of $I \subseteq \{1, \ldots, N\}$ such that $\{\phi_i, i \in I\}$ generates F and such that τ is contained in the cone generated by $\{\phi_i, i \in I\}$.

Let $h(x)$ be a polynomial function on $\mathbb{R}^N$. Fix a Lebesgue measure on the subspace V and let $dm_\Phi(x)$ be the corresponding Lebesgue measure on $V(\Phi, \lambda)$. Define

$$I(\Phi, \tau, h)(\lambda) = \int_{V(\Phi,\lambda)} \mathcal{X}(\Phi, \tau)(x) h(x) dm_\Phi(x). \tag{5.2}$$

In the case where F is a rational space with lattice Λ and that the ϕ_i are lattice vectors, define

$$S(\Phi, \tau, h)(\lambda) = \sum_{x \in V(\Phi,\lambda)\cap\mathbb{Z}^N} \mathcal{X}(\Phi, \tau)(x) h(x). \tag{5.3}$$

Then

(i) $\lambda \mapsto I(\Phi, \tau, h)(\lambda)$ *is a polynomial function on F.*
(ii) $\lambda \mapsto S(\Phi, \tau, h)(\lambda)$ *is a quasi-polynomial function on the lattice $\Lambda \subset F$.*
(iii) *If λ belongs to the closure of the tope τ, we have*

$$I(\Phi, \tau, h)(\lambda) = \int_{\mathfrak{p}(\Phi,\lambda)} h(x) dm_\Phi(x). \tag{5.4}$$

Let $\mathfrak{b}(\Phi)$ be the zonotope generated by Φ. If $\lambda \in (\tau - \mathfrak{b}(\Phi)) \cap \Lambda$, we have

$$S(\Phi, \tau, h)(\lambda) = \sum_{x \in \mathfrak{p}(\Phi,\lambda)\cap\mathbb{Z}^N} h(x). \tag{5.5}$$

(iv) *Furthermore, we have the following wall-crossing formulas (with the notations of Theorem* 4.1*). For $\lambda \in \tau_2$, we have*

$$\begin{aligned} I(\Phi, \tau_1, h)(\lambda) = &\int_{\mathfrak{p}(\Phi,\lambda)} h(x) dm_\Phi(x) \\ &-(-1)^{|A|} \int_{\mathfrak{p}_{\mathrm{flip}}(\Phi,A,\lambda)} h(x) dm_\Phi(x). \end{aligned} \tag{5.6}$$

$$\begin{aligned} S(\Phi, \tau_1, h)(\lambda) = &\sum_{x \in \mathfrak{p}(\Phi,\lambda)\cap\mathbb{Z}^N} h(x) \\ &-(-1)^{|A|} \sum_{x \in \mathfrak{p}_{\mathrm{flip}}(\Phi,A,\lambda)\cap\mathbb{Z}^N} h(x). \end{aligned} \tag{5.7}$$

Proof. (iii) follows immediately from Corollary 3.1 and (iv) from the wall crossing formulas of Corollary 4.3, together with Corollary 3.1.

The proof of the polynomiality in (i) and (ii) relies on the properties of generating functions, as we explain in [2] for the weighted Ehrhart theory.

To begin with, observe that it is enough to prove the theorem in the case where the weight $h(x)$ is a power of a linear form

$$h(x) = \frac{\langle \xi, x\rangle^M}{M!},$$

for $\xi \in (\mathbb{R}^N)^*$. This is the term of ξ-degree M of the exponential $e^{\langle \xi, x\rangle}$. Thus, we consider the functions of $\xi \in (\mathbb{R}^N)^*$

$$I(\Phi, \tau)(\xi, \lambda) = \int_{V(\Phi,\lambda)} \mathcal{X}(\Phi, \tau)(x) e^{\langle \xi, x\rangle} dm_\Phi(x). \tag{5.8}$$

$$S(\Phi, \tau)(\xi, \lambda) = \sum_{x \in V(\Phi,\lambda)\cap \mathbb{Z}^N} \mathcal{X}(\Phi, \tau)(x) e^{\langle \xi, x\rangle}. \tag{5.9}$$

As $\mathcal{X}(\Phi, \tau)(x)[V(\Phi, \lambda)](x)$ has bounded support by Corollary 3.2, (5.8) and (5.9) are holomorphic functions of ξ. We recover $I(\Phi, \tau, h)(\lambda)$ and $S(\Phi, \tau, h)(\lambda)$ by taking their term of ξ-degree M.

However the dependence on λ can be analyzed by looking at each summand in

$$\mathcal{X}(\Phi, \tau) \cap [V(\Phi, \lambda)] = \sum_{K \in \mathcal{G}(\Phi,\tau)} (-1)^{|K| - \dim F} [\mathfrak{t}_K(\Phi, \lambda)]. \tag{5.10}$$

Indeed, it is immediate to extend the valuation $I(\mathfrak{p})$ defined in the preceding section to a valuation $I(\mathfrak{p}, \lambda)$ defined on polyhedrons contained in the affine space $V(\Phi, \lambda)$.

Namely, there exists a unique valuation $I(\mathfrak{p}, \lambda)$ associating to every polyhedron $\mathfrak{p} \subseteq V(\Phi, \lambda)$ a meromorphic function $I(\mathfrak{p}, \lambda)(\xi)$ on $\mathbb{C}^N$, so that the following properties hold:

(i) If $\mathfrak{p}$ contains a straight line, then $I(\mathfrak{p}, \lambda) = 0$.
(ii) If $\xi \in \mathbb{C}^N$ is such that $\mathrm{e}^{\langle \xi, x\rangle}$ is integrable over $\mathfrak{p}$ for the measure dm_Φ, then
$$I(\mathfrak{p}, \lambda)(\xi) = \int_{\mathfrak{p}} \mathrm{e}^{\langle \xi, x\rangle}\, dm_\Phi(x).$$

Moreover, for every point $s \in \mathbb{R}^N$, one has

$$I(s + \mathfrak{p}, \lambda + \sum_{i=1}^N s_i \phi_i)(\xi) = \mathrm{e}^{\langle \xi, s\rangle} I(\mathfrak{p}, \lambda)(\xi).$$

Similarly, if F is a space with a lattice Λ, and the elements ϕ_i belongs to Λ, then, for $\lambda \in \Lambda$, there exists a unique valuation $\mathfrak{p} \mapsto S(\mathfrak{p}, \lambda)$ associating to any $\lambda \in \Lambda$ and every rational polyhedron $\mathfrak{p} \subseteq V(\Phi, \lambda)$ a meromorphic function $S(\mathfrak{p}, \lambda)(\xi)$ on $\mathbb{C}^N$, so that
(i) if $\mathfrak{p}$ contains a straight line, then $S(\mathfrak{p}, \lambda) = 0$;
(ii) if $\xi \in \mathbb{C}^N$ is such that $\mathrm{e}^{\langle \xi, x\rangle}$ is summable over the set $\mathfrak{p} \cap \mathbb{Z}^N$, then

$$S(\mathfrak{p}, \lambda)(\xi) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^N} \mathrm{e}^{\langle \xi, x\rangle}.$$

Moreover, for every point $s \in \mathbb{Z}^N$, one has

$$S(s + \mathfrak{p}, \lambda + \sum_i s_i \phi_i)(\xi) = \mathrm{e}^{\langle \xi, s\rangle} S(\mathfrak{p}, \lambda)(\xi).$$

Look at Equation (5.10). The polyhedron $\mathfrak{t}_K(\Phi, \lambda)$ contains a straight line as soon as if $K \in \mathcal{G}(\Phi, \tau)$ is not a basic subset. Thus, in terms of the valuations $I(\mathfrak{p}, \lambda)$, $S(\mathfrak{p}, \lambda)$, we have

$$I(\Phi, \tau)(\xi, \lambda) = \sum_{K \in \mathcal{B}(\Phi, \tau)} I(\mathfrak{t}_K(\Phi, \lambda), \lambda)(\xi),$$

and

$$S(\Phi, \tau)(\xi, \lambda) = \sum_{K \in \mathcal{B}(\Phi, \tau)} S(\mathfrak{t}_K(\Phi, \lambda), \lambda)(\xi).$$

Each of these equations expresses a holomorphic function as a sum of meromorphic ones whose poles cancel out. Furthermore, we can recover the term of ξ-degree M by taking the homogeneous de degree in ξ in each of these functions of ξ.

