VECTOR PARTITION FUNCTIONS AND INDEX OF TRANSVERSALLY ELLIPTIC OPERATORS.

C. DE CONCINI,   C. PROCESI,   M. VERGNE

ABSTRACT. Let G be a torus acting linearly on a complex vector space M, and let X be the list of weights of G in M. We determine the equivariant K-theory of the open subset M^f of M consisting of points with finite stabilizers. We identify it to the space $DM(X)$ of functions on the lattice $\hat{G}$, satisfying the cocircuit difference equations associated to X, introduced by Dahmen–Micchelli in the context of the theory of splines in order to study vector partition functions (cf. [7]).

This allows us to determine the range of the index map from G-transversally elliptic operators on M to generalized functions on G and to prove that the index map is an isomorphism on the image. This is a setting studied by Atiyah-Singer [1] which is in a sense *universal* for index computations.

1. INTRODUCTION

In 1968 appears the fundamental work of Atiyah and Singer on the index theorem of elliptic operators, a theorem formulated in successive steps of generality ([2], [3]). One general and useful setting is for operators on a manifold M which satisfy a symmetry with respect to a compact Lie group G and are elliptic in directions transverse to the G-orbits. The values of the index are generalized functions on G.

In his Lecture Notes [1] describing joint work with I.M. Singer, Atiyah explains how to reduce general computations to the case in which G is a torus, and the manifold M is a complex linear representation $M_X = \oplus_{a \in X} L_a$, where $X \subset \hat{G}$ is a finite list of characters and L_a the one dimensional complex line where G acts by the character $a \in X$. He then computes explicitly in several cases and ends his introduction saying

" ... for a circle (with any action) the results are also quite explicit. However for the general case we give only a reduction process and one might hope for something explicit. This probably requires the development of an appropriate algebraic machinery, involving cohomology but going beyond it."

The purpose of this paper is to provide this algebraic machinery, which turns out to be a spinoff of the theory of splines, and complete this computation. In fact we show that, once one has guessed the right formulas, the actual computations are obtained using directly the basic properties of

K- theory and turn out to be quite simple. In particular they neither involve computations of the index of specific operators nor any sophisticated commutative algebra. Let us now explain in some detail the content of the paper.

We denote by $T_G^* M$ the closed subset of $T^* M$, union of the conormals to the G orbits. The symbol $\sigma(x, \xi)$ of a pseudo-differential transversally elliptic operator A on M determines an element of the equivariant $K-$theory group $K_G^0(T_G^* M)$ which is defined in a topological fashion. The analytic index of the operator A is the virtual trace class representation of G obtained as difference of the spaces of solutions of A and its adjoint A^* in an appropriate Sobolev space. This index depends only of the class defined by σ in $K_G^0(T_G^* M)$, so that the index defines a $R(G)$ module homomorphism from $K_G^0(T_G^* M)$ to virtual trace class representations of G.

Let V be the dual of the Lie algebra of G, so that $\hat{G}$ is identified with the set Γ of weights of G, a lattice in V. Denote by $\mathcal{C}[\Gamma]$ the space of $\mathbb{Z}$ valued functions on Γ. A trace class representation Θ of G can be decomposed as a direct sum $\Theta = \oplus_{\gamma \in \Gamma} f(\gamma) L_\gamma$, where $f \in \mathcal{C}[\Gamma]$ is a function on Γ with at most polynomial growth. We say that f is the multiplicity function of Θ. We also denote by $\Theta(g)$ the trace of the representation Θ. It is a generalized function on G. In this article, we determine the subspace $\tilde{\mathcal{F}}(X)$ of $\mathcal{C}[\Gamma]$ arising from multiplicities of indices of transversally elliptic operators on M_X. In fact, we study the more general case of a non necessarily connected compact abelian group G, as, in recurrence steps, non connected subgroups of G appear. However in this introduction, we stay with the case of a torus, that is a direct product of circle groups S^1. It is also harmless to assume that G acts on M without non zero fixed vectors, and we do so.

In [1], the image of the index map is described explicitly for S^1 and implicitly in general by a constructive algorithm. It is useful to recall Atiyah-Singer's result for the simple case where $G = S^1$ acts by homotheties on $M_X = \mathbb{C}^{k+1}$. We denote by t the basic character of $S^1 := \{t \mid |t| = 1\}$, so that $R(G) = \mathbb{Z}[t, t^{-1}]$ and $X = [t, t, \ldots, t]$, $k+1$ times.

First Atiyah-Singer constructed a "pushed" $\overline{\partial}$ operator on M_X, with index the trace of the representation of G in the symmetric algebra $S(M_X)$. As $\binom{n+k}{k} = \frac{(n+1)\cdots(n+k)}{k!}$ is the dimension of the space of homogeneous polynomials in $k+1-$variables and degree n, the character of S^1 in $S(M_X) = \oplus_{n=0}^\infty S^n(M_X)$ is the generalized function

$$\Theta_X(t) := \sum_{n=0}^\infty \binom{n+k}{k} t^n.$$

Remark that $n \mapsto \binom{n+k}{k}$ extends to a polynomial function on $\mathbb{Z}$. For any n positive or negative, the function $n \mapsto \binom{n+k}{k}$ represents the dimension of a *virtual space*, the alternate sum of the cohomology spaces of the sheaf $\mathcal{O}(n)$

on $k-$ dimensional projective space. In particular, the tangential Cauchy-Riemann operator on the unit sphere S_{2k+1} of $\mathbb{C}^{k+1}$ is a transversally elliptic operator with index

$$\theta_X(t) := \sum_{n=-\infty}^{\infty} \binom{n+k}{k} t^n,$$

a generalized function on G supported at $t = 1$. Then it is proved in [1] that the index map is an isomorphism from $K_G^0(T_G^*M_X)$ to the space $\tilde{\mathcal{F}}(X)$ of generalized functions on G generated by Θ_X and θ_X under multiplication by an element of $R(G) = \mathbb{Z}[t, t^{-1}]$. In fact the $R(G)$ module generated by Θ_X is free over $R(G)$. The $R(G)$ module generated by θ_X is the torsion submodule. This submodule is the module of polynomial functions on $\mathbb{Z}$ of degree at most k so it is a free $\mathbb{Z}$-module of rank $k + 1$. It corresponds to indices of operators on $\mathbb{C}^{k+1} - \{0\}$, the set where S^1 acts freely and is the space of solutions of the difference equation $\nabla^k f = 0$ where we define the operator ∇ by $(\nabla f)(n) = f(n) - f(n-1)$.

In higher dimensions, the single difference equation $\nabla^k f = 0$ must be replaced by a system of difference equations, discovered by Dahmen–Micchelli as the natural generalization of a system of differential equations associated to splines in approximation theory.

This is the system of difference equations $\nabla_Y f = 0$ associated to cocircuits Y of X ($X \subset \Gamma$, a sequence of weights of G).

Let us recall the definitions introduced in [6].

- A subspace $\underline{r}$ of V is called *rational* if $\underline{r}$ is the span of a sublist of X.
- A *cocircuit* Y in X is a sublist of the form $Y = X \setminus H$ where H is a rational hyperplane.
- Given $a \in \Gamma$, the *difference operator* ∇_a is the operator on functions f on Γ defined by $\nabla_a(f)(b) := f(b) - f(b-a)$.
- For a list Y of vectors, we set $\nabla_Y := \prod_{a \in Y} \nabla_a$.
- The Dahmen-Micchelli space introduced in [7] is the space

$$DM(X) := \{f \,|\, \nabla_Y f = 0, \text{ for every cocircuit } Y \text{ in } X\}.$$

It is easy to see that $DM(X)$ is finite dimensional and consists of quasi–polynomial functions. We describe it in detail in §3. In [5], we introduced non homogeneous difference equations and defined

$$\mathcal{F}(X) := \{f \,|\, \nabla_{X\setminus\underline{r}}f \text{ is supported on } \underline{r} \text{ for every rational subspace } \underline{r}\}.$$

Clearly $DM(X)$ is contained in $\mathcal{F}(X)$.

We denote by $\tilde{\mathcal{F}}(X)$ the space of functions generated by $\mathcal{F}(X)$ under translations by elements of Γ. Thus, $\tilde{\mathcal{F}}(X)$ is the space of functions $f \in \mathcal{C}[\Gamma]$ such that $\nabla_{X\setminus\underline{r}}f$ is supported on a finite number of Γ translates of $\underline{r}$ for every proper rational subspace $\underline{r}$. Both $\mathcal{F}(X)$ and $\tilde{\mathcal{F}}(X)$ have a very precise description given by Formulas (14) and (16). Our main theorem is:

Theorem 1.1. *Let $M_X := \oplus_{a \in X} L_a$. Then the index multiplicity map induces an isomorphism from $K_G^0(T_G^* M_X)$ to $\tilde{\mathcal{F}}(X)$.*

Our proof is inspired by Atiyah-Singer strategy. We start by determining the equivariant K-theory of $T_G M_X^f$, M_X^f being the open subset of M_X of points with finite stabilizers. Here we assume that X spans V. If s denotes the dimension of G, we prove

Theorem 1.2.   *i) We have isomorphisms*

$$K_G^s(M_X^f) \cong K_G^0(T_G^* M_X^f) \overset{ind_m}{\cong} DM(X)$$

where ind_m is the index multiplicity map.
ii) $K_G^{s+1}(M_X^f) = 0$.

We then study the equivariant K-theory of the space $T_G^* M_X$ by induction over the natural stratification of $M_X = \cup_i F_i$, using as strata F_i the union of all orbits of dimension i.

Let us comment on our method. The proof of the exactness of several important sequences is based on the purely combinatorial fact that, if $X := [a, Z]$ is formed by adding a vector a to a list Z, then the map

$$\nabla_a : DM(X) \to DM(Z)$$

is surjective and its kernel is also a space $DM(\tilde{Z})$. This is just one of the several incarnations of the deletion and restriction principle in the theory of hyperplane arrangements (see [9]). It also corresponds to a natural geometric description of the zonotope $B(X)$ (cf. [5]) which we do not discuss here.

We give an elementary proof of this result based on a construction of generators for $DM(X)$. This is done in the first two sections. Although hidden in the proof, this system of generators corresponds to the generators given by Atiyah-Singer of $K_G^*(M_X^f)$.

We can deduce easily generators for $K_G^0(T_G^* M_X)$ from Theorem 1.1. Consider all possible complex structures J on M_X obtained in the following way. We consider any decomposition of X in A and B so that the cone $C(F) := C(A, -B)$ spanned by A and $-B$ is a pointed cone. We then take the conjugate complex structures on the lines L_b, with $b \in B$. Then, to this complex structure J is associated a "pushed" transversally elliptic $\overline{\partial}$ operator on M_J with symbol $At_J \in K_G^0(T_G^* M_X)$. We recall the definition of At_J in (5.9). As a consequence of our theorem, we obtain a simple proof of Atiyah-Singer system of generators for $K_G^0(T_G^* M_X)$.

Theorem 1.3. *The symbols At_J generates the $R(G)$ module $K_G^0(T_G^* M_X)$.*

The generalized functions associated to $DM(X)$ are distributions supported on a certain finite subset $P(X)$ of G, namely the points $p \in G$ having a fixed point in M_X^f. We will pursue in a subsequent article the description of the equivariant cohomology of M_X^f and of the natural strata of

M_X, and relate this to the local components of the distributions associated to $DM(X)$ at a point $p \in P(X)$ and other fixed points.

2. Combinatorics

2.1. **Notations.** Let G be a compact abelian Lie group of dimension s with character group Γ. If $\gamma \in \Gamma$, we denote by $\chi_\gamma(g) := g^\gamma$ the corresponding homomorphism $\chi_\gamma : G \to S^1$, where $S^1 := \{u \in \mathbb{C} \mid |u| = 1\}$.

This suggestive notation for characters is justified by the following. If $H = (S^1)^N$ is the product of N copies of the circle, then every character of H is of the form $(u_1, u_2, \ldots, u_N) \to u_1^{\gamma_1} \cdots u_N^{\gamma_N}$ where the γ_i are integers. We denote this character by the multi-index notation $u \mapsto u^\gamma$, where $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_N] \in \mathbb{Z}^N$. Take now a closed subgroup G of H, then every character of G is also of this form where now γ is in a quotient Γ of the lattice $\mathbb{Z}^N$.

We consider here G as an abstract compact abelian Lie group, so that Γ is a discrete finitely generated abelian group. The group law on the character group Γ of G will be denoted additively. If $\gamma_1, \gamma_2 \in \Gamma$, the character associated to $\gamma_1 + \gamma_2$ is the pointwise multiplication

$$\chi_{\gamma_1 + \gamma_2}(g) = \chi_{\gamma_1}(g)\chi_{\gamma_2}(g) = g^{\gamma_1}g^{\gamma_2} = g^{\gamma_1 + \gamma_2}$$

of characters.

Let $U := \mathfrak{g}$ be the Lie algebra of G and $V := \mathfrak{g}^*$ the dual vector space. The spaces U, V are real vector spaces of dimension s. Denote by G^0 the connected component of the identity of G, so that G^0 is a connected torus with Lie algebra U, the group G/G^0 is a finite group, and G is isomorphic, although not in a canonical way, to $G^0 \times G/G^0$. We denote by Γ_t the torsion subgroup of Γ, identified with the character group of the group G/G^0. Then the quotient group Γ/Γ_t may be identified with the weight lattice Λ of G^0, a lattice in V, as follows. Let $\gamma \in \Gamma$ and $g \in G^0$ and write $g = \exp(X)$, where $X \in \mathfrak{g}$. Then we can write $g^\gamma = e^{2i\pi\langle\lambda, X\rangle}$, where $\lambda \in \Lambda$ is an element of the weight lattice. In other words, the linear form $\lambda \in \Lambda$ is the differential $d\chi_\gamma : \mathfrak{g} \to \mathrm{Lie}(S^1) \sim \mathbb{R}$ of the character $\chi_\gamma : G \to S^1$, where the exponential map $\mathbb{R} \to S^1$ is given by $u \to e^{2i\pi u}$.

If $\gamma \in \Gamma$, then we denote by $\overline{\gamma} \in V$ the corresponding element of the weight lattice $\Gamma/\Gamma_t \equiv \Lambda \subset V$.

Let $\mathcal{C}[\Gamma]$ be the space of functions on Γ with values in $\mathbb{Z}$. We consider $\mathcal{C}[\Gamma]$ as a Γ module, by associating to $\gamma \in \Gamma$ the translation operator $\tau_\gamma f(\nu) := f(\nu - \gamma)$. We set $\nabla_\gamma = 1 - \tau_\gamma$ and call it a *difference operator*.

Given a finite list Y of elements of Γ, we set $\nabla_Y := \prod_{a \in Y} \nabla_a$. Also we set $a_Y := \sum_{a \in Y} a$ and $\overline{a}_Y := \sum_{a \in Y} \overline{a}$.

3. The space $DM(X)$

We introduce here the basic space of functions $DM(X)$ on Γ, which can be thought of as a multi dimensional generalization of suitable binomial coefficients.

Let us fix a finite non empty list X of characters in Γ. Associated to X, we get a list of vectors $\overline{X} := \{\overline{a} \mid a \in X\}$ in $\Lambda \subset V$.

Any non–zero element $\overline{a} \in \overline{X}$ defines a hyperplane in U. Thus, we associate to X the hyperplane arrangement $\mathcal{H}_X$ in U whose elements are all the intersections of the hyperplanes associated to the non–zero elements of $\overline{X}$. Given a set or list of vectors A in V, we denote by $\langle A \rangle$ the subspace spanned by A.

Definition 3.1. A subspace $\underline{r}$ of V is called *rational* (relative to X) if it is spanned by a sublist $\overline{Y}$ of $\overline{X}$.

The set of all rational subspaces associated to X will be denoted by S_X, and the subset of rational subspaces of dimension i by $S_X^{(i)}$.

If Y is a sublist in X, the rational subspace spanned by $\overline{Y}$ will be sometimes denoted, by abuse of notations, by $\langle Y \rangle := \langle \overline{Y} \rangle$. Similarly, by abuse of notations, we will often transfer to X definitions which make sense only for $\overline{X}$. For example, if $a \in X$ and F is a subset of U, we will say that a *is positive on* F if $\overline{a}$ is positive on every element u of F. If A is a list of elements of X, we will say that A *spans a pointed cone* if $\overline{A}$ spans a pointed cone. We then denote by $C(A)$ the elements of Γ which are non negative integral linear combinations of elements of A. If $\underline{r}$ is a rational subspace of V, we write $X \cap \underline{r}$ (resp. $X \setminus \underline{r}$) for the set of elements $a \in X$ such that $\overline{a} \in \underline{r}$ (resp. $\overline{a} \notin \underline{r}$).

