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BOX SPLINES AND THE EQUIVARIANT INDEX THEOREM

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Abstract In this article, we begin by recalling the inversion formula for the convolution with the box spline. The equivariant cohomology and the equivariant K-theory with respect to a compact torus G of various spaces associated to a linear action of G in a vector space M can both be described using some vector spaces of distributions, on the dual of the group G or on the dual of its Lie algebra $\mathfrak{g}$. The morphism from K-theory to cohomology is analyzed, and multiplication by the Todd class is shown to correspond to the operator (deconvolution) inverting the semi-discrete convolution with a box spline. Finally, the multiplicities of the index of a G-transversally elliptic operator on M are determined using the infinitesimal index of the symbol.

Keywords: splines; box splines; deconvolution; index theory; equivariant K-theory; equivariant cohomology; Riemann–Roch; elliptic operators

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Introduction

0.1. Motivations

The motivation of this work is to understand the multiplicities of a representation of a torus G in the virtual representation space $\mathrm{Ker}(A) - \mathrm{Coker}(A)$ obtained as the index of a G-invariant, elliptic, or more generally transversally elliptic, pseudo-differential operator A, in terms of the symbol. For basic definitions and results, we refer to the Lecture Notes of Atiyah (see [1]).

We shall restrict ourselves to the case of a torus G (with Lie algebra $\mathfrak{g}$) acting on a vector space (to some extent this is the essential case). Let Λ be the lattice of characters of G; for $\lambda \in \Lambda$, we denote by $g \to g^{\lambda}$ the corresponding function on G. According to the theory of Atiyah and Singer (see [1]), in the case of a transversally elliptic operator, $\mathrm{Ker}(A)$ and $\mathrm{Coker}(A)$ might be infinite dimensional, but the multiplicity of a character is finite, and the difference of the two multiplicities in $\mathrm{Ker}(A)$ and $\mathrm{Coker}(A)$ is the Fourier coefficient of the generalized function $\mathrm{index}(A)$ on G. We thus obtain a function $\mathrm{ind}_m(A)$

on Λ such that $\mathrm{index}(A)(g) = \sum_{\lambda\in\Lambda} \mathrm{ind}_m(A)(\lambda)g^{\lambda}$. We call $\mathrm{ind}_m(A)$ the multiplicity index map.

A cohomological formula for the equivariant index of elliptic operators was obtained by Atiyah, Bott, Segal, and Singer. Using integrals of equivariant cohomology classes, a formula for the equivariant index of transversally elliptic operators was obtained in [7, 8, 24]. These formulae define (generalized) functions on G in terms of the Chern character of the symbol of A. However, the behavior of multiplicities is our main interest. Note that, even in the case of elliptic operators where we deal with finite-dimensional representations, a 'formula' for the multiplicities is not easy to deduce from the Atiyah–Bott–Segal–Singer fixed-point formulae, as $\mathrm{index}(A)(g)$ is given by different formulae for each $g \in G$. A similar drawback of the formulae of [7, 8, 24] is that, for each $g \in G$, they are defined only on a neighborhood of g (and with different formulae for each $g \in G$). Thus, known formulae for the equivariant index were not adapted to the study of the Fourier transform.

Our point of view is new. Instead of functions on G, we consider directly the multiplicity index of an operator A, a function on $\hat{G}$. Similarly, we associate directly to the Chern character of the symbol of A a spline function on $\mathfrak{g}^*$. Here, splines (called also multisplines in several variables) are the familiar objects in approximation theory: piecewise polynomial functions with respect to a polyhedral subdivision of $\mathfrak{g}^*$ (see [12]). Our main theorem (Theorem 5.17) says (essentially) that the multiplicity index is the restriction of a suitable spline function to Λ, a lattice in $\mathfrak{g}^*$. Our inspiration comes from the 'continuous analogue' of the index: the Duistermaat–Heckman measure, a piecewise polynomial function on $\mathfrak{g}^*$, and from the 'quantization commutes with reduction' results on multiplicities of twisted Dirac operators. The key point of our approach is explicit computation of the index of some transversally elliptic operators in terms of vector partition functions. We construct two piecewise polynomial functions, one obtained from the multiplicity index and box splines, and the other from the Chern character using our theory of the infinitesimal index [17], and we compare them on generators. Our final theorem (Theorem 5.17) follows from a remarkable inversion formula, basically due to Dahmen and Micchelli [10], for multisplines.

Let us first recall the basic formalism of our approach. Let $M := M_X = \bigoplus_{a\in X} L_a$ be a complex vector space with a linear action of G, where $a \in X \subset \Lambda$ is a character and L_a denotes the corresponding one-dimensional representation of G.

The vector partition function $\mathcal{P}_X$, a function on Λ which describes the multiplicity of the action of the torus G on polynomial functions on M, is approximated by a multispline distribution T_X: the convolution of the Heaviside functions associated to the half-line $\mathbb{R}^+a$, where a runs through the sequence X of weights of G in M (we assume here in the introduction that all weights a are on one side of a half-space and span $\mathfrak{g}^*$). The locally polynomial measure T_X on $\mathfrak{g}^*$ is the Duistermaat–Heckman measure of the Hamiltonian vector space M_X.

In approximation theory, one introduces another special distribution, *the box spline* B_X, defined as convolution of the intervals $[0, 1]a$ (thought of as measures or distributions). An immediate relation between $\mathcal{P}_X$ and B_X is the fact that the convolution of the box spline B_X with the partition function $\mathcal{P}_X$ is the multispline T_X. The Todd operator, an infinite series of constant-coefficient differential operators, acts on spline functions. It enters naturally in the 'deconvolution' formula, leading to a 'Riemann-Roch formula' for $\mathcal{P}_X$ in term of T_X (at least in the special case of X unimodular). We apply a series of constant coefficient operators to the piecewise polynomial function T_X and then restrict the function so obtained to the lattice. In this way, we obtain the vector partition function $\mathcal{P}_X$.

These algebraic formulae are well known (see Khovanskiĭ and Pukhlikov [19], Dahmen and Micchelli [11], Brion and Vergne [9], De Concini and Procesi [14]), and they are equivalent to the Riemann–Roch theorem for line bundles over toric varieties.

Our aim in this article is to show that the same deconvolution formula allows us to compute the index of any transversally elliptic operator on M_X in term of a piecewise polynomial function on $\mathfrak{g}^*$ associated to its symbol by applying to it the Todd differential operator.

0.2. Summary of results

Let M be a manifold, T^*M its cotangent bundle, and $p : T^*M \to M$ the canonical projection. Given now a pseudo-differential operator A between the sections of two vector bundles $\mathcal{E}^+, \mathcal{E}^-$, one constructs its symbol $\Sigma = \Sigma(x,\xi)$, which is a bundle map $\Sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$.

If M has a G action, we denote by $T^*_G M$ the union of the conormals to the G orbits. Thus $T^*_G M$ is a closed subset of T^*M. Then a G-equivariant pseudo-differential operator A is called G-transversally elliptic if the symbol Σ restricted to $T^*_G M$ minus the zero section is an isomorphism of bundles.

The symbol $\Sigma(x,\xi)$ of the pseudo-differential transversally elliptic operator A on M determines two topological objects.

(1) An element of the equivariant K-theory group $K^0_G(T^*_G M)$.
(2) The Chern character $\mathrm{ch}(\Sigma)$ of Σ, which is an element of the G-equivariant cohomology with compact supports of $T^*_G M$.

The index of A, denoted $\mathrm{index}(A)$, depends only on the symbol, and defines a map from $K^0_G(T^*_G M)$ to the space of generalized functions on G. The Fourier transform of $\mathrm{index}(A)$ is the multiplicity index map $\mathrm{ind}_m(A)$, a function on $\Lambda \subset \mathfrak{g}^*$.

In [17], we have associated to $\mathrm{ch}(\Sigma)$ a distribution on $\mathfrak{g}^*$, its infinitesimal index, denoted $\mathrm{infdex}(\mathrm{ch}(\Sigma))$.

Assume now that M is a real vector space with a linear action of G. The list of weights of G in the complex vector space $M \otimes_{\mathbb{R}} \mathbb{C}$ is $X \cup -X$ for some list $X \subset \Lambda$. For simplicity, assume that X generates $\mathfrak{g}^*$.

For any $\Sigma \in K^0_G(T^*_G M)$, we prove that the distribution $\mathrm{infdex}(\mathrm{ch}(\Sigma))$ is piecewise polynomial on $\mathfrak{g}^*$. Furthermore (Proposition 5.14), the following identity of locally L^1-functions of $\xi \in \mathfrak{g}^*$ holds:

$$\sum_{\lambda \in \Lambda} \mathrm{ind}_m(A)(\lambda) B_{X\cup -X}(\xi - \lambda) = \frac{1}{(2i\pi)^{\dim M}} \mathrm{infdex}(\mathrm{ch}(\Sigma))(\xi). \qquad (1)$$

In other words, infdex(ch(Σ)) is (up to a multiplicative constant) the spline function obtained from convoluting a box spline with the discrete measure $\sum_\lambda \mathrm{ind}_m(A)(\lambda)\delta_\lambda$. It is easy to check this formula on the Atiyah symbols At^F, and these elements generate the $R(G)$-module $K_G(T^*_G M)$. In some sense, as explained in Part 2, this formula is the Fourier transform of the Berline–Paradan–Vergne formulae.

Apply the Todd operator associated to $X \cup -X$ to the spline function infdex(ch(Σ))(ξ). We obtain again a spline function p on $\mathfrak{g}^*$. Our final result (Theorem 5.17) says essentially that the restriction of this function p to Λ is the multiplicity index. This follows from a 'deconvolution' formula for splines functions produced by convolution with box splines.

0.3. Outline of the article

• In Part 1, we recall some results obtained by Dahmen and Micchelli in the purely combinatorial context of semi-discrete convolution with the box spline. Let $X = [a_1, a_2, \dots, a_N]$ be a finite list of vectors in a vector space V. The box spline B_X is the image measure of the hypercube $[0, 1]^N$ by the map $(t_1, t_2, \dots, t_N) \to \sum_i t_i a_i$.

We assume next that X spans a lattice Λ. Convoluting a discrete measure supported on the lattice Λ by B_X produces a locally polynomial function on V. The first step is to prove a *deconvolution formula*. Define the Todd operator $\prod_{a\in X} \frac{\partial_a}{1-e^{-\partial_a}}$. Then we prove in Theorem 2.15 that, if we apply (in an appropriate sense) the Todd operator Todd(X) to the box spline and restrict the resulting spline to the lattice, we obtain the δ function of the lattice Λ in the case of a unimodular system (a slightly more complicated formula is obtained for any X):

$$\mathrm{Todd}(X) * B_X|_\Lambda = \delta_0.$$

We prove this result using our knowledge of Dahmen–Micchelli spaces (see [14, 15]).

We show that the Khovanskiĭ–Pukhlikov formula and more generally the Brion–Vergne formula for the partition function $\mathcal{P}_X$ are particular cases of the deconvolution formula.

• In Part 2, we consider $M := M_X$ as a real G-manifold, and we recall our description of $K^0_G(T^*_G M)$ as well as of $H^*_{G,c}(T^*_G M)$ as vector spaces of distributions, on the dual of the group G or on the dual of its Lie algebra $\mathfrak{g}$ (see [16] and [18], respectively). We compute the infinitesimal index of the Chern character of the Atiyah symbol At^F and descent formulae associated to a finite subset of elements of $g \in G$. We use all these ingredients to give a general formula, in (34), for the index of a transversally elliptic operator A in term of the infinitesimal index of the Chern character of the symbol of A.

As already pointed out, our results are motivated by previous results of Berline and Vergne and of Paradan and Vergne. But our point of view is dual. We work with functions on $\hat{G}$ or $\mathfrak{g}^*$. This way, we are dealing with very familiar objects: partition functions and multispline functions. It is remarkable that the transversally elliptic operators having multiplicity index Partition functions are the building blocks of 'all index theory'.

1. Notation and preliminaries

Let V be an s-dimensional real vector space equipped with a lattice $\Lambda \subset V$. We choose the Lebesgue measure dv on V for which V/Λ has volume 1. With the help of dv, we can freely identify generalized functions on V and distributions on V. We denote by $\mathbb{C}[V]$ the space of (complex-valued) polynomial functions on V.

We denote by S^1 the circle group of complex numbers of modulus 1. The character group of Λ, $G := \hom(\Lambda, S^1)$, is a compact torus, and V can be identified to $\mathfrak{g}^*$, the dual of the Lie algebra $\mathfrak{g}$ of G. Of course, if we choose a basis of the lattice Λ, then we may identify Λ with $\mathbb{Z}^s$, V with $\mathbb{R}^s$, and G with $(S^1)^s$.

Dually, if $\Gamma \subset \mathfrak{g} = V^*$ is the lattice of elements $x \in \mathfrak{g}$ such that $\langle x \mid \lambda\rangle \in 2\pi\mathbb{Z}$ for all $\lambda \in \Lambda$, then the torus G is $\mathfrak{g}/\Gamma$. If $x \in \mathfrak{g}$, we shall denote by e^x its class in G, and the duality pairing $G \times \Lambda \to S^1$ will be given by

$$(e^x, \lambda) \mapsto e^{i\langle x \mid \lambda\rangle}.$$

Under this duality, Λ is the character group of G, and will sometimes be denoted by $\check{G}$.

We identify the space $C^\infty(G)$ with the subspace of $C^\infty(\mathfrak{g})$ formed by functions periodic under Γ. If $g = e^x \in G$, we will sometimes write g^λ for $e^{i\langle x \mid \lambda\rangle}$.

L_λ will denote the one-dimensional complex vector space with action of G given by g^λ. Notice that, as a real G linear representation, L_λ is isomorphic to $L_{-\lambda}$ by changing the complex structure with the conjugate one.

More generally, we have the following definition.

Definition 1.1. Let X be a finite sequence of non-zero elements of Λ. Define the vector space

$$M_X := \bigoplus_{a \in X} L_a. \tag{2}$$

Thus M_X is a complex representation space for G, and every finite-dimensional complex representation of G is of this form for a well-defined X.

Again, the space M_X as a real G-representation depends only on the sequence X up to sign changes. In fact, it will be very important to consider for a real representation space M (with no G-invariant non-zero vector) all possible G-invariant complex structures on M.

The space of $\mathbb{C}$-valued functions on $\Lambda = \check{G}$ will be denoted by $\mathcal{C}[\Lambda]$, while we shall set $\mathcal{C}_{\mathbb{Z}}[\Lambda]$ to be the subgroup of $\mathbb{Z}$-valued functions. We display such a function $f(\lambda)$ also as a formal series:

$$\Theta(f) := \sum_{\lambda \in \Lambda} f(\lambda) e^{i\lambda}.$$

We denote by δ_0 the function on Λ identically equal to 0 on Λ, except for $\delta_0(0) = 1$.

The subspace $\mathbb{C}[\Lambda]$ of the functions with finite support is the group algebra of Λ, but it can be also considered as the coordinate ring of the complex torus $\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}/\Gamma$ as an algebraic group. Finally, $\mathbb{Z}[\Lambda] := \mathbb{C}[\Lambda] \cap \mathcal{C}_{\mathbb{Z}}[\Lambda]$ is the group ring of Λ, but it can also be considered as the character ring of G or the Grothendieck group of finite-dimensional representations of G. Due to this, we shall sometimes denote it by $R(G)$. Indeed, if T is a representation of G in a finite-dimensional complex vector space, then $\mathrm{Tr}\, T(g)$ is a finite linear combination of characters, and this gives the desired homomorphism.

If $f(\lambda)$ is of at most polynomial growth, the series $g \to \sum_{\lambda\in\Lambda} f(\lambda)g^\lambda$ defines a generalized function on the torus G. We denote by $R^{-\infty}(G)$ the subspace of $\mathcal{C}[\Lambda]$ consisting of these $f(\lambda)$.

Let us point out that Λ acts on $\mathcal{C}[\Lambda]$ by translations; namely, if $a \in \Lambda$ and $f \in \mathcal{C}[\Lambda]$, $(t_a f)(\lambda) := f(\lambda - a)$. This clearly corresponds to multiplication by e^{ia} on $\Theta(f)$. It follows that both $\mathcal{C}[\Lambda]$ and $R^{-\infty}(G)$ are $\mathbb{C}[\Lambda]$-modules, and of course $\mathcal{C}_{\mathbb{Z}}[\Lambda]$ is a $\mathbb{Z}[\Lambda]$-module. We also define the difference operator

$$\nabla_a := id - t_a.$$

Passing to the continuous setting, if we take the space of polynomial functions $\mathbb{C}[\mathfrak{g}]$ on $\mathfrak{g}$ (equal to the symmetric algebra $S[\mathfrak{g}^*]$), we are going to consider the space of distributions $\mathcal{D}'(\mathfrak{g}^*)$ on $\mathfrak{g}^*$ as a $S[\mathfrak{g}^*]$-module, using differentiation. We denote by ∂_a the partial derivative in the $a \in \mathfrak{g}^*$ direction.