Regarding the dependence on λ, we have already observed the following crucial fact: the cone $\mathfrak{t}_K(\Phi, \lambda)$ is the shift $s_K(\Phi, \lambda) + \mathfrak{a}_0(K)$ of the **fixed cone** $\mathfrak{a}_0(K)$ by the vertex $s_K(\Phi, \lambda)$ depending linearly of λ.

Let us first study the integral. Using the translation property of the valuation $I(\mathfrak{p}, \lambda)$, we can express $I(\mathfrak{p}, \lambda)$ in function of the valuation $I(\mathfrak{p})$ defined on polyhedrons contained in the fixed space V. We then have

$$I(\mathfrak{t}_K(\Phi, \lambda), \lambda)(\xi) = e^{\langle \xi, s_K(\Phi, \lambda)\rangle} I(\mathfrak{a}_0(K))(\xi).$$

Only the first factor $e^{\langle \xi, s_K(\Phi, \lambda)\rangle}$ depends on λ. Actually, it is easy to see that $I(\mathfrak{a}_0(K))(\xi)$ is homogeneous of degree $-d$. Hence, the term of ξ-degree M of $I(\mathfrak{t}_K(\Phi, \lambda), \lambda)(\xi)$ is given by

$$I(\mathfrak{t}_K(\Phi, \lambda), \lambda)(\xi)_{[M]} = \frac{\langle \xi, s_K(\Phi, \lambda)\rangle^{M+d}}{(M+d)!} I(\mathfrak{a}_0(K))(\xi)_{[-d]}.$$

Thus we have, for $h(x)=\frac{\langle \xi, x\rangle^M}{M!}$,

$$I(\Phi,\tau,h)(\lambda)=\sum_{K\in\mathcal{B}(\Phi,\tau)}\frac{\langle \xi, s_K(\Phi,\lambda)\rangle^{M+d}}{(M+d)!}I(\mathfrak{a}_0(K))(\xi)_{[-d]}.$$

The right hand side of this formula is a polynomial function of λ of degree $M+d$, with coefficients which are polynomial functions of ξ of degree M, (although each K summand has poles). Thus we have proved (i).

Let us now study the discrete sum

$$S(\Phi,\tau)(\xi,\lambda)=\sum_{K\in\mathcal{B}(\Phi,\tau)} S(\mathfrak{t}_K(\Phi,\lambda),\lambda)(\xi).$$

Consider a sublattice Λ' of Λ such that all elements $s_K(\Phi,\lambda')$ have integral coefficients for $\lambda'\in\Lambda'$ and all $K\in\mathcal{B}(\Phi,\tau)$. If D is the least common multiple of all determinants of the Φ-basic subsets I in $\mathcal{B}(\Phi)$, we can choose $\Lambda'=D\Lambda$.

Thus, if $\lambda=\lambda_0+\lambda'$, with $\lambda_0\in\Lambda$,$\lambda'\in\Lambda'$, we obtain

$$S(\mathfrak{t}_K(\Phi,\lambda_0+\lambda'),\lambda_0+\lambda')(\xi)=\mathrm{e}^{\langle\xi,s_K(\Phi,\lambda')\rangle}S(\mathfrak{t}_K(\Phi,\lambda_0),\lambda_0)(\xi).$$

Indeed $\mathfrak{t}_K(\Phi,\lambda_0+\lambda')=s_K(\Phi,\lambda')+\mathfrak{t}_K(\Phi,\lambda_0)$.

Here again the dependence in λ' is only through the factor $\mathrm{e}^{\langle\xi,s_K(\Phi,\lambda')\rangle}$ and $s_K(\Phi,\lambda')$ depends linearly on λ'.

If $f_{\lambda_0}(K,\xi)=S(\mathfrak{t}_K(\Phi,\lambda_0),\lambda_0)(\xi)$, a meromorphic function of ξ of degree greater or equal to $-d$, we obtain:

$$S(\mathfrak{t}_K(\Phi,\lambda_0+\lambda'),\lambda_0+\lambda')(\xi)_{[M]}=\sum_{k=0}^{M+d}\frac{\langle\xi,s_K(\Phi,\lambda')\rangle^k}{k!}f_{\lambda_0}(K,\xi)_{[M-k]}.$$

This is a polynomial function of λ' of degree $M+d$.

Adding up the contributions, we see that we obtain that

$$\lambda'\to S(\Phi,\tau)(\xi,\lambda_0+\lambda')$$

is a polynomial function of λ' and ξ. □

Remark 5.1. Consider Equation (4.15):

$$\mathcal{X}(\Phi,\tau)[V(\Phi,\lambda)]=\sum_{\{B,\lambda\in\mathfrak{c}(\Phi^B_{\text{flip}})\}} z(\Phi,\tau,B)[Q^B_{\text{neg}}][V(\Phi,\lambda)]. \quad (5.11)$$

Consider the case where the elements ϕ_i are in a lattice Λ of F. Summing up the function $h=1$ over $V(\Phi,\lambda)\cap\mathbb{Z}^N$ on both sides, we obtain an expression for the quasi polynomial function $S(\Phi, \tau, h)(\lambda)$ in function of the partition functions associated to the flipped systems Φ^B_{flip}.

The functions $S(\Phi, \tau, h)(\lambda)$ are elements of the Dahmen-Micchelli space associated to Φ and Λ. It was proved in [13] that any Dahmen-Micchelli quasi polynomial can be expressed as a linear combination of partition functions associated to flipped systems Φ^B_{flip}. The equation (5.11) can be considered as a "set-theoretic" generalization of this theorem.

5.3 Paradan's convolution wall-crossing formulas

We assume that F is equipped with a lattice Λ.

The convolution of two functions f_1, f_2 (satisfying adequate support conditions) on Λ is defined by

$$(f_1 * f_2)(\mu) = \sum_{\lambda_1+\lambda_2=\mu} f_1(\lambda_1) f_2(\lambda_2).$$

If $\mu \in \Lambda$, we write δ_μ for the function on Λ such that $f(\lambda) = \delta^\lambda_\mu$.

Let h be a polynomial function on $\mathbb{R}^N$, and consider

$$E(\Phi, h)(\lambda) = \sum_{x \in \mathfrak{p}(\Phi,\lambda)\cap\mathbb{Z}^N} h(x).$$

When h is the constant function 1, then

$$k(\Phi)(\lambda) = E(\Phi, 1)(\lambda) = \mathrm{Card}(\mathfrak{p}(\Phi, \lambda) \cap \mathbb{Z}^N)$$

is the partition function associated to the sequence Φ. The function $k(\Phi)(\lambda)$ is the convolution product $f_{\phi_1} * \cdots * f_{\phi_N}$, where, for $\phi \in F$,

$$f_\phi := \sum_{n=0}^{\infty} \delta_{n\phi}.$$

Indeed, by definition $k(\Phi)(\lambda)$ is the number of solutions in integers $n_i \geq 0$ of the equation $\sum_i n_i\phi_i = \lambda$.

The case of a polynomial function h can be treated similarly. Assume h is a product $h(x_1, x_2, \ldots, x_N) = \prod_{i=1}^N h_i(x_i)$ where h_i are polynomial functions on $\mathbb{R}$. For h a polynomial function on $\mathbb{R}$, and ϕ a non zero element in Λ, introduce

$$f^h_\phi = \sum_{n=0}^{\infty} h(n)\delta_{n\phi}.$$

Then we see that

$$E(\Phi, h) = f_{\phi_1}^{h_1} * \cdots * f_{\phi_N}^{h_N}.$$

With the notations of Theorem 5.1, for each tope τ, the function $S(\Phi, \tau, h)(\lambda)$ is a quasi-polynomial function on the lattice Λ such that $E(\Phi, h)(\lambda) = S(\Phi, h, \tau)(\lambda)$ for $\lambda \in (\tau - \mathfrak{b}(\Phi)) \cap \Lambda$.

Let τ_1, τ_2 be two adjacent topes separated by a wall H. Let τ_{12} be the unique tope of $\Phi \cap H$ such that $\overline{\tau_1} \cap \overline{\tau_2} \subset \overline{\tau_{12}}$. Paradan's formula is a formula for $S(\Phi, h, \tau_1) - S(\Phi, h, \tau_2)$ when τ_1, τ_2 are adjacent topes in terms of the convolution of the quasi-polynomial function $S(\Phi \cap H, h, \tau_{1,2})$ on $\Lambda \cap H$ with some functions f_ϕ^h.

