INFINITESIMAL INDEX: COHOMOLOGY COMPUTATIONS

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1. Introduction

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, M a manifold with G-action and equipped with a G–equivariant 1–form σ.

From this setting, one has a moment map $\mu^\sigma : M \to \mathfrak{g}^*$. A particularly important case is that of $M = T^*N$, the cotangent bundle of a manifold with a G action, equipped with the canonical action form. In this case, the zeroes of the moment map is a subspace $T^*_G N$ whose equivariant K–theory is strongly related to the index of transversally elliptic operators as shown in [1].

In order to understand explicit formulas for such an index, in [8] we have introduced the *infinitesimal index* infdex, a map from the equivariant cohomology with compact support of the zeroes of the moment map to distributions on $\mathfrak{g}^*$.

We have proved several properties for this map which, at least in the case of the space $T^*_G N$, in principle allow us to reduce the computations to the case in which G is a torus and the manifold is a complex linear representation of G. A finite dimensional complex representation of a torus is the direct sum of one dimensional representations given by characters. If X is a list of characters, we denote by M_X the corresponding linear representation which is naturally filtered by open sets $M_{X,\geq i}$ where the dimension of the orbit is $\geq i$.

In this paper, we first compute the equivariant cohomology of the open sets $M_{X,\geq i}$, and also of some slightly more general open sets in M_X. This part of our paper, namely Sections 2 and 3, does not use the notion of infinitesimal index. The results are obtained from the structure of the algebra $S[\mathfrak{g}^*][(\prod_{a\in X} a)^{-1}]$ as a module over the Weyl algebra studied in [5].

In Section 4 we apply the results we have obtained to the equivariant cohomology of the open set M_X^{fin} of points with finite stabilizer. Using Poincaré duality, we remark that the equivariant cohomology with compact support $H^*_{G,c}(T^*_G M_X^{fin})$ of $T^*_G M_X^{fin}$ is isomorphic as a $S[\mathfrak{g}^*]$-module to a remarkable finite dimensional space $D(X)$ of polynomial functions on $\mathfrak{g}^*$, where $S(\mathfrak{g}^*)$ acts by differentiation. The space $D(X)$ is defined as the space of solutions of a set of linear partial differential equations combinatorially associated to X and has been of importance in approximation theory (see for example [2], [3]).

At this point the notion of infinitesimal index comes into play. We show in Theorem 4.18 that the infinitesimal index gives an isomorphism between $H^*_{G,c}(T^*_G M^{fin}_X)$ and $D(X)$. After this, we show that, for each i, the infinitesimal index establishes an isomorphism between $H^*_{G,c}(T^*_G M_{X,\geq i})$ and a space of *splines* $\tilde{\mathcal{G}}_i(X)$, introduced in [6], (cf. (15)) and generalizing $D(X)$.

It should be mentioned that, in the previous paper [7], similar results have been proved, using the index of transversally elliptic differential operators, in order to compute the equivariant K-theory of the spaces $T^*_G M_{X,\geq i}$).

This paper represents a sort of "infinitesimal" version of [7] and will be used, in a forthcoming paper [9], to give explicit formulas for the index of transversally elliptic operators.

Finally it is a pleasure to thank Michel Brion for a number of useful conversations and remarks.

2. A special module

2.1. **A module filtration.** Let G be a compact torus with Lie algebra $\mathfrak{g}$ and character group Λ. We are going to consider Λ as a lattice in $\mathfrak{g}^*$.

We need to recall some general results proved in [5]. Let us fix a list $X = (a_1, \dots, a_m)$ of non zero characters in $\Lambda \subset \mathfrak{g}^*$. For a list Y of vectors, let us set $d_Y := \prod_{a \in Y} a \in S[\mathfrak{g}^*]$.

Definition 2.2. A subspace $\underline{s}$ of $\mathfrak{g}^*$ is called *rational* (relative to X) if $\underline{s} = \langle X \cap \underline{s} \rangle$.

We shall denote by $\mathcal{S}_X$ the set of rational subspaces and, for a given $0 \leq k \leq s$, by $\mathcal{S}_X(k)$ the set of the rational subspaces of dimension k.

We need to recall that a *cocircuit* in X is a sublist of X of the form $Y := X \setminus H$ where H is a rational hyperplane.

Let $S[\mathfrak{g}^*]$ be the symmetric algebra on $\mathfrak{g}^*$ or in other words the algebra of polynomial functions on $\mathfrak{g}$. The polynomials $d_Y := \prod_{a \in Y} a \in S[\mathfrak{g}^*]$, as Y runs over the cocircuits, give a system of polynomial equations $d_Y = 0$.

Definition 2.3. We denote by I_X the ideal in $S[\mathfrak{g}^*]$ generated by the elements d_Y's, as Y runs over the cocircuits.

One knows that I_X defines a scheme V_X supported at 0 and of length $d(X) = \dim(S[\mathfrak{g}^*]/I_X)$ (see [5], Theorem 11.13). $d(X)$ equals the number of bases extracted from X.

Consider the localized algebra $R_X := S[\mathfrak{g}^*][d_X^{-1}]$, which is the coordinate ring of the complement of the hyperplane arrangement defined by the equations $a = 0,\ a \in X$, in $\mathfrak{g}$.

This algebra is a cyclic module over the Weyl algebra $W[\mathfrak{g}]$ of differential operators with polynomial coefficients, generated by d_X^{-1}.

In [5], we have seen that this $W[\mathfrak{g}]$-module has a canonical filtration, the *filtration by polar order*, where we put in degree of filtration $\leq k$ all fractions in which the denominator is a product of elements in X spanning a rational subspace of dimension $\leq k$ (we say that k is the polar order on the boundary divisors). We denote this subspace by $R_{X,k}$. One of the important facts (Theorem 8.10 in [5]) is that

Theorem 2.4. *The module $R_{X,k}/R_{X,k-1}$ is semisimple, its isotypic components are in 1–1 correspondence with the rational subspaces of dimension k and such a isotypic component is generated by the class of $1/d_{X\cap \underline{s}}$.*

Consider the rank 1 free $S[\mathfrak{g}^*]$ submodule $L := d_X^{-1}S[\mathfrak{g}^*]$ in R_X generated by d_X^{-1}. Set $L_k := L\cap R_{X,k}$, that is the intersection of L with the $k-$filtration. We obtain for each k an ideal I_k of $S[\mathfrak{g}^*]$ defined by

$$I_k := L_k d_X.$$

For a given rational subspace $\underline{s}$ of dimension k, denote by $I_{\underline{s}} := S[\mathfrak{g}^*]d_{X\setminus\underline{s}}$ the principal ideal generated by $d_{X\setminus\underline{s}}$. Notice that

$$I_{\underline{s}}L = d_{X\cap\underline{s}}^{-1}S[\mathfrak{g}^*] \subset L_k.$$

Thus $I_{\underline{s}} \subset I_k$ and indeed from Theorem 11.29 of [5] one gets

$$I_k = \sum_{\underline{s}\in\mathcal{S}_X(k)} I_{\underline{s}}.$$

If $Q \subset \mathcal{S}_X$ is a set of rational subspaces, we set

$$I_Q = \sum_{\underline{s}\in Q} I_{\underline{s}}$$

for the ideal generated by the elements $d_{X\setminus\underline{s}}$ for $\underline{s} \in Q$.