Part 1. Algebra

2. Box splines

2.1. Splines

Let $X = [a_1, a_2, \ldots, a_N]$ be a sequence (a multiset) of N non-zero vectors in Λ.

The **zonotope** $Z(X)$ associated to X is the polytope

$$Z(X) := \left\{ \sum_{i=1}^{N} t_i a_i \mid t_i \in [0,1] \right\}.$$

In other words, $Z(X)$ is the Minkowski sum of the segments $[0, a_i]$ over all vectors $a_i \in X$.

Recall that the box spline B_X is the distribution on V such that, for a test function *test* on V, we have the equality

$$\langle B_X, test\rangle = \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} test\left(\sum_{i=1}^{N} t_i a_i\right) dt_1 \cdots dt_N. \tag{3}$$

The box spline is a compactly supported probability measure on V, and we have

$$\int_V e^{i\langle v,x\rangle} B_X(v) = \prod_{k=1}^{N} \frac{e^{i\langle a_k,x\rangle} - 1}{i\langle a_k, x\rangle}. \tag{4}$$

If X generates V, the zonotope is a full-dimensional polytope, and B_X is given by integration against a piecewise polynomial function on V, supported and continuous on $Z(X)$, that we still call B_X.

Example 2.2. Let $V = \mathbb{R}$ be one dimensional, and let $X_k = [1, 1, \ldots, 1]$, where 1 is repeated k times. Figure 1 gives the graphs of B_{X_1}, B_{X_2}, and B_{X_3}.

Let us describe more precisely where this function is given by a polynomial formula.

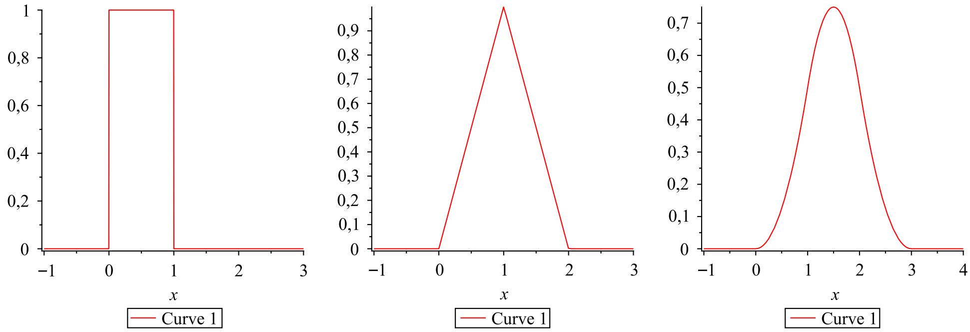

Figure 1. $B_{X_1}, B_{X_2}, B_{X_3}$.

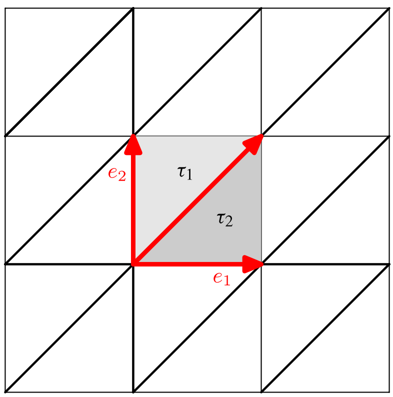

Figure 2. Topes for $X := [e_1, e_2, e_1 + e_2]$.

Definition 2.3. An hyperplane of V generated by a subsequence of elements of X is called *admissible.*

An *affine admissible hyperplane* is a translate $\lambda + H$ of an admissible hyperplane H by an element $\lambda \in \Lambda$.

Remark 2.4. The zonotope is bounded by affine admissible hyperplanes.

Definition 2.5. An element v of V is called *regular* if it does not lie in any admissible hyperplane. We denote by V_{reg} the open subset of V consisting of regular elements. A connected component $\mathfrak{c}$ of the set of regular elements will be called a *(conic) tope.*

An element v of V is called *affine regular* if it does not lie in any admissible affine hyperplane. We denote by $V_{\mathrm{reg,aff}}$ the open subset of V consisting of affine regular elements. A connected component τ of the set of affine regular elements will be called an *alcove* (see Figure 2).

Definition 2.6. We will say that a locally L^1 function b on V is piecewise polynomial (with respect to (X, Λ)) if, on each alcove τ, there exists a polynomial function b^τ on

V such that the restriction of b to τ coincides with the restriction of the polynomial b^τ to τ.

If b is a piecewise polynomial function, we will say that the distribution $b(v)dv$ is piecewise polynomial.

We denote by $\mathcal{PW}_{(X,\Lambda)}(V)$ the space of these piecewise polynomial functions on V.

When there is no ambiguity, we may drop Λ or X or both, and write simply $\mathcal{PW}_X(V)$ or $\mathcal{PW}(V)$.

Remark 2.7.
- $\mathcal{PW}_{(X,\Lambda)}(V)$ is preserved by the translation operators t_a, $a \in \Lambda$.
- The polynomial function b^τ is uniquely determined by b and τ.
- The support of a piecewise polynomial function is a union of closures of alcoves.
- If X generates V, the box spline B_X is a piecewise polynomial function supported on the zonotope $Z(X)$. Furthermore, if, for any a in X, $X \setminus \{a\}$ still spans V, this piecewise polynomial function extends continuously on V. In particular, this applies if 0 is an interior point in $Z(X)$.

A piecewise polynomial function h may be continuous. In this case, its restriction to the lattice Λ is well defined. If not, we may define the 'restriction' of h to Λ by a limit procedure, as follows.

Consider a piecewise polynomial function h on V. Let $\mathfrak{c}$ be an alcove in V containing 0 in its closure. Then, for any $\lambda \in \Lambda$, $\tau = \lambda + \mathfrak{c}$ is an alcove on which h is the polynomial h^τ. We thus define a map $\lim_\mathfrak{c} : \mathcal{PW}(V) \to \mathcal{C}[\Lambda]$ by setting

$$\lim_{\mathfrak{c}} h(\lambda) := h^\tau(\lambda).$$

Example 2.8. Consider the box spline B_{X_1} in Figure 1. If $\mathfrak{c}$ is the alcove $]0,1[$, then $\lim_\mathfrak{c} B_{X_1}(0) = 1$, as we take the limit from the right, while $\lim_{-\mathfrak{c}} B_{X_1}(0) = 0$ for the opposite alcove.

Notice that the operator $\lim_\mathfrak{c}$ on $\mathcal{PW}(V)$ commutes with translations by elements of Λ.

It is convenient to think of an element in $\mathcal{PW}_{(X,\Lambda)}(V)$ as a function only on the set of affine regular points. As such, differentiating alcove by alcove, this space is a module over the ring of formal differential operators of infinite order with constant coefficients.

Therefore, we may set the following definition.

Definition 2.9. Given an operator D of infinite order with constant coefficients and $b \in \mathcal{PW}_{(X,\Lambda)}(V)$, we shall denote by $D_{pw}b \in \mathcal{PW}_{(X,\Lambda)}(V)$ the element defined by the action of D on b alcove by alcove:

$$(D_{pw}b)^\tau = Db^\tau.$$

Notice that the action D_{pw} on $\mathcal{PW}(V)$ commutes with the action of translation by elements of Λ.

Warning We may apply to the function $b \in \mathcal{PW}_{(X,\Lambda)}(V)$, thought of as a distribution, a finite order differential operator. In general, we get a different result to that which we obtain by taking this function of affine regular points, applying the same operator, and then considering the result as an L^1 function. Indeed, the two coincide only on the set of affine regular points.

Example 2.10. Consider again the box spline B_{X_1} in Figure 1. Then, if we consider B_{X_1} as a distribution, we have $\partial B_{X_1} = \delta_0 - \delta_1$, a difference of two delta functions, while $\partial_{pw} B_{X_1} = 0$.

If K is a polynomial function on V, we will say that the function $\lambda \mapsto K(\lambda)$ is a polynomial function on Λ. The polynomial function K is determined by its restriction to Λ. A function k on Λ for which there exists a sublattice Λ' of Λ such that, for any $\xi \in \Lambda$, the function $\nu \to k(\xi + \nu)$ is polynomial on Λ' will be called a quasi-polynomial function on Λ.

Definition 2.11. If $f \in \mathcal{C}[\Lambda]$, define the distribution $B_X *_d f$ by

$$B_X *_d f = \sum_{\lambda \in \Lambda} f(\lambda) t_\lambda B_X.$$

When X spans the vector space V, this gives rise to the piecewise polynomial function

$$(B_X *_d f)(v) = \sum_{\lambda \in \Lambda} f(\lambda) B_X(v - \lambda).$$

The notation $*_d$ means discrete. $B_X *_d f$ is the convolution of B_X with the discrete measure $\sum_\lambda f(\lambda)\delta_\lambda$.

We denote (to emphasize the difference with the discrete case) the usual convolution of two distributions θ_1, θ_2 (with some support conditions so that their convolution exists) by $\theta_1 *_c \theta_2$.

Our aim is to write an inversion formula for $f \to B_X *_d f$. As this operator is not injective, we will need a few other data.

Remark 2.12. If $p \in \mathbb{C}[V]$ is a polynomial, then, by the Taylor formula, we have $t_b p = e^{-\partial_b} p$.

If Y is a sequence of vectors, we define the operator $I(Y)$ on $\mathbb{C}[V]$ by

$$I(Y) := \prod_{a \in Y} \frac{(1 - e^{-\partial_a})}{\partial_a}. \tag{5}$$

Then, by integrating the Taylor formula, we have

$$\int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} p\left(v - \left(\sum_{i=1}^{N} t_i a_i\right)\right) dt_1 \cdots dt_N = (I(Y)p)(v).$$

The operator $I(Y)$ is an invertible operator on $\mathbb{C}[V]$. We denote the inverse of $I(Y)$ by Todd(Y):

$$\mathrm{Todd}(Y) := \prod_{a \in Y} \frac{\partial_a}{(1 - e^{-\partial_a})}. \tag{6}$$

Notice that

$$\langle B_X *_c f, test\rangle = \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} \left\langle f, test\left(v - \left(\sum_{i=1}^{N} t_i a_i\right)\right)\right\rangle dt_1 \cdots dt_N$$

for any distribution f on V. Thus we have the following proposition.

Proposition 2.13. *For p a polynomial function on V, the usual convolution $B_X *_c p$ is still a polynomial given by the formula $B_X *_c p = I(X)p$, and its inverse is given by the operator* Todd(X).

2.14. Inversion formula: the unimodular case

Recall that a sequence X is unimodular if X spans V, and if any basis σ of V extracted from X is a basis of Λ. We will prove now that, if X is unimodular, then the inverse of the semi-discrete convolution by the box spline B_X

$$K \to B_X *_d K; \quad \mathbb{C}[\Lambda] \to \mathcal{PW}(V)$$

is obtained by applying the operator Todd $(X)_{pw}$ (see Definition 2.9) to the piecewise polynomial function $B_X *_d K$ and then passing to a suitable limit.

Theorem 2.15. *Assume that X is unimodular. Let $\mathfrak{c}$ be an alcove in V containing* 0 *in its closure and contained in $Z(X)$. Then*

(i) $\lim_{\mathfrak{c}}(\text{Todd}\,(X)_{pw} B_X) = \delta_0$.

(ii) *For any $K \in \mathcal{C}[\Lambda]$,*

$$K = \lim_{\mathfrak{c}}(\text{Todd}\,(X)_{pw}(B_X *_d K)).$$

Remark 2.16. If 0 belongs to the interior of the zonotope $Z(X)$, one can show that the function Todd $(X)_{pw} B_X$ extends to a continuous function on V. We will recall this result in Remark 3.15.

Remark 2.17. The two items in Theorem 2.15 are equivalent statements. The first item is the particular case of the second item applied to $K = \delta_0$, and the other is deduced from the first one by writing K as a linear combination of translates of δ_0. However, we list them independently, as we want to emphasize this striking property of the box spline function. Figure 3 describes Todd $(X)_{pw} B_X$ for $X = X_1, X_2, X_5$.

We will give the proof of this theorem in § 2.26, after having introduced some further notions.

2.18. Dahmen–Micchelli spaces

Let us recall some facts on Dahmen–Micchelli polynomials.

If I is a sequence of vectors, we define the operators $\partial_I := \prod_{a\in I} \partial_a$ and $\nabla_I := \prod_{a\in I} \nabla_a$. These operators are defined on distributions B, since by duality we can set

$$\langle t_a B, test\rangle = \langle B, t_{-a} test\rangle, \quad \langle \partial_a B, test\rangle = -\langle B, \partial_a test\rangle.$$

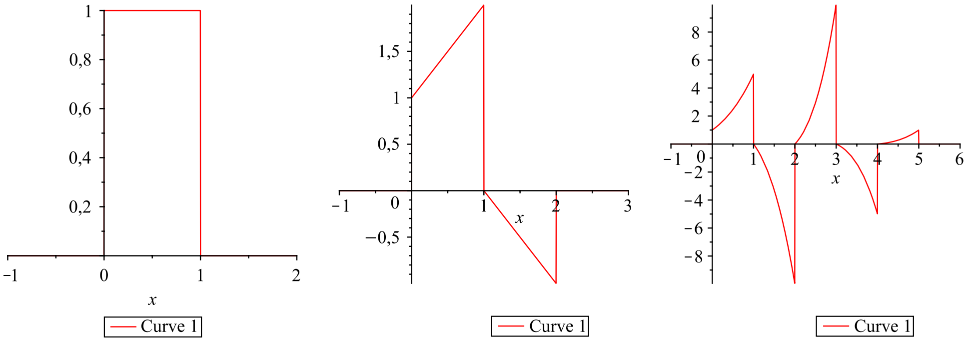

Figure 3. Todd $(X_1)_{pw} B_{X_1}$, Todd $(X_2)_{pw} B_{X_2}$, Todd $(X_5)_{pw} B_{X_5}$.

If Y is a subsequence of X, by $X \setminus Y$ we mean the complement in X of the sequence Y. If S is a subset of V, we also employ the notation $X \setminus S$ for the sequence of elements of X not lying in S, and $X \cap S$ for the sequence of elements of X lying in S. We have the following equality of distributions (see [14], Proposition 7.14):

$$\partial_Y B_X = \nabla_Y B_{X \setminus Y}. \tag{7}$$

A subsequence Y of X will be called *long* if the sequence $X \setminus Y$ does not generate the vector space V. A long subsequence Y, minimal along the long subsequences, is also called a *cocircuit*. In this case, $Y = X \setminus H$, where H is an admissible hyperplane.

In particular, if $Y = X \setminus H$, using equation (7), we have that $\partial_Y B_X = \nabla_Y B_{X \cap H}$ is supported on the union of the affine admissible hyperplanes which are translates of H by elements of Y. So the restriction of $\partial_Y B_X$ to any alcove τ is equal to 0.

Recall the following definitions.

Definition 2.19. (1) The space $D(X)$ is the space of (generalized) functions B on V such that $\partial_Y B = 0$ for all long subsequences Y of X.

(2) The space $DM(X)$ is the space of integral-valued functions K on Λ such that $\nabla_Y K = 0$ for all long subsequences Y of X.

Of course, it is sufficient to impose these equations for Y running along all cocircuits.

If X spans V, it is easy to see that $D(X)$ is a finite-dimensional space of polynomial functions on V, and that $DM(X)$ is a free abelian group of finite rank of quasi-polynomial functions on Λ [10] (see [14]).

In this paper, we often need to compare $D(X)$ with $DM(X)$. In order to do this, it is more convenient to extend $DM(X)$ to the spaces $DM(X)_{\mathbb{R}} := DM(X) \otimes_{\mathbb{Z}} \mathbb{R}$, $DM(X)_{\mathbb{C}} := DM(X) \otimes_{\mathbb{Z}} \mathbb{C}$, respectively, of real or complex valued functions satisfying the same difference equations as $DM(X)$ (see Theorem 2.25). Sometimes, by abuse of notation, we shall drop the subscript $\mathbb{C}$, and just write $DM(X)$ for $DM(X)_{\mathbb{C}}$. Similarly, we could take complex-valued solutions of the differential equations getting the space $D(X)_{\mathbb{C}} := DM(X) \otimes_{\mathbb{R}} \mathbb{C}$.

The restriction of a function $p \in D(X)$ (a polynomial) to Λ is in $DM(X)_{\mathbb{R}}$. If X is a unimodular system, this restriction map is an isomorphism.

The space $D(X)$ is invariant by differentiations. The space $DM(X)$ is invariant by translations by elements of Λ.

The following lemma follows from the definitions.