Before stating the formula, we note one property of the function $S(\Phi, h, \tau)$.

Assume that H is a face of the cone $\mathfrak{c}(\Phi)$ and that τ is a tope with one of its wall equal to H. Let τ_H be the unique tope of $\Phi \cap H$ so that $\overline{\tau} \cap H$ is contained in τ_H. Let $S(\Phi \cap H, h, \tau_H)$ be the quasi-polynomial function on $\Lambda \cap H$ associated to this data.

Let us denote the subsequence of elements ϕ_i not in H by $\Phi \setminus H = (\phi_1, \ldots, \phi_M)$. Then if $n_1, n_2, \ldots, n_M$ are non negative integers, and $\lambda \in \Lambda$, there are only a finite number of n_i such that $\lambda - \sum_{i=1}^M n_i\phi_i$ belongs to H, as the elements ϕ_i are all on one side of H.

Let $H^{\geq 0}$ be the closed half space delimited by H and containing τ.

Proposition 5.1. *For $\lambda \in H^{\geq 0}$, $S(\Phi, h, \tau)(\lambda)$ is equal to*

$$\sum_{n_i \geq 0, \lambda - \sum_i n_i\phi_i \in H} (h_1(n_1) \cdots h_M(n_M)) S(\Phi \cap H, h, \tau_H)\left(\lambda - \sum_{i=1}^M n_i\phi_i\right).$$

In other words, on $H^{\geq 0} \cap \Lambda$,

$$S(\Phi, h, \tau) = S(\Phi \cap H, h, \tau_H) * f_{\phi_1}^{h_1} * \cdots * f_{\phi_M}^{h_M}.$$

Proof. As follows from [5], the right hand side, being the convolution of a quasi-polynomial function on the lattice $\Lambda \cap H$ with products $f_{\phi_1}^{h_1} * \cdots * f_{\phi_M}^{h_M}$, coincides with a quasi-polynomial function on the domain $H^{\geq 0} \cap \Lambda$.

Now, to prove that the left hand side coincide with the right hand side, we will use the fact that two quasi-polynomial functions agreeing on $\mathfrak{c} \cap \Lambda$, where $\mathfrak{c}$ is a cone with non empty interior, coincide on Λ.

If $\lambda \in \tau$ is sufficiently near a point of $\overline{\tau} \cap H$, then the set $(\lambda - \sum_{i=1}^M \mathbb{R}_{\geq 0}\phi_i) \cap H$ is contained in τ_H. We see that the set $\mathfrak{c} := \{\lambda \in \tau; (\lambda - \sum_{i=1}^M \mathbb{R}_{\geq 0}\phi_i) \cap H \subset \tau_H\}$ is an open cone in τ. On $\mathfrak{c} \cap \Lambda$, the function $S(\Phi, h, \tau)$ coincide with $E(\Phi, h)$. On the other hand, if we compute $E(\Phi, h)$ and $E(\Phi \cap H, h)$ by their respective convolution formulae, we obtain that the right hand side coincide also with $E(\Phi, h)$ for $\lambda \in \mathfrak{c} \cap \Lambda$. This proves the proposition. $\square$

Let A be the set of indices k such that ϕ_k belongs to the τ_2 open side of H. Let $\mathcal{X}(\Phi, A, \tau_2) = \mathrm{Geom}(\mathrm{Flip}_A\, X(\Phi^A_{\mathrm{flip}}, \tau_2))$ and let

$$S(\Phi, A, h, \tau_2)(\lambda) = \sum_{x \in V(\Phi,\lambda)\cap\mathbb{Z}^N} h(x)\mathcal{X}(\Phi, A, \tau_2)(x).$$

We then obtain

$$S(\Phi, h, \tau_1) - S(\Phi, h, \tau_2) = S(\Phi, A, h, \tau_2).$$

Observe that $\overline{\tau}_2 \cap H$ is contained in the boundary of the cone $\mathfrak{c}(\Phi^A_{\mathrm{flip}})$. Using a slight modification of Proposition 5.1 above, we then can give the following "convolution description" of the function $S(\Phi, A, h, \tau_2)$.

Let B be the set of indices k such that ϕ_k belongs to the τ_1 open side of H. Thus the vectors ϕ_a for $a \in A$ and $-\phi_b$ for $b \in B$ all belong to the τ_2 open side of H.

Define

$$\mathrm{Flip}\, f^h_\phi = -\sum_{n=1}^{\infty} h(-n)\delta_{-n\phi}.$$

Then we have

Proposition 5.2. *The quasi-polynomial function $S(\Phi, A, h, \tau_2)$ is given by the convolution formula:*

$$S(\Phi, A, h, \tau_2) =$$

$$S(\Phi \cap H, h, \tau_{1,2}) * \left(\prod_{a\in A} \mathrm{Flip}\, f^{h_a}_{\phi_a} * \prod_{b\in B} f^{h_b}_{\phi_b} - \prod_{b\in B} \mathrm{Flip}\, f^{h_b}_{\phi_b} * \prod_{a\in A} f^{h_a}_{\phi_a}\right).$$

This is the wall crossing convolution formula given by Paradan in [18, Theorem 5.2]. It expresses the wall crossing variation in terms of the function $S(\Phi \cap H, h, \tau_{1,2})$ associated to a lower dimensional system (see also [5]).

6 A refinement of Brion's theorem

Let $\mathfrak{p} \subset V$ be a full-dimensional polytope in a vector space V provided with a lattice $V_\mathbb{Z}$. Recall Brion's Formula (1.9) for the generating function of a polytope.

$$S(\mathfrak{p})(\xi) = \sum_{s\in\mathcal{V}(\mathfrak{p})} S(s + \mathfrak{c}_s)(\xi).$$

As $\mathfrak{p}$ is compact, the function $\xi \in V^* \mapsto S(\mathfrak{p})(\xi)$ is holomorphic, but the contribution of each cone is a meromorphic function with singularities along hyperplanes. More precisely, an element $\xi \in V^*$ is singular for $S(s + \mathfrak{c}_s)$ if and only if ξ is constant on some face $\mathfrak{f}$ of $\mathfrak{p}$ such that s is a vertex of $\mathfrak{f}$ and $\dim \mathfrak{f} > 0$.

It is well known that Brion's formula is the combinatorial translation of the localization formula in equivariant cohomology, in the case of isolated fixed points. In this section, we generalize (1.9) to the combinatorial case which corresponds to non isolated fixed points [4]. In this degenerate case, the connected components of the set of fixed points correspond to the faces of $\mathfrak{p}$ on which ξ is constant which are maximal with respect to this property. The contribution of such a face to the sum $S(\mathfrak{p})(\xi)$ is

$$\sum_{s \in \mathcal{V}(\mathfrak{f})} S(s + \mathfrak{c}_s)(\xi).$$

We will study this sum by relating it to a Brianchon-Gram continuation of the face $\mathfrak{f}$. We will assume that the polytope $\mathfrak{p}$ is simple. The general case needs more efforts.

We need to introduce some meromorphic functions similar to the function $S(s + \mathfrak{c})(\xi)$. Let $\mathfrak{q} = s + \mathfrak{c}$ be a polyhedral cone in V, where $\mathfrak{c}$ is a cone generated by elements $g_j \in V_{\mathbb{Z}}$. Let P be a quasi polynomial function on $V_{\mathbb{Z}}$. The following sum $\sum_{x \in V_{\mathbb{Z}} \cap \mathfrak{q}} P(\xi) e^{\langle \xi, x \rangle}$ defines a generalized function F of the variable $\xi \in iV^*$: the above sum converges when ξ is imaginary in the distribution sense.