Associated to $\underline{s}$, we also consider the list $X\cap\underline{s}$ consisting of those elements of X lying in $\underline{s}$ and we may consider the ideal $I_{X\cap\underline{s}} \subset S[\underline{s}]$, as defined in 2.3, and its extension $J_{X\cap\underline{s}} := I_{X\cap\underline{s}}S[\mathfrak{g}^*]$. The obvious map

$$S[\mathfrak{g}^*] \otimes_{S[\underline{s}]} I_{X\cap\underline{s}} \to J_{X\cap\underline{s}}$$

is an isomorphism so that

$$S[\mathfrak{g}^*]/J_{X\cap\underline{s}} \simeq S[\mathfrak{g}^*] \otimes_{S[\underline{s}]} (S[\underline{s}]/I_{X\cap\underline{s}}). \tag{1}$$

Lemma 2.5. *If $\underline{s}$ is of dimension k, we have that $d_{X\cap\underline{s}}^{-1}S[\mathfrak{g}^*] \subset L_k$ and*

$$d_{X\cap\underline{s}}^{-1}S[\mathfrak{g}^*] \cap L_{k-1} \supset d_{X\cap\underline{s}}^{-1}J_{X\cap\underline{s}}. \tag{2}$$

Proof. We have already remarked the first statement. As for the second, by definition $J_{X\cap\underline{s}}$ is the ideal generated by the elements d_Z where Z is a cocircuit in $X\cap\underline{s}$. This means in particular that Z is contained in $X\cap\underline{s}$ and that $Y := (X\cap\underline{s})\setminus Z$ spans a subspace of dimension $k-1$. Hence $d_{X\cap\underline{s}}^{-1}d_Z S[\mathfrak{g}^*] = d_Y^{-1}S[\mathfrak{g}^*] \subset L_{k-1}$. □

Multiplying Formula (2) by d_X, we deduce that

$$I_{\underline{s}} \cap I_{k-1} \supset J_{X\cap\underline{s}}d_{X\setminus\underline{s}} = \sum_{\underline{t}\subset\underline{s},\ \underline{t}\in\mathcal{S}_X(k-1)} I_{\underline{t}}. \tag{3}$$

In this way, multiplication by $d_{X\cap \underline{s}}^{-1}$ gives an homomorphism of $S[\mathfrak{g}^*]$–modules $j_{\underline{s}} : S[\mathfrak{g}^*]/J_{X\cap\underline{s}} \to L_k/L_{k-1}$ and hence, taking direct sums, a homomorphism $j := \oplus_{\underline{s}\in\mathcal{S}_X(k)} j_{\underline{s}}$

$$j : \oplus_{\underline{s}\in\mathcal{S}_X(k)} S[\mathfrak{g}^*]/J_{X\cap\underline{s}} \to L_k/L_{k-1}. \tag{4}$$

We have (Theorem 11.3.15 of [5]):

Theorem 2.6. *The homomorphism j is an isomorphism.*

Using (3), Theorem 2.6 tells us that the morphism

$$\tilde{j} : \oplus_{\underline{s}\in\mathcal{S}_X(k)} I_{\underline{s}}/J_{X_{\underline{s}}} d_{X\setminus\underline{s}} \to I_k/I_{k-1} \tag{5}$$

is an isomorphism.

Definition 2.7. A set $Q \subset \mathcal{S}_X$ of rational subspaces is called admissible if, for every $\underline{s} \in Q$, Q also contains all rational subspaces $\underline{t} \subset \underline{s}$.

From Theorem 2.6, we deduce

Proposition 2.8. *1) For any subset $\mathcal{G} \subset \mathcal{S}_X(k)$*

$$(\sum_{\underline{s}\in\mathcal{G}} I_{\underline{s}}) \cap I_{k-1} = \sum_{\underline{t}\subset\underline{s}\in\mathcal{G},\ \underline{t}\in\mathcal{S}_X(k-1)} I_{\underline{t}}. \tag{6}$$

2) Given an admissible subset $Q \subset \mathcal{S}_X$ and a rational subspace $\underline{s} \in Q$ of maximal dimension k, then

$$I_{\underline{s}} \cap I_{Q\setminus\{\underline{s}\}} = I_{\underline{s}} \cap I_{k-1} = \sum_{\underline{t}\subset\underline{s},\ \underline{t}\in\mathcal{S}_X(k-1)} I_{\underline{t}}. \tag{7}$$

Proof. 1) By (5), the restriction of $\tilde{j}$ to $\oplus_{\underline{s}\in\mathcal{G}} I_{\underline{s}}/J_{X_{\underline{s}}} d_{X\setminus\underline{s}}$ is injective. It follows that

$$(\sum_{\underline{s}\in\mathcal{G}} I_{\underline{s}}) \cap I_{k-1} = \sum_{\underline{s}\in\mathcal{G}} J_{X_{\underline{s}}} d_{X\setminus\underline{s}} = \sum_{\underline{t}\subset\underline{s}\in\mathcal{G},\ \underline{t}\in\mathcal{S}_X(k-1)} I_{\underline{t}}$$

as desired.

2) We first assume that $Q \supset \mathcal{S}_X(k-1)$ so that $Q \setminus \underline{s} = \mathcal{S}_X(k-1) \cup \mathcal{G}$ with $\mathcal{G} \subset \mathcal{S}_X(k)$. If $\mathcal{G}$ is empty, then $I_{Q\setminus\{\underline{s}\}} = I_{k-1}$ and our claim is a special case of 1).

Otherwise $I_{Q_{\setminus\underline{s}}} = I_{k-1} + (\sum_{\underline{t}\in\mathcal{G}} I_{\underline{t}}))$. Let $b \in I_{\underline{s}} \cap I_{Q_{\setminus\underline{s}}}$. Passing modulo I_{k-1}, we get an element lying in $I_{\underline{s}}/(I_{\underline{s}} \cap I_{k-1})$ and in $(\sum_{\underline{t}\in\mathcal{G}} I_{\underline{t}})/(\sum_{\underline{t}\in\mathcal{G}} I_{\underline{t}}) \cap I_{k-1})$. But the restriction of $\tilde{j}$ to $\oplus_{\underline{t}\in G\cup\{\underline{s}\}} I_{\underline{t}}/I_{\underline{t}} \cap I_{k-1}$ is injective. It follows that $b \in I_{k-1}$ as desired.

Passing to the general case, set $\tilde{Q} = Q \cup \mathcal{S}_X(k-1)$. We have

$$I_{\underline{s}} \cap I_{Q\setminus\{\underline{s}\}} \subset I_{\underline{s}} \cap I_{\tilde{Q}\setminus\{\underline{s}\}} = I_{\underline{s}} \cap I_{k-1} = \sum_{\underline{t}\subset\underline{s},\ \underline{t}\in\mathcal{S}_X(k-1)} I_{\underline{t}}.$$

On the other hand it is clear that

$$I_{\underline{s}} \cap I_{Q\setminus\{\underline{s}\}} \supset \sum_{\underline{t}\subset\underline{s},\ \underline{t}\in\mathcal{S}_X(k-1)} I_{\underline{t}}$$

and our claim follows. □

3. Equivariant cohomology

3.1. **Equivariant cohomology of $M_{X,\geq k}$.** Let G be a compact torus. Given a G space M, we denote for simplicity by $H^*_G(M)$ the G equivariant cohomology $H^*_G(M,\mathbb{R})$ of M with real coefficients.