Lemma 2.20. *If Y is a subsequence of X such that $X \setminus Y$ still generates V, then $\partial_Y D(X)$ is contained in $D(X \setminus Y)$, and $\nabla_Y DM(X)$ is contained in $DM(X \setminus Y)$.*

Remark 2.21. In fact, the operators ∂_Y and ∇_Y are surjective onto $D(X \setminus Y)$ and $DM(X \setminus Y)$, respectively. This is a stronger statement that we proved in [16] (and over $\mathbb{Z}$ for DM). We will not use this stronger statement, which is more delicate to prove.

If τ is an alcove contained in $Z(X)$, then the polynomial B_X^τ is a non-zero polynomial belonging to $D(X)$ (as seen from equation (7)).

Lemma 2.22. *If $K \in DM(X)$, then $B_X *_d K \in D(X)$.*

Proof. Let H be an admissible hyperplane. Then

$$\begin{aligned}\partial_{X\setminus H}(B_X *_d K) &= \partial_{X\setminus \underline{H}} B_X *_d K \\ &= B_{X\cap H} *_d (\nabla_{X\setminus H} K) = 0.\end{aligned}$$ □

The following result is proved in [10] (see also [14], [25]).

Theorem 2.23. *If K is the restriction to Λ of a polynomial $k \in D(X)$, then $B_X *_d K$ is equal to the polynomial $I(X)k$.*

Thus, on the space $D(X)$, the operator $B_X *' k := B_X *_d K$, called semi-discrete convolution, is an isomorphism with inverse Todd(X).

We will need some structure theory on $DM(X)$.

Let $\mathfrak{c}$ be an alcove. Let us consider any point $\epsilon \in \mathfrak{c}$. It is easy to see that the set $(\epsilon - Z(X)) \cap \Lambda$ depends only of $\mathfrak{c}$. We give the following definition.

Definition 2.24. Let $\mathfrak{c}$ be an alcove. We denote $\delta(\mathfrak{c} \mid X) := (\epsilon - Z(X)) \cap \Lambda$, where ϵ is any element of $\mathfrak{c}$.

We finally recall the following important theorem of Dahmen and Micchelli [10, 11] (see [14]).

Theorem 2.25. *Let $\mathfrak{c}$ be an alcove. For any $\xi \in \delta(\mathfrak{c} \mid X)$, there exists a unique Dahmen–Micchelli element $k_{\mathfrak{c}}^{(\xi)} \in DM(X)$ such that*

$$k_{\mathfrak{c}}^{(\xi)}(\xi) = 1,$$
$$k_{\mathfrak{c}}^{(\xi)}(\nu) = 0$$

if $\nu \in \delta(\mathfrak{c} \mid X)$ and $\nu \neq \xi$.

2.26. Proof of the inversion formula

After these definitions, let us return to the proof of the inversion formula in the unimodular case. We consider an alcove $\mathfrak{c}$ contained in $Z(X)$ and containing 0 in its closure.

Proof. By Remark 2.17, it is enough to prove (i).

By definition, $\lim_{\mathfrak{c}}(\text{Todd}\,(X)_{pw}\,B_X)(\lambda) = (\text{Todd}(X)B_X^{\lambda+\mathfrak{c}})(\lambda)$. If $(\lambda+\mathfrak{c}) \cap Z(X) = \emptyset$, then $B_X^{\lambda+\mathfrak{c}} = 0$, so $\lim_{\mathfrak{c}}(\text{Todd}\,(X)_{pw}\,B_X)(\lambda) = 0$.

We now fix a point λ such that the alcove $\lambda+\mathfrak{c}$ does intersect $Z(X)$. The point $\lambda = 0$ is such a point, by our assumption on $\mathfrak{c}$.

The condition $(\lambda+\mathfrak{c}) \cap Z(X) \neq \emptyset$ is equivalent to the fact that $0 \in \delta(\lambda+\mathfrak{c}\,|\,X)$. Note that $\lambda = \lambda+\epsilon-\epsilon$, $\epsilon \in \mathfrak{c}$ is also in $\delta(\lambda+\mathfrak{c}\,|\,X)$. By Theorem 2.25, there is a unique element $p_{\lambda,\mathfrak{c}} := k^{(0)}_{\lambda+\mathfrak{c}}$ in $DM(X)$ coinciding with δ_0 on $\delta(\lambda+\mathfrak{c}\,|\,X)$.

Let us compute $(B_X *_d p_{\lambda,\mathfrak{c}})(v)$ with $v \in \lambda+\mathfrak{c}$. Using the definitions, for such a v, we have

$$(B_X *_d p_{\lambda,\mathfrak{c}})(v) = \sum_{\eta\in\Lambda} p_{\lambda,\mathfrak{c}}(\eta) B_X(v-\eta) = \sum_{\eta\in\delta(\lambda+\mathfrak{c}\,|\,X)} p_{\lambda,\mathfrak{c}}(\eta) B_X(v-\eta).$$

The second equality follows from the fact that the support of B_X is $Z(X)$. Since, on $\delta(\lambda+\mathfrak{c}\,|\,X)$, $p_{\lambda,\mathfrak{c}}$ vanishes except at 0, we obtain from Lemma 2.22

$$B_X^{\lambda+\mathfrak{c}} = B_X *_d p_{\lambda,\mathfrak{c}}.$$

At this point, we use the fact that X is a unimodular system, so that the restriction map from $D(X)$ to $DM(X)$ is an isomorphism. Thus $p_{\lambda,\mathfrak{c}}$ is the restriction to Λ of a polynomial still denoted by $p_{\lambda,\mathfrak{c}}$ belonging to $D(X)$ and, by Theorem 2.23, $B_X^{\lambda+\mathfrak{c}} = I(X)p_{\lambda,\mathfrak{c}}$. It follows that $\text{Todd}(X)B_X^{\lambda+\mathfrak{c}} = p_{\lambda,\mathfrak{c}}$ and

$$p_{\lambda,\mathfrak{c}}(\lambda) = \lim_{\mathfrak{c}}(\text{Todd}\,(X)_{pw}\,B_X)(\lambda).$$

Since $p_{\lambda,\mathfrak{c}}(\lambda) = 0$, when $\lambda \neq 0$ and $p_{\lambda,\mathfrak{c}}(0) = 1$, this proves our claim. □

2.27. Inversion formula: the general case

We keep the notation of § 1. G is a torus with a group of characters of Λ. For $g \in G$ and $\lambda \in \Lambda$, define

$$X^g := \{a \in X \mid g^a = 1\}, \quad G_\lambda := \{g \in G \mid g^\lambda = 1\}.$$

For each $a \in X$, the set G_a is a subgroup of codimension 1; these groups generate a *toric arrangement*, $\mathcal{A}_X$, formed by all connected components of the intersections of these groups G_a. Of particular importance are the vertices of the arrangement, which can also be described as follows.

Definition 2.28. We say that a point $g \in G$ is a *toric vertex* of the arrangement $\mathcal{A}_X$ if X^g generates V. We denote by $\mathcal{V}(X) \subset G$ the set of toric vertices of the arrangement $\mathcal{A}_X$.

If g is a vertex, there is a basis σ of V extracted from X such that $g^a = 1$, for all $a \in \sigma$. We thus see that the set $\mathcal{V}(X)$ is finite. We also see that, if X is unimodular, then $\mathcal{V}(X)$ is reduced to $g = 1$.

For $g \in G$, we think of $g^\lambda \in \mathcal{C}[\Lambda]$, and denote by $\hat{g}$ the operator on $\mathcal{C}[\Lambda]$, given by multiplication by g^λ: $(\hat{g}K)(\lambda) = g^\lambda K(\lambda)$. If $\nu \in \Lambda$, then $\hat{g} t_\nu \hat{g}^{-1} = g^\nu t_\nu$.

We introduce next the *twisted* difference and differential operators.

We set, for a vector a or for a sequence Y of elements of Λ,

$$\nabla_a^g := 1 - g^{-a} t_a, \quad \nabla(g, Y) = \prod_{a \in Y} \nabla_a^g, \tag{8}$$

$$D_a^g := 1 - g^{-a} e^{-\partial_a}, \quad D(g, Y) = \prod_{a \in Y} D_a^g. \tag{9}$$

The operator $\nabla(g, Y)$ acts on functions on Λ. One has the formula

$$\hat{g}^{-1} \nabla_Y \hat{g} = \nabla(g, Y). \tag{10}$$

The operator $\nabla(g, Y)$, being a linear combination of translation operators, also acts on piecewise polynomial functions on V. The operator $D(g, Y)$ acts on piecewise polynomial functions on V by its local action $D(g, Y)_{pw}$.

Be careful: The operators $D(g, Y)$ and $\nabla(g, Y)$ coincide on $\mathbb{C}[V]$, but their action is not the same on piecewise polynomial functions. Indeed, the operator $f \mapsto D(g, Y)_{pw} f$ respects the support of f, while the operator $\nabla(g, Y) f$ may move the support of f.

If $g^a \neq 1$, then $D_a^g = (1 - g^{-a}) + g^{-a}(1 - e^{-\partial_a})$ is an invertible operator on polynomial functions with inverse given by the series of differential operators

$$(D_a^g)^{-1} = (1 - g^{-a})^{-1} \sum_{k=0}^{\infty} (-1)^k \left(\frac{g^{-a}}{1 - g^{-a}} \right)^k (1 - e^{-\partial_a})^k.$$

If $Y \subset X \setminus X^g$, we also have $D(g, Y)^{-1} = \prod_{a \in Y} (D_a^g)^{-1}$, an infinite series of differential operators.

For $K \in \mathcal{C}[\Lambda]$ and $g \in \mathcal{V}(X)$, we define the function

$$\omega_g(K) := B_{X^g} *_d (\hat{g}^{-1} \nabla_{X \setminus X^g} K). \tag{11}$$

The function $\omega_g(K)$ is a piecewise polynomial function on V with respect to (X^g, Λ), thus a fortiori with respect to (X, Λ).

Theorem 2.29. *Let $\mathfrak{c}$ be an alcove in V containing 0 in its closure and contained in $Z(X)$. Then*

(i) $\delta_0 = \sum_{g \in \mathcal{V}(X)} \hat{g} \lim_{\mathfrak{c}} \left(D(g, X \setminus X^g)^{-1} \operatorname{Todd}(X^g) \right)_{pw} (\nabla(g, X \setminus X^g) B_{X^g})$.

(ii) *For any $K \in \mathcal{C}[\Lambda]$, one has the following inversion formula:*

$$K = \sum_{g \in \mathcal{V}(X)} \hat{g} \lim_{\mathfrak{c}} (D(g, X \setminus X^g)^{-1} \operatorname{Todd}(X^g))_{pw}\, \omega_g(K).$$

Remark 2.30. The first assertion is a particular case of the second. Indeed, for $K = \delta_0$, as $\hat{g}\delta_0 = \delta_0$, we have

$$\omega_g(K) = B_{X^g} *_d (\hat{g}^{-1} \nabla_{X \setminus X^g} \hat{g} \delta_0) = B_{X^g} *_d \nabla(g, X \setminus X^g) \delta_0;$$

thus

$$\omega_g(\delta_0) = \nabla(g, X \setminus X^g) B_{X^g}. \tag{12}$$

We will see later that the second assertion is a consequence of the first.

In order to prove Theorem 2.29, we shall follow essentially the same method of proof used in the unimodular case.

Notice that the restriction K of a function $k \in D(X)$ to Λ is an element of $DM(X)$. Recall the following structure theorem.

Theorem 2.31 (See [14] Formula 16.1 and Theorem 17.15).

(1) *If $g \in \mathcal{V}(X)$ is a toric vertex of the arrangement X and $k \in D(X^g)$, the function $\hat{g}K$ belongs to $DM(X)$.*

(2) *We have $DM\,(X)_{\mathbb{C}} = \bigoplus_{g\in\mathcal{V}(X)} \hat{g}D\,(X^g)_{\mathbb{C}}$.*

(3) *Let $E(X) = \bigoplus_{g\neq 1} \hat{g}D(X^g)$. Then, for any $K \in E(X)$, $B_X *_d K = 0$.*

Given an element $g \in \mathcal{V}(X)$, recall that the map $\nabla_{X\setminus X^g}$ sends $DM(X)$ to $DM(X^g)$. We have the following lemma.

Lemma 2.32. *Take $g, h \in \mathcal{V}(X)$.*

(i) *If $h \in \mathcal{V}(X^g)$, then*

$$\nabla_{X\setminus X^g} \hat{h} D(X^h) \subset \hat{h} D(X^h \cap X^g). \tag{13}$$

(ii) *If $h \notin \mathcal{V}(X^g)$, then*

$$\nabla_{X\setminus X^g} \hat{h} D(X^h) = 0.$$

Proof. Let K be a function on Λ. An operator ∇_Y acting on $\hat{h}K$ can be analyzed by decomposing $Y = Z \cup R$ into the part Z of elements $a \in Y$ such that $h^a = 1$ and the complement R. Then

$$\nabla_Y \hat{h} K = \hat{h}(\hat{h}^{-1} \nabla_Z \hat{h})(\hat{h}^{-1} \nabla_R \hat{h}) K = \hat{h} \nabla_Z \nabla(h, R) K. \tag{14}$$

In particular, we apply this to $Y = X \setminus X^g$, which, for a point h of the arrangement, we separate into the subsequences $Z := X^h \cap (X \setminus X^g) = X^h \setminus X^g$ and $R := X \setminus (X^h \cup X^g)$. We obtain, from (14),

$$\nabla_{X\setminus X^g} \hat{h} D(X^h) = \hat{h} \nabla_{X^h\setminus X^g} \nabla(h, R) D(X^h).$$

The operator $\nabla(h, R)$, a finite combination of translations, preserves the space $D(X^h)$. Thus, we get

$$\nabla_{X\setminus X^g} \hat{h} D(X^h) \subset \hat{h} \nabla_{X^h\setminus X^g} D(X^h) = \hat{h} \nabla_{X^h\setminus X^g} D(X^h), \tag{15}$$

and (i) follows from Lemma 2.20.

Furthermore, by definition, a point h is a vertex of the arrangement X^g if and only if the vectors in $X^h \cap X^g$ span V. So, if $h \notin \mathcal{V}(X^g)$, $X^h \setminus X^g$ is a long subsequence of X^h and $\nabla_{X^h\setminus X^g} D(X^h) = 0$, showing (ii). $\square$

Proposition 2.33. *If* $K \in DM(X)$, *then*

$$\omega_g(K) = B_{X^g} *_d (\hat{g}^{-1} \nabla_{X \setminus X^g} K) \in D(X^g).$$

Proof. Indeed, $\nabla_{X \setminus X^g} K \in DM(X^g)$, and $\hat{g}^{-1}$ preserves $DM(X^g)$. Thus $\omega_g(K)$ is a polynomial belonging to $D(X^g)$ by Lemma 2.22. □

Proposition 2.34. *Let* $K \in DM(X)$. *Write* $K = \sum_{g \in \mathcal{V}(X)} \hat{g} K_g$, *with* $k_g \in D(X^g)_{\mathbb{C}}$ *restricting to* K_g. *Then we have*

$$k_g = D(g, X \setminus X^g)^{-1} \operatorname{Todd}(X^g) \omega_g(K).$$

Proof. Let $k_h \in D(X^h)_{\mathbb{C}}$. Let us compute $\omega_g(\hat{h} K_h)$ for each $g \in \mathcal{V}(X)$.

By Lemma 2.32, $\nabla_{X \setminus X^g} \hat{h} K_h$ is zero unless h is a vertex of X^g.

Assume now that h is a vertex of X^g. Then $\nabla_{X \setminus X^g} \hat{h} K_h = \hat{h} Z$, where Z is the restriction of a polynomial z lying in $D(X^g \cap X^h)_{\mathbb{C}}$.

Clearly, $g^{-1}h$ is also a vertex of X^g, and $X^g \cap X^{g^{-1}h} = X^g \cap X^h$. We deduce using Lemma 2.32(i) that, if $g \neq h$,

$$\hat{g}^{-1} \nabla_{X \setminus X^g} \hat{h} K_h \in \widehat{g^{-1}h} D(X^g \cap X^{g^{-1}h})_{\mathbb{C}} \subset E(X^g).$$

So, by Theorem 2.31,

$$\omega_g(\hat{h} K_h) = B_{X^g} *_d (\hat{g}^{-1} \nabla_{X \setminus X^g} \hat{h} K_h) = 0.$$

Finally, if $h = g$, we obtain that $\hat{g}^{-1} \nabla_{X \setminus X^g} \hat{g} K_g$ is the restriction to Λ of the polynomial $D(g, X \setminus X^g) k_g \in D(X^g)_{\mathbb{C}}$. By Theorem 2.23, the semi-discrete convolution acts by the operator $I(X^g)$ on $D(X^g)$, so that we get

$$\omega_g(\hat{g} K_g) = D(g, X \setminus X^g) I(X^g) k_g.$$

In conclusion,

$$\omega_g(\hat{h} K_h) = \begin{cases} 0 & \text{if } h \neq g \\ D(g, X \setminus X^g) I(X^g) k_g & \text{if } h = g. \end{cases}$$

This implies our claims. □

We are now ready to prove our main theorem, Theorem 2.29.