It is easy to see that $\prod_i (1 - e^{\langle \xi, g_j \rangle}) F(\xi)$ is an analytic function of ξ. Thus, outside the affine hyperplanes in iV^* defined by $\langle \xi, g_j \rangle \in 2i\pi\mathbb{Z}$, the generalized function $F(\xi)$ is equal to $S(\mathfrak{q}, P)(\xi)$, where $S(\mathfrak{q}, P)(\xi)$ is a meromorphic function of ξ with poles on $\langle g_j, \xi \rangle \in 2i\pi\mathbb{Z}$. In particular this function belongs to the space $\mathcal{M}_\ell(V^*)$ introduced before. We write

$$S(\mathfrak{q}, P)(\xi) = \sum_{x \in V_{\mathbb{Z}} \cap \mathfrak{q}} P(\xi) e^{\langle \xi, x \rangle}$$

and depending on the context, we consider $S(\mathfrak{q}, P)$ either as a generalized function of $\xi \in iV^*$ or as a meromorphic function of $\xi \in V_{\mathbb{C}}$. If $\mathfrak{q}$ is a cone invariant by translation by a vector $v \in V_{\mathbb{Z}}$, it is easy that the generalized function $S(\mathfrak{q}, P)(\xi)$ is annihilated by a power of $(1 - e^{\langle v, \xi \rangle})$. The simplest case is when $P = 1$, $V = \mathfrak{c} = \mathbb{R}$, $V_{\mathbb{Z}} = \mathbb{Z}$, so that the equality is simply $(1 - e^{i\theta}) \sum_{n \in \mathbb{Z}} e^{in\theta} = 0$. In particular, if $\mathfrak{q}$ is a flat cone, the meromorphic function $S(\mathfrak{q}, P)(\xi)$ is equal to 0.

If $\mathfrak{f}$ is a face of $\mathfrak{p}$, we denote by $\mathrm{aff}(\mathfrak{f})$ the affine space generated by $\mathfrak{f}$ and by $\mathrm{lin}\, \mathfrak{f}$ the linear space parallel to $\mathrm{aff}(\mathfrak{f})$, that is the space spanned by elements $x - y$ with $x, y \in \mathfrak{f}$. The projection $\mathfrak{t}_{\text{trans}}(\mathfrak{p}, \mathfrak{f})$ of $\mathfrak{t}_{\text{aff}}(\mathfrak{p}, \mathfrak{f})$ in $V/\operatorname{lin}\mathfrak{f}$ is called the transverse cone. Note that this transverse cone is a salient cone in $V/\operatorname{lin}\mathfrak{f}$ with vertex y_0 the projection of any $y \in \mathfrak{f}$.

Theorem 6.1. *Let V be a rational vector space with lattice $V_{\mathbb{Z}}$. Let $\mathfrak{p} \subset V$ be a simple rational polytope and let $\mathfrak{f}$ be a face of $\mathfrak{p}$. Let $\mathfrak{t}_{\text{trans}}(\mathfrak{p}, f) \subset V/\operatorname{lin}\mathfrak{f}$ be the transverse cone. The tangent cone to $\mathfrak{p}$ at the vertex s is denoted by $s + \mathfrak{c}_s$. For $\xi \in V^*$, let*

$$S(s + \mathfrak{c}_s)(\xi) = \sum_{x \in (s+\mathfrak{c}_s)\cap V_{\mathbb{Z}}} e^{\langle \xi, x\rangle}.$$

The set of vertices of $\mathfrak{f}$ is denoted by $\mathcal{V}(\mathfrak{f})$.

(i) *The sum $\sum_{s\in\mathcal{V}(\mathfrak{f})} S(s + \mathfrak{c}_s)(\xi)$ restricts to a meromorphic function on $\operatorname{lin}\mathfrak{f}^{\perp} \subset V^*$, which is given by*

$$\sum_{s\in\mathcal{V}(\mathfrak{f})} S(s + \mathfrak{c}_s)(\xi) = \sum_{y\in \mathfrak{t}_{\text{trans}}(\mathfrak{p},\mathfrak{f})\cap(V/\operatorname{lin}\mathfrak{f})_{\mathbb{Z}}} e^{\langle \xi, y\rangle} P(y),$$

where $P(y)$ is a quasi-polynomial function on the projected lattice $(V/\operatorname{lin}\mathfrak{f})_{\mathbb{Z}} \subset V/\operatorname{lin}\mathfrak{f}$. Moreover if ξ is regular with respect to the cone $\mathfrak{t}_{\text{trans}}(\mathfrak{p}, \mathfrak{f})$, that is if ξ is not constant on a face strictly containing $\mathfrak{f}$, then $\sum_{s\in\mathcal{V}(\mathfrak{f})} S(s + \mathfrak{c}_s)(\xi)$ is holomorphic at ξ.

(ii) *For y close enough to the vertex y_0 of the transverse cone, $P(y)$ is the number of lattice points of the slice $\mathfrak{p} \cap (\operatorname{lin}\mathfrak{f} + y)$.*

Proof. We compute the signed sum of the generating functions of the tangent cones $\mathfrak{t}_{\text{aff}}(\mathfrak{p}, \mathfrak{g})$ where $\mathfrak{g}$ runs over the set $\mathcal{F}(\mathfrak{f}) \subset \mathcal{F}(\mathfrak{p})$ of faces of $\mathfrak{f}$. Since $\mathfrak{t}_{\text{aff}}(\mathfrak{p}, \mathfrak{g})$ contains lines if $\mathfrak{g}$ is not a vertex, we have

$$\sum_{s\in\mathcal{V}(\mathfrak{f})} S(s + \mathfrak{c}_s)(\xi) = \sum_{\mathfrak{g}\in\mathcal{F}(\mathfrak{f})} (-1)^{\dim\mathfrak{g}} S(\mathfrak{t}_{\text{aff}}(\mathfrak{p}, \mathfrak{g}))(\xi). \tag{6.1}$$

We will relate the right hand side to sums over slices of $\mathfrak{p}$ by affine subspaces parallel to $\mathfrak{f}$.

We define

$$\mathcal{T}(y)(x) = \sum_{\mathfrak{g}\in\mathcal{F}(\mathfrak{f})} (-1)^{\dim\mathfrak{g}} [\mathfrak{t}_{\text{aff}}(\mathfrak{p}, \mathfrak{g}) \cap (\operatorname{aff}(\mathfrak{f}) + y)](x). \tag{6.2}$$

The support of $\mathcal{T}(y)$ is illustrated in Figure 6.2.

Let us only observe that, as $\mathfrak{t}_{\text{aff}}(\mathfrak{p}, \mathfrak{g}) \cap \operatorname{aff}(\mathfrak{f})$ is the tangent cone of the polytope $\mathfrak{f} \subset \operatorname{aff}(\mathfrak{f})$ along its face $\mathfrak{g}$, we have, by Brianchon-Gram theorem,

$$\mathcal{T}(0) = [\mathfrak{f}].$$

Moreover, if y is small enough, then $\mathcal{T}(y)$ is the characteristic function of the intersection $\mathfrak{p} \cap (\mathrm{aff}(\mathfrak{f}) + y)$. This result can be deduced from the Euler relations. In the next section, in the case where $\mathfrak{p}$ is simple, we will obtain it as a consequence of Corollary 3.1 which is of course itself based on the Euler relations via the Brianchon-Gram theorem.

Let us compute the right hand side of Equation (6.1). If $\xi \in \mathrm{lin}\,\mathfrak{f}^{\perp}$ then $e^{\langle \xi, x\rangle}$ is constant on $\mathrm{lin}\,\mathfrak{f} + y$. Identifying $\mathrm{lin}\,\mathfrak{f}^{\perp}$ with $(V/\mathrm{lin}\,\mathfrak{f})^*$, we denote this constant value by $e^{\langle \xi, y\rangle}$.

Thus, we slice the lattice $V_{\mathbb{Z}}$ in slices parallel to the subspace $\mathrm{lin}\,\mathfrak{f}$. The slices are indexed by the projected lattice $(V/\mathrm{lin}\,\mathfrak{f})_{\mathbb{Z}}$. We write

$$\begin{aligned}\sum_{\mathfrak{g}\in\mathcal{F}(\mathfrak{f})} (-1)^{\dim \mathfrak{g}} S(\mathfrak{t}_{\mathrm{aff}}(\mathfrak{p}, \mathfrak{g}))(\xi) &= \sum_{\mathfrak{g}\in\mathcal{F}(\mathfrak{f})} (-1)^{\dim \mathfrak{g}} \sum_{x\in \mathfrak{t}_{\mathrm{aff}}(\mathfrak{p},\mathfrak{g})} e^{\langle \xi, x\rangle} \\ &= \sum_{y\in(V/\mathrm{lin}\,\mathfrak{f})_{\mathbb{Z}}} \sum_{\mathfrak{g}\in\mathcal{F}(\mathfrak{f})} (-1)^{\dim \mathfrak{g}} \sum_{x\in \mathfrak{t}_{\mathrm{aff}}(\mathfrak{p},\mathfrak{g})\cap(\mathrm{lin}\,\mathfrak{f}+y)\cap V_{\mathbb{Z}}} e^{\langle \xi, x\rangle}. \end{aligned} \tag{6.3}$$