Definition 3.2. If $\underline{r}$ has codimension 1, that is $\underline{r}$ is a rational hyperplane, the sublist $X \setminus \underline{r}$ will be called a *cocircuit*.

Our list X of elements in Γ defines a set of subgroups of G. If $a \in X$, we denote by G_a the kernel of the corresponding character $g \mapsto g^a : G \to S^1$. The set of all connected components of all intersections of these subgroups is a finite family of cosets of certain tori and is called the *real toric arrangement* associated to X. We let $\mathcal{T}_X^i$ be the set of tori of dimension i contained in this arrangement.

Notice that there is a 1–1 correspondence between the tori in $\mathcal{T}_X^i$ and the rational subspaces of codimension i by associating to T the space $\langle X_T \rangle$, X_T being the characters in X containing T in their kernel.

We are ready to introduce the space of functions $DM(X)$. This space is a straightforward generalization of the space introduced by Dahmen–Micchelli for vectors in a lattice. The extension to any finitely generated abelian group is very convenient for inductive constructions as we shall see in Theorem 3.22.

Definition 3.3. The space $DM(X)$ is the space of functions $f \in \mathcal{C}[\Gamma]$ which are solutions of the linear equations $\nabla_{X\setminus\underline{r}} f = 0$ as $\underline{r}$ runs over all rational hyperplanes contained in V.

In other words, the space $DM(X)$ is the space of functions $f \in \mathcal{C}[\Gamma]$ which are solutions of the linear equations $\nabla_Y f = 0$ as Y runs over sublists of X such that $X \setminus Y$ do not generate V

In particular, if X' is a sublist of X, the space $DM(X')$ is a subspace of $DM(X)$.

Example 3.4. If $G = S^1 \times \mathbb{Z}/2\mathbb{Z}$, $\Gamma = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Take the list $X = \{1 + \epsilon, 2 + \epsilon\}$, where $n = 1, 2$ denotes the character $\chi_n(t) = t^n$ of S^1 and ϵ the non trivial character of $\mathbb{Z}/2\mathbb{Z}$. Then $DM(X)$ consists of the functions f such that for every $\gamma \in \Gamma$

$$f(\gamma) - f(\gamma - 1 - \epsilon) - f(\gamma - 2 - \epsilon) + f(\gamma - 3) = 0.$$

It is then clear that f is uniquely determined by the values it takes on the points $0, -1, -2, -\epsilon, -1 - \epsilon, -2 - \epsilon$ and it is a free abelian group of rank 6.

Notice that, if $\Gamma = \Gamma_t$ is finite, then $DM(X) = \mathbb{Z}[\Gamma]$ and does not depend on X. Also, if the joint kernel of all the elements in X is of positive dimension, $DM(X) = 0$ since $\nabla_\emptyset = \nabla_{X\setminus\langle X\rangle} = 1$. In view of this, we are often going to tacitly make the *non degeneracy* assumption that $\langle X\rangle = V \neq \{0\}$.

Under this assumption, $DM(X)$ is the space of solutions of the equations $\nabla_Y f = 0$ as Y runs over all cocircuits.

Remark 3.5. If $X = [A, B]$ is decomposed into two lists, and if $X' = [A, -B]$ is obtained from the list X by changing the signs of the elements b in the sublist B, then $DM(X) = DM(X')$.

3.6. **Convolutions in the space $\mathcal{C}[\Gamma]$.** The purpose of this and the following section is to exhibit explicit elements in $DM(X)$, which have a natural interpretation in index theory.

If $\gamma \in \Gamma$, we denote by δ_γ the function on Γ identically equal to 0 on Γ, except for $\delta_\gamma(\gamma) = 1$. With this notation, we also write an element $f \in \mathcal{C}[\Gamma]$ as

$$f = \sum_{\gamma \in \Gamma} f(\gamma)\delta_\gamma.$$

The *support* of a function $f \in \mathcal{C}[\Gamma]$ is the set of elements $\gamma \in \Gamma$ with $f(\gamma) \neq 0$. If S is a subset of V, we will say, by abuse of notations, that f is supported on S if $f(\gamma) = 0$ except if $\overline{\gamma} \in S$.

Remark 3.7. If S_1, S_2 are the supports of f_1, f_2, the convolution product $f_1 * f_2$ is defined when, for every $\gamma \in \Gamma$, we have only finitely many pairs $(\gamma_1, \gamma_2) \in S_1 \times S_2$ with $\gamma = \gamma_1 + \gamma_2$. Then

$$(f_1 * f_2)(\gamma) = \sum_{\gamma_1 \in S_1, \gamma_2 \in S_2 \,|\, \gamma_1 + \gamma_2 = \gamma} f_1(\gamma_1)f_2(\gamma_2).$$

In particular, if $\gamma \in \Gamma$, $(\delta_\gamma * f)(x) = f(x-\gamma) = (\tau_\gamma f)(x)$. Thus convolution with elements of finite support, i.e. elements in $\mathbb{Z}[\Gamma] = R(G)$, gives the $R(G)$ module structure on $\mathcal{C}[\Gamma]$ already introduced.

Recall that, for $\gamma \in \Gamma$, we denoted by $g \to g^\gamma$ the corresponding function on G.

If $f \in \mathcal{C}[\Gamma]$ has finite support, the function $g \mapsto \sum_{\gamma \in \Gamma} f(\gamma)g^\gamma$ is a complex-valued function on G with Fourier coefficients $f(\gamma)$. When f is arbitrary, we also write the formal series of functions on G defined by

$$\Theta(f)(g) = \sum_{\gamma \in \Gamma} f(\gamma)g^\gamma.$$

This is usually formal although, in the cases of interest in this paper, it may give a convergent series in the sense of generalized functions on G. We denote by $C^{-\infty}(G)$ the space of generalized functions on G. The space $C^\infty(G)$ of smooth functions on G is naturally a subspace of $C^{-\infty}(G)$. We will often use the notation $\Theta(g)$ to denote a generalized function Θ on G although (in general) the value of Θ on a particular point g of G does not have a meaning. By definition, Θ is a linear form on the space of smooth densities on G (satisfying suitable continuity conditions).

In fact, all the elements $f = \sum_\gamma f(\gamma)\delta_\gamma$ that we shall consider have the property that the function $f(\gamma)$ has polynomial growth at infinity. This implies that the series $\Theta(f)$ converges in the distributional sense, and f can be interpreted analytically as the Fourier series of a $C^{-\infty}$ function $\Theta(f)$ on G. The convolution product on f_1, f_2 (if defined) corresponds to the multiplication (if defined) of the functions $\Theta(f_1)(g)\Theta(f_2)(g)$.

If $\gamma \in \Gamma$ is of infinite order, we set:

$$H_\gamma := \sum_{k=0}^\infty \delta_{k\gamma}.$$

The function H_γ is supported on the "half line" $\mathbb{Z}_{\geq 0}\gamma$ and is the discrete analogue of the Heaviside function. In fact $\nabla_\gamma H_\gamma = \delta_0$.

Let $A = [a_1, a_2, \ldots, a_n]$ be a finite list of vectors in $\Gamma \setminus \Gamma_t$ such that $\overline{A}$ spans a pointed cone in V, and consider the *generalized cone*

$$C(A) := \{\sum_{i=1}^n m_i a_i, \ m_i \in \mathbb{Z}_{\geq 0}\} \subset \Gamma.$$

which, according to our conventions, we treat as *a pointed cone*.

We easily see that every element $\gamma \in C(A)$ can be written in the form $\gamma = \sum_{i=1}^n m_i a_i, \ m_i \in \mathbb{Z}_{\geq 0}$ only in finitely many ways. It follows that the convolution H_A given by

$$H_A := H_{a_1} * H_{a_2} * \cdots * H_{a_n}$$

is well defined and supported on $C(A)$.

3.8. The function $\mathcal{P}^{F_{\underline{r}}}_{X\backslash\underline{r}}$. Given a rational subspace $\underline{r}$, $X \setminus \underline{r}$ defines a hyperplane arrangement in the space $\underline{r}^\perp \subset U$ orthogonal to $\underline{r}$. Take an open face $F_{\underline{r}}$ in $\underline{r}^\perp$ with respect to this hyperplane arrangement. By definition a vector $u \in \underline{r}^\perp$ and such that $\langle u, \overline{a} \rangle \neq 0$ for all $a \in X \setminus \underline{r}$ lies in a unique such face $F_{\underline{r}}$.

We get a decomposition of $X \setminus \underline{r}$ into the two lists A, B of elements which are positive (respectively negative) on $F_{\underline{r}}$. We denote by $C(F_{\underline{r}}, X)$ the cone $C(A, -B)$ generated by the list $[A, -B]$. Then $C(A, -B)$ is a pointed cone.

Notice that, given $a \in \Gamma$, we have $\nabla_a = -\tau_a \nabla_{-a}$.

We are going to consider the function $\mathcal{P}^{F_{\underline{r}}}_{X}$ which is characterized by the following two properties.

Lemma 3.9. *There exists a unique element $\mathcal{P}^{F_{\underline{r}}}_{X\backslash\underline{r}} \in \mathcal{C}[\Gamma]$ such that*

    i) $(\prod_{a \in X\backslash\underline{r}} \nabla_a)\mathcal{P}^{F_{\underline{r}}}_{X\backslash\underline{r}} = \delta_0$.

    ii) $\mathcal{P}^{F_{\underline{r}}}_{X\backslash\underline{r}}$ *is supported in* $-b_B + C(A, -B)$.

Proof. We define $\mathcal{P}^{F_{\underline{r}}}_{X\backslash\underline{r}}$ by convolution product of "Heaviside functions":

$$(1) \qquad \mathcal{P}^{F_{\underline{r}}}_{X\backslash\underline{r}} = (-1)^{|B|}\delta_{-b_B} * H_A * H_{-B}$$

where $b_B = \sum_{b \in B} b$. As $(\prod_{a \in X\backslash\underline{r}} \nabla_a) = (-1)^{|B|}\tau_{b_B}(\prod_{a \in A} \nabla_a)(\prod_{b \in B} \nabla_{-b})$, we see that $\mathcal{P}^{F_{\underline{r}}}_{X\backslash\underline{r}}$ is well defined, satisfies the two properties and is unique. $\square$

Remark 3.10. When G is connected so that Γ is a lattice, this function was constructed in [6].

It is easily seen that the series $\sum_{\gamma \in \Gamma} \mathcal{P}^{F_{\underline{r}}}_{X\backslash\underline{r}}(\gamma)g^\gamma$ defines a generalized function $\Theta^{F_{\underline{r}}}_{X\backslash\underline{r}}$ on G such that $\prod_{a \in X\backslash\underline{r}}(1 - g^a)\Theta^{F_{\underline{r}}}_{X\backslash\underline{r}}(g) = 1$ on G. Thus $\Theta^{F_{\underline{r}}}_{X\backslash\underline{r}}(g)$ is the inverse of $\prod_{a \in X\backslash\underline{r}}(1-g^a)$, in the space of generalized functions on G, with Fourier coefficients in the "pointed cone" $C(A, -B)$.

We can generalize this as follows. Choose an orientation of each $\underline{r} \in S_X$. Take a pair of rational subspaces $\underline{r} \in S_X^{(i)}$ and $\underline{t} \in S_X^{(i+1)}$ with $\underline{r} \subset \underline{t}$. We say that a vector v in $\underline{t} \setminus \underline{r}$ is positive if the orientation on $\underline{t}$ induced by v and the orientation on $\underline{r}$ coincides with that chosen of $\underline{t}$. Set

$$A = \{a \in X \cap \underline{t} \,|\, \overline{a} \text{ is positive}\}, \quad B = \{b \in X \cap \underline{t} \,|\, \overline{b} \text{ is negative}\}.$$

We define

$$(2) \qquad \mathcal{P}^{t+}_{\underline{r}} := (-1)^{|B|}\delta_{-b_B} * H_A * H_{-B},$$

$$(3) \qquad \mathcal{P}^{t-}_{\underline{r}} := (-1)^{|A|}\delta_{-a_A} * H_B * H_{-A}.$$

The function $\mathcal{P}^{t+}_{\underline{r}}$ is supported on the cone $-\sum_{b \in B} b + C(A, -B)$ while $\mathcal{P}^{t-}_{\underline{r}}$ is supported on the cone $-\sum_{a \in A} a + C(-A, B)$.

Definition 3.11. Take a pair of oriented rational subspaces $\underline{r} \in S_X^{(i)}$ and $\underline{t} \in S_X^{(i+1)}$ with $\underline{r} \subset \underline{t}$. We define

$$\mathcal{Q}_{\underline{r}}^{t}(X) = \mathcal{P}_{\underline{r}}^{t+} - \mathcal{P}_{\underline{r}}^{t-}.$$

If X is fixed, we will write simply $\mathcal{Q}_{\underline{r}}^{t}$ instead of $\mathcal{Q}_{\underline{r}}^{t}(X)$.

Similarly $\mathcal{P}_{\underline{r}}^{t\pm}, \mathcal{Q}_{\underline{r}}^{t}$ define generalized functions on G.

Definition 3.12. We set

(4)
$$\theta_{\underline{r}}^{t\pm}(g) := \sum_{\gamma \in \Gamma} \mathcal{P}_{\underline{r}}^{t\pm}(\gamma) g^{\gamma},$$

(5)
$$\theta_{\underline{r}}^{t}(g) := \theta_{\underline{r}}^{t+}(g) - \theta_{\underline{r}}^{t-}(g) = \sum_{\gamma \in \Gamma} \mathcal{Q}_{\underline{r}}^{t}(\gamma) g^{\gamma}.$$

Let us use the notation $D_Y(g) := \prod_{a \in Y}(1 - g^a)$. Then

(6)
$$D_{(X \setminus \underline{t}) \setminus \underline{r}}(g) \theta_{\underline{r}}^{t\pm}(g) = 1,$$

so that

$$D_{(X \setminus \underline{t}) \setminus \underline{r}}(g) \theta_{\underline{r}}^{t}(g) = 0.$$

Set $\Gamma_{\underline{r}} := \Gamma \cap \underline{r}$ equal to the pre-image of $\Lambda \cap \underline{r}$ under the quotient $\Gamma \to \Lambda$. If f is a function on $\Gamma_{\underline{r}}$, the hypotheses of Remark 3.7 are satisfied and we can perform the convolutions $\mathcal{P}_{\underline{r}}^{t\pm} * f$. So convolution by $\mathcal{Q}_{\underline{r}}^{t}$ induces a map

$$\Pi_{\underline{r}}^{t} : \mathcal{C}[\Gamma_{\underline{r}}] \to \mathcal{C}[\Gamma_{\underline{t}}], \quad f \mapsto \mathcal{Q}_{\underline{r}}^{t} * f.$$

Given a rational subspace $\underline{r}$, let us consider the space $DM(X \cap \underline{r})$ inside $\mathcal{C}[\Gamma_{\underline{r}}]$.

The following statement is similar to Proposition 3.6 in [6].

Proposition 3.13. $\Pi_{\underline{r}}^{t}$ maps $DM(X \cap \underline{r})$ to $DM(X \cap \underline{t})$.

Proof. Let $Y := X \cap \underline{r}$, $Y' := X \cap \underline{t}$. If $T \subset Y'$ is a cocircuit in Y', we need to see that $\nabla_T(\mathcal{Q}_{\underline{r}}^{t} * f) = 0$ for each $f \in DM(X \cap \underline{r})$.