We compute the function j on Λ given by

$$j = \sum_{g \in \mathcal{V}(X)} \hat{g} \lim_{\mathfrak{c}} (D(g, X \setminus X^g)^{-1} \operatorname{Todd}(X^g))_{pw} (\nabla(g, X \setminus X^g) B_{X^g}).$$

We proceed as in the proof of Theorem 2.15. The support of $\nabla(g, X \setminus X^g) B_{X^g}$ is contained in $Z(X)$. Indeed, if I is a subsequence of $X \setminus X^g$, $\sum_{i \in I} a_i + Z(X^g) \subset Z(X)$. So, if $\lambda \in \Lambda$ and $(\lambda + \mathfrak{c}) \cap Z(X)$ is empty, we see that $j(\lambda) = 0$.

Now, assume that $(\lambda + \mathfrak{c}) \cap Z(X)$ is not empty. Then, the points 0 and λ belong to $\delta(\lambda + \mathfrak{c} \,|\, X)$. Let $p_{\lambda,\mathfrak{c}}$ be the element of $DM(X)$ which coincides with δ_0 on $\delta(\lambda + \mathfrak{c} \,|\, X)$. Let us show that $j(\lambda) = p_{\lambda,\mathfrak{c}}(\lambda)$, so that we will obtain (i).

We decompose $p_{\lambda,\mathfrak{c}}$ according to Proposition 2.34. It is sufficient to prove that the polynomial $\omega_g(p_{\lambda,\mathfrak{c}})(v)$ coincides with the piecewise polynomial function $\nabla(g, X \setminus X^g)B_{X^g}$ on $\lambda + \mathfrak{c}$. That is, if $v \in \lambda + \mathfrak{c}$,

$$\omega_g(p_{\lambda,\mathfrak{c}})(v) = (\nabla(g, X \setminus X^g)B_{X^g})(v). \tag{16}$$

Indeed, by Proposition 2.34, we have

$$\omega_g(p_{\lambda,\mathfrak{c}}) = B_{X^g} *_d \hat{g}^{-1}\nabla(X \setminus X^g)p_{\lambda,\mathfrak{c}} = B_{X^g} *_d \nabla(g, X \setminus X^g)\hat{g}^{-1}p_{\lambda,\mathfrak{c}},$$

so, since semi-discrete convolution commutes with translation,

$$\omega_g(p_{\lambda,\mathfrak{c}}) = \nabla(g, X \setminus X^g)(B_{X^g} *_d \hat{g}^{-1}p_{\lambda,\mathfrak{c}}).$$

For any subsequence I of $X \setminus X^g$, set $a_I = \sum_{i \in I} a_i$. In order to see (16), we need to show that, for $v \in \lambda + \mathfrak{c}$,

$$B_{X^g}(v - a_I) = (B_{X^g} *_d \hat{g}^{-1}p_{\lambda,\mathfrak{c}})(v - a_I). \tag{17}$$

By definition, the right hand side of this expression equals

$$\sum_{\eta \in \Lambda} g^{-\eta} p_{\lambda,\mathfrak{c}}(\eta) B_{X^g}(v - a_I - \eta).$$

Now the summand $B_{X^g}(v - a_I - \eta)$ is zero except if $v - a_I - \eta$ is in the zonotope $Z(X^g)$. But if this is the case, $v - \eta \in Z(X^g) + a_I \subset Z(X)$, so necessarily η lies in $\delta(\lambda + \mathfrak{c} \,|\, X)$.

Let us prove the second item. Define

$$\begin{aligned} j(K) &= \sum_{g \in \mathcal{V}(X)} \hat{g} \lim_{\mathfrak{c}} \left(D\,(g, X \setminus X^g)^{-1}\,\mathrm{Todd}(X^g)\right)_{pw} B_{X^g} * \hat{g}^{-1}\nabla_{X \setminus X^g} K \\ &= \sum_{g \in \mathcal{V}(X)} \hat{g} \lim_{\mathfrak{c}} \left(D\,(g, X \setminus X^g)^{-1}\,\mathrm{Todd}(X^g)\right)_{pw} B_{X^g} * \nabla(g, X \setminus X^g)\hat{g}^{-1}K. \end{aligned}$$

Then $K \to j(K)$ is an operator of the form $\sum_g \hat{g} R_g \hat{g}^{-1} K$, where R_g is an operator commuting with translations by elements of Λ. Thus the operator $K \to j(K)$ commutes with translation. Furthermore, it is clear that the formula for $j(K)(\lambda)$ involves only a finite number of values of $K(v)$ (contained in $\lambda - Z(X)$). Thus, to prove that $j(K) = K$, it is sufficient to prove it for K with finite support. By translation invariance, this case follows from the formula for δ_0.

3. Partition functions and splines

3.1. The formula of Brion and Vergne

If Y is a sequence of elements of Λ generating a pointed cone $\mathrm{Cone}(Y)$, then we can define the series

$$\Theta_Y = \prod_{a \in Y} \sum_{k=0}^{\infty} e^{ka}.$$

We write

$$\Theta_Y = \sum_{\lambda \in \Lambda} \mathcal{P}_Y(\lambda) e^{\lambda},$$

where $\mathcal{P}_Y \in \mathcal{C}_{\mathbb{Z}}[\Lambda]$ is, by definition, the *partition function* associated to Y. For any subsequence S of Y, we then have

$$\nabla_S \mathcal{P}_Y = \mathcal{P}_{Y \setminus S}. \tag{18}$$

In particular, $\nabla_Y \mathcal{P}_Y = \delta_0$, the delta function on Λ.

Similarly, we can define the multivariate spline T_Y, which is the tempered distribution on V, by

$$\langle T_Y \,|\, f \rangle = \int_0^\infty \dots \int_0^\infty f\left(\sum_{i=1}^k t_i a_i\right) dt_1 \dots dt_k, \tag{19}$$

where $Y = [a_1, a_2, \dots, a_k]$.

For any subsequence S of Y, we then have

$$\partial_S T_Y = T_{Y \setminus S}. \tag{20}$$

In particular, $\partial_Y T_Y = \delta_0$, the δ-distribution on V.

Decomposing a ray as a sum of intervals, the following formula of Dahmen and Micchelli follows.

Proposition 3.2.

$$T_Y(v) = (B_Y *_d \mathcal{P}_Y)(v) = \sum_{\lambda \in \Lambda} B_Y(v - \lambda)\mathcal{P}_Y(\lambda).$$

Let us see that the formulae obtained in [9] are a corollary of the general inversion formula of Theorem 2.29. We assume that X generates the vector space V and spans a pointed cone (thus with non-empty interior). It follows that T_X is a piecewise polynomial distribution on V.

Let us apply the inversion formula of Theorem 2.29 to the partition function $\mathcal{P}_X$. From the equations

$$\hat{g}^{-1} \nabla_{X \setminus X^g} \mathcal{P}_X = \hat{g}^{-1} \mathcal{P}_{X^g} = \mathcal{P}_{X^g}, \quad B_{X^g} *_d \mathcal{P}_{X^g} = T_{X^g},$$

we deduce the following from Theorem 2.29.

Theorem 3.3. *Let $\mathfrak{c}$ be an alcove contained in* Cone(X) *and having 0 in its closure. Then $\mathcal{P}_X$ coincides with*

$$\sum_{g \in \mathcal{V}(X)} \hat{g} \lim_{\mathfrak{c}} \left(\prod_{b \in X^g} \frac{\partial_b}{1 - e^{-\partial_b}} \prod_{a \in X \setminus X^g} \frac{1}{1 - g^{-a} e^{-\partial_a}} T_{X^g} \right). \tag{21}$$

More generally, given a sequence of non-zero vectors Y in Λ, where we do not necessarily assume that Y spans a pointed cone, we can define *polarized partition functions* as follows. Consider the open subset $\{u \in V^* \mid \langle u, a \rangle \neq 0 \text{ for all } a \in Y\}$ of V^*. A connected component F of this open set will be called a *regular face* for Y. An element $\phi \in F$ decomposes $Y = A \cup B$, where ϕ is positive on A and negative on B. This decomposition depends only upon F. We define

$$\Theta_Y^F = (-1)^{|B|} \prod_{a \in A} \sum_{k=0}^{\infty} e^{ka} \prod_{b \in B} \sum_{k=1}^{\infty} e^{-kb}.$$

Thus $\Theta_Y^F = (-1)^{|B|} e^{-\sum_{b \in B} b} \Theta_{A \cup -B}$.

The idea behind the denition of Θ_Y^F is clear. Write

$$\prod_{y \in Y} \frac{1}{1-e^y} = \prod_{a \in A} \frac{1}{1-e^a} \prod_{b \in B} \frac{1}{1-e^b} = \prod_{a \in A} \frac{1}{1-e^a} \prod_{b \in B} \frac{-e^{-b}}{1-e^{-b}}$$

and then expand in geometric series. We needed to reverse the sign of the vectors in B in order to insure that the product Θ_Y^F of the corresponding geometric series makes sense.

We write

$$\Theta_Y^F = \sum_{\lambda \in \Lambda} \mathcal{P}_Y^F(\lambda) e^{\lambda},$$

where $\mathcal{P}_Y^F \in \mathcal{C}[\Lambda]$ is the *polarized (by F) partition function*. The polarized partition function is a $\mathbb{Z}$-valued function on Λ.

We define similarly

$$T_Y^F = (-1)^{|B|} T_{A,-B}$$

a distribution on V, and call it the polarized multispline function.

As for Proposition 3.2, it is easy to verify the following proposition.

Proposition 3.4. *For any regular face F for Y, one has $B_Y *_d \mathcal{P}_Y^F = T_Y^F$.*

Proposition 3.4 implies that $B_X = \nabla_X T_X^F$. Thus B_X is a linear combination of translates of multisplines T_X^F.

3.5. The spaces $\mathcal{F}(X)$ and $\mathcal{G}(X)$

In this subsection, we recall the definitions of the subspaces $\mathcal{F}(X)$, a subspace of functions on Λ, and $\mathcal{G}(X)$, a subspace of distributions on V introduced in [15]. They will be central objects in Part 2 as these spaces are related to the equivariant K-theory and cohomology of some G-spaces.

We use the convolution sign $*$ for convolutions between functions on Λ, or distributions on V, or semi-discrete convolution between a function on Λ and distributions on V. The meaning will be clear in the context.

Definition 3.6. A subspace $\underline{s}$ of V is called *rational* (relative to X) if $\underline{s}$ is the vector space generated by $X \cap \underline{s}$.

We shall denote by $\mathcal{S}_X$ the set of rational subspaces.

Denote by $\mathcal{C}_{\mathbb{Z}}[\Lambda]$ the $\mathbb{Z}$-module of $\mathbb{Z}$-valued functions on Λ. Define the following subspace of $\mathcal{C}_{\mathbb{Z}}[\Lambda]$.

Definition 3.7.

$$\mathcal{F}(X) := \{f : \Lambda \to \mathbb{Z} \mid \nabla_{X \setminus \underline{r}} f \text{ is supported on } \Lambda \cap \underline{r} \text{ for all } \underline{r} \in \mathcal{S}_X\}.$$

We set $\tilde{\mathcal{F}}(X)$ to be the $\mathbb{Z}[\Lambda]$ module generated by $\mathcal{F}(X)$.

The space $\mathcal{F}(X)$ clearly contains the space of Dahmen–Micchelli quasi-polynomials $DM(X)$ and all polarized partition functions $\mathcal{P}_X^F$.

Remark 3.8. Assume that X is unimodular. Let τ be a tope. We have proven in [15] that a function f in $\mathcal{F}(X)$ coincides on $(\tau - Z(X)) \cap \Lambda$ with the restriction of a polynomial function h^τ. Thus we see that this collection of functions $h^\tau|_\tau$ extends to a continuous function on the cone $\mathrm{Cone}(X)$ generated by X.

We have a precise description of $\mathcal{F}(X)$ and $\tilde{\mathcal{F}}(X)$ in Theorem 4.5 of [15].

Theorem 3.9. • *Choose, for every rational space $\underline{r}$, a regular face $F_{\underline{r}}$ for $X \setminus \underline{r}$. Then*

$$\mathcal{F}(X) = \bigoplus_{\underline{r} \in S_X} \mathcal{P}_{X \setminus \underline{r}}^{F_{\underline{r}}} * DM(X \cap \underline{r}). \tag{22}$$

• *$\tilde{\mathcal{F}}(X)$ is spanned over $\mathbb{Z}[\Lambda]$ by the elements $\mathcal{P}_X^F$ as F runs over all regular faces for X.*

Thus any Dahmen–Micchelli quasi-polynomial p in $DM(X)$ can be written as a linear combination of polarized partition functions $\mathcal{P}_X^F$, for various F. It is not so easy to do this explicitly.

The easiest instance of this result is when $\Lambda = \mathbb{Z}$. In this case, one has only two rational spaces: $\mathbb{R}$ and $\{0\}$. Thus, our results says that, if we take as $F_{\{0\}}$ the positive half-line $\mathbb{R}^{\geqslant 0}$, every element in $\mathcal{F}(X)$ can be written uniquely as the sum of a quasi-polynomial in $DM(X)$ and a multiple of the partition function $\mathcal{P}_X^{\mathbb{R}^{\geqslant 0}}$.

In general, the decomposition (22) is not so easy to compute explicitly.

We defined in [15] an analogue space of piecewise polynomial distributions on V.

Denote by $\mathcal{D}'(V)$ the space of distributions on V. Let $\underline{r}$ be a vector subspace in V. We have an embedding $j : \mathcal{D}'(\underline{r}) \to \mathcal{D}'(V)$ by $\langle j(\theta), f\rangle = \langle \theta, f|\underline{r}\rangle$ for any $\theta \in \mathcal{D}'(\underline{r})$, f a test function on V. We denote the image $j(\mathcal{D}'(\underline{r}))$ by $\mathcal{D}'(V, \underline{r})$ (sometimes we even identify $\mathcal{D}'(\underline{r})$ with $\mathcal{D}'(V, \underline{r})$ if there is no ambiguity). We next define the vector space.

Definition 3.10.

$$\mathcal{G}(X) := \{f \in \mathcal{D}'(V) \mid \partial_{X \setminus \underline{r}} f \in \mathcal{D}'(V, \underline{r}), \text{ for all } \underline{r} \in \mathcal{S}_X\}.$$

We set $\tilde{\mathcal{G}}(X)$ to be the module generated by $\mathcal{G}(X)$ under the action of the algebra $S[V]$ of differential operators with constant coefficients.

It is clear that $\mathcal{G}(X)$ contains the space $D(X)$ of Dahmen–Micchelli polynomials (we identify freely a locally L^1-function p and the distribution $p(v)dv$ using our choice of Lebesgue measure) as well as the polarized multisplines T_X^F.

Theorem 3.11. • *Choose, for every rational space $\underline{r}$, a regular face $F_{\underline{r}}$ for $X \setminus \underline{r}$. Then*

$$\mathcal{G}(X) = \bigoplus_{\underline{r} \in S_X} T_{X \setminus r}^{F_{\underline{r}}} * D(X \cap \underline{r}). \tag{23}$$

- *The space $\tilde{\mathcal{G}}(X)$ is generated as an $S[V]$ module by the distributions T_X^F as F runs over all regular faces for X.*

It follows from this theorem that any θ in $\mathcal{G}(X)$ is a piecewise polynomial distribution on V.

3.12. An isomorphism

We want to show now the strict relationship between the two spaces $\mathcal{F}(X)$ and $\mathcal{G}(X)$. We may use real valued functions $DM(X)_{\mathbb{R}}$ and $\mathcal{F}(X)_{\mathbb{R}} := \mathcal{F}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ defined by the same difference equations.

The spaces $\mathcal{F}(X)$ and $\mathcal{G}(X)$ are related by semi-discrete convolution with the box spline B_X. Indeed, the following lemma generalizes the fact that a Dahmen–Micchelli quasi-polynomial becomes a polynomial in $D(X)$ by semi-discrete convolution.

Lemma 3.13. *If $f \in \mathcal{F}(X)$, then $f * B_X \in \mathcal{G}(X)$.*

Proof. If $f \in \mathcal{F}(X)$, then $\nabla_{X \setminus \underline{r}} f$ is supported on $\underline{r}$ for every rational subspace $\underline{r}$. We need to show that $\partial_{X \setminus \underline{r}} f * B_X \in \mathcal{D}(\underline{r})$ for every rational subspace $\underline{r}$.

We have, from formula (7),

$$\partial_{X \setminus \underline{r}} f * B_X = f * \partial_{X \setminus \underline{r}} B_X = f * \nabla_{X \setminus \underline{r}} B_{X \cap \underline{r}} = (\nabla_{X \setminus \underline{r}} f) * B_{X \cap \underline{r}}.$$

Since $\nabla_{X \setminus \underline{r}} f$ is supported on $\Lambda \cap \underline{r}$, we have that $(\nabla_{X \setminus \underline{r}} f) * B_{X \cap \underline{r}} \in \mathcal{D}(\underline{r})$, as desired. □

Theorem 3.14. *The map $f \mapsto f * B_X$ induces a surjective map*

$$i : \mathcal{F}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathcal{G}(X),$$

compatible with the two decompositions (22) and (23).