Let $y_0 \in V/\mathrm{lin}\,\mathfrak{f}$ be the projection of the face $\mathfrak{f}$. From (6.3), we obtain

$$\sum_{s\in\mathcal{V}(\mathfrak{f})} S(s + \mathfrak{c}_s)(\xi) = \sum_{y\in(V/\mathrm{lin}\,\mathfrak{f})_{\mathbb{Z}}} e^{\langle \xi, y\rangle} \sum_{x\in V_{\mathbb{Z}}} \mathcal{T}(y - y_0)(x). \tag{6.4}$$

The shift by y_0 is there to make further notations simpler. At this point, we postpone the proof of Theorem 6.1 until the next section, where we will relate $\mathcal{T}(y)$ to the Brianchon-Gram continuation of the face $\mathfrak{f}$, under the assumption that $\mathfrak{p}$ is simple. ☐

6.1 Brianchon-Gram continuation of a face of a partition polytope

Let $\mathfrak{p} = \mathfrak{p}(\Phi, \lambda) \subset \mathbb{R}^N$. Let $\mathfrak{f}$ be a face of $\mathfrak{p}$. If λ is regular and belongs to a tope τ, then there is a unique $I \in \mathcal{G}(\Phi, \tau)$ such that $\mathfrak{f} = \mathfrak{f}(\Phi, \lambda, I)$ is the corresponding face. We have $\dim \mathfrak{f} = |I| - \dim F$. If λ is on a wall, there may be several such pairs (τ, I).

Definition 6.1. Let $I \subseteq \{1, \ldots, N\}$ be such that the sequence Φ_I generates F. Let $\widetilde{\Phi}_I = (\widetilde{\phi}_i)$, $1 \leq i \leq N$, be the sequence of elements in $F \oplus \mathbb{R}^{I^c}$ defined by $\widetilde{\phi}_i = \phi_i$ if $i \in I$ and $\widetilde{\phi}_i = \phi_i \oplus e_i$, if $i \in I^c$.

Lemma 6.1.

(i) *The sequence $\widetilde{\Phi}_I$ generates a salient cone of full dimension in $F \oplus \mathbb{R}^{I^c}$.*
(ii) $V(\widetilde{\Phi}_I, (\lambda, y)) = \{x \in \mathbb{R}^N;\ \sum_{i=1}^N x_i\phi_i = \lambda,\ x_i = y_i$ *for* $i \in I^c\}$.
(iii) *Let τ be a Φ_I-tope. Let R be an open quadrant in $\mathbb{R}^{I^c}$. Then $\{(\lambda, y) \in F \oplus \mathbb{R}^{I^c};\ y \in R, \lambda - \sum_{i\in I^c} y_i\phi_i \in \tau\}$ is a $\widetilde{\Phi}_I$-tope and all $\widetilde{\Phi}_I$-topes are of this form.*

(iv) *Let τ be a Φ_I-tope, let τ_I be the $\widetilde{\Phi}_I$-tope which consists of (λ, y) such that $y_i > 0$ for $i \in I^c$ and $\lambda - \sum_{i\in I^c} y_i\phi_i \in \tau$. Then $\mathcal{G}(\widetilde{\Phi}_I, \tau_I)$ is the set of $K \cup I^c$, where $K \subseteq I$ and $K \in \mathcal{G}(\Phi_I, \tau)$. Hence*

$$X(\widetilde{\Phi}_I, \tau_I) = \sum_{K\in\mathcal{G}(\Phi_I,\tau)} (-1)^{|K|-\dim F} \prod_{i\in I\setminus K} p_i \prod_{i\in K\cup I^c} (p_i + q_i).$$

Proof. (i) follows from the fact that Φ_I generates F. (ii) is immediate. Consider the linear bijection from $F \oplus \mathbb{R}^{I^c}$ to itself defined by $(\lambda, y) \mapsto (\lambda - \sum_{i\in I^c} y_i\phi_i, y)$. The image of $\widetilde{\phi}_i$ is ϕ_i if $i \in I$, and e_i if $i \in I^c$. Therefore the $\widetilde{\Phi}_I$-topes are the pull-backs of the topes relative to the sequence $\psi_i = \phi_i$ if $i \in I$ and $\psi_i = e_i$ if $i \in I^c$. The latter are the products of Φ_I-topes in F with the quadrants in $\mathbb{R}^{I^c}$. This proves (iii).

Let $\widetilde{K} \subseteq \{1, \ldots, N\}$. Then $\widetilde{\Phi}_{\widetilde{K}}$ generates $F \oplus \mathbb{R}^{I^c}$ if and only if $\widetilde{K} = K \cup I^c$, where $K \subseteq I$ is such that Φ_K generates F. Moreover $\tau_I \subset c((\widetilde{\Phi}_I)_{\widetilde{K}})$ if and only if $\tau \subset \mathfrak{c}(\Phi_K)$, whence (iv). □

Proposition 6.1. *Let τ be a Φ-tope and let $\lambda \in \overline{\tau}$. Let $\mathfrak{p} = \mathfrak{p}(\Phi, \lambda)$ and $\mathfrak{f} = \mathfrak{f}_I(\Phi, \lambda)$ be a face of $\mathfrak{p}$. Assume that* $\dim \mathfrak{f} = |I| - \dim F$. *We identify the quotient space $V/\operatorname{lin}\mathfrak{f}$ with $\mathbb{R}^{I^c}$ by the projection parallel to $\mathbb{R}^I$. Let τ_I be the Φ_I-tope which contains τ. If $y_i \geq 0$ for $i \in I^c$ and $\lambda - \sum_{i\in I^c} y_i\phi_i \in \overline{\tau_I}$, then*

$$\begin{aligned} \mathcal{X}(\widetilde{\Phi}_I, \tau_I)[V(\widetilde{\Phi}_I, (\lambda, y))] &= [\mathfrak{p}(\widetilde{\Phi}_I, (\lambda, y))] \\ &= [\mathfrak{p}(\Phi, \lambda) \cap (\operatorname{aff}(\mathfrak{f}) + y)]. \end{aligned} \tag{6.5}$$

In particular, if λ is regular, the conditions $y_i \geq 0$ for $i \in I^c$ and $\lambda - \sum_{i\in I^c} y_i\phi_i \in \overline{\tau_I}$ define a neighborhood of $y = 0$ in $\mathbb{R}^{I^c}_{\geq 0}$ on which (6.5) *holds.*

Proof. The conditions $y_i \geq 0$ for $i \in I^c$ and $\lambda - \sum_{i\in I^c} y_i\phi_i \in \overline{\tau_I}$ mean that (λ, y) belongs to the closure of the $\widetilde{\Phi}_I$-tope τ_I associated to τ_I. Therefore by Corollary 3.1, we have

$$\mathcal{X}(\widetilde{\Phi}_I, \tau_I)[V(\widetilde{\Phi}_I, (\lambda, y))] = [\mathfrak{p}(\widetilde{\Phi}_I, (\lambda, y))].$$

Moreover, as $\dim \mathfrak{f} = |I| - \dim F$, the affine span $\operatorname{aff}(\mathfrak{f})$ is given by $\operatorname{aff}(\mathfrak{f}) = \{x \in V(\Phi, \lambda); x_i = 0 \text{ for } i \in I^c\}$. It follows that $V(\widetilde{\Phi}_I, (\lambda, y)) = \operatorname{aff}(\mathfrak{f}) + y$, hence $\mathfrak{p}(\widetilde{\Phi}_I, (\lambda, y)) = \mathfrak{p}(\Phi, \lambda) \cap (\operatorname{aff}(\mathfrak{f}) + y)$. □

Remark 6.1. Define $\mathfrak{q}_0(\mathfrak{p}, \mathfrak{f}, \tau) \subseteq \mathbb{R}^{I^c}_{\geq 0}$ by

$$\mathfrak{q}_0(\mathfrak{p}, \mathfrak{f}, \tau) = \{y = (y_i) \in \mathbb{R}^{I^c}\ ;\ y_i \geq 0 \text{ for } i \in I^c, \lambda - \sum_{i\in I^c} y_i\phi_i \in \overline{\tau_I}\}.$$