By definition

$$\nabla_{A \cup B} \mathcal{P}_{\underline{r}}^{t+} = \nabla_{A \cup B} \mathcal{P}_{\underline{r}}^{t-} = \delta_0$$

so that, if $T = A \cup B = Y' \setminus Y$,

$$\nabla_T(\mathcal{Q}_{\underline{r}}^{t} * f) = (\nabla_T \mathcal{Q}_{\underline{r}}^{t}) * f = 0.$$

Otherwise, the set $Y \cap T$ is a cocircuit in Y and then $\nabla_{Y \cap T} f = 0$. So

$$\nabla_T(\mathcal{Q}_{\underline{r}}^{t} * f) = (\nabla_{T \setminus Y} \mathcal{Q}_{\underline{r}}^{t}) * (\nabla_{Y \cap T} f) = 0.$$

$\square$

This proposition gives a way to construct explicit elements of $DM(X)$. Take a flag ϕ of oriented rational subspaces $0 = \underline{r}_0 \subset \underline{r}_1 \subset \underline{r}_2 \subset \cdots \subset \underline{r}_s = V$ with $\dim(\underline{r}_i) = i$. Set $\mathcal{Q}_i := \mathcal{P}_{\underline{r}_{i-1}}^{r_i+} - \mathcal{P}_{\underline{r}_{i-1}}^{r_i-}$, then

Proposition 3.14.

$$(7) \qquad\qquad \mathcal{Q}_\phi^X := \mathcal{Q}_1 * \mathcal{Q}_2 * \cdots * \mathcal{Q}_s$$

lies in $DM(X)$.

Set $\theta_i^\pm(g) := \theta_{\underline{r}_{i-1}}^{\underline{r}_i \pm}(g)$ the corresponding generalized functions on G, then their product is well defined. We set

$$(8) \qquad\qquad \theta_\phi^X(g) := \prod_{i=1}^s (\theta_i^+(g) - \theta_i^-(g)).$$

3.15. **Removing a vector.** In this section, we prove the key technical result of this article, that is Theorem 3.17.

Given a subgroup $\Psi \subset \Gamma$, we identify the space $\mathcal{C}[\Gamma/\Psi]$ of $\mathbb{Z}$ valued functions on Γ/Ψ with the subspace of $\mathcal{C}[\Gamma]$ formed by the functions constant on the cosets of Ψ. In particular, given $a \in \Gamma$, a function is constant on the cosets of $\mathbb{Z}a$ if and only if $\nabla_a f = 0$. Therefore the space $\mathcal{C}[\Gamma/\mathbb{Z}a]$ is identified with $\ker(\nabla_a)$.

We define

$$(9) \qquad i_a : \mathcal{C}[\Gamma/\mathbb{Z}a] \to \ker(\nabla_a), \quad i_a(f)(x) := f(x + \mathbb{Z}a).$$

The space $DM(X \cap \underline{r})$ is a space of functions on $\Gamma_{\underline{r}} := \Gamma \cap \underline{r}$. We embed $\mathcal{C}[\Gamma_{\underline{r}}]$ into $\mathcal{C}[\Gamma]$ by extending each function by 0 to the entire Γ outside $\Gamma \cap \underline{r}$.

Definition 3.16. Define $DM^{(G)}(X \cap \underline{r})$ to be the $R(G)$ submodule of $\mathcal{C}[\Gamma]$ generated by the image of $DM(X \cap \underline{r})$ inside $\mathcal{C}[\Gamma]$.

Let us assume now that X is non degenerate. Using Proposition 3.13, we obtain, for each $\underline{r} \in S_X^{(s-1)}$, a homomorphism

$$\Pi_{\underline{r}} : DM^{(G)}(X \cap \underline{r}) \to DM(X), \qquad \Pi_{\underline{r}} := \Pi_{\underline{r}}^V$$

of $R(G)$ modules given by convolution with $\mathcal{Q}_{\underline{r}}^V(X)$, the function given by Definition 3.11.

We can thus consider the map

$$\Pi_X : H^X \to DM(X)$$

where

$$\Pi_X := \oplus_{\underline{r} \in S_X^{(s-1)}} \Pi_{\underline{r}}, \quad H^X := \oplus_{\underline{r} \in S_X^{(s-1)}} DM^{(G)}(X \cap \underline{r}).$$

Take $a \in X$ of infinite order. Write $X = [Z, a]$ and set $\tilde{Z}$ to be the image of Z in $\Gamma/\mathbb{Z}a$.

Theorem 3.17. *1) We have an exact sequence:*

$$(10) \qquad\qquad 0 \to DM(\tilde{Z}) \xrightarrow{i_a} DM(X) \xrightarrow{\nabla_a} DM(Z) \to 0.$$

2) The map

$$\Pi_X : H^X \to DM(X)$$

is surjective.

Proof. We proceed by induction on the number of elements in X.

We know that the kernel of ∇_a in $\mathcal{C}[\Gamma]$ equals $i_a(\mathcal{C}[\Gamma/\mathbb{Z}a])$. We first show that $\ker(\nabla_a) \cap DM(X) = i_a(DM(\tilde{Z}))$. This is clear since a cocircuit in $\tilde{Z}$ is the image of a cocircuit in X not containing a.

It is also clear that ∇_a maps $DM(X)$ to $DM(Z)$. In fact, let $T \subset Z$ be a cocircuit in Z, then $T \cup \{a\}$ is a cocircuit in X. Thus $\nabla_T(\nabla_a f) = 0$ if $f \in DM(X)$.

Thus in order to finish the proof of the exactness of the sequence (10), it only remains to show that the map $\nabla_a : DM(X) \to DM(Z)$ is surjective.

If $\{a\}$ is a cocircuit, that is if Z is degenerate, then $DM(Z) = 0$, the map i_a is an isomorphism and there is nothing more to be proven.

Assume that Z is non degenerate. Given $\underline{r} \in S_X^{(s-1)}$, we have three possibilities.

In the first two cases, we assume that $a \in \underline{r}$ so that $X \cap \underline{r} = (Z \cap \underline{r}) \cup \{a\}$ and set $\tilde{\underline{r}}$ equal to the hyperplane in $V/\langle a \rangle$ image of $\underline{r}$.

> i) $Z \cap \underline{r}$ does not span $\underline{r}$, that is $\underline{r} \in S_X^{(s-1)} \setminus S_Z^{(s-1)}$. In this case, we get the isomorphism
>
> $$DM^{(G)}(\tilde{Z} \cap \tilde{\underline{r}}) \xrightarrow{i_a} DM^{(G)}(X \cap \underline{r}).$$
>
> Furthermore $\Pi_{\underline{r}} \circ i_a = i_a \circ \Pi_{\tilde{\underline{r}}}$.
>
> ii) $Z \cap \underline{r}$ spans $\underline{r}$. In this case, by induction, we may assume that we have the exact sequence
>
> $$0 \to DM^{(G)}(\tilde{Z} \cap \tilde{\underline{r}}) \xrightarrow{i_a} DM^{(G)}(X \cap \underline{r}) \xrightarrow{\nabla_a} DM^{(G)}(Z \cap \underline{r}) \to 0.$$
>
> Furthermore $\Pi_{\underline{r}} \circ i_a = i_a \circ \Pi_{\tilde{\underline{r}}}$.
>
> iii) $a \notin \underline{r}$. In this case, we get an equality $DM^{(G)}(X \cap \underline{r}) = DM^{(G)}(Z \cap \underline{r})$. Denote by $\Pi_{\underline{r}}(X), \Pi_{\underline{r}}(Z)$ respectively the two maps given by convolution by the functions $\mathcal{Q}_{\underline{r}}^V(X), \mathcal{Q}_{\underline{r}}^V(Z)$ associated to the two lists X and Z. We clearly have $\nabla_a \mathcal{Q}_{\underline{r}}^V(X) = \mathcal{Q}_{\underline{r}}^V(Z)$ and hence the identity $\Pi_{\underline{r}}(Z) = \nabla_a \Pi_{\underline{r}}(X)$.

Define a map $p : H^X \to H^Z$ as follows. If $\underline{r} \in S_X^{(s-1)} \setminus S_Z^{(s-1)}$, we are in case i) and we set $p = 0$ on $DM^{(G)}(X \cap \underline{r})$. In case ii) $\underline{r} \in S_Z^{(s-1)}$ and $a \in \underline{r}$, we set $p := \nabla_a : DM^{(G)}(X \cap \underline{r}) \to DM^{(G)}(Z \cap \underline{r})$, a surjective map by induction. Finally in case iii) $\underline{r} \in S_Z^{(s-1)}$ and $a \notin \underline{r}$, then $DM^{(G)}(X \cap \underline{r}) = DM^{(G)}(Z \cap \underline{r})$ and we take $p = Id$ the identity.

We have thus that p is surjective and its kernel is the direct sum of $DM^{(G)}(X \cap \underline{r})$ for $\underline{r} \in S_X^{(s-1)} \setminus S_Z^{(s-1)}$ plus the direct sum of the kernels of the maps $\nabla_a : DM^{(G)}(X \cap \underline{r}) \to DM^{(G)}(Z \cap \underline{r})$, over the $\underline{r} \in S_Z^{(s-1)}$ with $a \in \underline{r}$.

Let us next compare $\ker(p)$ with $H^{\tilde{Z}}$. The preimage in V of a subspace $\underline{v} \in S_{\tilde{Z}}^{(s-2)}$ is a subspace $\underline{r} \in S_X^{(s-1)}$ with $a \in \underline{r}$, thus a space of the first two types. By all the previous remarks, taking direct sums and assuming

by induction on dimension our result true for each list $X \cap \underline{r}$, $\underline{r} \in S_X^{(s-1)}$, we deduce an exact sequence

$$0 \to H^{\tilde{Z}} \to H^X \to H^Z \to 0.$$

Furthermore the diagram

(11)
$$
\begin{array}{ccccc}
0 \to H^{\tilde{Z}} & \longrightarrow & H^X & \longrightarrow & H^Z \to 0 \\
\Pi_{\tilde{Z}} \downarrow & & \Pi_X \downarrow & & \Pi_Z \downarrow \\
0 \to DM(\tilde{Z}) & \xrightarrow{i_a} & DM(X) & \xrightarrow{\nabla_a} & DM(Z) \to 0
\end{array}
$$

commutes.

By induction, the map Π_Z is surjective. We immediately deduce the surjectivity of $\nabla_a : DM(X) \to DM(Z)$, hence part 1).

2) To finish, we have to prove the surjectivity of Π_X. By induction, both $\Pi_{\tilde{Z}}$ and Π_Z are surjective so that also Π_X is, as desired. $\qquad \square$

Let $s := \dim V$, we say that a sublist $\underline{b} := \{b_1, \ldots, b_s\}$ of X is *a basis* if the elements $\overline{\underline{b}} := \{\overline{b}_1, \ldots, \overline{b}_s\}$ form a basis for V. Let us denote by $\mathcal{B}_X$ the set of all such bases extracted from X.

For each $\underline{b} \in \mathcal{B}_X$, the elements $\{\overline{b}_1, \ldots, \overline{b}_s\}$ generate a lattice $\langle \overline{b}_1, \ldots, \overline{b}_s \rangle$ in Λ of some index $d(\underline{b})$ (equal to the absolute value of the determinant or the volume of the corresponding parallelepiped).

The elements $\underline{b}$ generate a free abelian group $\mathbb{Z}_{\underline{b}}$ and the subgroup $\Gamma_t \times \mathbb{Z}_{\underline{b}}$ is still of index $d(\underline{b})$ in Γ. Set

$$\delta(X) := \sum_{\underline{b} \in \mathcal{B}_X} d(\underline{b}).$$

The number $\delta(X)$ has a geometric meaning, it is the volume of the *zonotope* $B(X) := \{\sum_{a \in X} t_a \overline{a} \,|\, 0 \le t_a \le 1\}$. This is a consequence of the decomposition of the zonotope into parallelepipeds of volume $d(\underline{b})$ (cf. [5]).

Corollary 3.18. $DM(X)$ *is a free* $\mathbb{Z}[\Gamma_t]$*-module of rank* $\delta(X)$.

Proof. Notice that, if Γ is finite or X consists of torsion elements, our statement is trivially true.

We now proceed by induction using the exact sequence (10). Thus, take $a \in X$ an element of infinite order and write $X = [Z, a]$. By induction, the space $DM(Z)$ is a free $\mathbb{Z}[\Gamma_t]$-module of rank $\delta(Z)$, thus it suffices to see that $DM(\tilde{Z})$ is a free $\mathbb{Z}[\Gamma_t]$-module of rank $\delta(X) - \delta(Z)$.

Set $\tilde{\Gamma} = \Gamma/\mathbb{Z}a$. By induction, $DM(\tilde{Z})$ is a free $\mathbb{Z}[\tilde{\Gamma}_t]$ module of rank $\delta(\tilde{Z})$. On the other hand, since a is not a torsion element, $\mathbb{Z}[\tilde{\Gamma}_t]$ is a free $\mathbb{Z}[\Gamma_t]$-module of rank $|\tilde{\Gamma}_t/\Gamma_t|$. Thus it suffices to see that $\delta(X) - \delta(Z) = |\tilde{\Gamma}_t/\Gamma_t| \delta(\tilde{Z})$.

The number $\delta(X) - \delta(Z)$ is the sum, over all the bases $\underline{b}$ of X containing a, of the index $d(\underline{b})$ of $\Gamma_t \times \mathbb{Z}_{\underline{b}}$ in Γ. Pass modulo $a \in \underline{b}$ and denote by $\tilde{\underline{b}}$ the

induced basis in $\tilde{\Gamma}$. We see that

$$|\Gamma_t|d(\underline{b}) = |\Gamma/\mathbb{Z}_{\underline{b}}| = |\tilde{\Gamma}/\mathbb{Z}_{\tilde{\underline{b}}}| = |\tilde{\Gamma}_t|d(\tilde{\underline{b}})$$

as desired.   □

Remark 3.19. One could have also followed the approach of Dahmen–Micchelli to this theorem. Then one needs to show that the space $DM(X)$ restricts isomorphically to the space of $\mathbb{Z}$ valued functions on any set $\delta(u\,|\,X) := \Gamma \cap (u - B(X))$ when u is a generic vector. In turn, this set is a union of $\delta(X)$ cosets of Γ_t and this statement can be interpreted as saying that:

The values on $\delta(u\,|\,X)$ are initial values for the recursion equations given by the cocircuits. A function in $DM(X)$ is completely determined by these values that can be assigned arbitrarily (see [5]).

Proceeding by induction and using the surjectivity of $\Pi_X : H^X \to DM(X)$ we immediately get

Corollary 3.20. *The elements $\mathcal{Q}_\phi^X$, as ϕ varies among complete flags of rational subspaces, span $DM(X)$ as $R(G)$ module.*

3.21. **A basis for $DM(X)$.** This section is not needed for the paper but it gives a very precise description of $DM(X)$.

We assume that X is totally ordered in such a way that the torsion elements follow the elements of infinite order.

Take a basis $\underline{b} \in \mathcal{B}_X$. Given $c \in X$, set $\underline{b}_{\geq c} = \{b \in \underline{b}\,|\,c \leq b\}$. Define $X_{\underline{b}} = \{c \in X\,|\,\overline{c} \in \langle \underline{b}_{\geq c}\rangle\}$. Since $X_{\underline{b}}$ is a sublist of X, we have that $DM(X_{\underline{b}})$ is a subset of $DM(X)$.

Now consider the complete flag $\phi_{\underline{b}}$ of rational subspaces $\underline{r}_i$ generated by $\{\overline{b}_1, \ldots, \overline{b}_i\}$, $i = 1, \ldots, s$. We obtain the element $\mathcal{Q}_{\phi_{\underline{b}}}^{X_{\underline{b}}} \in DM(X_{\underline{b}}) \subset DM(X)$.

Choose a complete set $C_{\underline{b}} \subset \Gamma$ of coset representatives modulo the subgroup $\Gamma_t \times \mathbb{Z}_{\underline{b}}$. To every pair $(\underline{b}, \gamma)$ with $\underline{b} \in \mathcal{B}_X$ and $\gamma \in C_{\underline{b}}$, we associate the element

$$\mathcal{Q}_{\phi_{\underline{b}}, \gamma}^{X_{\underline{b}}} = \tau_\gamma \mathcal{Q}_{\phi_{\underline{b}}}^{X_{\underline{b}}}$$

in $DM(X)$. We can now state our

Theorem 3.22. *The set of elements $\mathcal{Q}_{\phi_{\underline{b}}, \gamma}^{X_{\underline{b}}}$, as $\underline{b}$ varies in $\mathcal{B}_X$ and γ in $C_{\underline{b}}$, is a basis of $DM(X)$ as a $\mathbb{Z}[\Gamma_t]$ module.*

Proof. Let a in X be the least element. As usual, we can assume that it is of infinite order. As in the previous section, write $X = [a, Z]$ and set $\tilde{Z}$ to be the image of Z in $\tilde{\Gamma} := \Gamma/\mathbb{Z}a$.