If X is unimodular, i is a linear isomorphism.

Proof. We have

$$\mathcal{G}(X) = \bigoplus_{\underline{r} \in S_X} T_{X \setminus r}^{F_{\underline{r}}} * D(X \cap \underline{r}),$$

while

$$\mathcal{F}(X) = \bigoplus_{\underline{r} \in S_X} \mathcal{P}_{X \setminus \underline{r}}^{F_{\underline{r}}} * DM(X \cap \underline{r}).$$

Under the mapping $f \to f * B_X$, we have that $DM(X) \otimes_{\mathbb{Z}} \mathbb{R}$ maps surjectively to $D(X)$. Furthermore, in the unimodular case, it induces a linear isomorphism onto $D(X)$.

Then, consider an element $\mathcal{P}_{X \setminus \underline{r}}^{F_{\underline{r}}} * u,\ u \in DM(X \cap \underline{r})$. We have

$$\mathcal{P}_{X \setminus \underline{r}}^{F_{\underline{r}}} * u * B_X = \mathcal{P}_{X \setminus \underline{r}}^{F_{\underline{r}}} * u * B_{X \setminus \underline{r}} * B_{X \cap \underline{r}} = T_{X \setminus \underline{r}}^{F_{\underline{r}}} * (B_{X \cap \underline{r}} * u),$$

and the claim follows. □

Remark 3.15. If X is unimodular, the inverse of i is given by the following deconvolution formula:

$$i^{-1}h = \lim_{\mathfrak{c}} \mathrm{Todd}(X)*_{pw}h,$$

where $\mathfrak{c}$ is an alcove contained in $Z(X)$.

Consider a function $h \in \mathcal{G}(X)$. Then the locally polynomial function $\mathrm{Todd}(X)*_{pw}h$ coincides on topes $\tau \cap \Lambda$ with the function g^{τ}, with $g = i^{-1}(h) \in \mathcal{F}(X)$. It follows from the continuity properties of elements of $\mathcal{F}(X)$ that the locally polynomial function $\mathrm{Todd}(X)*_{pw}h$ extends continuously on the cone generated by X. In particular, if $Z(X)$ contains 0 in its interior, $\mathrm{Todd}(X)*_{pw}h$ extends continuously on V.

The box spline B_X is a combination of translates of elements T_X^F which belongs to $\mathcal{G}(X)$. It follows that Todd $(X)_{pw}\, B_X$ extends continuously on V.

Part 2. Geometry

4. Equivariant K-theory and equivariant cohomology

4.1. Preliminaries

Although the theory we are going to review exists for a general compact Lie group G, we restrict our treatment to the case in which G is a compact torus, and denote by Λ its character group.

We want to apply the preceding purely algebraic results in order to compare the infinitesimal index, and the index associated to the symbol of a transversally elliptic operator on a linear representation of G.

Let N be a G manifold provided with a real G-invariant form σ. We assume that N is oriented. If v_x is the vector field on N associated to $x \in \mathfrak{g}$, the moment map $\mu : N \to \mathfrak{g}^*$ is defined by $\langle \mu(n), x\rangle = -\langle \sigma, v_x\rangle$, ($n \in N, x \in \mathfrak{g}$, and the sign convention is that $v_x = \frac{d}{d\epsilon}\exp(-\epsilon x)n|_{\epsilon=0}$). Define Z as the zero fiber of the moment map.

Remark 4.2. We will mainly apply the construction described below to the case where $N = T^*M$ is the cotangent bundle to a G-manifold M, ω is the Liouville 1-form defined for $m \in M$, $\xi \in T_m^*M$, and V is a tangent vector at the point $(m, \xi) \in T^*M$, by

$$\omega_{m,\xi}(V) = \langle \xi, p_*V\rangle. \tag{24}$$

By definition, the symplectic form $\Omega = -d\omega$ is the symplectic form of T^*M, and we will use the corresponding orientation of T^*M to compute integrals of differential forms on T^*M.

If v_x is the vector field on M associated to $x \in \mathfrak{g}$, the moment map $\mu : T^*M \to \mathfrak{g}^*$ is then $\langle \mu(m, \xi), x\rangle = -\langle \xi, v_x\rangle$, and the zero fiber Z of the moment map is denoted by T_G^*M. In a point $m \in M$, the fiber of the projection $p : T_G^*M \to M$ is the space of covectors conormal to the G orbit through m.

For reasons explained later, we will use the opposite form $-\omega$, the opposite moment map $-\mu$, and, as stated before, the orientation given by $-d\omega$.

4.3. Transversally elliptic symbols and their index

Let Z be a topological space provided with an action of G. Let $\mathcal{E}^+$ and $\mathcal{E}^-$ be two complex equivariant vector bundles over Z. Let $\Sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a G-equivariant morphism, and, for $z \in Z$, denote by $\Sigma_z : \mathcal{E}_z^+ \to \mathcal{E}_z^-$ the corresponding linear map. Recall that *the support of* Σ is the subset of Z consisting of elements z, where Σ_z is not invertible.

Let us now recall the definition of the multiplicity index map.

Let $\mathcal{E}^+$ and $\mathcal{E}^-$ be two complex equivariant vector bundles over M. If $\Sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ is a G-equivariant morphism, we say that Σ is a G-equivariant symbol. Thus, for $m \in M$, $\xi \in T_m^*M$, $\Sigma_{m,\xi}$ is a linear map: $\mathcal{E}_m^+ \to \mathcal{E}_m^-$. If the support of Σ is a compact set, we say that Σ is an elliptic symbol, and Σ determines an element $[\Sigma]$ in $K_G^0(T^*M)$. If the support of Σ intersects T_G^*M in a compact set, we say that Σ is a transversally elliptic symbol (it is elliptic in the directions transverse to the G-orbits). Then Σ determines an element $[\Sigma]$ in $K_G^0(T_G^*M)$, and all elements of this group are obtained in this way (see [1]).

Recall that Atiyah and Singer [1] have associated to any transversally elliptic symbol a virtual representation of G of trace class. The induced trace of operators associated to smooth functions on G is its index, a generalized function on G. By taking Fourier coefficients, one then gets a homomorphism of $R(G) = \mathbb{Z}[\Lambda]$ modules,

$$\mathrm{ind}_m : K_G^0(T_G^*M) \to \mathcal{C}_{\mathbb{Z}}[\Lambda],$$

called the index multiplicity function. When the symbol Σ is elliptic, one gets that $\mathrm{ind}_m(\Sigma)$ has finite support, and the index is a virtual character of G.

In [16], we have studied in particular the case $M = M_X$, the linear representation associated to a list X of characters, and proved that $\mathrm{ind}_m : K_G^0(T_G^*M_X) \to \mathcal{C}_{\mathbb{Z}}[\Lambda]$ gives an isomorphism of $\mathbb{Z}[\Lambda]$-modules onto $\tilde{\mathcal{F}}(X)$ (see Definition 3.7). Moreover, if M_X^f denotes the open set of points with finite stabilizer, we have that the map ind_m establishes an isomorphism of $K_G^0(T_G^*M_X^f)$ with $DM(X)$.

Since, as we have recalled in Theorem 3.9, $\tilde{\mathcal{F}}(X)$ is spanned over $\mathbb{Z}[\Lambda]$ by the elements $\mathcal{P}_X^F$ as F runs over all regular faces for X, in order to find generators of $K_G^0(T_G^*M_X)$, we are going to construct certain symbols whose index multiplicity gives $\mathcal{P}_X^F$.

4.4. Some explicit computations for K-theory

Let Y be a sequence of vectors in Λ and $M_Y = \bigoplus_{a\in Y} L_a$ the corresponding complex G-representation space. We write $z = \oplus z_a$, an element of M_Y, with $z_a \in L_a$.

We choose a G-invariant Hermitian structure $\langle , \rangle$ on M_Y.

We first recall here the description of the generator $\mathrm{Bott}(M_Y)$ of $K_G^0(M_Y)$, a free module of rank 1 over $R(G)$. Let $E := \bigwedge M_Y$ with the Hermitian structure induced by that of M_Y, graded as $E^+ \oplus E^-$ by even and odd degree. Then, for $z \in M_Y$, consider the exterior multiplication $m(z) : E \to E$, $m(z)(\omega) := z \wedge \omega$, and the Clifford action,

$$c(z) = m(z) - m(z)^*, \tag{25}$$

of M_Y on E. One has $c(z)^2 = -\|z\|^2$, so that $c(z)$ is an isomorphism, if $z \neq 0$, exchanging the summands E^+ and E^-. Consider the complex G-equivariant vector bundles $\mathcal{E}^\pm = M_Y \times E^\pm$. The G-equivariant morphism from $\mathcal{E}^+$ to $\mathcal{E}^-$, defined by $\Sigma_z(\omega) = c(z)\omega$, is supported at 0, and thus defines an element $\mathrm{Bott}(M_Y)$ of $K_G^0(M_Y)$, a generator of $K_G^0(M_Y)$ over $R(G) = \mathbb{Z}[\Lambda]$.

Notice that, if $Y = Y_1 \cup Y_2$, $\bigwedge M_Y = \bigwedge M_{Y_1} \otimes \bigwedge M_{Y_2}$, and $\mathrm{Bott}(M_Y)$ is the external tensor product of the symbols $\mathrm{Bott}(M_{Y_1})$ and $\mathrm{Bott}(M_{Y_2})$.

Definition 4.5. Given a G-invariant Hermitian structure $\langle , \rangle$ on M_Y, we define the G-invariant real 1-form

$$\sigma_Y = -\frac{1}{2}\Im\langle z, dz\rangle.$$

Here, $\Im : \mathbb{C} \to \mathbb{R}$ is the imaginary part.

For example, if $M = L_a \sim \mathbb{R}^2$ and $z = v_1 + iv_2$, then $\sigma = \frac{1}{2}(v_1 dv_2 - v_2 dv_1)$.

We consider the moment map μ_Y for σ_Y. Thus

$$\mu_Y(z) = \frac{1}{2}\sum_{a \in Y} |z_a|^2\, a.$$

Definition 4.6. We define Z_Y to be the zero fiber of the moment map μ_Y:

$$Z_Y := \left\{ z \in M_Y \mid \sum_{a \in Y} |z_a|^2\, a = 0 \right\}.$$

This set Z_Y was also denoted M_Y^0 in [18]. However, as when $Y = X \cup -X$, we will use different moment maps on M_Y. We keep the notation Z_Y for the set of zeros of the moment map μ_Y defined above.

The following construction of some elements of $K_G^0(Z_Y)$ is due to Boutet de Monvel.

Of course the Bott element $\mathrm{Bott}(M_Y)$ restricts to Z_Y as an element of $K_G^0(Z_Y)$. Let us construct some genuine elements not coming from the space $K_G^0(M_Y)$.

We start with a simple case. Assume first that there is an element $\phi \in \mathfrak{g}$ which is strictly positive on all the characters $a \in Y$. It follows that $Z_Y = \{0\}$, and the equivariant K-theory of Z_Y is the $\mathbb{Z}[\Lambda]$-module generated by the class of the trivial vector bundle $\mathbb{C}$ over the point Z_Y (which is in fact $\mathrm{Bott}(\{0\})$).

We come now to the case of an arbitrary sequence Y of weights. Let F be a regular face for Y. We take a linear form $\phi \in F$, which is non-zero on each element of Y. We write $Y = A \cup B$, A being the subsequence of elements on which ϕ takes positive values, and B being the subsequence of elements on which ϕ takes negative values (notice that A and B depend only on F and not on the choice of ϕ). Accordingly, we write $M_Y = M_A \oplus M_B$. Thus every $z \in M_Y$ can be uniquely decomposed as $z = z_A \oplus z_B$, with $z_A \in M_A$, $z_B \in M_B$.

Let $E_A = \bigwedge M_A$ graded as $E_A^+ \oplus E_A^-$ taking the odd and even degree parts.

Definition 4.7. The morphism Σ^F between the trivial bundles $M_Y \times E_A^+$ and $M_Y \times E_A^-$ is defined by

$$\Sigma_z^F = c(z_A) \quad \forall z \in M_Y.$$

It is clear that the support of Σ^F is the subspace M_B. We deduce the following lemma.

Lemma 4.8. *The intersection of the support of Σ^F with Z_Y reduces to the zero vector.*

Proof. Indeed, in Z_Y, $\sum_{a\in A} |z_a|^2 a = -\sum_{b\in B} |z_b|^2 b$. If we are in the support of Σ^F, each $z_a = 0$, so we deduce that $-\sum_{b\in B} |z_b|^2 b = 0$. Since ϕ takes a negative value on each $b \in B$, this implies that $z_b = 0$ for all b. □

We deduce that the restriction of Σ^F to Z_Y defines an element of $K_G^0(Z_Y)$ still denoted by Σ^F.

Let us now consider the case $Y = X \cup -X$. In this case, $M_Y = M_X \oplus M_X^* = T^*M_X$, so that $\mathrm{Bott}(T^*M_X)$ gives a class in $K^0(T^*M_X)$, whose index is the trivial representation of G (see [3]).

We may restrict $\mathrm{Bott}(T^*M_X)$ to $T_G^*M_X$, getting a class whose index multiplicity is the delta function δ_0 on Λ.

In order to get further elements of $K_G^0(T_G^*M_X)$, we follow the construction of Boutet de Monvel, as follows.

Consider the $\mathbb{R}$-linear G-isomorphism h of M_X^*, with M_X given by

$$\xi(z) := \Re(\langle z, h(\xi)\rangle).$$

Here, $\Re(\langle z_1, z_2\rangle)$ is the real part of the Hermitian product on M_X, a positive definite inner product. The isomorphism h induces a $\mathbb{R}$-linear G-isomorphism, still denoted by h, of T^*M_X, with $M_X \oplus M_X$.

Let F be a regular face of the arrangement X (and hence also of $Y = X \cup -X$). Let $\phi \in F$ so that $X = A \cup B$, with ϕ positive in A and negative on B. We denote by J the standard complex structure on M_X, and by J_F the complex structure on M_X defined as J_F is J on M_A and $-J$ on M_B. Then the list of weights of G for this new complex structure on M_X is $A \cup -B$.

We consider the associated 1-form σ^F on M, which has moment map

$$\nu_F(z) = \frac{1}{2}\left(\sum_{a\in A} |z_a|^2 a - \sum_{b\in B} |z_b|^2 b\right).$$

Clearly, the zero fiber is reduced to $\{0\}$.

Lemma 4.9. *Consider the isomorphism of T^*M_X with $M_X \oplus M_X$, given by*

$$(z, \xi) \to [h(\xi) + J_F z, h(\xi) - J_F z].$$

In this isomorphism, the moment map μ on T^*M_X associated to the Liouville form becomes the moment map $[\frac{1}{2}\nu_F, -\frac{1}{2}\nu_F]$ for $[\frac{1}{2}\sigma_F, -\frac{1}{2}\sigma_F]$.

In particular, under the above isomorphism, the space $T_G^*M_X$ is identified with the zeros of the moment map $[\frac{1}{2}\nu_F, -\frac{1}{2}\nu_F]$.

We define the map

$$p_F : T^*M_X \to M_X \tag{26}$$

by $p_F(z, \xi) = h(\xi) + J_F z$.

Definition 4.10. We set

$$\Sigma^F(z, \xi) = c(h(\xi) + J_F z).$$

In other words, $\Sigma^F = p_F^* \mathrm{Bott}(M_X)$ is the pull-back of the morphism $\mathrm{Bott}(M_X)$ by p_F.

By Lemmas 4.9 and 4.8, the intersection of the support of Σ^F with T_G^*M is reduced to the zero vector. Thus Σ^F determines an element of $K_G^0(T_G^*M_X)$ which depends only on the connected component F of ϕ in the set of regular elements.

Denote by ρ the representation of G in M_X, and also by ρ the infinitesimal action of $\mathfrak{g}$ in M_X. If $\phi \in \mathfrak{g}$, and $z = \sum_a z_a$ is in M_X, then $\rho(\phi)z = \sum_a i\langle a, \phi\rangle z_a$.

Lemma 4.11. *The symbol Σ^F is equal in K-theory to the Atiyah symbol*

$$At^F(z, \xi) = c(h(\xi) + \rho(\phi)z)$$

(see [16]).

Proof. Indeed, for $t \in [0, 1]$, it is easy to see that

$$At_t^F(z, \xi) = c(h(\xi) + (t\rho(\phi) + (1-t)J_F)z)$$

is transversally elliptic. Thus At^F and Σ^F being homotopic coincide in K-theory. □

Let us consider

$$\Theta_X^F(e^x) = \sum_\lambda \mathcal{P}_X^F(\lambda) e^{i\langle\lambda, x\rangle}$$

as a generalized function on G.