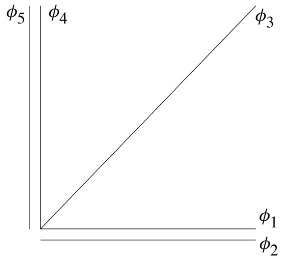
The set $\mathfrak{q}_0(\mathfrak{p}, \mathfrak{f}, \tau)$ is a polytope in $V/\operatorname{lin}\mathfrak{f} \simeq \mathbb{R}^{I^c}$. Let us denote its cone at vertex 0 by $\mathfrak{t}_0(\mathfrak{p}, \mathfrak{f}, \tau)$.

$$\mathfrak{t}_0(\mathfrak{p}, \mathfrak{f}, \tau) = \{y = (y_i) \in \mathbb{R}^{I^c} \; ; y_i \geq 0 \text{ for } i \in I^c, \\ \lambda - \epsilon \sum_{i \in I^c} y_i \phi_i \in \overline{\tau_I} \text{ for } \epsilon > 0 \text{ small enough } \}.$$

Then $\mathfrak{t}_0(\mathfrak{p}, \mathfrak{f}, \tau)$ is a subcone of the transverse cone $\mathfrak{t}_0(\mathfrak{p}, \mathfrak{f})$. If $\lambda \in \tau$ is regular, then $\mathfrak{t}_0(\mathfrak{p}, \mathfrak{f}, \tau) = \mathfrak{t}_0(\mathfrak{p}, f) = \mathbb{R}^{I^c}_{\geq 0}$.

If λ lies on a wall of a tope τ, then $\mathfrak{t}_0(\mathfrak{p}, \mathfrak{f}, \tau)$ may be strictly contained in the transverse cone $\mathfrak{t}_0(\mathfrak{p}, \mathfrak{f})$. When we consider all the topes τ such that $\lambda \in \overline{\tau}$, the cones $\mathfrak{t}_0(\mathfrak{p}, \mathfrak{f}, \tau')$ form a subdivision of $\mathfrak{t}_0(\mathfrak{p}, f)$. An example is illustrated in Figure 6.1. The polytope $\mathfrak{p} \subset \mathbb{R}^3$ is a tipi with four poles, with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$ and $\mathfrak{f}$ is the vertical edge with vertices $(0, 0, 0)$, $(0, 0, 1)$. The picture shows also the corresponding system Φ such that $\mathfrak{p}$ corresponds to a partition polytope $\mathfrak{p}(\Phi, \lambda)$. In this case, λ belongs to the wall generated by ϕ_3, thus λ belongs to two tope closures τ_1 and τ_2. We identify the quotient $V/\operatorname{lin}(\mathfrak{f})$ with the ground. Then the sets $\mathfrak{q}_0(\mathfrak{p}, \mathfrak{f}, \tau_i)$ are the two triangles which subdivide the ground face of the tipi.

This remark suggests how to modify Proposition 6.2 in the case of a non simple polytope.



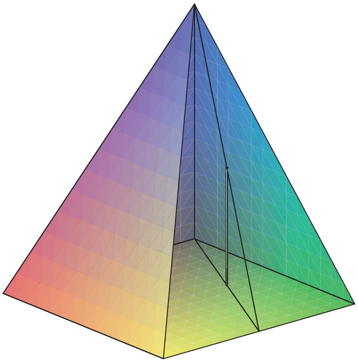


Figure 6.1.

Proposition 6.2. *Let $\Phi = (\phi_j)_{1 \leq j \leq N}$ be a sequence of non zero elements of a vector space F, generating F, and spanning a salient cone. Let τ be a Φ-tope, $\lambda \in \tau$ a regular element and $I \in \mathcal{G}(\Phi, \tau)$. Let $\mathfrak{p} = \mathfrak{p}(\Phi, \lambda)$ and $\mathfrak{f} = \mathfrak{f}(\Phi, \lambda, I)$.*

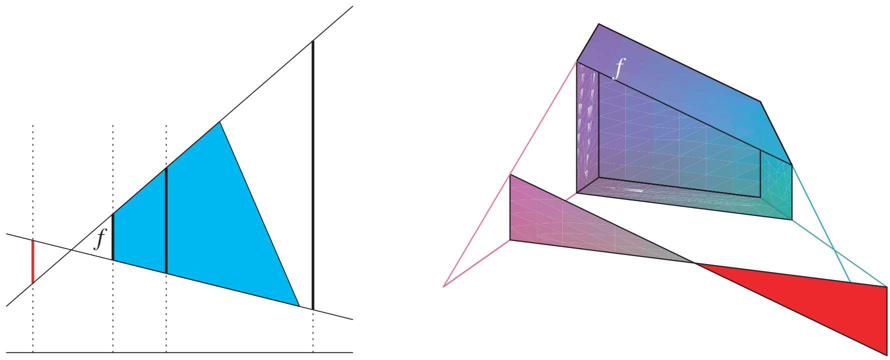


Figure 6.2. Brianchon-Gram continuation of a face. The segment and the triangle in red come with a minus sign. The end-points of the segment have to be deleted and two edges of the triangle also.

We identify the quotient space $V/\operatorname{lin}\mathfrak{f}$ *with* $\mathbb{R}^{I^c}$ *by the projection parallel to* $\mathbb{R}^I$. *For* $y \in \mathbb{R}^{I^c}$, *let*

$$\mathcal{T}(y) = \sum_{\mathfrak{g}\in\mathcal{F}(\mathfrak{f})} (-1)^{\dim\mathfrak{g}}[\mathfrak{t}_{\mathrm{aff}}(\mathfrak{p},\mathfrak{g}) \cap (\operatorname{aff}(\mathfrak{f}) + y)], \tag{6.6}$$

where the set of faces of $\mathfrak{f}$ *is denoted by* $\mathcal{F}(\mathfrak{f})$.

Let $\widetilde{\Phi}_I = (\widetilde{\phi}_i)$, $1 \le i \le N$, *be the sequence of elements in* $F \oplus \mathbb{R}^{I^c}$ *defined by* $\widetilde{\phi}_i = \phi_i$ *if* $i \in I$ *and* $\widetilde{\phi}_i = \phi_i \oplus e_i$, *if* $i \in I^c$. *Let* τ_I *be the* $\widetilde{\Phi}_I$*-tope which consists of elements* $(\lambda, y) \in F \oplus \mathbb{R}^{I^c}$ *such that* $y_i > 0$ *for* $i \in I^c$ *and* $\lambda - \sum_{i\in I^c} y_i\phi_i \in \tau_I$, *where* τ_I *is the unique* Φ_I*-tope which contains* τ. *Then*

$$\mathcal{T}(y)(x) = \mathcal{X}(\widetilde{\Phi}_I, \tau_I)(x)[V(\widetilde{\Phi}_I, (\lambda, y))](x) \prod_{i\in I^c}[y_i \ge 0], \tag{6.7}$$

Proof. The faces $\mathfrak{g}$ of $\mathfrak{f} = \mathfrak{f}(\Phi, \lambda, I)$ are indexed by the subsets $K \in \mathcal{G}(\Phi, \tau)$ which are contained in I. For $\mathfrak{g} = \mathfrak{f}(\Phi, \lambda, K)$, we have

$$\mathfrak{t}_{\mathrm{aff}}(\mathfrak{p}, \mathfrak{g}) = \{x \in V(\Phi, \lambda);\, x_i \ge 0 \text{ for } i \in K^c\}.$$

We write (6.6) as

$$\mathcal{T}(y) = \sum_{K\in\mathcal{G}(\Phi,\tau),\, K\subseteq I} (-1)^{|K|-\dim F} \prod_{i\in K^c}[x_i \ge 0]\, [\operatorname{aff}(\mathfrak{f}) + y]. \tag{6.8}$$

We observe that $\mathcal{G}(\Phi_I, \tau_I) = \{K \in \mathcal{G}(\Phi, \tau), K \subseteq I\}$. Therefore, by Lemma 6.1, we have

$$\begin{aligned}&\mathcal{X}(\widetilde{\Phi}_I, \tau_I)[V(\widetilde{\Phi}_I, (\lambda, y))]\\ &\qquad = \sum_{K \in \mathcal{G}(\Phi,\tau), K \subseteq I} (-1)^{|K|-\dim F} \prod_{i \in I \setminus K} [x_i \geq 0][V(\widetilde{\Phi}_I, (\lambda, y))].\end{aligned}$$

We factor out $\prod_{i \in I^c} [x_i \geq 0]$ in each summand of (6.8).
As $V(\widetilde{\Phi}_I, (\lambda, y)) = \mathrm{aff}(\mathfrak{f}) + y$, we obtain

$$\mathcal{T}(y)(x) = \mathcal{X}(\widetilde{\Phi}_I, \tau_I)(x)[V(\widetilde{\Phi}_I, (\lambda, y))](x) \prod_{i \in I^c} [x_i \geq 0].$$

As $x_i = y_i$ for $i \in I^c$ if $x \in V(\widetilde{\Phi}_I, (\lambda, y))$, we obtain (6.7) □

We resume the proof of Theorem 6.1.