Take a basis $\underline{b} \in \mathcal{B}_X$. If a does not lie in $\underline{b}$, then $\underline{b} \in \mathcal{B}_Z$. Observe that $Z_{\underline{b}} = X_{\underline{b}} \setminus \{a\}$ and that for every $\gamma \in C_{\underline{b}}$

$$\mathcal{Q}_{\phi_{\underline{b}}, \gamma}^{Z_{\underline{b}}} = \nabla_a(\mathcal{Q}_{\phi_{\underline{b}}, \gamma}^{X_{\underline{b}}}).$$

Thus, by induction, the elements $\nabla_a(\mathcal{Q}^{X_{\underline{b}}}_{\phi_{\underline{b}},\gamma})$, as $\underline{b}$ varies in $\mathcal{B}_Z$ and γ in $C_{\underline{b}}$, are a basis of $DM(Z)$ as a $\mathbb{Z}[\Gamma_t]$ module.

If on the other hand a lies in $\underline{b}$, consider the corresponding basis $\underline{\tilde{b}} \in \mathcal{B}_{\tilde{Z}}$. Take the quotient map $p : \Gamma \to \tilde{\Gamma}$. It is clear that p maps $C_{\underline{b}}$ injectively onto a set of representatives of the cosets of $\Gamma_t \times \mathbb{Z}_{\underline{\tilde{b}}}$ in $\tilde{\Gamma}$. Define an equivalence relation on $p(C_{\underline{b}})$ setting $\gamma \simeq \gamma'$ if $\gamma - \gamma' \in \tilde{\Gamma}_t \times \mathbb{Z}_{\underline{\tilde{b}}}$. If we choose an element in each equivalence class, we obtain a subset of $\tilde{\Gamma}$ which we can take as $C_{\underline{\tilde{b}}}$.

By induction, it is then clear that the set of elements $\mathcal{Q}^{\tilde{Z}_{\underline{\tilde{b}}}}_{\phi_{\underline{\tilde{b}}},p(\gamma)}$, as $\underline{\tilde{b}}$ varies in $\mathcal{B}_{\tilde{Z}}$ and γ in $C_{\underline{b}}$, is a basis of $DM(\tilde{Z})$ as a $\mathbb{Z}[\Gamma_t]$ module. Now observe that

$$(12) \qquad i_a(\mathcal{Q}^{\tilde{Z}_{\underline{\tilde{b}}}}_{\phi_{\underline{\tilde{b}}},p(\gamma)}) = \mathcal{Q}^{X_{\underline{b}}}_{\phi_{\underline{b}},\gamma}.$$

In fact, whenever one multiplies a function $f = i_a(g) \in \ker \nabla_a$ by any element δ_c (or by a series formed by these elements), one sees that

$$(13) \qquad \delta_c * i_a(g) = i_a(\delta_{p(c)} * g), \ \forall c \in \Gamma, \ \forall g \in \mathcal{C}[\tilde{\Gamma}].$$

Furthermore the elements $(X_{\underline{b}} \setminus \{a\}) \cap \underline{r}_i$, $i \geq 2$, map surjectively under p to the elements of $\tilde{Z}_{p(\underline{b})} \cap \underline{r}_i / \underline{r}_1$ and thus the formulas defining the two functions on both sides of Equation (12) coincide, factor by factor, for all steps of the flag from Formula (13).

Thus everything follows from the exact sequence (10). $\qquad \square$

3.23. **Support of** $DM(X)$. In this subsection, we are going to assume that X spans V. Otherwise $DM(X) = \{0\}$ and our discussion is trivial.

Recall that we have denoted by $\mathcal{B}(X)$ the set of all bases extracted from X. Let $\underline{b} = [b_1, b_2, \ldots, b_s]$ be in $\mathcal{B}_X$. Correspondingly we get an inclusion $j_{\underline{b}} : \mathbb{Z}^s \to \Gamma$ defined by $j_{\underline{b}}((n_1, \ldots, n_s)) = \sum_{h=1}^s n_h b_h$ and a surjective homomorphism $j_{\underline{b}}^* : G \to (S^1)^s$. We denote its kernel by $G(\underline{b})$. This is a finite subgroup in G. We define

Definition 3.24.   i) $P(X)$ is the union of all the sets $G(\underline{b})$, when $\underline{b}$ varies in $\mathcal{B}_X$.
    ii) For $g \in G$, denote by X_g the sublist of elements of X taking value 1 at g.

Now consider the ring $R(G)$ as a ring of functions on G, or better its complexification $G_{\mathbb{C}}$.

Lemma 3.25. *Let $V(X)$ denote the set of elements $g \in G_{\mathbb{C}}$ such that $\prod_{a \in Y}(1 - g^a) = 0$, when Y runs over all cocircuits. Then $V(X) = P(X)$.*

Proof. If $g \in G(\underline{b})$ and Y is a cocircuit, necessarily there exists an element b_i of $\underline{b}$ belonging to Y. Otherwise the complement of Y would not be contained in a proper subspace of V. So we have $\prod_{a \in Y}(1 - g^a) = 0$.

Conversely take $g \in V(X)$, and consider X_g. We claim that X_g spans V, otherwise $X \setminus X_g$ contains a cocircuit Y, and $\prod_{a \in Y}(1 - g^a) \neq 0$. So there is a basis $\underline{b}$ in $X_g \subset X$ and therefore $g \in G(\underline{b})$ ☐

Since the ideal J_X defines the finite set of points $P(X)$, we have that $\mathbb{C}[\Gamma]/J_X$ is a semi–local algebra, direct sum of its local components at each point in $P(X)$.

Let us normalize the Haar measure on G to be of total mass 1. This allows us to identify generalized functions on G and distributions on G. Under this identification, call $\widehat{DM}(X)$ the space of distributions on G of which $DM(X)$ gives the Fourier coefficients. Thus, if $p \in G$, the delta distribution at p is identified to the generalized function

$$\delta_p(g) = \sum_{\gamma \in \Gamma} p^{-\gamma} g^\gamma,$$

the Fourier transform of the element $f_p \in \mathcal{C}[\Gamma]$ given by $f_p(\gamma) := p^{-\gamma}$.

Any generalized function on G supported at p is a derivative of the δ function δ_p, so is of the form

$$\theta(p, q) = \sum_{\gamma \in \Gamma} p^{-\gamma} q(\gamma) g^\gamma,$$

where q is a polynomial on V. We have written $q(\gamma)$ instead of $q(\overline{\gamma})$, where $\overline{\gamma}$ is the image of γ in V. The function $\theta(p, q)$ is the Fourier transform of

$$f(p, q)(\gamma) = p^{-\gamma} q(\gamma).$$

From the definitions, $DM_{\mathbb{C}}(X)$ is dual to the algebra $\mathbb{C}[\Gamma]/J_X$ thus, from Lemma 3.25, we deduce

Proposition 3.26. *The support of every element in $\widehat{DM}(X)$ is contained in the finite set $P(X)$.*

Moreover, once we take complex linear combinations of the elements in $\widehat{DM}(X)$, we obtain a space of distributions which is the direct sum of the local contributions to $\widehat{DM}(X)$ at each point $p \in P(X)$. In order to understand these local contributions, we need to recall [6] the *differentiable* Dahmen-Micchelli space $D(X)$.

Given a vector $v \neq 0$ in V, we denote by ∂_v the directional derivative associated to v. For a subset Y of X , we denote by $\partial_Y := \prod_{a \in Y} \partial_{\overline{a}}$.

Definition 3.27. The space $D(X)$ is the space of polynomial functions on V satisfying the system of differential equations $\partial_Y f = 0$ as Y varies among all cocircuits of X.

Remark 3.28. The space $D(X)$ is a space of polynomials of finite dimension $d(X)$: the number of bases of V extracted from X. It describes locally some important functions as the box–spline.

We will prove in a subsequent article that $D(X)$ is isomorphic to the G-equivariant cohomology with compact support of M_X^f. Similarly, for each

$g \in P(X)$, the space $D(X_g)$ is the G-equivariant cohomology with compact support of the submanifold $(M_X^f)^g$ consisting of fixed points by g in M_X^f.

The equivariant cohomology, as an algebra, is the algebra of differential operators with constant coefficients induced on $D(X)$.

The following proposition is proven in [7].

Proposition 3.29. *Any function f in $DM_{\mathbb{C}}(X)$ can be written uniquely as*

$$f = \sum_{p \in P(X)} f(p, q)$$

where $q \in D(X_p)$.

Proof. We recall briefly the proof. We have just seen that $f \in DM(X)$ can be written uniquely as $f = \sum f(p, q)$, where $p \in P(X)$ and q is a polynomial on V.

We verify that $\nabla_Y f(p, q) = f(p, \nabla(Y, p)q)$, where $\nabla(Y, p) = \prod_{a \in Y}(1 - p^a \tau_a)$. On polynomials, the operator $(1 - \tau_a)q(v) = q(v) - q(v - a)$ is nilpotent, so that $(1 - p^a \tau_a) = (1 - p^a) - (p^a(\tau_a - 1))$ is invertible when $p^a \neq 1$.

Thus we split X in $X_p \cup (X \setminus X_p)$ and see that $\nabla_Y f(p, q) = 0$ if and only $\nabla_{Y \cap X_p} q = 0$. If Y is a cocircuit of X, then $Y \cap X_p$ contains a cocircuit of X_p. Thus $f(p, q) \in DM(X)$, if and only if $\nabla_Z q = 0$ for all cocircuits Z of X_p. Similarly, if $v \in V$, the difference operator ∇_v and the differential operator ∂_v satisfy $\nabla_v = T_v \partial_v$ with $T_v = \frac{1 - e^{-\partial_v}}{\partial_v}$ invertible on the space of polynomials. Thus we obtain that $f(p, q) \in DM_{\mathbb{C}}(X)$ if and only if $q \in D(X_p)$. $\qquad\square$

Remark 3.30. Since each point $p \in P(X)$ is an element of finite order in G, we clearly see the quasi–polynomial nature of each summand $f(p, q)$.

4. The spaces $\mathcal{F}(X)$ and $\tilde{\mathcal{F}}(X)$

Following [6], we define

Definition 4.1. The space $\mathcal{F}(X)$ is the space of functions $f \in \mathcal{C}[\Gamma]$ such that $\nabla_{X \setminus \underline{r}} f$ is supported on $\underline{r}$ for every proper rational subspace $\underline{r}$.

Notice that, since $DM(X)$ is the space of integer valued functions on Γ such that $\nabla_{X \setminus \underline{r}} f = 0$ for each rational subspace $\underline{r}$, $DM(X) \subset \mathcal{F}(X)$.

In [6], we have introduced the space $\mathcal{F}(X)$ only when X is a finite list of characters of a connected torus G, but the discussion there repeats verbatim in this general case. We have a *canonical filtration* which we shall interpret geometrically in the next subsection. Consider $\nabla_{X \setminus \underline{r}}$ as an operator on $\mathcal{F}(X)$ with values in $\mathcal{C}[\Gamma]$. Define the spaces

$$\mathcal{F}_i(X) := \cap_{\underline{t} \in S_X^{(i-1)}} \ker \nabla_{X \setminus \underline{t}} \cap \mathcal{F}(X).$$

Notice that by definition: $\mathcal{F}_0(X) = \mathcal{F}(X)$, $\mathcal{F}_{\dim V}(X)$ is the space $DM(X)$ and $\mathcal{F}_{i+1}(X) \subset \mathcal{F}_i(X)$. It follows from the definitions that, for each $\underline{t} \in S_X^{(i)}$,

$\nabla_{X\setminus\underline{t}}$ maps $\mathcal{F}_i(X)$ to $DM(X \cap \underline{t})$ and $\mathcal{F}_{i+1}(X)$ to 0. Consider the map

$$\mu_i = (\oplus\nabla_{X\setminus\underline{t}})_{\underline{t}\in S_X^{(i)}} : \mathcal{F}_i(X) \to \bigoplus_{\underline{t}\in S_X^{(i)}} DM(X \cap \underline{t}).$$

With a proof entirely similar to the proof of Lemma 3.10 in [6], we obtain the following theorem.

Proposition 4.2. *The sequence*

$$0 \to \mathcal{F}_{i+1}(X) \to \mathcal{F}_i(X) \overset{\mu_i}{\to} \bigoplus_{\underline{r}\in S_X^{(i)}} DM(X \cap \underline{r}) \to 0$$

is split exact with splitting:

$$\mathcal{F}_i(X) = \mathcal{F}_{i+1}(X) \oplus \bigoplus_{\underline{r}\in S_X^{(i)}} \mathcal{P}_{X\setminus\underline{r}}^{F_{\underline{r}}} * DM(X \cap \underline{r}).$$

As a consequence we obtain a decomposition dependent upon the choices of the faces $F_{\underline{r}}$

(14) $\qquad \mathcal{F}(X) = DM(X) \oplus \left(\oplus_{\underline{r}\in S_X | \underline{r}\neq V} \mathcal{P}_{X\setminus\underline{r}}^{F_{\underline{r}}} * DM(X \cap \underline{r}) \right).$

Remark 4.3. From Formula (14) and Theorem 3.22, one can describe an explicit linear basis for the abelian group $\mathcal{F}(X)$.

While $DM(X)$ is a $R(G)$ module, in general $\mathcal{F}(X)$ is not stable under $R(G)$, so we consider the $R(G)$ submodule $\tilde{\mathcal{F}}(X)$ in $\mathcal{C}[\Gamma]$ generated by $\mathcal{F}(X)$ and similarly for $\tilde{\mathcal{F}}_i(X)$.

Remark 4.4. It is easy to see that, if X' is deduced from X as in Remark 3.5, then $\tilde{\mathcal{F}}(X) = \tilde{\mathcal{F}}(X')$.

Recall the definition 3.16 of the subspace $DM^{(G)}(X \cap \underline{r})$ generated by $DM(X \cap \underline{r})$ inside $\mathcal{C}[\Gamma]$.

The group $\Gamma_{\underline{r}} = \Gamma \cap \underline{r}$ is a direct summand in Γ containing Γ_t. Thus the group $G_{\underline{r}}$, kernel of all the elements in $\Gamma_{\underline{r}}$, is connected with character group $\Gamma/\Gamma_{\underline{r}}$. One gets a split exact sequence

$$1 \to G_{\underline{r}} \to G \to G/G_{\underline{r}} \to 1.$$

We deduce that

$$R(G/G_{\underline{r}}) = \mathbb{Z}[\Gamma_{\underline{r}}]$$

while $R(G)$ is isomorphic (in a non canonical way) to

$$\mathbb{Z}[\Gamma] \sim \mathbb{Z}[\Gamma_{\underline{r}}] \otimes \mathbb{Z}[\Gamma/\Gamma_{\underline{r}}] = R(G/G_{\underline{r}}) \otimes R(G_{\underline{r}}).$$

It is then immediate to verify that

(15) $DM^{(G)}(X \cap \underline{r}) = R(G) \otimes_{R(G/G_{\underline{r}})} DM(X \cap \underline{r}) = R(G_{\underline{r}}) \otimes DM(X \cap \underline{r}).$

Since both the maps μ_i and the convolutions $\Theta_{X\setminus\underline{r}}^{F_{\underline{r}}}$ obviously extend to maps of $R(G)$ modules, we easily deduce

Corollary 4.5. *The sequence*

$$0 \to \tilde{\mathcal{F}}_{i+1}(X) \to \tilde{\mathcal{F}}_i(X) \xrightarrow{\mu_i} \bigoplus_{\underline{r} \in S_X^{(i)}} DM^{(G)}(X \cap \underline{r}) \to 0$$

is split exact with splitting:

$$\tilde{\mathcal{F}}_i(X) = \tilde{\mathcal{F}}_{i+1}(X) \oplus \bigoplus_{\underline{r} \in S_X^{(i)}} \mathcal{P}_{X \setminus \underline{r}}^{F_{\underline{r}}} * DM^{(G)}(X \cap \underline{r}).$$

From this, we obtain a decomposition dependent upon the choices of the faces $F_{\underline{r}}$

$$(16) \qquad \tilde{\mathcal{F}}(X) = DM(X) \oplus \left(\oplus_{\underline{r} \in S_X | \underline{r} \neq V} \mathcal{P}_{X \setminus \underline{r}}^{F_{\underline{r}}} * DM^{(G)}(X \cap \underline{r}) \right).$$

One easily verifies that $\tilde{\mathcal{F}}(X)$ can also be intrinsically defined as:

Definition 4.6. The space $\tilde{\mathcal{F}}(X)$ is the space of functions $f \in \mathcal{C}[\Gamma]$ such that $\nabla_{X \setminus \underline{r}} f$ is supported on a finite number of Γ translates of $\underline{r}$ for every proper rational subspace $\underline{r}$.