Recall Atiyah's theorem (in Section 6 in [1]; see also Appendix 2 in [8]).

Theorem 4.12.

$$\mathrm{index}(At^F)(g) = (-1)^{|X|}\, g^{a_X}\, \Theta_X^F(g) = \Theta_{-X}^F(g).$$

We immediately translate this theorem as follows.

Theorem 4.13. *Let F be a regular face. Let*

$$\Sigma^F = p_F^* \mathrm{Bott}(M_X).$$

Then $\mathrm{ind}_m(\Sigma^F) = \mathcal{P}_{-X}^F$

This identity is in accord with a simple remark.

Remark 4.14. In the K-theory of $K_G^0(T_G^*M)$, we have the identity

$$(\Lambda M_{-X})\Sigma^F = \mathrm{Bott}(M_X \oplus M_{-X}).$$

Applying the index, we obtain the identity

$$\prod_{a\in X}(1-e^{-i\langle a,x\rangle})\mathrm{index}(At^F)(e^x)=1.$$

4.15. The infinitesimal index

Consider N, an oriented G-manifold, equipped with a G-invariant 1-form σ. Recall that Z is the set of zeros of the moment map $\mu : N \to \mathfrak{g}^*$.

We denote by ι_x the contraction of a differential form on N by the vector field v_x associated to x on N. We have defined in [17] a Cartan model for the equivariant cohomology with compact supports $H^*_{G,c}(Z)$ of the subset Z of N. This is a $\mathbb{Z}$-graded space. A representative of this group is an equivariant form $\alpha(x)$ with compact support, $\alpha : \mathfrak{g} \to \mathcal{A}_c(N)$, such that $D(\alpha)$ is equal to 0 in a neighborhood of Z. The dependence of α on x is polynomial. Here, $\mathcal{A}_c(N)$ is the space of differential forms with compact supports on N, while the equivariant differential D is defined by

$$D\alpha(x) = d\alpha(x) - \iota_x\alpha(x).$$

Clearly, an element $\alpha \in H^*_{G,c}(N)$ of the equivariant cohomology with compact supports of N defines a class in $H^*_{G,c}(Z)$. Indeed, $D\alpha = 0$ on all N.

Remark 4.16. If α is an equivariantly closed form on N such that the support of α intersected with Z is a compact set, we associate to α an element $[\alpha]_c$ in the equivariant cohomology with compact supports of Z defined as follows. Take a G-invariant function χ equal to 1 in a neighborhood of Z and supported sufficiently near Z. Then $\chi\alpha$ is compactly supported on N, and $D\alpha = 0$ in the neighborhood of Z where $\chi = 1$, and thus defines a class $[\alpha]_c$ in $H^*_{G,c}(Z)$ independent of the choice of χ.

Let us now recall the definition of the infinitesimal index of $[\alpha] \in H^*_{G,c}(Z)$. Let $\Omega = d\sigma$, and set $\Omega(x) = D\sigma(x) = \mu(x) + \Omega$. $\Omega(x)$ is a closed (in fact exact) equivariant form on N. If f is a smooth function on $\mathfrak{g}^*$ with compact support, let

$$\hat{f}(x) := \int_{\mathfrak{g}^*} e^{-i\langle \xi \mid x\rangle} f(\xi)d\xi$$

be the Fourier transform of f. Choose the measure dx on $\mathfrak{g}$ so that the inverse Fourier transform is $f(\xi) = \int_{\mathfrak{g}} e^{i\langle \xi \mid x\rangle}\hat{f}(x)dx$; thus $\hat{f}(x)dx$ is independent of the choices.

The double integral

$$\int_N \int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx$$

is independent of s for s sufficiently large.

We then define

$$\langle \mathrm{infdex}^{\mu}_G([\alpha]), f\rangle := \lim_{s\to\infty} \int_N \int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx. \tag{27}$$

This is a well-defined map from $H^*_{G,c}(Z)$ to distributions on $\mathfrak{g}^*$. It is a map of $S[\mathfrak{g}^*]$ modules, where $\xi \in \mathfrak{g}^*$ acts on forms by multiplication by $\langle \xi, x\rangle$ and on distributions by $i\partial_\xi$.

As the notation indicates, infdex_G^{μ} depends only on μ, and not on the choice of the real invariant form σ with moment map μ. We can obviously also change μ to a positive multiple of μ without changing the infinitesimal index. However, the change of μ to $-\mu$ usually changes the map infdex radically.

Remark 4.17. Clearly, if α is a closed equivariant form with compact support on N, then the infinitesimal index is just the Fourier transform of the equivariant integral $\int_N \alpha(x)$ of the equivariant form α, as $e^{is\Omega(x)} = e^{isD\sigma(x)}$ is equivalent to 1 for any s.

Using the same formula (27), we have extended the definition of infdex_G^{μ} to a $\mathbb{Z}/2\mathbb{Z}$-graded cohomology space $\mathcal{H}_{G,c}^{\infty,m}(Z)$, which we now define.

A representative of a class $[\alpha]$ in $\mathcal{H}_{G,c}^{\infty,m}(Z)$ is a smooth map $\alpha : \mathfrak{g} \to \mathcal{A}_c(N)$, such that the dependence of $\alpha(x)$ on x is of at most polynomial growth, and such that $D(\alpha)$ is equal to 0 in a neighborhood of Z. The index m indicates moderate growth on $\mathfrak{g}$.

As we will recall in § 5.2, the equivariant Chern character $\mathrm{ch}(\Sigma)$ of an element $\Sigma \in K_G^0(Z)$ belongs to the space $\mathcal{H}_{G,c}^{\infty,m}(Z)$.

We note the following.

Proposition 4.18. *Let $b(x) = \int_{\mathfrak{g}^*} e^{i\langle\xi,x\rangle} m(\xi) d\xi$ be the Fourier transform of a compactly supported distribution $m(\xi)$ on $\mathfrak{g}^*$. Then $b(x)$ is a function on $\mathfrak{g}$ of moderate growth, and the space $\mathcal{H}_{G,c}^{\infty,m}(Z)$ is stable by multiplication by $b(x)$.*

Furthermore,

$$\mathrm{infdex}_G^{\mu}(b\alpha) = m * \mathrm{infdex}_G^{\mu}(\alpha). \tag{28}$$

The character group Λ acts on forms with moderate growth by multiplication by $e^{i\langle\lambda,x\rangle}$, for $\lambda \in \Lambda$, inducing an action on cohomology. It also acts on distributions on $\mathfrak{g}^*$ by translations:

$$t_\lambda D(f) = D(t_{-\lambda} f) = D(f(\xi + \lambda)).$$

Proposition 4.19. *The map* infdex *is equivariant with the respect to the previous actions of Λ.*

Proof. The proposition follows from the definition of infdex, once we notice that

$$\widehat{t_{-\lambda} f} = e^{i\langle\lambda,x\rangle} \hat{f}$$

for any function f on $\mathfrak{g}^*$ lying in the Schwartz space. □

4.20. Some explicit computations in cohomology

Recall that, if N is a vector space provided with an action of G, $H_{G,c}^*(N)$ is a free $S[\mathfrak{g}^*]$ module with a generator $\mathrm{Thom}(N)$ with equivariant integral $\int_N \mathrm{Thom}(N)(x)$ identically equal to 1 (see, for example, [22]).

Notice that in particular, by Remark 4.17, if N is a vector space with a (any) real 1-form σ, then $\mathrm{Thom}(N)$ defines an element of $H_{G,c}^*(Z)$ with infinitesimal index equal to the δ-function of $\mathfrak{g}^*$. The form $\mathrm{Thom}(N)$ depends on the choice of an orientation of N.

Let Y be a sequence of vectors in Λ, and let $M := M_Y$ be the corresponding complex G-representation space. We give to M the orientation given by its complex structure.

We want to describe $\mathrm{Thom}(M_Y)$. For this, it is sufficient to give the formula of the Thom form for $M = L_a$, a complex line. The formula for $\mathrm{Thom}(M_Y)$ is then obtained by taking the exterior product of the corresponding equivariant differential forms.

If $\alpha(x)$ is an equivariant form on a G-manifold M, fixing $m \in M$, we may write $\alpha[m](x)$ for the element $\alpha\,(x)_m \in \bigwedge T_m^*M$ defined by the differential form $\alpha(x)$ at the point m.

Let $L_a = \mathbb{C}$, with coordinate z. The infinitesimal action of $x \in \mathfrak{g}$ is given by $i\langle a, x\rangle$. Choose a function χ on $\mathbb{R}$ with compact support and identically 1 near 0. Then

$$\mathrm{Thom}(L_a)[z](x) = -\frac{1}{2\pi}\Big(\chi\,(|z\,|^2)\langle a, x\rangle + \chi'(|z\,|^2)i(dz \wedge d\bar{z})\Big) \tag{29}$$

is the required closed equivariant differential form on L_a with equivariant integral identically equal to 1.

As in § 4.4, we fix an G-invariant Hermitian structure on M_Y and take the G-invariant real 1-form $\sigma_Y = -1/2\Im\langle z, dz\rangle$ with moment map $\mu_Y(z) = \frac{1}{2}\sum_{a\in Y}|z_a|^2\,a$ and zero fiber Z_Y.

We define some elements of $H^*_{G,c}(Z_Y)$ by an analogous procedure to the K-theory case. Of course, the restriction of $\mathrm{Thom}(M_Y)$ to Z_Y defines an element in $H^*_{G,c}(Z_Y)$.

Let us construct some further elements of $H^*_{G,c}(Z_Y)$, not coming by restriction to Z_Y of a compactly supported class on M_Y.

As before, we start with the basic case, in which there is $\phi \in \mathfrak{g}$ strictly positive on all the characters $a \in Y$. Then $Z = \{0\}$, and its equivariant cohomology with compact supports is the algebra $S[\mathfrak{g}^*]$ generated by 1.

Do not assume any more that the weights Y generate a pointed cone. Let F be a regular face for Y. Let A be the subsequence of Y where ϕ takes positive values, and let B be the subsequence where ϕ takes negative values. Write $M_Y = M_A \oplus M_B$. Then the pull-back of $\mathrm{Thom}(M_A)$ by the projection $M_Y \to M_A$ is supported near M_B. Thus the support of the pull-back of $\mathrm{Thom}(M_A)$ intersected with Z_Y is compact, and therefore, as explained in Remark 4.16, $\mathrm{Thom}(M_A)$ defines a class in $H^*_{G,c}(Z_Y)$. We can write an explicit representative of this class by choosing a G-invariant function χ on M_Y identically equal to 1 in a neighborhood of Z_Y and supported near Z_Y. Then our representative will be given by

$$t^F[z](x) := \chi\,(z)\mathrm{Thom}(M_A)[z_A](x).$$

Consider the inclusion $i_B : M_B \to M_Y$. In [18], we have defined a map $(i_B)_! : H^*_{G,c}(Z_B) \to H^*_{G,c}(Z_Y)$ preserving the infinitesimal index. In this setting, we see that the class $t^F \in H^*_{G,c}(Z_Y)$ is by definition a representative of $(i_B)_!(1)$.

We now compute the infinitesimal index $\mathrm{infdex}^{\nu}_G([\alpha])$ of the elements t^F.

The equivariant Thom form has equivariant integral equal to 1, so that, by Fourier transformation,

$$\mathrm{infdex}^{\nu}_G(\mathrm{Thom}(M_Y)) = \delta_0,$$

where δ_0 is the δ-distribution on $\mathfrak{g}^*$.

Consider now the element t^F associated to F. The subsequence B spans a pointed cone. We can then define the partial multispline distribution T_B on $\mathfrak{g}^*$.

Theorem 4.21. $\operatorname{infdex}_G^{\nu}(t^F) = (2i\pi)^{|B|}\, T_B$, *where T_B is the multivariate spline.*

Proof. Since the class $t^F \in H^*_{G,c}(Z_Y)$ is a representative of $(i_B)_!(1)$ and $i_!$ preserves infdex, we are reduced to proving our theorem when $M_A = 0$. The computation is reduced to the one-dimensional case by the product axiom; see [18]. Let us make the computation in this case, that is, when $M_Y = L_b$.

The action form σ in coordinates $z = v_1 + iv_2$ is $\frac{1}{2}(v_1 dv_2 - v_2 dv_1)$, so $D\sigma(x) = dv_1 \wedge dv_2 + \frac{1}{2}\langle b, x\rangle \|v\|^2$.

Let $\chi(t)$ be a function on $\mathbb{R}$ with compact support and identically equal to 1 in a neighborhood of $t = 0$. Then, by definition, we get

$$\begin{aligned}\langle \operatorname{infdex}_G^{\nu}(1), f\rangle &= \lim_{s\to\infty} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \chi\left(\frac{\|v\|^2}{2}\right) e^{is\langle b,x\rangle \frac{\|v\|^2}{2}} e^{is dv_1 dv_2} \hat{f}(x) dx \\ &= \lim_{s\to\infty} is \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \chi\left(\frac{\|v\|^2}{2}\right) e^{is\langle b,x\rangle \frac{\|v\|^2}{2}} dv_1 dv_2 \hat{f}(x) dx.\end{aligned}$$

Use polar coordinates on $\mathbb{R}^2$, and take $t = \frac{\|v\|^2}{2}$ as a new variable. We obtain

$$\lim_{s\to\infty} 2i\pi s \int_{t=0}^{\infty} \int_{x=-\infty}^{\infty} \chi(t) e^{i\langle stb, x\rangle} \hat{f}(x) dx dt.$$

Changing t to t/s, and using the Fourier inversion formula, we obtain

$$\langle \operatorname{infdex}_G^{\nu}(1), f\rangle = 2i\pi \lim_{s\to\infty} \int_0^{\infty} \chi(t/s) f(tb) dt.$$

Passing to the limit, as χ is identically 1 in a neighborhood of 0 and f is compactly supported, we obtain our formula:

$$\langle \operatorname{infdex}_G^{\nu}(1), f\rangle = 2i\pi \int_0^{\infty} f(tb) dt. \qquad \square$$

Exactly as in the K-theory case, when $Y = X \cup -X$, that is, $(T^*M_X)^0 = T^*_G M_X$, we can use this construction to get classes in $t^F \in H^*_{G,c}(T^*_G M_X)$ and compute their infdex. Let F be a regular face for X. Consider the map $p_F : T^*M_X \to M_X$ defined by $p_F(z, \xi) = h(\xi) + J_F z$ (see § 4.4). We have the following theorem.

Theorem 4.22. *Let χ be a G-invariant function on T^*M_X identically equal to 1 in a neighborhood of $T^*_G M_X$ and supported near $T^*_G M_X$. Then*

$$t^F = \chi\, p_F^* \operatorname{Thom}(M_X)$$

defines a class in $H^*_{G,c}(T^*_G M_X)$ such that $\operatorname{infdex}_G^{-\mu}(t^F) = (-1)^{|X|}\,(2i\pi)^{|X|}\, T_X^F$.

Proof. This is obtained from the preceding calculations. Indeed, the infinitesimal index depends only on the moment map. So, using Lemma 4.9, we are reduced to the calculation performed in Theorem 4.21. The sign comes from taking into account the orientations on T^*M_X. Indeed, in the isomorphism of T^*M_X with $M_X \oplus M_X$, the orientation of T^*M_X is $(-1)^{|X|}$, the orientation given by the complex structure on $M_X \oplus M_X$. $\square$

5. The equivariant Chern character and the index theorem

In this section, we compare the equivariant K-theory and equivariant cohomology via the Chern character.

5.1. Motivations

Our goal is to compute the multiplicity index of a symbol $\Sigma \in K^0_G(T^*_G M)$ in terms of the infinitesimal index of the equivariant Chern character of Σ for a general G-manifold M. We are going to provide a direct formula at least in the case where $M = M_X$. This construction is motivated by taking the Fourier transform of the formula of Berline and Vergne for the equivariant index of a transversally elliptic operator ([8, 7, 24], where one can also find the notation and various definitions). We first recall this formula in the simple case of elliptic symbols.

In this case, the equivariant Chern character $\mathrm{ch}(\Sigma)$ of an element $\Sigma \in K^0_G(T^*M)$ is an element in $\mathcal{H}^\infty_{G,c}(T^*M)$, and the index of Σ is a regular function on G. For $x \in \mathfrak{g}$ small enough,

$$\mathrm{index}(\Sigma)(e^x) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \mathrm{ch}_c(\Sigma)(x)\hat{A}(T^*M)(x), \tag{30}$$

where $\hat{A}(T^*M)(x)$ is the equivariant $\hat{A}$ genus of T^*M, $\mathrm{ch}_c(\Sigma)(x)$ is the Chern character with compact support, and $\hat{A}(T^*M)(x)$ is defined for x small enough. For any element $g \in G$, similar 'descent formulae' are given for $\mathrm{index}(\Sigma)(ge^x)$, where the integral is over T^*M^g, M^g being the fixed-point set of the action of $g \in G$ on M.