Proof of Theorem 6.1. We identify $\mathfrak{p}$ with a partition polytope $\mathfrak{p}(\Phi, \lambda)$ by an affine map $V \simeq V(\Phi, \lambda)$. We can assume that λ is regular. Some care is needed with respect to the lattice $V_{\mathbb{Z}}$. In general, its image $V(\Phi, \lambda)_{\mathbb{Z}}$ in $V(\Phi, \lambda)$ is not $\mathbb{Z}^N \cap V(\Phi, \lambda)$. However we can always write $V(\Phi, \lambda)_{\mathbb{Z}} = (b + \Gamma) \cap V(\Phi, \lambda)$, where $b \in \mathbb{Q}^N$ and Γ is a lattice in $\mathbb{R}^N$, (Γ is a fixed lattice and b projects on λ). Let τ be the Φ-tope which contains λ and let $I \in \mathcal{G}(\Phi, \tau)$ such that $\mathfrak{f}$ is identified with the face $\mathfrak{f}(\Phi, \lambda, I)$. Then $V/\operatorname{lin}\mathfrak{f}$ is identified with $V/\operatorname{lin}\mathfrak{f} \simeq \mathbb{R}^{I^c}$ and the projected lattice $(V/\operatorname{lin}\mathfrak{f})_{\mathbb{Z}}$ is identified with a lattice in $\mathbb{R}^{I^c}$. By Proposition 6.2, we have, for every $x \in \mathbb{R}^N$,

$$\mathcal{T}(y)(x) = \mathcal{X}(\widetilde{\Phi}_I, \tau_I)[V(\widetilde{\Phi}_I, (\lambda, y))](x) \prod_{i \in I^c} [y_i \geq 0].$$

So we define

$$P(y) = \sum_{x \in b+\Gamma} \mathcal{X}(\widetilde{\Phi}_I, \tau_I)[V(\widetilde{\Phi}_I, (\lambda, y - y_0))](x).$$

Then $P(y)$ is a quasi-polynomial function of $y \in (V/\operatorname{lin}\mathfrak{f})_{\mathbb{Z}}$. This fact follows from a minor generalization of Theorem 5.1 (ii). We only have to take care of the shifts: the summation is over $x \in b + \Gamma$ and the parameter $y - y_0$ in the Brianchon-Gram function runs over the shifted lattice $(V/\operatorname{lin}\mathfrak{f})_{\mathbb{Z}} - y_0$.

The equalities (6.3) and (6.4) of generalized functions imply equalities of holomorphic functions of ξ in an open subset of $(\operatorname{lin}\mathfrak{f})^{\perp}$, hence

$\sum_{s\in\mathcal{V}(\mathfrak{f})} S(s+\mathfrak{c}_s)(\xi)$ restricts to a meromorphic function on $(\operatorname{lin}\mathfrak{f})^{\perp}$, given by

$$\sum_{s\in\mathcal{V}(\mathfrak{f})} S(s+\mathfrak{c}_s)(\xi) = \sum_{y\in\mathfrak{t}_{\text{trans}}(\mathfrak{p},\mathfrak{f})\cap(V/\operatorname{lin}\mathfrak{f})_{\mathbb{Z}}} e^{\langle\xi,y\rangle}P(y). \tag{6.9}$$

So we have proved (i).

By Proposition 6.1, for $y\in\mathfrak{t}_{\text{trans}}(\mathfrak{p},\mathfrak{f})$ close to the vertex, $\mathcal{T}(y-y_0)$ is the characteristic function of the slice $\mathfrak{p}\cap(\operatorname{aff}(\mathfrak{f})+y)$, hence (ii). □

7 Cohomology of line bundles over a toric variety

Let us indicate the relation of our work with toric varieties. Let $\Phi=(\phi_j)_{1\le j\le N}$ be a sequence of non zero elements of a vector space F, generating F, and spanning a salient cone. Assume that the ϕ_i's belong to a lattice Λ and let T be the torus with character group Λ embedded in $T^N=S_1^N$ by the characters of T associated to (ϕ_i). This determines an action of T in the complex space $\mathbb{C}^N$. Each tope τ determines a toric variety M_τ (with orbifold singularities) for the quotient torus T^N/T, in the following way. If $\lambda\in\tau$, M_τ is the reduced manifold $\mathbb{C}^N//_\lambda T$ at $\lambda\in\mathfrak{t}^*$. Then the vectors ϕ_i parameterize the boundary divisors D_i in M_τ and each element $\lambda\in\Lambda$ determines a T^N-equivariant sheaf $\mathcal{O}(\lambda)$ on M_τ.

The lattice of characters of the d-dimensional torus T^N/T is identified with $V\cap\mathbb{Z}^N$.

The torus T^N acts on the cohomology groups $H^i(M_\tau,\mathcal{O}(\lambda))$. When $\lambda\in\tau$, then all the cohomology groups H^i for $i>0$ vanish, and a weight $m\in\mathbb{Z}^N$ of T^N occurs in $H^0(M_\tau,\mathcal{O}(\lambda))$ if and only if $m\in\mathfrak{p}(\Phi,\lambda)\cap\mathbb{Z}^N$. Thus the dimension of the space $H^0(M_\tau,\mathcal{O}(\lambda))$ is just the number of integral points in $\mathfrak{p}(\Phi,\lambda)$.

If $\lambda\in\Lambda$ does not belong to the tope τ, and $i>0$, the cohomology space $H^i(M_\tau,\mathcal{O}(\lambda))$ is in general not zero. It is natural to introduce the virtual space

$$\mathcal{H}(\tau,\lambda):=\sum_{i=0}^{d}(-1)^iH^i(M_\tau,\mathcal{O}(\lambda)).$$

It follows from the Kawasaki-Riemann-Roch theorem that the virtual dimension of $\mathcal{H}(\tau,\lambda)$ is a quasi polynomial function of λ.

More precisely, we can use the fixed point theorem to compute the character of T^N in $\mathcal{H}(\tau,\lambda)$ (see [7]). As the construction of the present article reproduces this fixed point theorem at the level of sets, we obtain the weight decomposition of the T^N-module

$$\mathcal{H}(\tau,\lambda)=\sum_{m\in\mathbb{Z}^N\cap V(\Phi,\lambda)}\mathcal{X}(\Phi,\tau)(m)e^m.$$

In other words, the function $\mathcal{X}(\Phi, \tau)$ on $\mathbb{Z}^N$ computes simultaneously (for all sheaves $\mathcal{O}(\lambda)$) the multiplicity of a weight m in the alternating sum of cohomology spaces. In particular, the function $\mathcal{X}(\Phi, \tau) \cap [V(\Phi, \lambda)]$ is the constructible function on $V(\Phi, \lambda)$ associated by Morelli [16] to the sheaf $\mathcal{O}(\lambda)$.

Recall the formula

$$\mathcal{X}(\Phi, \tau) = \sum_B z(\Phi, \tau, B)[Q^B_{\text{neg}}].$$

Let us comment on the explicit computation of the coefficients $z(\Phi, \tau, B)$ of $\mathcal{X}(\Phi, \tau)$. We wrote a brute force Maple program to compute $X(\Phi, \tau)$, out of its definition (Equation (2.3)), by enumerating the generating subsets of the system Φ and checking which ones are in $\mathcal{G}(\Phi, \tau)$. It would be certainly more efficient to use Theorem 3.3, and then determine $\mathcal{B}(\Phi, \tau)$ using the reverse-search algorithm of Avis-Fukuda [1]. Anyway, we obtain the decomposition as a sum of monomials

$$X(\Phi, \tau) = \sum_B z(\Phi, \tau, B) \prod_{i\in B^c} p_i \prod_{j\in B} q_j.$$

If $m \in \mathbb{Z}^N$, we denote by B_m the set of indices i such that $m_i < 0$. Then the multiplicity of m in the T^N module $\mathcal{H}(\tau, \lambda)$ is obtained by computing the coefficient $z(\Phi, \tau, B_m)$ of the monomial $\prod_{i\in B_m^c} p_i \prod_{i\in B_m} q_i$ in $X(\Phi, \tau)$.