Remark 4.7. By definition, the filtration $\tilde{\mathcal{F}}_i(X)$ is defined by a condition on torsion or in the language of modules by support. One can verify that in particular $DM(X)$ is the part supported in dimension 0, that is of maximal possible torsion.

4.8. **Generators of $\tilde{\mathcal{F}}(X)$.** In this subsection, we assume that $\overline{a} \neq 0$ for any $a \in X$. Thus every open face F produces a decomposition $X = A \cup B$ into positive and negative vectors and can define as in (1):

$$\mathcal{P}_X^F := (-1)^{|B|} \delta_{-b_B} * H_A * H_{-B}.$$

Theorem 4.9. *The elements $\mathcal{P}_X^F$, as F runs on all open faces, generate $\tilde{\mathcal{F}}(X)$ as $R(G)$ module.*

Proof. Denote by M the $R(G)$ module generated by the elements $\mathcal{P}_X^F$, as F runs on all open faces. In general, from the description of $\tilde{\mathcal{F}}(X)$ given in 4.2, it is enough to prove that elements of the type $\mathcal{P}_X^{F_{\underline{r}}} * g$ with $g \in DM(X \cap \underline{r})$ are in M. As $DM(X \cap \underline{r}) \subset \mathcal{F}(X \cap \underline{r})$, it is sufficient to prove by induction that each element $\mathcal{P}_X^{F_{\underline{r}}} * \mathcal{P}_{X \cap \underline{r}}^K$ is in M, where K is any open face for the system $X \cap \underline{r}$. We choose a linear function u_0 in the face $F_{\underline{r}}$. Thus u_0 vanishes on $\underline{r}$ and is non zero on every element $a \in X$ not in $\underline{r}$. We choose a linear function u_1 such that the restriction of u_1 to $\underline{r}$ lies in the face K. In particular, u_1 is non zero on every element $a \in X \cap \underline{r}$. We can choose ϵ sufficiently small such that $u_0 + \epsilon u_1$ is non zero on every element $a \in X$. Then $u_0 + \epsilon u_1$ defines an open face F in the arrangement $\mathcal{H}_X$. We see that $\mathcal{P}_X^{F_{\underline{r}}} * \mathcal{P}_{X \cap \underline{r}}^K$ is equal to $\mathcal{P}_X^F$. $\square$

4.10. **Support of $\tilde{\mathcal{F}}_i(X)$.** For a fixed $0 \leq i \leq s$, take a linearly independent sublist $[\underline{b}_1, \ldots, b_i]$ in X. Correspondingly we get an inclusion $j_{\underline{b}} : \mathbb{Z}^i \to \Gamma$ defined by $j_{\underline{b}}((n_1, \ldots, n_i)) = \sum_{h=1}^{h} n_h b_h$ and a surjective homomorphism $j_{\underline{b}}^* : G \to (S^1)^i$. We denote its kernel by $G(\underline{b})$. This is a subgroup in G of dimension $s - i$. We define

Definition 4.11. $P_{s-i}(X)$ is the union of all the sets $G(\underline{b})$, when $\underline{b}$ varies among linearly independent sublists in X of cardinality i.

As before, normalize the Haar measure on G to be of total mass 1 and identify generalized functions on G and distributions on G. Call $\widehat{\tilde{\mathcal{F}}}_i(X)$ the space of distributions on G of which $\tilde{\mathcal{F}}_i(X)$. gives the Fourier coefficients.

Then the decomposition (16) together with Proposition 3.26 immediately implies

Proposition 4.12. *The support of every element in* $\widehat{\tilde{\mathcal{F}}}_i(X)$ *is contained in* $P_{s-i}(X)$.

5. Index theory

5.1. **K-theory.** We briefly review the notations for K-theory that we will use.

Let G be a compact Lie group acting on a locally compact space N. One has the notion of the equivariant topological $K-$ theory group $K_G^0(N)$. $K_G^0(N)$ is a contravariant functor for proper maps and covariant for open embeddings. We recall that representatives of the $K-$theory group $K_G^0(N)$ can be described in the following way. Given two G-equivariant complex vector bundles E^0, E^1 on N and a G-equivariant bundle map $f : E^0 \to E^1$, the support $supp(f)$ of f is the set of points where $f_x : E_x^0 \to E_x^1$ is not an isomorphism. A G-equivariant bundle map f with compact support defines an element $[f]$ of $K_G^0(N)$.

Let $f : E^0 \to E^1$ and $g : F^0 \to F^1$ be two G-equivariant bundle maps. Using G-invariant Hermitian metrics on the bundles E^i, F^i we can define:

$$f \odot g : E^0 \otimes F^0 \oplus E^1 \otimes F^1 \to E^1 \otimes F^0 \oplus E^0 \otimes F^1$$

by

$$f \odot g := \begin{pmatrix} f \otimes 1 & -1 \otimes g^* \\ 1 \otimes g & f^* \otimes 1 \end{pmatrix}.$$

The support of $f \odot g$ is the intersection of the supports of f, g thus $f \odot g$ induces an element in $K_G^0(N)$ as soon as one of the two f, g has compact support.

In particular this defines a product $[f][g] := [f \odot g]$ on $K_G^0(N)$.

If $N = $ pt is a point, then $K_G^0(\text{pt})$ is isomorphic to the Grothendieck ring $R(G)$ of finite dimensional representations of G.

In general, tensor product with finite dimensional representations of G induces a $R(G)$ module structure on $K_G^0(N)$. Take the projection $\pi : N \to$

pt. Given $\tau \in R(G)$ and $\sigma \in K_G^0(N)$, we have that $[\pi^*(\tau) \odot \sigma] \in K_G^0(N)$ and this gives a $R(G)$ module structure to $K_G^0(N)$.

We will need also the groups defined inductively as

$$K_G^{i+1}(N) := K_G^i(N \times \mathbb{R}).$$

One has $K_G^0(\mathbb{R}) = K_G^1(\text{pt}) = 0$.

There is a natural isomorphism $K_G^i(N) \to K_G^{i+2}(N) = K_G^i(N \times \mathbb{R}^2)$ given by Bott periodicity, that we describe below.

• Let W be a Hermitian vector space and let $E = \bigwedge W$. Then, for $w \in W$, consider the exterior multiplication $m(w) : E \to E$ and the Clifford action

(17) $$c(w) = m(w) - m(w)^*, \qquad (m(w)(\omega) := w \wedge \omega)$$

of W on $\bigwedge W$. Then one has $c(w)^2 = -\|w\|^2$, so that $c(w)$ is an isomorphism, if $w \neq 0$.

If $p : W \to M$ is a $G-$equivariant complex vector bundle over a $G-$space M, we can consider also W as $G-$space and we have a Thom isomorphism

$$C_W : K_G^0(M) \to K_G^0(W).$$

In order to make this explicit, we use a G-invariant Hermitian metric on W. Then the fiberwise Clifford action $c(w_x) : \bigwedge^{even} W_x \to \bigwedge^{odd} W_x$ defines a morphism $\mathbf{c}_W : p^* \bigwedge^{even} W \to p^* \bigwedge^{odd} W$ of vector bundles over W, that we call the *Bott symbol*. Take a bundle map $f : E \to F$ of complex equivariant vector bundles on M which is an isomorphism outside a compact set, and denote still by f its pull back $f : p^*E \to p^*F$. Then $f \odot \mathbf{c}_W$ is a bundle map of bundles over W, which is an isomorphism outside the support of f embedded in W via the zero section. We set

(18) $$C_W([f]) = [f \odot \mathbf{c}_W].$$

• If F is a G-invariant closed subset of N, denote by $i : F \to N$ the closed embedding and $j : N \setminus F \to N$ the open embedding. There is a long exact sequence of $R(G)$ modules:

(19) $$\cdots \to K_G^i(N \setminus F) \xrightarrow{j_*} K_G^i(N) \xrightarrow{i^*} K_G^i(F) \xrightarrow{\delta} K_G^{i+1}(N \setminus F) \to \cdots$$

5.2. **Transversally elliptic operators.** To define the index of a transversally elliptic operator on M, we require the following hypothesis on M. If G is a compact Lie group acting on M, we assume that M can be embedded as a G-invariant open subset of a compact G-manifold $\tilde{M}$.

Given such a manifold M with a C^∞ action of a compact Lie group G, one has the notion of *transversally elliptic operator* between two equivariant complex vector bundles E, F on M. Such an operator is a pseudo-differential operator $A : \Gamma(M, E) \to \Gamma(M, F)$ from the space $\Gamma(M, E)$ of smooth sections of E to the space $\Gamma(M, F)$ of smooth sections of F, which commutes with the action of G, is *elliptic* in the directions transversal to the orbits of G and is "trivial " at infinity.

In more technical terms, let T^*M denote the cotangent bundle of M and $p : T^*M \to M$ the canonical projection. Inside T^*M, there is a special closed subset denoted by T_G^*M. Its fiber over a point $x \in M$ is formed by all the cotangent vectors $\xi \in T_x^*M$ which vanish on the tangent space to the orbit of x under G, in the point x. Thus each fiber $(T_G^*M)_x$ is a linear subspace of T_x^*M. In general the dimension of $(T_G^*M)_x$ is not constant and this space is not a vector bundle.

Definition 5.3. By a *symbol*, one means a smooth section on T^*M of the bundle $\hom(p^*(E), p^*(F))$: in other words, for each point $(x, \xi), x \in M, \xi \in T_x^*M$, we have a linear map $\sigma(x, \xi) : E_x \to F_x$.

Assume first that M is a compact manifold. To the pseudo-differential operator A, one associates its principal *symbol* σ_p which is defined outside the zero section of T^*M. The operator A is said to be G-transversally elliptic if its principal symbol $\sigma_p(x, \xi)$ is invertible for all $(x, \xi) \in T_G^*M$ such that $\xi \neq 0$. Using a G- invariant function χ on T^*M identically equal to 1 in a neighborhood of M and compactly supported, then $\sigma(x, \xi) := (1 - \chi(x, \xi))\sigma_p(x, \xi)$ is defined on the whole space T^*M. Furthermore $\sigma(x, \xi)$ restricted to T_G^*M is an isomorphism outside a compact G-invariant subset of T_G^*M. Thus, by restriction to T_G^*M, the symbol σ defines a K-theory class $[\sigma]$ in the topological equivariant K -theory group $K_G^0(T_G^*M)$. This class does not depend of the choice of χ. We still say that this class σ is the symbol of A.

Let $\hat{G}$ be the set of equivalence classes of finite dimensional irreducible representations of G, and let $\mathcal{C}[\hat{G}]$ be the group of $\mathbb{Z}$-valued functions on $\hat{G}$. Let $\chi_\tau(g) = \mathrm{Tr}(\tau(g))$ be the character of the representation $\tau \in \hat{G}$ of G. We associate to an element $f \in \mathcal{C}[\hat{G}]$ a formal (virtual) character $\Theta(f) = \sum_\tau f(\tau)\chi_\tau$, that is a formal combination of the characters χ_τ with multiplicities $f(\tau) \in \mathbb{Z}$. When $f(\tau)$ satisfies certain moderate growth conditions, then the series $\Theta(f)(g) = \sum_\tau f(\tau)\chi_\tau(g)$ converges, in the distributional sense, to a generalized function on G.

The *index map* associates to a transversally elliptic operator A an element of $\mathcal{C}[\hat{G}]$ constructed as follows. For every $\tau \in \hat{G}$, the space $\hom_G(\tau, \ker(A))$ is finite dimensional of dimension $m(\tau, A)$. Thus $m(\tau, A)$ is the multiplicity of τ in the space $\ker(A)$ of smooth solutions of A. We choose a G-invariant metric on M and G-invariant Hermitian structures on E, F. Then $A^* : \Gamma(M, F) \to \Gamma(M, E)$ is also transversally elliptic.

Definition 5.4. The index multiplicity of the pseudo-differential operator A is the function $ind_m(A) \in \mathcal{C}[\hat{G}]$ defined by

$$ind_m(A)(\tau) := m(\tau, A) - m(\tau, A^*).$$

It follows also from Atiyah-Singer [1] that the series $\sum_\tau m(\tau, A)\chi_\tau(g)$ defines a generalized function on G. Thus we may also associate to A the

generalized function

$$ind(A)(g) = \sum_\tau ind_m(A)(\tau)\chi_\tau(g)$$

on G with integral Fourier coefficients. One of the main points in the index theory consists in showing that the index factors through the symbols and defines a homomorphism of $R(G)$ modules from $K_G^0(T_G^*M)$ to $\mathcal{C}[\hat{G}]$.

If $j : U \to M$ is an open G-invariant set of a compact G manifold M, we still denote by j the corresponding open embedding from T_G^*U to T_G^*M. Then j_* defines a map from $K_G^0(T_G^*U)$ to $K_G^0(T_G^*M)$. The index of $\sigma \in K_G^0(T_G^*U)$ is defined to be the index of $j_*(\sigma)$. The excision property of the index shows that this is independent of the choice of the open embedding j and thus allows us to define the index map also for manifolds which can be embedded as open sets of compact ones.

In particular, if V is a vector space with a linear action of a compact group G, then V is diffeomorphic to the sphere, minus a point. Thus we can define the index of any $\sigma \in K_G^0(T_G^*V)$. More generally, if U is an open G-invariant subset of a vector space, we can define the index of $\sigma \in K_G^0(T_G^*U)$.

The problem of computing the index can be reduced, at least theoretically, to the case in which G is a torus. For a given compact manifold M, one embeds M into a linear representation and then is reduced to perform the computations in the representation.

In this article, the group G is an abelian compact Lie group. An irreducible representation a of G is a one dimensional complex vector space L_a, where G acts via a character $\chi_a : G \to S^1$, so that $\hat{G}$ is identified with the abelian group of characters, denoted by Γ.

Definition 5.5. Let X be a finite list of elements of Γ. Define the complex vector space

(20) $$M_X := \oplus_{a\in X} L_a.$$

The space M_X is a G-manifold and our goal is the determination of $K_G^0(T_G^*M_X)$. The basic tool that we shall use is the space of functions $DM(X)$ on Γ.

Remark 5.6. Let G be a torus acting on a real vector space M without fixed non zero subspace. Then M can be given a complex structure, so that the G-manifold M is isomorphic to the space M_X for some list X of weights. It will be clear that our description of $K_G^0(T_G^*M_X)$ depends only of the list X up to signs.

5.7. **Examples of transversally elliptic symbols.** Let M be a real vector space provided with a linear action of a compact abelian Lie group G, which we may assume to be orthogonal with respect to some chosen Euclidean structure (v, w). Let U be the Lie algebra of G. We assume that there exists $u \in U$ such that the infinitesimal action $\rho(u)$ of u on M is invertible. Such a u will be called regular. Since $-\rho(u)^2$ is a symmetric

and positive operator, in particular it is semisimple with positive eigenvalues and we can take its unique square root with positive eigenvalues. We choose on M the complex structure $J_u = \rho(u)/(-\rho(u)^2)^{1/2}$. This complex structure depends only of the connected component F of u in the space of regular elements. The given Euclidean structure is the real part of a positive Hermitian structure $(v,w) + i(v, J_u w)$ for which the action is unitary. Let us write $M = \oplus_{a \in X} L_a$, where X is the list of weights of G in the complex vector space M. The connected components of the space of regular elements are the open faces of the arrangement $\mathcal{H}_X$. By definition of the complex structure, all weights a are positive on $u \in U$. We can then define the generalized functions $\Theta_X^{\pm F}(g)$ on G as explained in subsection 3.8.

• **The tangential Cauchy-Riemann operator.** With the given Hermitian structure on M, let S be the unit sphere of M. Let $\mathbb{P}(M)$ be the complex projective space of M. Consider on S the differential operator δ acting on the pull back of the Dolbeault complex on the associated projective space $\mathbb{P}(M)$ using $\overline{\partial} + \overline{\partial}^* : \sum \Omega^{0,2p} \to \sum \Omega^{0,2p+1}$. Then δ is a G-transversally elliptic differential operator (the tangential Cauchy-Riemann operator) on S. Indeed, using the Hermitian structure, identify T^*S with its tangent bundle $TS \subset TM$, the subspace H_p of T_p^*S orthogonal to the line $\mathbb{R}J_u p$ is then identified to the complex subspace of M, orthogonal under the Hermitian form to p. We call it the horizontal cotangent space. The symbol of δ is $\sigma(p, \xi) = c(\xi^1)$ where ξ^1 is the projection of ξ on H_p, and c the Clifford action of H_p on $\bigwedge H_p$. This morphism is invertible if $\xi^1 \neq 0$. We have also $H_p \oplus \mathbb{R}\rho(u)p = T_p^*S$, as the eigenvalues of $-iu$ on M are all positive. Thus we see that $\sigma(p, \xi)$ restricted to T_G^*S is invertible outside the zero section.