For a character $a \in \Lambda$, we denote by L_a the one dimensional complex G module on which G acts via a. Given a list X in Λ, we set

$$M_X = \oplus_{a\in X} L_a.$$

Our purpose is to compute the equivariant cohomology of various G stable open sets in M_X.

To begin with, since M_X is a vector space, $H^*_G(M_X)$ equals the equivariant cohomology of a point and thus $H^*_G(M_X) = S[\mathfrak{g}^*]$, and $\mathfrak{g}^* = H^2_G(M_X)$.

Let X and M_X be as before and Y a sublist of X. We have $M_Y \subset M_X$.

Lemma 3.2. $H^*_G(M_X \setminus M_Y) = S[\mathfrak{g}^*]/(d_{X\setminus Y})$.

Proof. Write $M_X = M_{X\setminus Y} \oplus M_Y$ and denote by $\pi : M_X \to M_Y$ the projection onto the second factor. This is a G equivariant vector bundle on M_Y with fiber $M_{X\setminus Y}$. Thus its equivariant Euler class in $H^*_G(M_Y) = S[\mathfrak{g}^*]$ is given by $d_{X\setminus Y}$. The space $M_X \setminus M_Y$ is obtained by removing the zero section of π. It is a standard fact that $H^*_G(M_X \setminus M_Y)$ equals the equivariant cohomology of M_Y modulo the ideal generated by the Euler class, that is $S[\mathfrak{g}^*]/(d_{X\setminus Y})$. □

Take a subset $Q \subset \mathcal{S}_X$ of rational subspaces and set

$$\mathcal{A}_Q = M_X \setminus \cup_{\underline{s}\in Q} M_{X\cap\underline{s}}.$$

Theorem 3.3. *$H^*_G(\mathcal{A}_Q)$ is isomorphic as a graded ring to $S[\mathfrak{g}^*]/I_Q$.*

In particular $\mathcal{A}_Q$ has no G equivariant odd cohomology.

Proof. Let us add to Q all the rational subspaces $\underline{t}$ which are contained in at least one of the elements of Q. In this way, we get a new subset $\overline{Q} \supset Q$ which is now admissible and is such that $\mathcal{A}_Q = \mathcal{A}_{\overline{Q}}$. Also it is clear that $I_Q = I_{\overline{Q}}$.

Having made this remark, we may without loss of generality assume that Q is admissible. If $Q = \emptyset$, then $\mathcal{A}_\emptyset = M_X$, the ideal $I_\emptyset = \{0\}$ and there is nothing to prove. Thus we can proceed by induction on the cardinality of Q and assume that Q is nonempty.

Notice that $Q \setminus \{\underline{s}\}$ is also admissible. Furthermore the set

$$\mathcal{S}_{<\underline{s}} = \{\underline{t} \in \mathcal{S}_X \mid \underline{t} \subsetneq \underline{s}\}$$

is also admissible and strictly contained in Q.

We have

$$\mathcal{A}_Q = \mathcal{A}_{Q\setminus\{\underline{s}\}} \cap (M_X \setminus M_{X\cap\underline{s}})$$

and

$$\mathcal{A}_{Q\setminus\{\underline{s}\}} \cup (M_X \setminus M_{X\cap\underline{s}}) = \mathcal{A}_{\mathcal{S}_{<\underline{s}}}.$$

Thus, by induction, we have

(8) $H^*_G(\mathcal{A}_{Q\setminus\{\underline{s}\}}) = S[\mathfrak{g}^*]/I_{Q\setminus\{\underline{s}\}}, \quad H^*_G(\mathcal{A}_{Q\setminus\{\underline{s}\}} \cup (M_X \setminus M_{X\cap\underline{s}}) = S[\mathfrak{g}^*]/I_{\mathcal{S}_{<\underline{s}}}.$

Consider the homomorphism

$$\psi : H^*_G(\mathcal{A}_{Q\setminus\{\underline{s}\}} \cup (M_X \setminus M_{X\cap\underline{s}})) \to H^*_G(\mathcal{A}_{Q\setminus\{\underline{s}\}}) \oplus H^*_G(M_X \setminus M_Y)$$

induced by inclusion. Using the isomorphisms (8) and Lemma 3.2, we get a commutative diagram

$$\begin{array}{ccc} H^*_G(\mathcal{A}_{Q\setminus\{\underline{s}\}} \cup (M_X \setminus M_{X\cap\underline{s}})) & \xrightarrow{\psi} & H^*_G(\mathcal{A}_{Q\setminus\{\underline{s}\}}) \oplus H^*_G(M_X \setminus M_Y) \\ \simeq\downarrow & & \simeq\downarrow \\ S[\mathfrak{g}^*]/I_{\mathcal{S}_{<\underline{s}}} & \longrightarrow & S[\mathfrak{g}^*]/I_{Q\setminus\{\underline{s}\}} \oplus S[\mathfrak{g}^*]/(d_{X\setminus Y}) \end{array}$$

where the vertical arrows are isomorphisms. Now by Proposition 2.8 2)

$$I_{\underline{s}} \cap I_{Q\setminus\{\underline{s}\}} = I_{\underline{s}} \cap I_{k-1} = \sum_{\underline{t}\subset\underline{s},\ \underline{t}\in\mathcal{S}_X(k-1)} I_{\underline{t}} = I_{\mathcal{S}_{<\underline{s}}}.$$

Thus ψ is injective. We immediately deduce from the Mayer-Vietoris sequence that the homomorphism

$$\phi : H^*_G(\mathcal{A}_{Q\setminus\{\underline{s}\}}) \oplus H^*_G(M_X \setminus M_Y) \to H^*_G(\mathcal{A}_Q)$$

is surjective and that $H^*_G(\mathcal{A}_Q) \simeq S[\mathfrak{g}^*]/I_Q$ as desired. □

Remark 3.4. There is a parallel theorem for the algebraic counterpart of equivariant cohomology, that is the equivariant Chow ring (see Edidin and Graham [10]).

3.5. **Equivariant cohomology of $M_{X,\geq k}$.** Let us look at some special cases of Theorem 3.3.

If $Q = \mathcal{S}_X(k-1)$,

$$\mathcal{A}_{\mathcal{S}_X(k-1)} = M_X \setminus \cup_{\underline{s}\in\mathcal{S}_X(k-1)} M_{X\cap\underline{s}} := M_{X,\geq k}$$

is the set of points whose orbits have dimension at least k.

Definition 3.6. For $k = s$, $M_{X,\geq s}$ is the open set of points with finite stabilizer that we also denote by M_X^{fin}.