Y. Karshon and S. Tolman [14] have studied the representation space $\mathcal{H}(\tau, \lambda)$ associated to a non-ample line bundle on the manifold M_τ, and they have given an algorithm to compute a weight in this representation space by wall crossing. Our algorithm (Theorem 4.3) to determine $z(\Phi, \tau, B)$ is probably very similar. However, as we deal with arbitrary "weights" ϕ_i (not assumed rational), our methods use "only linear algebra", not geometry.

By summing up the multiplicities of the weights in $\mathcal{H}(\tau, \lambda)$, we obtain the expression of the function

$$\lambda \mapsto \dim \mathcal{H}(\tau, \lambda) = \sum_B z(\Phi, \tau, B)\text{cardinal}(\mathfrak{p}_{\text{flip}}(\Phi, B, \lambda) \cap \mathbb{Z}^N)$$

as a sum of partition functions with respect to particular flipped systems.

Remark that if $\lambda \in (\tau - \mathfrak{b}(\Phi)) \cap \Lambda$, the continuity property asserts that the dimension of $\mathcal{H}(\tau, \lambda)$ is still equal to the dimension of H^0, that is the cardinal of $\mathfrak{p}(\Phi, \lambda) \cap \mathbb{Z}^N$. This is in accordance with the following vanishing theorem [17].

Theorem 7.1. *If $\lambda \in (\tau - \mathfrak{b}(\Phi)) \cap \Lambda$ then $H^i(M_\tau, \mathcal{O}(\lambda)) = 0$ for $i > 0$.*

It would be interesting to study the locally quasi polynomial function $h_i(\Phi,\tau)(\lambda) = \dim H^i(M_\tau, \mathcal{O}(\lambda))$ for each i. From Demazure's description of the individual cohomology groups $H^i(M_\tau, \mathcal{O}(\lambda))$ (see for example the forthcoming book [11, Chapter 9]), we see that it is a locally quasi-polynomial function, sum of partition functions of flipped systems. Thus each locally quasi polynomial function $h_i(\Phi,\tau)$ is a particular element of the generalized Dahmen-Micchelli space $\mathcal{F}(\Phi)$ introduced in [13]. It would be interesting to study the relations between these different locally quasi polynomial functions on Λ.

Let us give a last example to illustrate the method. We consider the hexagon defined by the following inequalities in $\mathbb{R}^2$. $x_1 \geq 0$, $x_1 \leq 2$, $x_2 \geq 0$, $x_1 + x_2 \geq 1$, $x_1 + x_2 \leq 4$, $x_1 - x_2 \geq -2$. The corresponding toric variety M_{hex} of dimension 2 is defined by the fan with edges $(1,0)$, $(1,1)$, $(0,1)$, $(-1,0)$, $(-1,-1)$, $(1,-1)$.

We can also describe M_{hex} as a reduced Hamiltonian manifold, with the help of an ample line bundle. We consider the standard torus of dimension 4 acting in $\mathbb{C}^6$ with the following list Φ of weights

$$((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (-1,-1,1,1), (1,-1,0,1)).$$

If τ is the tope which contains the vector $[2,-1,2,4]$, then the reduced manifold M_τ is the manifold M_{hex}.

We compute $X(\Phi,\tau)$ (by brute force) and obtain:

$$\begin{aligned}
X(\Phi,\tau) = {} & \\
& p_1p_2p_3p_4p_5p_6 - p_1p_2p_3p_4q_5q_6 - p_1p_2p_3p_5q_4q_6 - p_1p_2p_3p_6q_4q_5 \\
& - 2\, p_1p_2p_3q_4q_5q_6 \\
& - p_1p_2p_4p_6q_3q_5 - p_1p_2p_4q_3q_5q_6 - p_1p_2p_6q_3q_4q_5 - p_1p_2q_3q_4q_5q_6 - p_1p_3p_5p_6q_2q_4 \\
& - p_1p_3p_5q_2q_4q_6 - p_1p_3p_6q_2q_4q_5 - p_1p_3q_2q_4q_5q_6 - p_1p_4p_5p_6q_2q_3 - p_1p_4p_6q_2q_3q_5 \\
& - p_1p_5p_6q_2q_3q_4 - p_1p_6q_2q_3q_4q_5 - p_2p_3p_4p_5q_1q_6 - p_2p_3p_4q_1q_5q_6 - p_2p_3p_5q_1q_4q_6 \\
& - p_2p_3q_1q_4q_5q_6 - p_2p_4p_5p_6q_1q_3 - p_2p_4p_5q_1q_3q_6 - p_2p_4p_6q_1q_3q_5 - p_2p_4q_1q_3q_5q_6 \\
& - p_3p_4p_5p_6q_1q_2 - p_3p_4p_5q_1q_2q_6 - p_3p_5p_6q_1q_2q_4 - p_3p_5q_1q_2q_4q_6 \\
& - 2\, p_4p_5p_6q_1q_2q_3 \\
& - p_4p_5q_1q_2q_3q_6 - p_4p_6q_1q_2q_3q_5 - p_5p_6q_1q_2q_3q_4 + q_1q_2q_3q_4q_5q_6.
\end{aligned}$$

We can immediately read on this expression the multiplicity of a weight $m = (m_1, m_2, m_3, m_4, m_5, m_6)$ in the space $\mathcal{H}(\tau,\lambda)$ for any m and any λ. We see that the multiplicities of m can be 0,1,-1,-2 depending on the quadrant in which m lies.

For example, for $\lambda = (200, 434, 378, -400)$, the weight

$$m = (200, 234, 478, -200, -100, -100)$$

has multiplicity -2 in the space $\mathcal{H}(\tau,\lambda)$. Indeed the coefficient of $p_1p_2p_3q_4q_5q_6$ in $X(\Phi,\tau)$ is -2.

Given $\lambda \in \mathbb{Z}^4$, we parameterize the integral points in $V(\Phi, \lambda)$ by $(x_1, x_2) \in \mathbb{Z}^2$, with corresponding $m \in \mathbb{Z}^6$ given by

$$m = (\lambda_1 + x_1 - x_2, \lambda_2 + x_1 + x_2, \lambda_3 - x_1, \lambda_4 - x_1 - x_2, x_1, x_2).$$

With this parametrization, the Figures 7.1, 7.2 and 7.3 describe the support of the module $\mathcal{H}(\tau, \lambda)$ as λ moves along the line joining $\lambda_0 =$ (200, -100, 200, 400 (in the ample cone) to $\lambda_1 = (200, 434, 378, -400)$. The line crosses six walls.

We assign colors to the multiplicities: $blue = 1$, $yellow := -1$, $red := -1$, $green := -1$, $magenta := -2$, $black := -1$, $khaki = -1$. In the first figure 7.1, λ is in the starting tope (the ample cone). In the last three steps, a polygon with multiplicity -2 (colored in magenta) has appeared in the middle of the picture.

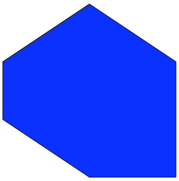


Figure 7.1. At the beginning $\lambda = (200, -100, 200, 400)$ is in the ample cone. The partition polytope is an hexagon.

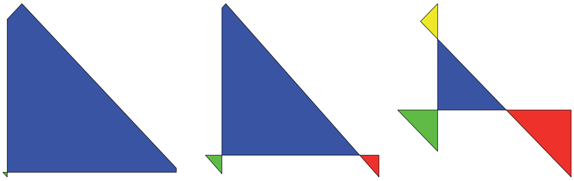


Figure 7.2. From left to right, λ crosses three walls, one at a time. The new triangles have multiplicity -1.

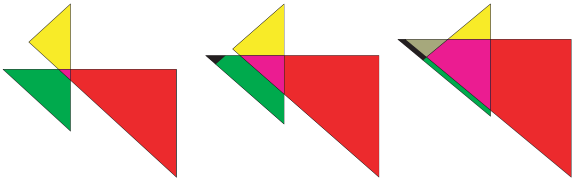


Figure 7.3. λ crosses three more walls. The polytope colored in magenta has multiplicity -2.

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