The following formula is proven in [1] (Proposition 5.4).

Theorem 5.8. *Let M be provided with the complex structure J_u and let δ be the tangential Cauchy-Riemann operator on the unit sphere of M. Then*

$$index(\delta)(g) = (-1)^{|X|} g^{a_X} (\Theta_X^{-F}(g) - \Theta_X^F(g))$$

where $a_X = \sum_{a \in X} a$.

Proof. We recall briefly the proof. Let S^1 be the circle group acting by homotheties on M. We decompose solution spaces with respect to characters $t \mapsto t^n$ of S^1. The group G acts on $\mathbb{P}(M)$ and on every line bundle $\mathcal{O}(n)$ on $\mathbb{P}(M)$. Thus the index as an index of $G \times S^1$ is the sum of the index of G in the cohomology on $\mathbb{P}(M)$ of the line bundles $\mathcal{O}(n)$. Define $\chi_n(g)$ as the virtual character (as a representation of G) in the virtual finite dimensional vector space $\sum (-1)^i H^{0,i}(\mathbb{P}(M), \mathcal{O}(n))$. Then

$$index(\delta)(g) = \sum_{n \in \mathbb{Z}} \chi_n(g).$$

Let us show that

$$(21) \qquad\qquad \sum_{n \geq 0} \chi_n(g) = (-1)^{|X|} g^{a_X} \Theta_X^{-F}(g),$$

$$(22) \qquad \sum_{n<0} \chi_n(g) = (-1)^{|X|+1} g^{a_X} \Theta_X^F(g).$$

For $n \geq 0$, $\mathcal{O}(n)$ has only 0-cohomology and $H^{0,0}(\mathbb{P}(M), \mathcal{O}(n))$ is just the space of homogeneous polynomials on M of degree n. So $\sum_{n=0}^{\infty} \chi_n$ is the character of the symmetric algebra $S[M^*] = \prod_{a \in X} S[L_{-a}]$. The function $\sum_{k=0}^{\infty} g^{-ka}$ is the character of the action of G in $S[L_{-a}]$. The function Θ_X^{-F} is the product of the functions $-g^{-1} \sum_{k=0}^{\infty} g^{-ka}$. Thus we obtain Formula (21). On the other hand, if $n < 0$, we have two cases. If $-|X|-1 < n \leq -1$, then $H^{0,i}(\mathbb{P}(M), \mathcal{O}(n)) = 0$ for every i. Otherwise we apply Serre's duality and we obtain the second equality (22).

$\square$

• **Atiyah–Singer pushed symbol.** We identify T^*M with $M \times M$, using the Hermitian metric on M. Let $c(v) : \bigwedge^{even} M \to \bigwedge^{odd} M$ be the Clifford action (17) of M on $\bigwedge M$. Given, as before, a regular element u in the Lie algebra of G and letting $\rho(u)$ denote its infinitesimal action on M, we define

Definition 5.9.
$$At_u(v, \xi) = c(\xi + \rho(u)v).$$

The morphism $At_u(v, \xi)$ is invertible except if $\xi + \rho(u)v = 0$. If furthermore ξ is in T_G^*M, ξ is orthogonal to the tangent vector $\rho(u)v$. Thus the support of $At_u(v, \xi)$ restricted to T_G^*M is the unique point $v = 0, \xi = 0$ and At_u determines an element of $K_G^0(T_G^*M)$, which depends only of the connected component F of u in the set of regular elements. We denote it by At_F. The index of At_F is computed in [1] (Theorem 8.1). In more detail, in the Appendix of [4], it is constructed an explicit G-transversally elliptic pseudo-differential operator A on the product of the projective lines $\mathbb{P}(L_a \oplus \mathbb{C})$. If $j : M_X \to \prod_{a \in X} \mathbb{P}(L_a \oplus \mathbb{C})$ is the natural open embedding, it is shown that $j_*(At_F)$ is homotopic to the symbol of A. By definition, the index of At_F is that of A and one has the explicit formula:

Theorem 5.10. *Let M be provided with the complex structure J_u and let $At_F \in K_G^0(T_G^*M)$ be the "pushed" $\overline{\partial}$ symbol. Then*
$$index(At_F)(g) = (-1)^{|X|} g^{a_X} \Theta_X^F(g).$$

5.11. **Some properties of the index map.** We now recall briefly some properties of the index map: $K_G^0(T_G^*M) \to C^{-\infty}(G)$ that we will use. Here again G is an abelian compact Lie group and M a G-manifold, which can be embedded as a. G-invariant open subset of a compact G-manifold $\tilde{M}$. This hypothesis on M is in place in the rest of the article. It is stable under the following operations:

i)   If U is an open G-invariant subset of M, then U satisfies our hypothesis.

ii)   If W is a real vector space with a linear representation of G, then $M \times W$ satisfies our hypothesis.

iii)   Let H be a closed subgroup of G. Let M be a space with H action, open H-invariant subset of a compact H-manifold. Let $N = G \times_H M$ be the G space with typical fiber M over G/H. Then N satisfies our hypothesis.

We now list some properties of the index.

i)   Any element $\sigma \in K_G^0(T_G^*M)$ arises from the restriction to T_G^*M of a G- bundle morphism $\sigma(x, \xi) : E_x \to F_x$, such that $supp(\sigma) \cap T_G^*M$ is a compact set. Here E, F are G-equivariant complex vector bundles over M.

ii)   If $j : \mathcal{U} \to M$ is an open embedding, the map $j_* : K_G^0(T_G^*\mathcal{U}) \to K_G^0(T_G^*M)$ is compatible with the index.

iii)   Let W be a real vector space with a linear representation of G and W' be the dual vector space. We identify $TW = W \times W$ with $W_\mathbb{C}$ by $(v, w) \to v + iw$. Furthermore, we identify $TW = W \times W$ with $T^*W = W \times W'$ using an Euclidean structure on W. Thus the Bott symbol $c_{W_\mathbb{C}}(v + i\xi)$ acting on $\bigwedge W_\mathbb{C}$ defines a G-equivariant elliptic symbol on W. Its G-equivariant index is identically equal to 1.

Let $i : N \to N \times W$ be the injection of a G-manifold N into $N \times W$. Then we obtain a map $i_! : K_G^0(T_G^*N) \to K_G^0(T_G^*(N \times W))$ given at the level of symbols by $\sigma \mapsto \sigma \odot c_{W_\mathbb{C}}$. The index of σ is equal to the index of $i_!\sigma$.

iv)   In case $W = \mathbb{R}$ with the trivial action, $T_G^*(N \times \mathbb{R}) = T_G^*N \times T^*\mathbb{R}$ and thus $i_!$ is an isomorphism by Bott periodicity.

v)   Let H be a closed subgroup of G. Then there is a surjective map $\hat{G} \to \hat{H}$ induced by the restriction of characters. The dual map induces an injection $\mathrm{Ind}_H^G : \mathcal{C}[\hat{H}] \to \mathcal{C}[\hat{G}]$.

Let M be a space with H action (open subset a compact H-manifold), and let $N = G \times_H M$ be the G space with typical fiber M over G/H. It is easy to see that there is an isomorphism

(23) $$i_H^G : K_H^i(T_H^*M) \to K_G^i(T_G^*N)$$

and, by ([1], Theorem 4.1), for any $\sigma \in K_H^0(T_H^*M)$,

(24) $$ind_m(i_H^G(\sigma)) = \mathrm{Ind}_H^G(ind_m(\sigma)).$$

We shall also need the following simple consequence of the previous facts:

Lemma 5.12. *Let G be a compact Lie group and $\chi : G \to S^1$ be a surjective character. Set $H := \ker \chi$ be the kernel of χ.*

Take a manifold M over which G acts and consider the product $\mathbb{C}^* \times M$, with the action of G on the first factor induced by χ.

There is an isomorphism

(25) $$k : K_H^i(T_H^*M) \cong K_G^i(T_G^*(\mathbb{C}^* \times M)).$$

Moreover, if $\sigma \in K_H^0(T_H^*M)$, we have $ind_m(k(\sigma)) = \mathrm{Ind}_H^G(ind_m(\sigma))$.

Proof. Since $\mathbb{C}^* = S^1 \times \mathbb{R}^+$, we get by iv) that the inclusion $i : S^1 \times M \to \mathbb{C}^* \times M$ induces the isomorphism

$$i_! : K_G^i(T_G^*(S^1 \times M)) \to K_G^i(T_G^*(\mathbb{C}^* \times M))$$

which at the level of K^0 is compatible with the index.

On the other hand, the space $G \times_H M$ identifies with $S^1 \times M$ via the map $[g, m] \mapsto [\chi(g), g \cdot m]$. So (23) gives us the isomorphism

$$i_H^G : K_H^i(T_H^*(M)) \to K_G^i(T_G^*(S^1 \times M)).$$

If $\sigma \in K_H^0(T_H^*M)$, then $ind_m(i_H^G(\sigma)) = \mathrm{Ind}_H^G(ind_m(\sigma))$ by Formula (24) .

Thus we can take $k := i_! i_H^G$. $\qquad\qquad\square$

6. Equivariant K-theory and Dahmen-Micchelli spaces

This section contains the main results of this paper, that is Theorem 6.16 and Theorem 6.20.

6.1. **Two exact sequences.** Let G be, as before, a compact abelian Lie group of dimension s and $M_X := \oplus_{a \in X} L_a$ as in (20). We assume that X is a non degenerate list of characters of G.

Given a vector $v \in M_X$, its *support* is the sublist of elements $a \in X$ such that v has a non zero coordinate in the summand L_a.

If Y is the support of v, an element t of G stabilizes v if and only if $t^a = 1$ for all $a \in Y$. If Y spans a rational subspace of dimension k, the G−orbit of v has dimension k.

For any rational subspace $\underline{r}$, we may consider the subspace $M_{\underline{r}} := \oplus_{a \in X \cap \underline{r}} L_a$ of M_X. We set

$$M_{\leq i} := \cup_{\underline{r} \in S_X^{(i)}} M_{\underline{r}}, \quad M_{\geq i} := M_X \setminus M_{\leq i-1},$$

(26) $$F_i := M_{\leq i} \setminus M_{\leq i-1} = M_{\geq i} \setminus M_{\geq i+1} = M_{\leq i} \cap M_{\geq i}.$$

Notice that

$$M_X = M_{\geq 0} \supset M_{\geq 1} \supset M_{\geq 2} \supset \cdots \supset M_{\geq s} := M_X^f.$$

The set $M_{\leq i}$ is the closed set of points in M with the property that the orbit has dimension $\leq i$ while $M_{\geq i}$ is the open set of points in M with the property that the orbit has dimension $\geq i$. The set F_i is open in $M_{\leq i}$ and closed in $M_{\geq i}$ and it is the set of points in M whose orbit under G has dimension exactly i. In particular, $F_s = M_{\geq s} = M_X^f$ is the open set of points in M with finite stabilizer under the action of G, which plays a particular role.

Given a rational subspace $\underline{r}$, we have denoted by $G_{\underline{r}}$ the subgroup of G joint kernel of the elements in $\Gamma \cap \underline{r}$. The group $G_{\underline{r}}$ is a torus and acts trivially on $M_{\underline{r}} := \oplus_{a \in X \cap \underline{r}} L_a$ inducing an action of $G/G_{\underline{r}}$.

Definition 6.2. We define the set $F(\underline{r})$ to be the open set of $M_{\underline{r}}$ where $G/G_{\underline{r}}$ acts with finite stabilizers.

In other words, the connected component of the stabilizer of an element of $F(\underline{r})$ is exactly the group $G_{\underline{r}}$.

Remark 6.3. By definition of $G_{\underline{r}}$, the set $F(\underline{r})$ is non–empty. The set F_i is the disjoint union of the sets $F(\underline{r})$ as $\underline{r}$ runs over all rational subspaces of dimension i. Thus the space M_X is the disjoint union of the locally closed strata $F(\underline{r})$.

We now analyze the equivariant K theory of $T_G^* M_X^f$.

Let $a \in X$ be an element of infinite order so that the homomorphism $g^a : G \to S^1$ is surjective. Set $Z := X \setminus \{a\}$ and $G_a := \ker g^a$. Denote by $\tilde{Z}$ the list of the restrictions to G_a of the elements of Z. For $v \in M_X$, denote by $v_a \in \mathbb{C}$ its coordinate in L_a with respect to a choice of a basis of the one dimensional vector space L_a.

The set $M_Z^f := \{v \in M_X^f \,|\, v_a = 0\}$ is closed in M_X^f. Denote by $i : M_Z^f \to M_X^f$ the closed embedding and by $j : M_X^f \setminus M_Z^f \to M_X^f$ the open embedding of the complement.

Lemma 6.4. *There exists an isomorphism*

$$k : K_{G_a}^i(T_{G_a}^* M_{\tilde{Z}}^f) \to K_G^i(T_G^*(M_X^f \setminus M_Z^f)).$$

If $\sigma \in K_{G_a}^0(T_{G_a}^* M_{\tilde{Z}}^f)$, we have $ind_m(k(\sigma)) = \mathrm{Ind}_{G_a}^G(ind_m(\sigma))$.

Proof. Take an element $(v_a, w) \in L_a \times M_Z$ with $v_a \neq 0$, its stabilizer in G is the subgroup of G_a stabilizing w, therefore the space $M_X^f \setminus M_Z^f$ is isomorphic to $\mathbb{C}^* \times M_{\tilde{Z}}^f$. Thus we are in the setting of Lemma 5.12. $\square$

For a real vector space W, we shall denote by W' its dual. Consider the projection $p : T_G^* M_X^f \to M_X^f$. Then $p^{-1} M_Z^f$ is a closed subset of $T_G^* M_X^f \subset M_X^f \times M_X'$ and $T_G^* M_X^f \setminus p^{-1} M_X^f$ is equal to $T_G^*(M_X^f \setminus M_Z^f)$. We use the same notations i, j also in this setting for the closed and open embedding associated. Remark the following fact.

Lemma 6.5.   *i) We have $p^{-1} M_Z^f = T_G^* M_Z^f \times L_a'$.*
          ii) We have an isomorphism $C_a : K_G^i(T_G^* M_Z^f) \to K_G^i(p^{-1} M_Z^f)$.

Proof. The first assertion is immediate to verify. The second follows from the first and Thom isomorphism. $\square$

The first theorem is:

Theorem 6.6.   *i) $K_G^1(T_G^* M_X^f) = 0$.*
          ii) If $a \in X$ has infinite order, there is a short exact sequence:

$$(27) \qquad 0 \to K_{G_a}^0(T_{G_a}^* M_{\tilde{Z}}^f) \xrightarrow{j_* k} K_G^0(T_G^* M_X^f) \xrightarrow{C_a^{-1} i^*} K_G^0(T_G^* M_Z^f) \to 0.$$

Proof. If G is finite, then $M_X^f = M_X$ and the first statement is Bott periodicity while the second statement does not exist. So we assume that G has positive dimension. Therefore we can choose $a \in X$ of infinite order. By induction we can assume that $K_G^1(T_G^* M_Z^f) = 0 = K_{G_a}^1(T_G^* M_{\tilde{Z}}^f)$.

If Z is degenerate, then M_Z^f is empty, j is the identity and our claims reduce to Lemma 6.4. Otherwise both statements follow from Lemma 6.4, Lemma 6.5 and the long exact sequence of $K-$theory. $\qquad\square$

We choose an Hermitian product $\langle v, w \rangle$ on M_X and we identify the (real) vector space M_X', dual to M_X, with M_X using the real part of the Hermitian product. Thus $T^* M_X^f$ is identified with $M_X^f \times M_X'$, where M_X' is the space M_X with the opposite complex structure, so that $T^* M_X^f$ is a complex vector bundle over M_X^f. The following proposition allows us to reduce the computation of the equivariant K-theory of M_X^f to that of $T_G^* M_X^f$.