Corollary 3.7. *The equivariant cohomology of $M_{X,\geq k}$ is isomorphic as a graded algebra to $S[\mathfrak{g}^*]$ modulo the ideal I_{k-1}. In particular $H^*_G(M_X^{fin}) = S[\mathfrak{g}^*]/I_X$ with I_X the ideal generated by the elements d_Y as Y runs over the cocircuits.*

Remark 3.8. Assume that X spans an acute cone in $\mathfrak{g}^*$. Write $z \in M_X$ as $z = \sum_a z_a$ with $z_a \in L_a$. Let $\xi \in \mathfrak{g}^*$ not lying in any rational hyperplane. Then the set $P_\xi := \{z \in M_X \mid \sum_a |z_a|^2 a = \xi\}$ is smooth, contained in M_X^{fin}, and P_ξ/G is a toric variety. Generators and relations for the ring

$H_G^*(P_\xi) = H^*(P_\xi/G)$ are well known (see for example [4]). Consider the restriction map $H_G^*(M_X^{fin}) \to H_G^*(P_\xi)$. Then this map is surjective for any ξ and its kernel is generated by elements $d_{X\setminus\sigma} \in S(\mathfrak{g}^*)$, where $\sigma \subset X$ runs over the bases of $\mathfrak{g}^*$ such that ξ is not in the cone generated by σ.

Remark 3.9. It may be interesting to observe that to X, as to any matroid, is associated a two variable polynomial, the Tutte polynomial, that describes the statistics of external and internal activity. Then the statistic of external activity gives rise to the Betti numbers of equivariant cohomology of M_X^{fin} while from internal activity one deduces the characteristic polynomial that describes Betti numbers of the complement of the complex hyperplane arrangement deduced from X. It may be interesting to give a direct topological interpretation of the Tutte polynomial.

4. Equivariant cohomology of T_G^*M

4.1. **The space $D(X)$.** In order to perform our cohomology computations, we need first to introduce some new spaces. We keep the notation of the previous sections.

Given $a \in \mathfrak{g}^*$, let us denote by ∂_a the derivative in the a direction. We identify $S(\mathfrak{g}^*)$ to the space of differential operators with constant coefficients on $\mathfrak{g}^*$.

To a cocircuit Y, we associate the differential operator $\partial_Y := \prod_{a\in Y} \partial_a$.

Definition 4.2. The space $D(X)$ is given by

$$D(X) := \{f \in S[\mathfrak{g}] \,|\, \partial_Y f = 0, \quad \text{for every cocircuit } Y\}. \tag{9}$$

The space $D(X)$ is stable by the action of $S(\mathfrak{g}^*)$.

Notice that, by its definition, $D(X)$ is the (graded) vector space dual to the algebra $D^*(X) = S[\mathfrak{g}^*]/I_X$, that is the cohomology ring $H_G^*(M_X^{fin})$ by Corollary 3.7. To be consistent with grading in cohomology, we double the degrees in $S[\mathfrak{g}]$ and hence in $D(X)$ and we set for each $i \geq 0$, $D(X)^{2i+1} = \{0\}$.

Using the Lebesgue measure associated to the lattice Λ, we will in what follows freely identify polynomial functions on $\mathfrak{g}^*$ with polynomial densities on $\mathfrak{g}^*$.

The polynomials in $D(X)$, dual to the algebra $D^*(X) := S[\mathfrak{g}^*]/I_X$, can be naturally interpreted as Laplace–Fourier transforms of the finite dimensional space $\hat{D}(X)$ of those generalized functions which vanish on the functions vanishing at V_X.

Denote by $\mathcal{S}'(\mathfrak{g}^*)$ the space of tempered distributions on $\mathfrak{g}^*$. Assume now that there is an element $x \in \mathfrak{g}$ such that $\langle x, a\rangle > 0$ for every a in X. Recall that the *multivariate spline* T_X is the tempered distribution defined by:

$$\langle T_X \,|\, f\rangle = \int_0^\infty \dots \int_0^\infty f(\sum_{i=1}^m t_i a_i) dt_1 \dots dt_m. \tag{10}$$

Its Laplace transform is $d_X^{-1} := 1/\prod_{a\in X} a$. Notice that, if $a \in X$, $\partial_a T_X = T_{X\setminus a}$. In particular $\partial_X T_X = T_\emptyset = \delta_0$.

Let $\underline{r}$ be a vector subspace in $\mathfrak{g}^*$. We have an embedding $j : \mathcal{S}'(\underline{r}) \to \mathcal{S}'(\mathfrak{g}^*)$ by $j(\phi)(f) = \phi(f|\underline{r})$ for any $\phi \in \mathcal{S}'(\underline{r})$, f a Schwartz function on $\mathfrak{g}^*$. We denote the image $j(\mathcal{S}'(\underline{r}))$ by $\mathcal{S}'(\mathfrak{g}^*, \underline{r})$ (sometimes we even identify $\mathcal{S}'(\underline{r})$ with $\mathcal{S}'(\mathfrak{g}^*, \underline{r})$ if there is no ambiguity). We next define the vector space:

Definition 4.3.

$$\mathcal{G}(X) := \{f \in \mathcal{S}'(\mathfrak{g}^*) \mid \partial_{X\setminus\underline{r}} f \in \mathcal{S}'(\mathfrak{g}^*, \underline{r}), \text{ for all } \underline{r} \in \mathcal{S}_X\}. \tag{11}$$

Example 4.4. Let $G = S^1$ and identify Λ with $\mathbb{Z}$ and $\mathfrak{g}^*$ with $\mathbb{R}$. Let $X = 1^{k+1} = \underbrace{(1, 1, \dots, 1)}_{k+1}$.

Then there are two rational subspaces: $\mathbb{R}$ and the origin. The only cocircuit is X itself and $\partial_X = \frac{d^{k+1}}{dx^{k+1}}$. The space $D(X)$ consists of the polynomials of degree $\leq k$ and $T_X = x^k/k!$ if $x \geq 0$ and 0 otherwise. It is easy to see that $\mathcal{G}(X) = D(X) \oplus \mathbb{R}T_X$.

We are now going to recall a few properties of $\mathcal{G}(X)$ (see also [7]). For this, given a list of non zero vectors Z in $\mathfrak{g}^*$, we consider the dual hyperplane arrangement, $a^\perp \subset \mathfrak{g}$, $a \in Z$. Any connected component F of the complement of this arrangement is called a *regular face* for Z. An element $\phi \in F$ decomposes $Z = A \cup B$ where ϕ is positive on A and negative on B. This decomposition depends only upon F. We define

$$T_Z^F = (-1)^{|B|} T_{A,-B}.$$

Notice that T_Z^F is supported on the cone $C(A, -B)$ of non negative linear combinations of the vectors in the list $(A, -B)$.

Take the subset $\mathcal{S}_X(i)$ of subspaces $\underline{r} \in \mathcal{S}_X$ of dimension i. Consider $\partial_{X\setminus\underline{r}}$ as an operator on $\mathcal{G}(X)$ with values in $\mathcal{S}'(\mathfrak{g}^*, \underline{r})$. Define the spaces

$$\mathcal{G}(X)_i := \cap_{\underline{t}\in\mathcal{S}_X(i-1)} \ker(\partial_{X\setminus\underline{t}}). \tag{12}$$

Notice that by definition $\mathcal{G}(X)_0 = \mathcal{G}(X)$, that $\mathcal{G}(X)_{\dim\mathfrak{g}^*}$ is the space $D(X)$ and that $\mathcal{G}(X)_{i+1} \subseteq \mathcal{G}(X)_i$.