Let ω be the canonical (Liouville) 1-form on T^*M: $\omega_{m,\xi}(V) = \langle \xi, p_* V\rangle$. Let Σ be a transversally elliptic symbol. In [8, 7], it is shown that, although $\mathrm{ch}(\Sigma)$ is not compactly supported, the formula

$$\mathrm{index}(\Sigma)(e^x) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} e^{-iD\omega(x)}\mathrm{ch}(\Sigma)(x)\hat{A}(T^*M)(x) \tag{31}$$

still holds as a generalized function of x, for a sufficiently large class of transversally elliptic symbols. The factor $e^{-iD\omega(x)}$ is congruent to 1 in cohomology, but is crucial in defining a convergent oscillatory integral when $\mathrm{ch}(\Sigma)(x)$ is not compactly supported.

Let us write formula (31) more explicitly in the case where M is a manifold such that T^*M is stably equivalent to $M \times R$, a trivial vector bundle: here, R is a real representation space of G. Consider the sequence X_R of weights of G in $R \otimes \mathbb{C}$. Thus, if $a \in X_R$, then $-a \in X_R$. Consider the function

$$j_R(x) = \prod_{a \in X_R} \frac{1 - e^{-i\langle a, x\rangle}}{i\langle a, x\rangle}$$

on $\mathfrak{g}$. Then the equivariant class $\hat{A}(T^*M)(x)$ is just the function $j_R(x)^{-1}$, and it is defined only for x small enough. In this 'trivial tangent bundle case', formula (31) implies that

$$j_R(x)\mathrm{index}(\Sigma)(e^x) = (2i\pi)^{-\dim M}\int_{\mathbf{T}^*M} e^{-iD\omega(x)}\mathrm{ch}(\Sigma)(x).$$

Recalling formula (4) and the definition of $\mathrm{ind}_m(\Sigma)$, we obtain

$$\hat{B}_{X_R}(x)\sum_\lambda \mathrm{ind}_m(\Sigma)(\lambda)e^{i\langle\lambda,x\rangle} = (2i\pi)^{-\dim M}\int_{\mathbf{T}^*M} e^{-iD\omega(x)}\mathrm{ch}(\Sigma)(x).$$

Let μ be the moment map on T^*M associated to the 1-form ω. Thus, by Fourier transformation, this equality suggests that the following formula should hold:

$$B_{X_R} *_d \mathrm{ind}_m(\Sigma) = (2i\pi)^{-\dim M}\,\mathrm{infdex}_G^{-\mu}(\mathrm{ch}(\Sigma)). \tag{32}$$

All the terms of this formula make sense, since we will show in § 5.2 that $\mathrm{ch}(\Sigma)$ belongs to the cohomology $\mathcal{H}^{\infty,m}_{G,c}(T^*_GM)$ of classes with compact support on T^*_GM where the infinitesimal index is defined.

Consider now $M = M_X$, our representation space for G. We always will consider M_X as a real G-manifold, except if specified differently. Let $T^*M_X = M_X \times M_X^*$ be a trivial vector bundle. Here, M_X^* is thus considered as the real vector space dual to M_X. The sequence of weights of G in $M_X^* \otimes \mathbb{C}$ is the sequence $X \cup (-X)$. Note that the real dimension of M_X is $2|X|$. Our aim in the next subsections is to see that formula (32) indeed holds for any $\Sigma \in K^0_G(T^*_GM_X)$.

Finally, using 'descent formulae' for g running over the finite set of toric vertices of the system X, we will use Theorem 2.29 to give a formula for the multiplicity function $\mathrm{ind}_m(\Sigma)$ on Λ.

Let us point out that we could use the formulae of [8, 7] or [24]. However, we found it instructive to prove formula (32) directly in the case of M_X being a vector space, using the explicit description of the generators of $K^0_G(T^*_GM_X)$ given in[16]. Using the functoriality principle, we hope that it will be possible to describe directly $\mathrm{ind}_m(\Sigma)$, a function on Λ, in term of $\mathrm{infdex}_G^{\mu}(\mathrm{ch}(\Sigma))$, a function on $\mathfrak{g}^*$, for any transversally elliptic operator on a general G-manifold M.

5.2. The equivariant Chern character

We recall the construction of $\mathrm{ch}(\Sigma)$ for Σ a morphism of vector bundles on a general G-manifold N. We refer to [23] for the comparison between the different constructions of the Chern character.

Let $\mathcal{E}$ be a G-equivariant complex vector bundle on N. We choose a G-invariant Hermitian structure and a G-invariant Hermitian connection ∇ on $\mathcal{E}$. For any $x \in \mathfrak{g}$, let $\mathcal{L}_x$ be the action of x on the space $\Gamma(N, \bigwedge T^*N \otimes \mathcal{E})$ of $\mathcal{E}$-valued forms on N. The operator $j(x) := \mathcal{L}_x - \nabla_x$ is a bundle map called the *moment* of the connection ∇. At each point $n \in N$, $j(x)$ is an anti-Hermitian endomorphism of $\mathcal{E}_n$. Let F be the curvature of the connection ∇; thus F is a 2-form on N with values in the bundle of anti-Hermitian linear operators on $\mathcal{E}$.

The equivariant curvature of $\mathcal{E}$ at the point n is by definition $j(x) + F$. Then the equivariant Chern character $\mathrm{ch}(\mathcal{E}, \nabla)$ is the equivariant differential form

$$\mathrm{ch}(\mathcal{E}, \nabla)(x) = \mathrm{Tr}(e^{j(x)+F}).$$

This is a closed equivariant differential form on N with C^∞ coefficients (see [5], Chapter 7).

Lemma 5.3. *Over any compact subset of N, the Fourier transform of the equivariant Chern character $x \to \mathrm{ch}(\mathcal{E}, \nabla)(x)$ is a compactly supported distribution on $\mathfrak{g}^*$.*

Proof. Let us fix $n \in N$. Set $E = \mathcal{E}_n$ to be the fiber of the vector bundle $\mathcal{E}$ at n, and $A = \bigwedge^{2*} T_n^* N$ to be the even part of the exterior of the cotangent space at n. Then $(j(x) + F)(n) \in A \otimes \mathfrak{u}$, where $\mathfrak{u}$ is the Lie algebra of anti-Hermitian linear operators on E. The map $x \to j(x)$ defines a map $\mathfrak{g} \to \mathfrak{u}$, with dual map $j^* : \mathfrak{u}^* \to \mathfrak{g}^*$.

If $P(E)$ denotes the projective space of E, we define $\mu^P : P(E) \to \mathfrak{u}^*$ by

$$\mu^P(p)(iX) = \frac{\langle Xv, v\rangle}{\langle v, v\rangle}.$$

Here, p is the point of $P(E)$ associated to $v \in E - \{0\}$ and $iX \in \mathfrak{u}$.

By Corollary A.2 to the Nelson theorem (for completeness, in Theorem A.1 we give a proof of this fact based on localization formula in equivariant cohomology in the Appendix),

$$\mathrm{Tr}(e^{(j(x)+F)}) = \int_{P(E)} e^{i\langle j^*\mu^P(p), x\rangle} D(p, n),$$

where $D(p, n) = e^{\mu^P(p)(F)} \mathrm{Tr}(\beta(p, u))$ is a differential form on $P(E)$ with values in $\bigwedge T_n^* N$ (if $F = \sum_k F_k u_k$ with $F_k \in \bigwedge^2 T_n^* N$ and $u_k \in \mathfrak{u}$, $e^{\mu^P(p)(F)} = e^{\sum_k F_k \mu^P(p)(u_k)}$ is a smooth function on $P(E)$ with values in $\bigwedge T_n^* N$).

Integrating over the fiber of the map $j^*\mu^P : P(E) \to \mathfrak{g}^*$, we obtain

$$\mathrm{Tr}(e^{j(x)+F}) = \int_{\mathfrak{g}^*} e^{i\langle x, \xi\rangle} \gamma(\xi),$$

where $\gamma(\xi)$ is a distribution supported on the compact set $j^*\mu^P(P(E))$. Thus we see that, at each point n of N, the function $x \to \mathrm{Tr}(e^{j(x)+F})$ is the Fourier transform of a compactly supported distribution on $\mathfrak{g}^*$ (with values in $\bigwedge T_n^* N$). It is clear that our estimates are uniform on any compact neighborhood of the point $n \in N$. Thus we obtain our lemma. □

In particular, the closed equivariant differential form $\mathrm{ch}(\Sigma)(x)$ has moderate growth with respect to $x \in \mathfrak{g}$ over any compact subset of N.

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a Hermitian G-equivariant super-vector bundle over N. Let $\Sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a G-equivariant morphism. Outside the support of Σ, the complex vector bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ are 'the same', so that it is natural to construct representatives of $\mathrm{ch}(\mathcal{E}) := \mathrm{ch}(\mathcal{E}^+) - \mathrm{ch}(\mathcal{E}^-)$ which are zero 'outside' the support of Σ by the following identifications of bundle with connections.

Let U be a neighborhood of the support of Σ. We first may choose the Hermitian structures on $\mathcal{E}^+, \mathcal{E}^-$ so that Σ is an isomorphism of Hermitian vector bundles outside U.

A pair of connections ∇^+, ∇^- is said to be 'adapted' to the morphism Σ on U when the following holds:

$$\nabla^- \circ \Sigma = \Sigma \circ \nabla^+ \tag{33}$$

outside the neighborhood U of the support of Σ. A pair of adapted connections is easy to construct.

Proposition 5.4. *Let ∇^+, ∇^- be a pair of G-invariant Hermitian connections adapted to $\Sigma : \mathcal{E}^+ \to \mathcal{E}^-$. Then the differential form $\mathrm{ch}(\mathcal{E}^+, \nabla^+) - \mathrm{ch}(\mathcal{E}^-, \nabla^-)$ is a closed equivariant differential form on M supported near the support of Σ. We denote it as $\mathrm{ch}_s(\Sigma)$.*

The index s means with support condition.

In particular, if the support of Σ is compact, we can also choose the neighborhood U so that its closure is compact. We deduce that the cohomology class of $\mathrm{ch}_s(\Sigma)$ lies in $\mathcal{H}^{\infty,m}_{G,c}(N)$, that is, $\mathrm{ch}_s(\Sigma)$ is a compactly supported class on N with moderate growth in x. This class will be denoted simply by $\mathrm{ch}(\Sigma)$.

Let $g \in G$, and let N^g be the fixed-point submanifold of g. Then g acts by a fiberwise transformation on $\mathcal{E} \to N$ still denoted g. We still denote by F the curvature of the bundle $\mathcal{E}$ restricted to N^g. The equivariant twisted Chern character $\mathrm{ch}^g(\mathcal{E}, \nabla)$ is the equivariant differential form

$$\mathrm{ch}^g(\mathcal{E}, \nabla)(x) = \mathrm{Tr}(g e^{(j(x)+F)}).$$

This is a closed equivariant differential form on N^g. Similarly, we have the following proposition.

Proposition 5.5. *Let (∇^+, ∇^-) be a pair of G-invariant Hermitian connections adapted to $\Sigma : \mathcal{E}^+ \to \mathcal{E}^-$. Then the differential form*

$$\mathrm{ch}^g(\Sigma)(x) = \mathrm{ch}^g(\mathcal{E}^+, \nabla^+)(x) - \mathrm{ch}^g(\mathcal{E}^-, \nabla^-)(x)$$

is a closed G-equivariant form on N^g supported near the support of $\Sigma|_{N^g}$.

Over any compact subset of N^g, it has moderate growth with respect to $x \in \mathfrak{g}$.

If we change the choice of connections, we can see using the usual transgression formulae for Chern characters (see for example [23]) that the class $\mathrm{ch}^g(\Sigma)$ stays the same in the cohomology with moderate growth: that is, the boundary $\nu(x)$ expressing the change of $\mathrm{ch}^g(\Sigma)$ with respect to the connection remains with moderate growth with respect to x, over any compact subset of N^g.

We return to the situation where $N = T^*M$ is the conormal bundle to a G-manifold M and Σ is a transversally elliptic symbol. Let χ be a function identically equal to 1 near the set $T^*_G M$ and supported in a neighborhood of $T^*_G M$ whose closure has compact intersection with the support of Σ. We have the following proposition.

Proposition 5.6. *The equivariant form $\alpha^g(x) = \chi \mathrm{ch}^g(\Sigma)(x)$ on T^*M^g is compactly supported, and $D\alpha^g(x)$ is equal to 0 in a neighborhood of $T^*_G M^g$. Over any compact subset of $T^*_G M^g$, it has moderate growth with respect to $x \in \mathfrak{g}$.*

It follows that the Chern character gives a morphism

$$\mathrm{ch} : K^0_G(T^*_G M) \to \mathcal{H}^{\infty,m}_{G,c}(T^*_G M).$$

Similarly, for $g \in G$, the twisted Chern character is a morphism

$$\mathrm{ch}^g : K^0_G(T^*_G M) \to \mathcal{H}^{\infty,m}_{G,c}(T^*_G M^g).$$

As we have seen in § 4.15, the character group Λ acts on forms with moderate growth by multiplication by $e^{i\langle\lambda,x\rangle}$, for $\lambda \in \Lambda$, inducing an action on cohomology. Of course Λ also acts on G-equivariant vector bundles by tensoring with L_a and inducing an action in K-theory. We have the following proposition.

Proposition 5.7. *For any $g \in G$, the map ch^g is equivariant with respect to the previous actions of Λ.*

Proof. Let $\mathcal{E}$ be a G-equivariant complex vector bundle on N. We choose a G-invariant Hermitian structure and a G-invariant Hermitian connection ∇ with moment $j(x)$ and curvature F.

Then, for the vector bundle $L_\lambda \otimes \mathcal{E}$, with endomorphism bundle canonically isomorphic to that of $\mathcal{E}$, we can take the same connection ∇. By the definition of the moment, we see that the equivariant curvature of $L_\lambda \otimes \mathcal{E}$ equals $i\langle\lambda, x\rangle + j(x) + F$, giving our claim. □

5.8. Explicit computations of the Chern character

We consider our G-manifold $M = M_X$. Choose a Hermitian structure on M_X. Let $\Sigma_z = c(z)$ be the Clifford multiplication acting on the complex vector bundle $\bigwedge M_X$. The support of Σ is $\{0\}$, and Σ determines the class $\mathrm{Bott}(M_X) \in K^0_G(M_X)$.

The following result is well known.

$$\mathrm{ch}(\mathrm{Bott}(M_X))(x) = (2i\pi)^{|X|} \prod_{a\in X} \frac{e^{i\langle a,x\rangle} - 1}{i\langle a, x\rangle} \mathrm{Thom}(M_X)(x)$$

in the cohomology group of smooth equivariant differential forms, without moderate growth conditions.

In fact, this equality also holds in $\mathcal{H}^{\infty,m}_{G,c}(M_X)$.

Proposition 5.9. *We have the equality*

$$\mathrm{ch}(\mathrm{Bott}(M_X))(x) = (2i\pi)^{|X|} \prod_{a\in X} \frac{e^{i\langle a,x\rangle} - 1}{i\langle a, x\rangle} \mathrm{Thom}(M_X)(x)$$

in $\mathcal{H}^{\infty,m}_{G,c}(M_X)$.

Proof. Since $M_X = \bigoplus_{a\in X} L_a$, and both $\mathrm{Bott}(M_X)$ and $\mathrm{Thom}(M_X)$ are the external product of the various $\mathrm{Bott}(L_a)$ and $\mathrm{Thom}(L_a)$ for $a \in X$, it suffices to prove our claim when $M_X = L_a$.

In this case, $\mathcal{E}^+$ is the trivial bundle $L_a \times \mathbb{C}$, $\mathcal{E}^- = L_a \times L_a$, and the morphism is $\Sigma_z = z$. We choose $\nabla^+ = d$.

Let $\chi(t)$ be a function on $\mathbb{R}$ with compact support contained in $|t| < 1$ and identically equal to 1 near 0. Let $\beta = (\chi(|z|^2) - 1)\frac{dz}{z}$, a well-defined G-invariant 1-form. We consider

$$\nabla^- = d + \beta.$$

Outside $|z|^2 < 1$, the connections $\nabla^+ = d$, $\nabla^- = d - \frac{dz}{z}$ verify $\nabla^- z = z\nabla^+$, so that the pair (∇^+, ∇^-) is adapted for the morphism z.

We compute the corresponding difference of Chern characters.

The moment $j(x)$ of the connection ∇^+ is 0 and the equivariant curvature $F^+(x) = 0$. So $\mathrm{ch}(\mathcal{E}^+, \nabla^+) = 1$.