Proposition 6.7.   i) $K_G^{s+1}(M_X^f) = 0$.

    ii) There is a natural isomorphism: $K_G^s(M_X^f) \to K_G^0(T_G^* M_X^f)$.

Proof. Let M be a manifold with an action of G. Assume G has finite stabilizers on M. We claim that, for every i, there is a natural isomorphism between $K_G^{i+s}(T_G^* M)$ and $K_G^i(T^* M)$. In fact the infinitesimal action of $\mathrm{Lie}(G)$ determines a trivial vector bundle L in TM. Using a G-invariant Riemannian structure on M, we identify $T^* M$ with TM so that $T_G^* M$ is identified to the orthogonal of L. Thus we have the product decomposition $T_G^* M \times \mathrm{Lie}(G) = TM$. In this decomposition G acts trivially on the $s-$dimensional factor $\mathrm{Lie}(G)$. We can apply Bott periodicity

$$K_G^0 T_G^* M = K_G^s TM, \quad K_G^1 T_G^* M = K_G^{s+1} TM.$$

In our case, $T^* M_X^f$ is a complex vector bundle over M_X^f, so we can apply the Thom isomorphism for this bundle and we obtain, using Theorem 6.6, that $K_G^s(M_X^f)$ is isomorphic to $K_G^0(T_G^* M_X^f)$ and that $K_G^{s+1}(M_X^f)$ vanishes. $\qquad\square$

For a rational subspace $\underline{r}$, the action of G on $F(\underline{r})$ factors through $G/G_{\underline{r}}$ and, with respect to this action, $F(\underline{r}) = M_{X \cap \underline{r}}^f$. Thus

$$K_G^i(F(\underline{r})) = R(G) \otimes_{R(G/G_{\underline{r}})} K_{G/G_{\underline{r}}}^i(F(\underline{r})).$$

In particular, by Theorem 6.6, we deduce that $K_G^1(T_G^* F(\underline{r})) = 0$.

Now set $\tilde{T}_G^* F(\underline{r}) := T_G^* M_X | F(\underline{r})$, the restriction of $T_G^* M_X$ to $F(\underline{r})$. We see that $\tilde{T}_G^* F(\underline{r}) = T_G^* F(\underline{r}) \times M_{X \setminus \underline{r}}'$, so we have a Thom isomorphism

$$C_{\underline{r}} : K_G^0(T_G^* F(\underline{r})) \to K_G^0(\tilde{T}_G^* F(\underline{r})), \quad K_G^1(\tilde{T}_G^* F(\underline{r})) = 0.$$

Choose $0 \le i \le s$. We pass now to study the G-invariant open subspace $M_{\ge i}$ of M. The set $M_{\ge i+1}$ is open in $M_{\ge i}$ with complement the space F_i disjoint union of the spaces $F(\underline{r})$ with $\underline{r} \in S_X^{(i)}$. Denote by $\tilde{T}_G^* F_i$ the restriction of $T_G^* M$ to F_i, disjoint union of the spaces $\tilde{T}_G^* F(\underline{r})$. Denote $j : M_{\ge i+1} \to M_{\ge i}$ the open inclusion and $e : \tilde{T}_G^* F_i \to T_G^* M_{\ge i}$ the closed embedding. Let C_i be the Thom isomorphism from $K_G^0(T_G^* F_i)$ to $K_G^0(\tilde{T}_G^* F_i)$ direct sum of the Thom isomorphisms $C_{\underline{r}}$.

Theorem 6.8. *For each* $0 \leq i \leq s - 1$,

    i) $K_G^1(T_G^* M_{\geq i}) = 0$.
    ii) The following sequence is exact

$$(28) \qquad 0 \to K_G^0(T_G^* M_{\geq i+1}) \xrightarrow{j_*} K_G^0(T_G^* M_{\geq i}) \xrightarrow{C_i^{-1} e^*} K_G^0(T_G^* F_i) \to 0.$$

Proof. Since $M_{\geq s} = M_X^f$, we can assume by induction on $s - i$ that i) holds for each $j > i$. Also by Theorem 6.6 i) we get that $K_G^1(T_G^* F_i) = 0$ for each $0 \leq i \leq s - 1$. Using this both statements follow immediately from the long exact sequence of equivariant K-theory. $\qquad\square$

Remark 6.9. The fact that the sequence (28) is exact is proved in [1] using a splitting. We will comment on this point in Section 7.

6.10. **Two commutative diagrams.** Let N be a complex representation space for G. Recalling the structure of $R(G)$ module of the equivariant K-theory, the multiplication by the difference $\bigwedge^{even} N - \bigwedge^{odd} N$ will be denoted by $\bigwedge_{-1} N \otimes -$. This is by definition the action of the element $\det_N(1 - g) \in R(G)$ on the equivariant K-theory.

Lemma 6.11. *Take a sublist Y in X and decompose $M_X = M_{X \setminus Y} \oplus M_Y$. Let U be an open G-invariant set contained in $M_{X \setminus Y} \times (M_Y \setminus \{0\})$. Then if $\sigma \in K_G^i(U)$ or $\sigma \in K_G^i(T_G^* U)$, we have $\bigwedge_{-1} M_Y \otimes \sigma = 0$.*

Proof. We give the proof for U, the case of $T_G^* U$ being identical.

Take $v \in U$ and decompose it as $v = v_{X \setminus Y} + v_Y$ with $v_{X \setminus Y} \in M_{X \setminus Y}$ and $v_Y \in M_Y$. The component v_Y is not zero by assumption. Consider the complex G-equivariant vector bundle $V_Y = U \times M_Y$ on U. Set now $E^+ := U \times \bigwedge^{even} M_Y$, $E^- = U \times \bigwedge^{odd} M_Y$. Choosing an Hermitian metric on M_Y, for every $u \in M_Y$, we get the Clifford action $c(u) : \bigwedge^{even} M_Y \to \bigwedge^{odd} M_Y$ of M_Y on $\bigwedge M_Y$, which is an isomorphism as soon as $u \neq 0$.

Going back to our bundles E^+, E^-, for every $\epsilon \in [0, 1]$, define the bundle map $\mathbf{c}_\epsilon : E^+ \to E^-$ by

$$\mathbf{c}_\epsilon(v, \omega) = (v, \epsilon c(v_Y) \omega).$$

If $\sigma \in K_G^0(U)$, the element $\bigwedge_{-1} M_Y \otimes \sigma \in K_G^0(U)$ is represented by the morphism $\mathbf{c_0} \odot \sigma$, homotopic to $\mathbf{c_1} \odot \sigma$. This last bundle map is an isomorphism since $v_Y \neq 0$ on U. This implies that $\bigwedge_{-1} M_Y \otimes \sigma = 0$. $\qquad\square$

We apply this to the open set M_X^f where G acts with finite stabilizers. If Y is a cocircuit, $M_{X \setminus Y} \cap M_X^f = \emptyset$, thus $\bigwedge_{-1} M_Y \otimes \sigma = 0$ for all $\sigma \in K^0(T_G^* M_X^f)$. As the index map is a $R(G)$ module map, this implies that for cocircuit Y, the generalized function $index(\sigma)(g)$ on G satisfies the equation $\prod_{a \in Y}(1 - g^a)index(\sigma)(g) = 0$. The function $ind_m(\sigma)$ on $\hat{G}$ is the Fourier transform of the function $index(\sigma)$. It follows that $\nabla_Y ind_m(\sigma) = 0$.

Thus we obtain

Corollary 6.12. *The multiplicity index map ind_m maps $K^0_G(T^*_G M^f_X)$ to the space $DM(X)$.*

More generally the same argument shows that

Corollary 6.13. *Choose $0 \le i \le s$. If $\sigma \in K^0_G(T^*_G M_{\ge i})$ and $\underline{t}$ is a rational subspace of dimension strictly less than i, then $\bigwedge_{-1} M_{X \setminus \underline{t}} \otimes \sigma = 0$.*

Let us now split $X = A \cup B$ and $M_X = M_A \oplus M_B$. Let $p : T^*_G M_X \to M_X$ be the projection and consider $\tilde{T}^*_G M_A := p^{-1} M_A$. We have $\tilde{T}^*_G M_A = T^*_G M_A \times M'_B$. In particular, we get a Thom isomorphism

$$C_{M'_B} : K^0_G(T^*_G M_A) \to K^0_G(\tilde{T}^*_G M_A) \cong K^0_G(T^*_G M_A \times M'_B).$$

Denote by i the closed inclusion $M_A \to M_X$, and, by abuse of notation, also the inclusion $\tilde{T}^*_G M_A \to T^*_G M_X$ above i. Then i induces the morphisms $i^* : K^0_G(T^*_G M_X) \to K^0_G(\tilde{T}^*_G M_A)$ and $i_! : K^0_G(T^*_G M_A) \to K^0_G(T^*_G M_X)$. Combining these 3 maps, we claim that

Lemma 6.14. *Take $\sigma \in K^0_G(T^*_G M_X)$, then $i_! C^{-1}_{M'_B} i^*(\sigma) = \bigwedge_{-1} M_B \otimes \sigma$.*

Proof. Since we are working on vector spaces, we can assume that all vector bundles are topologically trivial. Thus we can represent σ as given by a variable linear map $\sigma(v, w, \xi, \eta) : E \to F$ where E, F are complex representation spaces, $v \in M_A, w \in M_B, \xi \in M'_A, \eta \in M'_B$. Now σ restricts to an element $\tilde{\sigma} := i^* \sigma$ in $K^0_G(T^*_G M_X | M_A)$ which is represented by the map $\sigma(v, 0, \xi, \eta)$. Since $T^*_G M_X | M_A = T^*_G M_A \times M'_B$, the element $\tilde{\sigma}$ is equivalent to $\mathbf{c}_{M'_B} \odot q^* \tau$, where $q : T^*_G M_A \times M'_B \to T^*_G M_A$ is the projection, τ a transversally elliptic symbol on M_A and $\mathbf{c}_{M'_B}$ the Bott symbol with support the zero section of the bundle $T^*_G M_A \times M'_B$ on $T^*_G M_A$. Thus we have to show that $\bigwedge_{-1} M_B \otimes \sigma$ and $i_!(\tau)$ are homotopic.

By definition, a representative of the symbol $i_!(\tau)$ on $M_A \times M_B$ is the product of the symbol $\mathbf{c}_{M_B \otimes_{\mathbb{R}} \mathbb{C}}$ by the symbol $q^* \tau$. As $\mathbf{c}_{M_B \otimes_{\mathbb{R}} \mathbb{C}} = \mathbf{c}_{M_B} \odot \mathbf{c}_{M'_B}$, we see that $i_!(\tau) = q^* \tau \odot \mathbf{c}_{M_B} \odot \mathbf{c}_{M'_B} = \tilde{\sigma} \odot \mathbf{c}_{M_B}$.

As we have seen before, the symbol defined as $\bigwedge_{-1} M_B \otimes \sigma$ on the manifold $M_A \times M_B$ is homotopic to the element $\mathbf{c}_{M_B} \odot \sigma$. Now consider the symbol $\sigma(t)(v, w, \xi, \eta) = \sigma(v, tw, \xi, \eta)$ on $T^* M$. The intersection of the support of $\mathbf{c}_{M_B} \odot \sigma(t)$ with $T^*_G M$ stays compactly supported for all t. Indeed its support remains constant: this is the intersection of the support of σ with $T^*_G M_A$. So we obtain the desired homotopy between $\bigwedge_{-1} M_B \otimes \sigma = \mathbf{c}_{M_B} \odot \sigma(1)$ and $i_!(\tau) = \mathbf{c}_{M_B} \odot \sigma(0)$ and the claim follows. $\square$

With the previous notations, $X = A \cup B$, $i : T^*_G M_A \times M'_B \to T^*_G M_X$.

Corollary 6.15. *Take $\sigma \in K^0_G(T^*_G M_X)$. Let $\sigma_0 = C^{-1}_{M'_B} i^*(\sigma) \in K^0_G(T^*_G M_A)$. Then, we have the equality of generalized functions on G:*

$$det_{M_B}(1 - g) index(\sigma)(g) = index(\sigma_0)(g).$$

We are now ready to compare the exact sequence (27) with the exact sequence given by Theorem (3.17) using the index. We get

Theorem 6.16. *The diagram*

$$
\begin{array}{ccccccccc}
0 \to & K^0_{G_a}(T^*_{G_a} M^f_{\tilde{Z}}) & \xrightarrow{\;j_* k\;} & K^0_G(T^*_G M^f_X) & \xrightarrow{\;C_a^{-1} i^*\;} & K^0_G(T^*_G M^f_Z) & \to 0 \\
& \quad ind_m \downarrow & & \quad ind_m \downarrow & & \quad ind_m \downarrow & \\
0 \longrightarrow & DM(\tilde{Z}) & \xrightarrow{\;i_a\;} & DM(X) & \xrightarrow{\;\nabla_a\;} & DM(Z) & \to 0
\end{array}
$$

(29)

is commutative. Its vertical arrows are isomorphisms.

In particular, the index multiplicity map gives an isomorphism between $K^0_G(T^*_G M^f_X)$ *and* $DM(X)$.

Proof. We start by remarking that, by Corollary 6.12, all the vertical maps in our diagram are indeed taking values in the corresponding Dahmen-Micchelli spaces.

So we need to show commutativity. To prove the commutativity of the square on the right hand side, using Fourier transform, we need to prove that $(1 - g^a)index(\sigma)(g) = index(C_a^{-1} i^*(\sigma))(g)$.

From the symbol σ on M^f_X, an open set in M_X, we deduce a symbol on M_X with same index, by the excision property of the index. Thus the commutativity follows from Corollary 6.15 applied to $A = Z$, $B = \{a\}$. As for the square on the left hand side, since j_* is an open embedding, it preserves indices. The statement thus follows from Proposition 6.4.

By induction we can then assume that the two external vertical arrows are isomorphism so, by the five Lemma, also the central one is and everything follows. $\square$

Summarizing we have isomorphisms

$$\boxed{K^{s+1}_G(M^f_X) \cong K^1_G(T^*_G M^f_X) = 0, \qquad K^s_G(M^f_X) \cong K^0_G(T^*_G M^f_X) \cong DM(X).}$$

Let us make two obvious remarks on the isomorphism $K^0_G(T^*_G M^f_X) \cong DM(X)$.

Remark 6.17. The space $K^0_G(T^*_G M^f_X)$ depends only of the manifold M_X considered as a real manifold. By Remark 3.5, the space $DM(X)$ depends only of the list X up to change of signs.

Remark 6.18. If U is a G manifold, the index of an element $\sigma \in K^0_G(T^*_G U)$ is a generalized function supported on the set of points $g \in G$ such that g has a fixed point in U.

We have seen in §3.23 that Fourier transforms of elements in $DM(X)$ are supported on the finite set of points $P(X)$. This is in agreement with the fixed point philosophy that we just recalled. In fact, an element $g \in G$ has a fixed point v in M^f_X if and only if $g \in P(X)$. Indeed if $g \in P(X)$, there exists a basis $\underline{b}$ of V extracted form X with $g^{b_i} = 1$, for all $b_i \in \underline{b}$. Thus any

element $v \in M_X$ with non zero coordinates on each L_{b_i} is fixed by g, and is in M_X^f.

We now come to our next commutative diagram.

Lemma 6.19. *For each $s \geq i \geq 0$, the index multiplicity map ind_m sends $K_G^0(T_G^* M_{\geq i})$ to the space $\tilde{\mathcal{F}}_i(X)$.*

Proof. Recall that $\tilde{\mathcal{F}}_i(X)$ is the subspace in $\tilde{\mathcal{F}}(X)$ such that $\nabla_{X \setminus \underline{t}} f = 0$ for all $\underline{t} \in S_X^{(i-1)}$. Denote by $\ell : \tilde{\mathcal{F}}_i(X) \to \tilde{\mathcal{F}}(X)$ the inclusion.