Remark 4.5. Consider a polynomial density $g \in D(X \cap \underline{r})$, a face $F_{\underline{r}}$ defining $X \setminus \underline{r} = A \cup B$ and $T_{X\setminus\underline{r}}^{F_{\underline{r}}}$. The convolution $T_{X\setminus\underline{r}}^{F_{\underline{r}}} * g$ is well defined since, for any $z \in \mathfrak{g}^*$, the set of pairs $x \in C(A, -B), y \in \underline{r}$ with $x + y = z$ is compact.

Lemma 4.6. *Let $\underline{r} \in \mathcal{S}_X(i)$.*

i) The image of $\partial_{X\setminus\underline{r}}$ restricted to $\mathcal{G}(X)_i$ is contained in $D(X \cap \underline{r})$.

ii) Take rational subspaces $\underline{t}$ and $\underline{r}$. For any $g \in D(X \cap \underline{r})$,

$$\partial_{X\setminus\underline{t}}(T_{X\setminus\underline{r}}^{F_{\underline{r}}} * g) = (\partial_{(X\setminus\underline{t})\setminus\underline{r}} T_{X\setminus\underline{r}}^{F_{\underline{r}}}) * (\partial_{(X\cap\underline{r})\setminus(\underline{t}\cap\underline{r})} g). \tag{13}$$

iii) If g is in $D(X \cap \underline{r})$, then $T_{X\setminus\underline{r}}^{F_{\underline{r}}} * g \in \mathcal{G}(X)_i$.

Proof. i) First we know, by the definition of $\mathcal{G}(X)$, that $\partial_{X\setminus\underline{r}}\mathcal{G}(X)_i$ is contained in the space $\mathcal{S}'(\mathfrak{g}^*,\underline{r})$. Let $\underline{t}$ be a rational hyperplane of $\underline{r}$, so that $\underline{t}$ is of dimension $i-1$. By definition, we have that for every $f \in \mathcal{G}(X)_i$

$$0 = \prod_{a\in X\setminus\underline{t}} \partial_a f = \prod_{a\in(X\cap\underline{r})\setminus\underline{t}} \partial_a\partial_{X\setminus\underline{r}} f.$$

This means that $\partial_{X\setminus\underline{r}} f$ satisfies the differential equations given by the cocircuits of $X\cap\underline{r}$, that is, it lies in $D(X\cap\underline{r})$.

ii) We have that $\partial_{X\setminus\underline{t}} = \partial_{(X\setminus\underline{t})\cap\underline{r}}\partial_{(X\setminus\underline{t})\setminus\underline{r}}$ but $\partial_{(X\setminus\underline{t})\cap\underline{r}} = \partial_{(X\cap\underline{r})\setminus(\underline{t}\cap\underline{r})}$. Thus

$$\partial_{X\setminus\underline{t}}(T^{F_{\underline{r}}}_{X\setminus\underline{r}} * g) = (\partial_{(X\setminus\underline{t})\setminus\underline{r}} T^{F_{\underline{r}}}_{X\setminus\underline{r}}) * (\partial_{(X\cap\underline{r})\setminus(\underline{t}\cap\underline{r})} g)$$

as desired.

iii) If $\underline{t}$ does not contain $\underline{r}$, we get that $(\partial_{(X\cap\underline{r})\setminus(\underline{t}\cap\underline{r})} g) = 0$ and hence, by (13),

$$\partial_{X\setminus\underline{t}}(T^{F_{\underline{r}}}_{X\setminus\underline{r}} * g) = 0.$$

□

Consider the map $\mu_i : \mathcal{G}(X)_i \to \oplus_{\underline{r}\in\mathcal{S}_X(i)} D(X\cap\underline{r})$ given by

$$\mu_i f := \oplus_{\underline{r}\in\mathcal{S}_X(i)}\partial_{X\setminus\underline{r}} f$$

and the map $\mathbf{P}_i : \oplus_{\underline{r}\in\mathcal{S}_X(i)} D(X\cap\underline{r}) \to \mathcal{G}(X)_i$ given by

$$\mathbf{P}_i(\oplus g_{\underline{r}}) := \sum T^{F_{\underline{r}}}_{X\setminus\underline{r}} * g_{\underline{r}}.$$

Theorem 4.7. *The sequence*

$$0 \longrightarrow \mathcal{G}(X)_{i+1} \longrightarrow \mathcal{G}(X)_i \xrightarrow{\mu_i} \oplus_{\underline{r}\in\mathcal{S}_X(i)} D(X\cap\underline{r}) \longrightarrow 0$$

is exact. Furthermore, the map $\mathbf{P}_i$ provides a splitting of this exact sequence, i.e. $\mu_i\mathbf{P}_i = \mathrm{Id}$.

Proof. By definition, $\mathcal{G}(X)_{i+1}$ is the kernel of μ_i, thus we only need to show that $\mu_i\mathbf{P}_i = \mathrm{Id}$. Given $\underline{r}\in\mathcal{S}_X(i)$ and $g\in D(X\cap\underline{r})$, by Formula (13) we have $\partial_{X\setminus\underline{r}}(T^{F_{\underline{r}}}_{X\setminus\underline{r}} * g) = g$. If instead we take another subspace $\underline{t}\neq\underline{r}$ of $\mathcal{S}_X(i)$, $\underline{r}\cap\underline{t}$ is a proper subspace of $\underline{t}$. As we have seen above, $g\in D(X\cap\underline{r})$, $\partial_{X\setminus\underline{t}}(T^{F_{\underline{r}}}_{X\setminus\underline{r}} * g) = 0$. Thus, given a family $g_{\underline{r}}\in D(X\cap\underline{r})$, the function $f = \sum_{\underline{t}\in\mathcal{S}_X(i)} T^{F_{\underline{t}}}_{X\setminus\underline{t}} * g_{\underline{t}}$ is such that $\partial_{X\setminus\underline{r}} f = g_{\underline{r}}$ for all $\underline{r}\in\mathcal{S}_X(i)$. This proves our claim that $\mu_i\mathbf{P}_i = \mathrm{Id}$. □

Putting together these facts, we immediately get

Theorem 4.8. *Choose, for every rational space $\underline{r}$, a regular face $F_{\underline{r}}$ for $X\setminus\underline{r}$. Then:*

$$\mathcal{G}(X) = \oplus_{\underline{r}\in\mathcal{S}_X} T^{F_{\underline{r}}}_{X\setminus r} * D(X\cap\underline{r}). \tag{14}$$

Corollary 4.9. *The dimension of $\mathcal{G}(X)$ equals the number of sublists of X which are linearly independent.*

Proof. . This follows immediately from (14) and the fact (see for example [5] Theorem 11.8) that $D(X)$ has dimension equal to the number of bases which can be extracted from X. □

We define

$$\tilde{\mathcal{G}}(X) = S[\mathfrak{g}^*]\mathcal{G}(X), \quad \tilde{\mathcal{G}}_i(X) = S[\mathfrak{g}^*]\mathcal{G}_i(X) \tag{15}$$

where the elements in $S[\mathfrak{g}^*]$ act on distributions as differential operators with constant coefficients.