The moment of the connection ∇^- is $i\langle a, x\rangle + i\langle a, x\rangle(\chi(|z|^2) - 1) = i\langle a, x\rangle\chi(|z|^2)$. Thus the equivariant curvature of ∇^- is

$$F^-(x) = i\langle a, x\rangle\chi(|z|^2) - \chi'(|z|^2)dz \wedge d\bar{z}.$$

Note that $F^-(x) = 0$, if $|z|^2 > 1$, so that $\mathrm{ch}(\mathcal{E}^+, \nabla^+) - \mathrm{ch}(\mathcal{E}^-, \nabla^-)$ is supported on $|z|^2 < 1$. Thus $\mathrm{ch}(\mathrm{Bott}(L_a)) := \mathrm{ch}(\mathcal{E}^+, \nabla^+) - \mathrm{ch}(\mathcal{E}^-, \nabla^-)$ is a closed equivariant form with compact support.

We have explicitly

$$\mathrm{ch}(\mathrm{Bott}(L_a))[z](x) = (1 - e^{i\langle a,x\rangle\chi(|z|^2)}) + e^{i\langle a,x\rangle\chi(|z|^2)}\chi'(|z|^2)dz \wedge d\bar{z}.$$

Let us see that $\mathrm{ch}(\mathrm{Bott}(L_a))[z](x)$ is equal to

$$(2i\pi)\frac{e^{i\langle a,x\rangle} - 1}{i\langle a, x\rangle}\mathrm{Thom}(L_a)[z](x) = \frac{e^{i\langle a,x\rangle} - 1}{i\langle a, x\rangle}(i\langle a, x\rangle\chi(|z|^2) - \chi'(|z|^2)dz \wedge d\bar{z})$$

modulo a boundary with moderate growth.

We consider the 1-form

$$\nu(x) = \left(\left(\frac{e^{i\chi(|z|^2)\langle a,x\rangle} - 1}{i\langle a, x\rangle}\right) - \left(\frac{e^{i\langle a,x\rangle} - 1}{i\langle a, x\rangle}\right)\chi(|z|^2)\right)\frac{dz}{z}.$$

We see that $\nu(x)$ is well defined and compactly supported on L_a. Indeed,

$$\left(\frac{e^{i\chi(|z|^2)\langle a,x\rangle} - 1}{i\langle a, x\rangle}\right) - \left(\frac{e^{i\langle a,x\rangle} - 1}{i\langle a, x\rangle}\right)\chi(|z|^2)$$

is equal to 0 if z is near 0, where $\chi(|z|^2)$ is equal to 1, and is also equal to 0 when $[z| > 1$, where $\chi(|z|^2)$ is equal to 0.

Furthermore, the Fourier transform of $(e^{i\chi(|z|^2)\langle a,x\rangle} - 1)/i\langle a, x\rangle$ is supported, at the point $z \in L_a$, on the interval $[0, -\chi(|z|^2)a]$. Thus we see that ν has moderate growth.

Since it is easily verified that

$$D\nu(x) = 2i\pi\frac{e^{i\langle a,x\rangle} - 1}{i\langle a, x\rangle}\mathrm{Thom}(L_a)(x) - \mathrm{ch}(\mathrm{Bott}(L_a))(x),$$

Proposition 5.9 follows. □

Let $g \in G$. The submanifold M^g for the action of g on M is M_{X^g}, where $X^g := [a \in X \mid g^a = 1]$ is a subsequence of X. Then the restriction of the symbol $\mathrm{Bott}(L_a)$ to M^g is equal to

$$\mathrm{Bott}(M^g) \otimes \bigwedge(M_{X\setminus X^g}).$$

Thus we obtain the following proposition.

Proposition 5.10. *We have*

$$\mathrm{ch}^g(\mathrm{Bott}(M_X))(x) = (2i\pi)^{|X^g|} \prod_{a\in X^g} \frac{e^{i\langle a,x\rangle}-1}{i\langle a,x\rangle} \prod_{b\notin X^g} (1-g^b e^{i\langle b,x\rangle})\mathrm{Thom}(M_{X^g})(x).$$

We now compute the Chern character of the symbol Σ^F. Recall the map $p_F(x,\xi) = h(\xi) + J_F x$ from T^*M to M. The morphism Σ^F is the pull-back of $\mathrm{Bott}(M_X)$ via this map. It defines a class with compact support. Comparing with the element $t^F \in H^*_{G,c}(T^*_G M)$ which is obtained as a pull-back of a Thom class, from Proposition 5.9, we deduce the following proposition.

Proposition 5.11. *We have the following equality in $\mathcal{H}^{\infty,m}_{G,c}(T^*_G M)$:*

$$\mathrm{ch}(\Sigma^F)(x) = (2i\pi)^{|X|} \prod_{a\in X} \frac{e^{i\langle a,x\rangle}-1}{i\langle a,x\rangle} t^F(x).$$

The computation of the infinitesimal index follows from this formula. Since

$$\prod_{a\in X} \frac{e^{i\langle a,x\rangle}-1}{i\langle a,x\rangle} = \int_{\mathfrak{g}^*} e^{i\langle \xi,x\rangle} B_X(\xi),$$

using formula (28), we obtain the following theorem.

Theorem 5.12.

$$\mathrm{infdex}_G^{-\mu}(\mathrm{ch}(\Sigma^F)) = (2\pi)^{2|X|}\, B_X *_c T_X^F.$$

Similarly, for any $g \in G$, we have

$$\mathrm{infdex}_G^{-\mu}(\mathrm{ch}^g(\Sigma^F)) = (2\pi)^{2|X^g|} \prod_{b\notin X^g} (1-g^b t_b)(B_{X^g} *_c T^F_{X^g}).$$

In particular, when g is a vertex on X (Definition 2.28), the system X^g spans $\mathfrak{g}^*$, and the infinitesimal index of $\mathrm{ch}^g(\Sigma^F)$ is a piecewise polynomial function on $\mathfrak{g}^*$ with respect to (X, Λ). In fact, we even see that this function is continuous on $\mathfrak{g}^*$.

5.13. The index theorem

We are now ready to compare the morphism index and the morphism index on $T^*_G M_X$ and to prove formula (32).

We denote by X_R the sequence $X \cup -X$ of characters. Note that the zonotope associated to X_R contains 0 in its closure.

Proposition 5.14. *Let $X \subset \Lambda$ be a system of characters of G. Let*

$$X_R = X \cup -X.$$

Let Σ be a G-invariant transversally elliptic symbol on M. Let $\mathrm{ind}_m(\Sigma) \in \mathcal{C}_{\mathbb{Z}}[\Lambda]$ be its multiplicity index. Let $\mathrm{infdex}_G^{-\mu}(\mathrm{ch}(\Sigma))$ be the infinitesimal index of its Chern character. Then

$$B_{X_R} *_d \mathrm{ind}_m(\Sigma) = (2i\pi)^{-2|X|}\,\mathrm{infdex}_G^{-\mu}\mathrm{ch}(\Sigma).$$

Proof. Using Propositions 4.19 and 5.7, we are reduced to proving our equality on generators of $K_G^0(T_G^*M_X)$. We thus consider the symbol Σ^F which, by Theorem 4.13, has infinitesimal index equal to the polarized partition function $\mathcal{P}_{-X}^F$. Recall that

$$B_{-X} *_d \mathcal{P}_{-X}^F = T_{-X}^F = (1)^{|X|}\,T_X^F.$$

Since $B_{X_R} = B_X * B_{-X}$, the theorem follows from Theorem 5.12. □

Using the deconvolution theorem in the unimodular case, Proposition 5.14 leads to the following theorem, which is strongly reminiscent of the Riemann–Roch theorem. Note that, as X_R contains 0 in its interior, we may use any alcove containing 0 in its closure in the limiting procedure.

We denote by $\mathrm{Todd}(X_R)$ the Todd operator associated to X_R. It acts on the space of piecewise polynomial functions for the system (X, Λ).

Theorem 5.15. *Let $X \subset \Lambda$ be a unimodular system of characters of G. Let*

$$\mathrm{Todd}(X_R) = \prod_{a \in X \cup -X} \frac{\partial_a}{1 - e^{-\partial_a}}$$

be the Todd *operator.*

Let Σ be a G-invariant transversally elliptic symbol on M, $\mathrm{ind}_m(\Sigma) \in \mathcal{C}_{\mathbb{Z}}[\Lambda]$ be its multiplicity index, and $\mathrm{infdex}_G^{-\mu}(\mathrm{ch}(\Sigma))$ be the infinitesimal index of its Chern character. Then the following hold.

- $\mathrm{infdex}_G^{-\mu}(\mathrm{ch}\Sigma)$ *is a piecewise polynomial measure on $\mathfrak{g}^*$.*
- *Let $\mathfrak{c}$ be an alcove having 0 in its closure. We have*

$$\mathrm{ind}_m(\Sigma) = (2i\pi)^{-2|X|} \lim_{\mathfrak{c}} \mathrm{Todd}\,(X_R)_{pw}\,\mathrm{infdex}_G^{-\mu}(\mathrm{ch}(\Sigma)).$$

Remark 5.16. It is possible to show in this unimodular case that the piecewise polynomial function $(2i\pi)^{-2|X|}\,\mathrm{Todd}\,(X_R)_{pw}\,\mathrm{infdex}_G^{-\mu}(\mathrm{ch}(\Sigma))$ extends to a continuous function on $\mathfrak{g}^*$. Thus its restriction to Λ gives the index multiplicity.

We now formulate the general index theorem. We denote by $\mathcal{V}(X) \subset G$ the set of toric vertices of the sequence of characters X of G (see Definition 2.28).

Theorem 5.17. *Let X be a sequence of elements in Λ, and let $M := M_X$. Let*

$$X_R = X \cup -X.$$

Let Σ be a G-invariant transversally elliptic symbol on M, and let $\mathrm{ind}_m(\Sigma) \in \mathcal{C}_{\mathbb{Z}}[\Lambda]$ be its multiplicity index.

For any $g \in \mathcal{V}(X)$, let $\mathrm{infdex}_G^{-\mu}(\mathrm{ch}^g(\Sigma))$ be the distribution on $\mathfrak{g}^*$ associated to the cohomology class $\mathrm{ch}^g(\Sigma) \in \mathcal{H}_{G,c}^{\infty,m}(T_G^* M^g)$ by the infinitesimal index. Then

- *$\mathrm{infdex}_G^{-\mu}(\mathrm{ch}^g(\Sigma))$ is a piecewise polynomial measure on $\mathfrak{g}^*$.*
- *Let $\mathfrak{c}$ be an alcove having 0 in its closure. We have*

$$\mathrm{ind}_m(\Sigma) = \sum_{g \in \mathcal{V}(X)} (2i\pi)^{-2|X^g|}\, \hat{g} \lim_{\mathfrak{c}} D\left(X_R \setminus X_R^g, g\right)^{-1} \times \mathrm{Todd}(X_R^g) *_{pw} \mathrm{infdex}_G^{-\mu}(\mathrm{ch}^{g^{-1}}(\Sigma)). \tag{34}$$

Proof. Again, using Propositions 4.19 and 5.7, we are reduced to proving our equality on generators of $K_G^0(T_G^* M_X)$.

Let $K = \mathrm{ind}_m(\Sigma^F) = (-1)^{|X|}\, t_{a_X} \mathcal{P}_X^F$. From the general inversion formula obtained in Part 1 of this paper (Theorem 2.29), we have only to show that, for every $g \in \mathcal{V}(X)$,

$$B_{X_R^g} *_d (\hat{g}^{-1} \nabla_{X_R \setminus X_R^g} K) = (-1)^{|X \setminus X^g|}\, (2\pi)^{-2|X^g|}\, \mathrm{infdex}_G^{-\mu}(\mathrm{ch}^{g^{-1}}(\Sigma)).$$

Now, observe that

$$(-1)^{|X/X_g|}\, \nabla_{-X \setminus (-X^g)} t_{a_X} \mathcal{P}_X^F = t_{a_{X^g}} \mathcal{P}_{X^g}^F$$

and

$$B_{-X^g} * t_{a_{X^g}} \mathcal{P}_{X^g}^F = T_{X^g}^F.$$

So, substituting, we get

$$B_{X_R^g} *_d (\hat{g}^{-1} \nabla_{X_R \setminus X_R^g} K) = (-1)^{|X^g|} \prod_{b \notin X^g} (1 - g^{-b} t_b)(B_{X^g} * T_{X^g}^F).$$

But, by Theorem 5.12,

$$\mathrm{infdex}_G^{-\mu}(\mathrm{ch}^{g^{-1}}(\Sigma^F)) = (2\pi)^{2|X^g|} \prod_{b \notin X^g} (1 - g^{-b} t_b)(B_{X^g} *_c T_{X^g}^F).$$

So our claim follows. □

5.17.1. The box spline again. It is quite amusing to verify this theorem on elliptic symbols. The infinitesimal index of $\mathrm{Bott}(T^* M_X)$, after multiplying by $1/(2i\pi)^{2|X|}$, is the double box spline $B_{X \cup -X}$. In the unimodular case, the 'mother formula'

$$\lim_{\mathfrak{c}} \mathrm{Todd}(X \cup -X) *_{pw} B_{X \cup -X} = \delta_0$$

simply expresses the fact that the index of the elliptic operator with symbol $\mathrm{Bott}(T^* M_X)$ is the trivial representation of G.

The infinitesimal index of the elliptic symbols is thus obtained by a finite number of translations of the double box spline.

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Appendix. Nelson formula

We here sketch a short and explicit proof of the Nelson formula, as suggested to us by Michel Duflo.

Let E be a Hermitian vector space. The projective space $P(E)$ is a Hamiltonian space for the action of the unitary group $U(E)$. Let $\mathfrak{u}$ be the space of anti-Hermitian matrices: $\mathfrak{u}$ is the Lie algebra of $U(E)$.

Let ω be the Kahler form on $P(E)$. We denote by $\omega(u) = \mu^P(u) + \omega$ the equivariant symplectic form. Here, $u \in \mathfrak{u}$, $\mu^P(p)(u) = \frac{\langle uv, v\rangle}{\langle v, v\rangle}$, and $p \in P(E)$ is the image of v. The form $\omega(u)$ is a closed equivariant form on $P(E)$. We have

$$\frac{1}{(2i\pi)^{\dim P(E)}} \int_{P(E)} e^{i\omega(u)} = 1.$$

Consider the $\mathrm{End}(E)$-valued polynomial function of (u, z):

$$Q(u, z) = \frac{\det(u - z)}{u - z}.$$

Here, $u \in \mathfrak{u}$ and z is a variable. We can substitute $\omega(u)$ by z, so that $\beta(p, u) := Q(u, \omega(u))$ is an $\mathrm{End}(E)$-valued differential form on $P(E)$ depending polynomially on u.

Theorem A.1. *For any $u \in \mathfrak{u}$, we have*

$$\int_{P(E)} e^{i\omega(u)} \beta(p, u) = e^{iu}. \tag{35}$$

Proof. Since the formula (35) is clearly equivariant under conjugation and analytic in u, it is sufficient to prove it when u is a generic diagonal matrix. In this case, the formula follows right away from the localization formula of Berline and Vergne applied to the action of the torus $\exp(tu)$. Let us for example do the calculation for u being the 3×3 matrix (the general case is identical)

$$u = \begin{pmatrix} i\theta_1 & 0 & 0 \\ 0 & i\theta_2 & 0 \\ 0 & 0 & i\theta_3 \end{pmatrix}.$$

Then

$$Q(u, z) = \begin{pmatrix} (i\theta_2 - z)(i\theta_3 - z) & 0 & 0 \\ 0 & (i\theta_1 - z)(i\theta_3 - z) & 0 \\ 0 & 0 & (i\theta_1 - z)(i\theta_2 - z) \end{pmatrix}.$$

To compute the integral of $e^{i\omega(u)}\beta(p, u)$ over $P(E)$, we can apply the localization theorem. At the point $p_k = \mathbb{C}e_k$, $\omega(u)$ restricts to $i\theta_k$; thus the first diagonal entry of $Q(u, \omega(u))(p_k)$ is zero except for $p_1 = \mathbb{C}e_1$, which is $(i\theta_2 - i\theta_1)(i\theta_3 - i\theta_1)$. Thus, by the localization formula, the first diagonal entry of the matrix $\int_{P(E)} e^{i\omega(u)}\beta(p, u)$ is just $e^{-\theta_1}$. The calculation is similar for all diagonal entries. $\square$

Using the fact that formula (35) is analytic in u, we immediately deduce the following corollary.

Corollary A.2. *Let A be a finite-dimensional commutative algebra over $\mathbb{R}$, and let $u \in A \otimes_{\mathbb{R}} \mathfrak{u}$. Then*

$$\int_{P(E)} e^{i\omega(u)} Q(u, \omega(u)) = e^{iu}. \tag{36}$$

Proof. We write $u = \sum_j x_j f_j$, where f_j is a chosen basis for $\mathfrak{u}$. The two sides of the formula are power series in the variables x_i which coincide for real values of the x_i, and hence coincide formally, and so we can substitute to the x_i any commuting values. $\square$

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