By Corollary 6.13, if $\sigma \in K_G^0(T_G^* M_{\geq i})$ and $\underline{t}$ is a rational subspace of dimension strictly less than i, we have $\bigwedge_{-1} M_{X \setminus \underline{t}} \otimes \sigma = 0$. Thus $\nabla_{X \setminus \underline{t}} ind_m(\sigma) = 0$. It follows that the only thing we have to show is that, if $\sigma \in K_G^0(T_G^* M_X)$, then $ind_m(\sigma)$ lies in $\tilde{\mathcal{F}}(X)$. Take a rational subspace $\underline{r}$. By Lemma 6.14, the index of $\bigwedge_{-1} M_{X \setminus \underline{r}} \otimes \sigma$ equals the index of an element $\sigma_0 \in K_G^0(T_G^* M_{X \cap \underline{r}})$. But the action of G on $M_{X \cap \underline{r}}$ factors though the quotient $G/G_{\underline{r}}$ whose character group is $\Gamma_{\underline{r}}$. Thus $K_G^0(T_G^* M_{X \cap \underline{r}}) \cong R(G) \otimes_{R(G/G_{\underline{r}})} K_{G/G_{\underline{r}}}^0(T_{G/G_{\underline{r}}}^* M_{X \cap \underline{r}})$, hence $\nabla_{X \setminus \underline{r}} ind_m(\sigma) = ind_m(\sigma_0)$ lies in $R(G) \otimes_{R(G/G_{\underline{r}})} \mathcal{C}[\Gamma_{\underline{r}}]$ as desired. $\square$

Our second commutative diagram and main theorem characterizes the values of the index on the entire M_X. This time, we use the notations and the exact sequences contained in Theorem 6.8 and Corollary 6.15

Theorem 6.20. *For each $0 \leq i \leq s$,*

- *the diagram*

$$
\begin{array}{ccccc}
0 \to K_G^0(T_G^* M_{\geq i+1}) & \xrightarrow{j_*} & K_G^0(T_G^* M_{\geq i}) & \xrightarrow{C_i^{-1} e^*} & K_G^0(T_G^* F_i) \to 0 \\
{\scriptstyle ind_m} \downarrow & & {\scriptstyle ind_m} \downarrow & & {\scriptstyle ind_m} \downarrow \\
0 \to \tilde{\mathcal{F}}_{i+1}(X) & \xrightarrow{\ell} & \tilde{\mathcal{F}}_i(X) & \xrightarrow{\mu_i} & \oplus_{\underline{r} \in S_X^{(i)}} DM^{(G)}(X \cap \underline{r}) \to 0
\end{array}
$$

   commutes.
- *Its vertical arrows are isomorphisms.*
- *In particular, the index gives an isomorphism between $K_G^0(T_G^* M_X)$ and $\tilde{\mathcal{F}}(X)$.*

Proof. Lemma 6.19 tells us that the diagram is well defined. We need to prove commutativity.

Again, we prove that the square on the right hand side is commutative using Corollary 6.15. The square on the left hand side is commutative since j_* is compatible with the index and ℓ is the inclusion.

Recall that $K_G^0(T_G^* M_{X \cap \underline{r}}) \cong R(G) \otimes_{R(G/G_{\underline{r}})} K_{G/G_{\underline{r}}}^0(T_{G/G_{\underline{r}}}^* M_{X \cap \underline{r}})$ and that $DM^{(G)}(X \cap \underline{r}) \cong R(G) \otimes_{R(G/G_{\underline{r}})} DM(X \cap \underline{r})$. Using Theorem 6.16, this implies that the right vertical arrow is always an isomorphism.

We want to apply descending induction on i. When $i + 1 = s$, since $M_{\geq s} = M_X^f$ and $\tilde{\mathcal{F}}_{s-1}(X) = DM(X)$, Theorem 6.16 gives that the left

vertical arrow is an isomorphism. So assume that the left vertical arrow is an isomorphism. We then deduce by the five Lemma that the central vertical arrow is an isomorphism and conclude by induction.   □

Remark 6.21. Fourier transforms of elements in $\tilde{\mathcal{F}}_i(X)$ are supported on the closed set $P_{s-i}(X)$ described in Proposition 4.12. This is again in agreement with the fixed point philosophy that we recalled in Remark 6.18. In fact, an element $g \in G$ has a fixed point v in $M_{\geq i}$ if and only if $g \in P_{s-i}(X)$.

7. GENERATORS OF $K_G^0(T_G^* M_X)$

In this section, we show that the generators of $\tilde{\mathcal{F}}(X)$ constructed in Section 4.8 corresponds via the index map to the generators of $K_G^0(T_G^* M_X)$ constructed by Atiyah-Singer in [1].

We assume that X does not contain any element of finite order. This is harmless as otherwise we need only to tensor our results with the Bott symbol on $\oplus_{a \in \Gamma_t} L_a$.

Recall that, given a manifold M, a way to construct elements of $K_G^0(T_G^* M)$ is to take a closed $G-$ manifold N embedded by $i : N \to M$. Then we have a map $i_! : K_G^0(T_G^* N) \to K_G^0(T_G^* M)$.

For our case $M = M_X$, we shall take the following manifolds. Take a flag ϕ of rational subspaces $0 = \underline{r}_0 \subset \underline{r}_1 \subset \underline{r}_2 \subset \cdots \subset \underline{r}_s$ with $\dim(\underline{r}_i) = i$ (and $s = \dim G$). Consider then the spaces $E_i := \oplus_{a \in (X \cap \underline{r}_i) \setminus \underline{r}_{i-1}} L_a$. We choose an orientation for each $\underline{r}_i$ and divide the set of characters $Z_i := (X \cap \underline{r}_i) \setminus \underline{r}_{i-1}$ into positive and negative elements A_i, B_i. Accordingly, we change the complex structure on each L_b for which b is negative into its conjugate structure. Let A be the union of the sets A_i and B the union of the sets B_i. Choose a G-invariant Hermitian metric h_i on E_i and consider the unit sphere $S_i(h_i)$ on E_i. The product $S_\phi(h) = \prod_{i=1}^s S_i(h_i)$ is a closed submanifold of M_X^f.

The tangent space at a point p of $S_\phi(h)$ decomposes as a vertical space generated by the rotations on each factor E_i and the horizontal space H_p, a lift of the tangent space to the corresponding product of projective spaces. The horizontal space is a Hermitian vector space. The tangential Cauchy-Riemann operator δ_ϕ is a differential operator on $S_\phi(h)$, the product of the operators δ_i described in Subsection 5.7. The index of δ_ϕ is the product of the indices of δ_i.

Let $c_\phi(p, \xi) = c(\xi^1)$ be the Clifford action on $\bigwedge H_p$ of the projection ξ^1 of ξ on the horizontal tangent space H_p.

We then have [1] the following theorem.

Theorem 7.1.

$$index(\delta_\phi)(g) = (-1)^s (-1)^{|B|} g^{\sum_{a \in A} a} \theta_\phi^X(g).$$

In fact $c_\phi = c_1 \odot c_2 \odot \cdots \odot c_s$ is the external product of the symbols c_i of the operators δ_i.

Let i_ϕ be the closed embedding of $S_\phi(h)$ in M_X^f. We can then give an "easy" proof of the following theorem of Atiyah-Singer (Theorem 7.9 of [1])

Theorem 7.2. *Let i_ϕ be the closed injection of $S_\phi(h)$ to M_X. Then the elements $(i_\phi)_! c_\phi$ generate $K_G^0(T_G^* M_X^f)$.*

Proof. This follows from Theorem 3.22 giving generators for $DM(X)$, and the fact that the index multiplicity map is an isomorphism onto $DM(X)$. $\square$

Remark 7.3. In fact Theorem 3.22 gives a basis of the space $DM(X)$, so that the previous theorem can be refined accordingly.

According to §5.7, if we consider M_X as a real representation of G, each connected component F of the space of regular elements $u \in U = Lie(G)$ gives us a complex structure J_F on M_X and a corresponding "pushed" symbol At_F with $index(At_F)(g) = (-1)^{|B|} g^{\sum_{a \in A} a} \Theta_X^F$ (Theorem 5.10).

Theorem 7.4. *The symbols $At_F \in K_G^0(T_G^* M_X)$, where F varies over all open faces of the arrangement $\mathcal{H}_X$, give us a set of generators for $K_G^0(T_G^* M_X)$*

Proof. This follows from Theorem 4.9 giving generators for the space $\tilde{\mathcal{F}}(X)$, and the fact that the index multiplicity map is an isomorphism onto $\tilde{\mathcal{F}}(X)$.
$\square$

Remark 7.5. After checking naturality axioms, Theorem 7.4 reduces the proof of the cohomological index formula given by [4] or [8] to the case of the symbols At_F. So Theorem 7.4 is crucial in establishing a cohomological formula valid for any transversally elliptic operator.

Remark 7.6. Consider the exact sequence in Theorem 6.8.

$$0 \to K_G^0(T_G^* M_{\geq i+1}) \xrightarrow{j_*} K_G^0(T_G^* M_{\geq i}) \xrightarrow{C_i^{-1} e^*} K_G^0(T_G^* F_i) \to 0.$$

In [1], the exactness of this sequence is proved using a splitting. We recall the proof of [1] for $i = s - 1$, the proof for any $0 \leq i \leq s$ being identical. Consider $\underline{r}$ of codimension 1 and write $M_X = M_{X \cap \underline{r}} \oplus M_{X \backslash \underline{r}}$. Choose a regular element $u_{\underline{r}}$ vanishing on $\underline{r}$. We modify the complex structure on $M_{X \backslash \underline{r}}$ so that all the weights of G on $M_{X \backslash \underline{r}}$ are positive on $u_{\underline{r}}$. We then construct the corresponding "pushed" $\overline{\partial}$ symbol on $M_{X \backslash \underline{r}}$:

$$At_u(v, \xi) = c(\xi + \rho(u)v).$$

Here $v, \xi \in M_{X \backslash \underline{r}}$, and the operator c is the Clifford action of $M_{X \backslash \underline{r}}$ on $\bigwedge M_{X \backslash \underline{r}}$.

Denote by p, q the projections of $M_X = M_{X \backslash \underline{r}} \times M_{X \cap \underline{r}}$ on the two factors. Consider $M_{X \backslash \underline{r}} \times M_{X \cap \underline{r}}$ as a $G \times G$ manifold. We use now the general multiplicative formula Theorem 3.5 of [1], applied to the groups $G \times G$.

If τ is a G-transversally elliptic symbol on $F(\underline{r})$, then $\sigma := p^* At_u \odot q^* \tau$ is a $G \otimes G$-transversally elliptic symbol on $M_{X \backslash \underline{r}} \times F(\underline{r})$ with index the product of the two indices on $G \times G$: $index(p^* At_u \odot q^* \tau)(g_1, g_2) =$

$index(p^*At_u)(g_1)index(q^*\tau)(g_2)$. Now, consider G embedded as the diagonal in $G \times G$. As τ is $G/G_{\underline{r}}$ transversally elliptic, and At_u is $G(\underline{r})$ transversally elliptic, the symbol σ remains G-transversally elliptic. We thus obtain that $p^*At_u \odot q^*\tau$ is a G-transversally elliptic symbol on M_X with index the product of the two indices. Let us restrict σ to $T^*_G M_X|_{F(\underline{r})}$. This space is the product $T^*_G F(\underline{r}) \times M'_{X \setminus \underline{r}}$ and thus, for each point in this product, the vector $v = 0$ and the symbol p^*At_u coincides with the Bott symbol $\mathbf{c}_{M'_{X \setminus \underline{r}}}$. In other words, by definition of the isomorphism $C_{\underline{r}}$, the restriction $e^*(\sigma)$ to $T^*_G M_X|_{F(\underline{r})}$ is $C_{\underline{r}}(\tau)$.

Thus we see that $C^{-1}_{\underline{r}} e^*(\sigma) = \tau$, so that $C^{-1}_{\underline{r}} e^*$ is surjective and the map $\tau \mapsto p^*At_u \odot q^*\tau$ is the desired splitting.

Finally remark that this splitting corresponds, under the index isomorphism, to the splitting of the spaces $\tilde{\mathcal{F}}_i(X)$ using convolution by the partial partition functions $\mathcal{P}^{F_{\underline{r}}}_X$. This follows from the explicit computation of the index of the pushed symbols in the multiplicative formula which we have previously recalled.

8. BACK TO PARTITION FUNCTIONS

Assume G is a torus and let $M_X := \oplus_{a \in X} L_a$.

Let $\mathrm{Cone}(X) := \{\sum_{a \in X} t_a a \,|\, t_a \geq 0\}$ be the cone generated by X in V. Assume $\mathrm{Cone}(X)$ is a pointed cone so that we can consider the partition function $\mathcal{P}_X$. This partition function counts the number of integral points in the partition polytope $P(\lambda) := \{t_a, t_a \in \mathbb{R}_{\geq 0}, \,|\, \sum_a t_a a = \lambda\}$.

Let V_{sing} be the union of the cones $\mathrm{Cone}(Y)$ generated by the sublists Y of X which do not span V. A connected component of $V \setminus V_{\mathrm{sing}}$ is called a big cell. Recall [6] that the partition function $\mathcal{P}_X$ coincide on a big cell $\mathfrak{c}$ with an element of $DM(X)$ denoted by $\mathcal{P}^{\mathfrak{c}}_X$.

Consider the moment map $\mu_X : M_X \to V$ given by $\mu_X(v) = \sum_{a \in X} \|v_a\|^2 a$. Then μ_X is a proper map. If $v \in \mathrm{Cone}(X)$ is in a big cell $\mathfrak{c}$, v is a regular value of μ_X, and the manifold $P_v := \mu_X^{-1}(v)$ is a compact closed submanifold of M_X^f. We recall that $M(\mathfrak{c}) := P_v/G$ is a toric manifold which depends only of $\mathfrak{c}$.

A neighborhood $\mathcal{U}_v$ of P_v in M_X^f is isomorphic to the product $V \times P_v$, so that we have $K^0_G(P_v) \sim K^s_G(\mathcal{U}_v)$. Via the open embedding $\mathcal{U}_v \to M_X^f$, we obtain a map $m_v : K^0_G(P_v) \to K^s_G(M_X^f)$.

Let I_v be the trivial bundle over P_v. It is easy to see that the element $m_v(I_v) \in K^s_G(M_X^f)$ depends only of the big cell $\mathfrak{c}$ where v leaves. We denote it by $I_{\mathfrak{c}}$.

Recall the isomorphism $r : K^s_G(M_X^f) \to DM(X)$ given by combining 6.7 and the index map 6.16. The following theorem will be proved in a subsequent article.

Theorem 8.1.   *i) When $\mathfrak{c}$ runs over all big cells contained in $\mathrm{Cone}(X)$, the elements $I_\mathfrak{c}$ generate $K_G^s(M_X^f)$ as a $R(G)$ module.*

ii) We have $r(I_\mathfrak{c}) = \mathcal{P}_X^\mathfrak{c}$.

The proof of the second item of this theorem follows right away from the free action property of the index when the toric manifold $M(\mathfrak{c})$ is smooth. Indeed, choose an Hermitian structure on M_X. The space $T_G^* P_v$ has the structure of an Hermitian vector bundle. Its fiber at the point $p \in P_v$ is isomorphic to the "horizontal tangent space" H_p. As in Example 5.7, we can construct the transversally elliptic symbol $\sigma_v(p, \xi) = c(\xi^1)$, where ξ^1 is the projection of ξ on H_p and c the Clifford action of H_p on ΛH_p. Via the closed embedding $i_v : P_v \to M_X^f$, we obtain an element $(i_v)_!(\sigma_v)$ in $K^0(T_G^* M_X^f)$. Thus the element $r(I_\mathfrak{c})$ is the index multiplicity of σ_v.

Assume $M(\mathfrak{c})$ smooth. Then $P_v \to P_v/G$ is a principal bundle. If $\lambda \in \Lambda$, we obtain a holomorphic line bundle $\mathcal{O}(\lambda) = P_v \times_G \mathbb{C}_\lambda$ over $M(\mathfrak{c})$. By the free action property of the index, $ind_m(\sigma_v)(\lambda)$ is the virtual dimension of the space $\sum_{i=0}^{\dim M(\mathfrak{c})} (-1)^i H^{0,i}(M(\mathfrak{c}), \mathcal{O}(\lambda))$. This dimension is polynomial in λ. On the other hand, when $\lambda \in \mathfrak{c}$, $H^{0,i}(M(\mathfrak{c}), \mathcal{O}(\lambda))$ vanishes if $i > 0$ and $H^{0,0}(M(\mathfrak{c}), \mathcal{O}(\lambda))$ is the number of integral points in the partition polytope $P(\lambda)$, which is given by the function $\mathcal{P}_X(\lambda)$.

When $M(\mathfrak{c})$ is an orbifold, we deduce Theorem 8.1 from the general cohomological index theorem for transversally elliptic operators. We will discuss this point in a subsequent article.

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