Remark 4.10. If we set

$$D^{\mathfrak{g}}(X \cap \underline{r}) = S[\mathfrak{g}^*]D(X \cap \underline{r}) \cong S[\mathfrak{g}^*] \otimes_{S[(\mathfrak{g}/\mathfrak{g}_{\underline{r}})^*]} D(X \cap \underline{r}),$$

Theorem 4.7, together with the fact that the maps μ_i and P_i extend to $S[\mathfrak{g}^*]$-module maps (which we denote by the same letter), immediately implies that we have an exact sequence of $S[\mathfrak{g}^*]$-modules

$$0 \to \tilde{\mathcal{G}}_{i+1}(X) \to \tilde{\mathcal{G}}_i(X) \xrightarrow{\mu_i} \oplus_{\underline{r} \in \mathcal{S}_X(i)} D^{\mathfrak{g}}(X \cap \underline{r}) \to 0.$$

Furthermore one can give generators for $\tilde{\mathcal{G}}(X)$ as a $S[\mathfrak{g}^*]$-module as follows:

Theorem 4.11.

$$\tilde{\mathcal{G}}(X) = \sum_F S[\mathfrak{g}^*]T_X^F$$

as F runs over all regular faces for X.

Proof. Denote by M the $S[\mathfrak{g}^*]$ module generated by the elements T_X^F, as F runs on all open faces. In general, from the description of $\tilde{\mathcal{G}}(X)$ given in Formula (14), it is enough to prove that elements of the type $T_{X\setminus\underline{r}}^{F_{\underline{r}}} * g$ with $g \in D(X \cap \underline{r})$ are in M. As $D(X \cap \underline{r}) \subset \mathcal{G}(X \cap \underline{r})$, it is sufficient to prove by induction that each element $T_{X\setminus\underline{r}}^{F_{\underline{r}}} * T_{X\cap\underline{r}}^{K}$ is in M, where K is any open face for the system $X \cap \underline{r}$. We choose a linear function u_0 in the face $F_{\underline{r}}$. Thus u_0 vanishes on $\underline{r}$ and is non zero on every element $a \in X$ not in $\underline{r}$. We choose a linear function u_1 such that the restriction of u_1 to $\underline{r}$ lies in the face K. In particular, u_1 is non zero on every element $a \in X \cap \underline{r}$. We can choose ϵ sufficiently small such that $u_0 + \epsilon u_1$ is non zero on every element $a \in X$. Then $u_0 + \epsilon u_1$ defines an open face F. We see that $T_{X\setminus\underline{r}}^{F_{\underline{r}}} * T_{X\cap\underline{r}}^{K}$ is equal to T_X^F. □

This construction has a discrete counterpart, thoroughly studied in [7] and related to the study of the index of transversally elliptic operators and of computations in equivariant K-theory in which differential operators are replaced by difference operators.

4.12. **Equivariant cohomology with compact support and the infinitesimal index.** Let us now recall that in [8] we have introduced a de Rham model for the equivariant cohomology $H^*_{G,c}(Z)$ with compact support of a G-stable closed subset $Z \subset N$ of a G-manifold N.

Furthermore assume that we have a G-equivariant one form σ on N called an action form. We define the corresponding moment map $\mu : N \to \mathfrak{g}^*$ by setting for any $u \in \mathfrak{g}$, $n \in N$, $\mu(n)(u) := \langle \sigma, v_u \rangle(n)$, v_u being the vector field on N corresponding to u.

If we take as Z the zeroes $N^0 = \mu^{-1}(0)$ of the moment map, we have then defined a map of $S[\mathfrak{g}^*]$-modules

$$\text{infdex} : H^*_{G,c}(Z) \to \mathcal{S}'(\mathfrak{g}^*)$$

called infinitesimal index. We refer to [8] for the proof of most of the properties of $H^*_{G,c}(Z)$ and of the infinitesimal index which we are going to use in what follows.

We are going to study the case in which we start with a G-variety M. We set $N = T^*M$ and we take the canonical one form σ. In this case it follows immediately from the definitions that $(T^*M)^0$ equals the space $T^*_G M$ whose fiber over a point $x \in M$ is formed by all the cotangent vectors $\xi \in T^*_x M$ which vanish on the tangent space to the orbit of x under G, in the point x. Thus each fiber $(T^*_G M)_x$ is a linear subspace of $T^*_x M$. In general the dimension of $(T^*_G M)_x$ is not constant and this space is not a vector bundle.

4.13. **The equivariant cohomology of $T^*_G M^{fin}_X$.** Our task is now to use the infinitesimal index to compute the equivariant cohomology with compact support of $T^*_G M_X$ and more generally of $T^*_G M_{X,\geq k}$. Notice that if we consider ordinary equivariant cohomology, it is immediate by G-homotopy equivalence to deduce

Proposition 4.14. *The equivariant cohomology of the space $T^*_G M_{X,\geq k}$ equals that of $M_{X,\geq k}$ for all k.*

We have already remarked that, in the case $k = s$, we have $M_{X,\geq k} = M^{fin}_X$ and that $H^*_G(M^{fin}_X) = D^*(X)$. Now, since G acts on M^{fin}_X with finite stabilizers, and we use cohomology with real coefficients, we get that $H^*_G(M^{fin}_X) = H^*(M^{fin}_X/G)$ and by Poincaré duality

(16)
$$H^h_{G,c}(M^{fin}_X) = H^h_c(M^{fin}_X/G) = (H^{2|X|-s-h}(M^{fin}_X/G))^* = D^{2|X|-s-h}(X).$$

Now, in order to compute the equivariant cohomology with compact support of $T^*_G M^{fin}_X$, we need some well known general considerations.

Let N be a G-manifold, $\mathcal{M}$ be a G-equivariant vector bundle on N of rank r with projection $p : \mathcal{M} \to N$. Then (see [11]), there is a Thom class $\tau_{\mathcal{M}} \in H^*_{G,c}(\mathcal{M})$ such that

Proposition 4.15. *The map*

$$C : H^*_{G,c}(N) \to H^{*+r}_{G,c}(\mathcal{M})$$

defined by $C(\alpha) = p^*(\alpha) \wedge \tau_{\mathcal{M}}$ *is an isomorphism*

Corollary 4.16. *Assume* $M = M_B$ *for some list* B *of non zero vectors in* Λ*. Let* $\mathcal{M} = N \times M_B$*. If* $\sigma \in H^*_{G,c}(N \times M_B)$ *and* $i : N \to N \times M_B$ *is the 0-section, we have:*

$$Ci^*(\sigma) = d_B\sigma.$$

Proof. Write $\sigma = C(\sigma_0)$ with $\sigma_0 \in H^*_{G,c}(N)$. Since the Euler class of M_B in $H^*_G(pt) = S[\mathfrak{g}^*]$ equals d_B and $\mathcal{M} = N \times M_B$, we get that

$$i^*(\sigma) = i^*C(\sigma_0) = d_B\sigma_0.$$

Since C is a map of $S[\mathfrak{g}^*]$-module, we deduce

$$Ci^*(\sigma) = d_BC(\sigma_0) = d_B\sigma$$

as desired. □

Let us now go back to our computations. The projection $p : T^*_G M^{fin}_X \to M^{fin}_X$ is a real vector bundle of rank $2|X| - s$ so that, applying Proposition 4.15, we get $H^*_{G,c}(T^*_G M^{fin}_X) = H^{*+2|X|-s}_{G,c}(M^{fin}_X)$. Thus putting together this with (16), we get

Proposition 4.17. *As a graded* $S[\mathfrak{g}^*]$*-module,*

$$H^*_{G,c}(T^*_G M^{fin}_X) \simeq D^{4|X|-2s-*}(X).$$

In particular $T^*_G M^{fin}_X$ *has no equivariant odd cohomology with compact support.*

It is now interesting to apply to it the theory of the infinitesimal index. To do this, we need to recall a few facts.

As we have already remarked, the action of G on $T^*_G M^{fin}_X$ is essentially free, so, denoting by Q the quotient $T^*_G M^{fin}_X/G$, we can take an equivariant $\mathfrak{g}$-valued curvature form R for the map $T^*_G M^{fin}_X \to Q$. We have the Chern-Weil map $c : S[\mathfrak{g}^*] \to H^*_G(T^*_G M^{fin}_X)$ defined by $p \mapsto [p(R)]$ (see [8] p.8).

Now we have by Proposition 4.20 of [8] that, for any $[\gamma] \in H^*_c(Q) \simeq H^*_{G,c}(T^*_G M^{fin}_X)$, the infinitesimal index is given by the polynomial density on $\mathfrak{g}^*$

$$(\int_Q \gamma e^{i\langle R,\xi\rangle})d\xi. \tag{17}$$

We have

Theorem 4.18. *The map* infdex *is a graded isomorphism of* $H^*_{G,c}(T^*_G M^{fin}_X)$ *onto* $D(X)$*.*

Proof. Given $p \in S[\mathfrak{g}^*]$, if we consider p as a differential operator with constant coefficients on $\mathfrak{g}$, we shall write it as $p(\partial)$. Now notice that the Poincaré duality pairing $([\gamma], c(p))$ is given by

$$\int_Q \gamma p(R) = \Big(\int_Q \gamma p(R) e^{i\langle R,\xi\rangle}\Big)_{|\xi=0} = (p(\partial)\mathrm{infdex}(\gamma))_{|\xi=0}.$$

Since in our case the Chern-Weil map is surjective with kernel I_X, everything follows. □

4.19. **The equivariant cohomology of $T^*_G M_{X,\geq i}$.** For a rational subspace $\underline{s}$, the action of G on $F(\underline{s})$ factors through $G/G_{\underline{s}}$ and, with respect to this action, $F(\underline{s}) = M^{fin}_{X\cap\underline{s}}$. Thus

$$H^*_{G,c}(F(\underline{s})) = S(\mathfrak{g}^*) \otimes_{S((\mathfrak{g}/\mathfrak{g}_{\underline{s}})^*)} H^*_{G/G_{\underline{s}},c}(F(\underline{s}))$$

where $\mathfrak{g}^*$ is in degree 2. In particular, by Proposition 4.17, we deduce that $H^{2i+1}_{G,c}(T^*_G F(\underline{s})) = 0$.

Now set $\tilde{T}^*_G F(\underline{s}) := T^*_G M_X | F(\underline{s})$, the restriction of $T^*_G M_X$ to $F(\underline{s})$. We see that $\tilde{T}^*_G F(\underline{s}) = T^*_G F(\underline{s}) \times M^*_{X\setminus\underline{s}}$, so we have a Thom isomorphism

$$C_{\underline{s}} : H^{2i}_{G,c}(T^*_G F(\underline{s})) \to H^{2(i+|X\setminus\underline{s}|)}_{G,c}(\tilde{T}^*_G F(\underline{s})), \quad H^{2i+1}_{G,c}(\tilde{T}^*_G F(\underline{s})) = 0.$$

Choose $0 \leq i \leq s$. We pass now to study the G-invariant open subspace $M_{X,\geq i}$ of M. The set $M_{X,\geq i+1}$ is open in $M_{X,\geq i}$ with complement the space F_i, disjoint union of the spaces $F(\underline{s})$ with $\underline{s} \in \mathcal{S}_X(i)$. Denote by $\tilde{T}^*_G F_i$ the restriction of $T^*_G M$ to F_i, disjoint union of the spaces $\tilde{T}^*_G F(\underline{s})$. Denote $j : M_{X,\geq i+1} \to M_{X,\geq i}$ the open inclusion and $e : \tilde{T}^*_G F_i \to T^*_G M_{X,\geq i}$ the closed embedding. Let C_i be the Thom isomorphism from $H^{2i}_{G,c}(T^*_G F_i)$ to $H^{2(i+|X\setminus\underline{s}|)}_{G,c}(\tilde{T}^*_G F_i)$, the direct sum of the Thom isomorphisms $C_{\underline{s}}$.

Theorem 4.20. *For each $0 \leq i \leq s-1$,*

i) $H^{2i+1}_{G,c}(T^*_G M_{X,\geq i}) = 0$.

ii) The following sequence is exact

(18)
$$0 \to H^{2j}_{G,c}(T^*_G M_{X,\geq i+1}) \stackrel{j_*}{\to} H^{2j}_{G,c}(T^*_G M_{X,\geq i}) \stackrel{C_i^{-1}e^*}{\to} \oplus_{\underline{s}\in\mathcal{S}_X(i)} H^{2j-2|X\setminus\underline{s}|}_{G,c}(T^*_G F_{\underline{s}}) \to 0.$$

Proof. Since $M_{X,\geq s} = M^{fin}_X$, we can assume by induction on $s-i$, that *i)* holds for each $j > i$. Also since F_i is the disjoint union of the spaces $F(\underline{s})$ which have no odd equivariant cohomology with compact support, we get that $H^{2i+1}_{G,c}(T^*_G F_i) = 0$ for each $0 \leq i \leq s-1$. Using this fact, both statements follow immediately from the long exact sequence of equivariant cohomology with compact support. □

Let us now make a simple but important remark.

Lemma 4.21. *Let $\underline{s} \in \mathcal{S}_X(j)$ with $j < k$. The element $d_{X\setminus s} \in S[\mathfrak{g}^*]$ lies in the annihilator of $H^*_{G,c}(T^*_G M_{X,\geq k})$.*

Proof. $H^*_{G,c}(T^*_G M_{X,\geq k})$ is a module over $H^*_G(T^*_G M_{X,\geq k})$ and hence also over $H^*_G(M_{X,\geq k})$. Thus this lemma follows from Lemma 3.2. □

Let us now split $X = A \cup B$ and $M_X = M_A \oplus M_B$. Let $p : T^*_G M_X \to M_X$ be the projection and consider $\tilde{T}^*_G M_A := p^{-1} M_A$. We have $\tilde{T}^*_G M_A = T^*_G M_A \times M^*_B$. In particular, we get a Thom isomorphism

$$C_{M^*_B} : H^*_{G,c}(T^*_G M_A) \to H^{*+2|B|}_{G,c}(\tilde{T}^*_G M_A) \cong H^{*+2|B|}_{G,c}(T^*_G M_A \times M^*_B).$$

Denote by $i : M_A \to M_X$ the closed inclusion, and, by abuse of notation, also the inclusion $\tilde{T}^*_G M_A \to T^*_G M_X$. Then i induces the morphisms $i^* : H^*_{G,c}(T^*_G M_X) \to H^*_{G,c}(\tilde{T}^*_G M_A)$ and $i_! : H^*_{G,c}(T^*_G M_A) \to H^*_{G,c}(T^*_G M_X)$. Combining these 3 maps, we claim that

Proposition 4.22. *Take $\sigma \in H^*_{G,c}(T^*_G M_X)$, then $i_! C^{-1}_{M'_B} i^*(\sigma) = (-1)^{|B|} d_B \sigma$.*

Proof. We use Corollary 4.16, and remark that, since M^*_B is dual to M_B, we have $M^*_B = \oplus_{a\in B} L_{-a}$ □

Corollary 4.23. *Take $\sigma \in H^*_{G,c}(T^*_G M_X)$. Let $\sigma_0 = C^{-1}_{M^*_B} i^*(\sigma) \in H^*_{G,c}(T^*_G M_A)$. Then, we have the equality of distributions:*

$$(-1)^{|B|} \partial_B(\mathrm{infdex}(\sigma)) = \mathrm{infdex}(\sigma_0).$$

Proof. We use the fact that the infinitesimal index commutes with $i_!$ and is a map of $S[\mathfrak{g}^*]$ modules. □

We have defined in Definition 4.3 the space of distributions $\mathcal{G}(X)$ as those distributions f on $\mathfrak{g}^*$ such that $\partial_{X\setminus \underline{r}} f \in \mathcal{S}'(\mathfrak{g}^*, \underline{r})$ for all $\underline{t} \in \mathcal{S}_X$ and $\tilde{\mathcal{G}}(X)$ as the $S[\mathfrak{g}^*]$ module generated by $\mathcal{G}(X)$. Then $\tilde{\mathcal{G}}_i(X)$ is the subspace in $\tilde{\mathcal{G}}(X)$ such that $\partial_{X\setminus \underline{t}} f = 0$ for all $\underline{t} \in \mathcal{S}_X(i-1)$.

Lemma 4.24. *For each $i \geq 0$,* infdex *maps $H^*_{G,c}(T^*_G M_{X,\geq i})$ to the space $\tilde{\mathcal{G}}_i(X)$.*

Proof. Denote by $\ell : \tilde{\mathcal{G}}_i(X) \to \tilde{\mathcal{G}}(X)$ the inclusion. By Lemma 4.21, if $\sigma \in H^*_{G,c}(T^*_G M_{X,\geq i})$ and $\underline{t}$ is a rational subspace of dimension strictly less than i, we have $d_{X\setminus \underline{t}}\sigma = 0$. Thus $\partial_{X\setminus \underline{t}}\mathrm{infdex}(\sigma) = 0$. It follows that the only thing we have to show is that, if $\sigma \in H^*_{G,c}(T^*_G M_X)$, then $\mathrm{infdex}(\sigma)$ lies in $\tilde{\mathcal{G}}(X)$. Take a rational subspace $\underline{s}$. By Corollary 4.23, the infinitesimal index of $d_{X\setminus \underline{s}}\sigma$ equals the infinitesimal index of an element $\sigma_0 \in H^*_{G,c}(T^*_G M_{X\cap \underline{s}})$. But the action of G on $M_{X\cap \underline{s}}$ factors though the quotient $G/G_{\underline{s}}$ whose Lie algebra is $\mathfrak{g}/\mathfrak{g}_{\underline{s}}$. Thus $H^*_{G,c}(T^*_G M_{X\cap \underline{s}}) \cong S[\mathfrak{g}^*] \otimes_{S[(\mathfrak{g}/\mathfrak{g}_{\underline{s}})^*]} H^*_{G/G_{\underline{s}},c}(T^*_{G/G_{\underline{s}}} M_{X\cap \underline{s}})$, hence $\partial_{X\setminus \underline{s}}\mathrm{infdex}(\sigma) \in S[\mathfrak{g}^*]\mathrm{infdex}(H^*_{G/G_{\underline{s}},c}(T^*_{G/G_{\underline{s}}} M_{X\cap \underline{s}}))$.

But $\mathrm{infdex}(H^*_{G/G_{\underline{s}},c}(T^*_{G/G_{\underline{s}}} M_{X\cap \underline{s}})) \subset \mathcal{S}'(\mathfrak{g}^*, \underline{r})$ hence the claim. □

The following theorem characterizes the values of the infinitesimal index on the entire M_X. We need to fix signs and set for each $\underline{s}$, $\epsilon_{\underline{s}} = (-1)^{|X\setminus \underline{s}|}$ and

$$\tilde{\mu}_i = \oplus_{\underline{s}\in \mathcal{S}_X(i)} \epsilon_{\underline{s}} \mu_{\underline{s}}.$$

This time, we use the notations and the exact sequences contained in Theorem 4.20 and Corollary 4.23

Theorem 4.25. *For each* $0 \leq i \leq s$*,*

- *the diagram*

$$\begin{array}{ccccc} 0 \to H^*_{G,c}(T^*_G M_{X,\geq i+1}) & \xrightarrow{j_*} & H^*_{G,c}(T^*_G M_{X,\geq i}) & \xrightarrow{C_i^{-1}e^*} & H^*_{G,c}(T^*_G F_i) \to 0 \\ \downarrow{\scriptstyle \text{infdex}} & & \downarrow{\scriptstyle \text{infdex}} & & \downarrow{\scriptstyle \text{infdex}} \\ 0 \to \tilde{\mathcal{G}}_{i+1}(X) & \xrightarrow{\ell} & \tilde{\mathcal{G}}_i(X) & \xrightarrow{\tilde{\mu}_i} & \oplus_{\underline{s} \in \mathcal{S}_X(i)} D^{\mathfrak{g}}(X \cap \underline{s}) \to 0 \end{array}$$

  commutes.
- *Its vertical arrows are isomorphisms.*
- *In particular, the infinitesimal index gives an isomorphism between* $H^*_{G,c}(T^*_G M_X)$ *and* $\tilde{\mathcal{G}}(X)$*.*

Proof. Lemma 4.24 tells us that the diagram is well defined. We need to prove commutativity.

Again, we prove that the square on the right hand side is commutative using Corollary 4.23. The square on the left hand side is commutative since j_* is compatible with the infinitesimal index and ℓ is the inclusion.

Recall that $H^*_{G,c}(T^*_G M_{X \cap \underline{s}}) \cong S[\mathfrak{g}^*] \otimes_{S[(\mathfrak{g}/\mathfrak{g}_{\underline{s}})^*]} H^*_{G/G_{\underline{s}},c}(T^*_{G/G_{\underline{s}}} M_{X \cap \underline{s}})$ and that

$$D^{\mathfrak{g}}(X \cap \underline{s}) = S[\mathfrak{g}^*] D(X \cap \underline{s}) \cong S[\mathfrak{g}^*] \otimes_{S[(\mathfrak{g}/\mathfrak{g}_{\underline{s}})^*]} D(X \cap \underline{s}).$$

Using Theorem 4.18, this implies that the right vertical arrow is always an isomorphism.

We want to apply descending induction on i. When $i + 1 = s$, since $M_{X,\geq s} = M_X^{fin}$ and $\tilde{\mathcal{G}}_{s-1}(X) = D(X)$, Theorem 4.18 gives that the left vertical arrow is an isomorphism. So assume that the left vertical arrow is an isomorphism. We then deduce from the five Lemma that the central vertical arrow is an isomorphism and conclude. ☐

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