

KIRILLOV'S FORMULA AND GUILLEMIN-STERNBERG CONJECTURE

MICHEL DUFLO AND MICHÈLE VERGNE

ABSTRACT. Let G be a connected reductive real Lie group, and H a compact connected subgroup. Harish-Chandra associates to a regular coadjoint admissible orbit M of G some unitary representations Π . Using the character formula for Π , we show that the multiplicities of Π restricted to H can be computed in terms of the fibers of the restriction map $p : M \rightarrow \mathfrak{h}^*$, if p is proper. In particular, this gives an alternate proof of a result of Paradan.

Résumé

Soit G un groupe de Lie réel réductif connexe, et H un sous-groupe compact connexe. Harish-Chandra associe à une orbite coadjointe régulière admissible M de G des représentations unitaires Π . Grâce aux formules de caractère pour Π , nous montrons que les multiplicités de la restriction de Π à H peuvent être calculées en fonction des fibres de la projection $p : M \rightarrow \mathfrak{h}^*$, si p est propre. En particulier, ceci donne une autre démonstration d'un résultat de Paradan.

1. INTRODUCTION

When M is a compact pre-quantizable Hamiltonian manifold for the action of a compact connected Lie group H with moment map $p : M \rightarrow \mathfrak{h}^*$, Guillemin and Sternberg defined a quantization of M , which is a virtual representation of H . They proposed formulas for the multiplicities in terms of the reduced “manifolds” $p^{-1}(v)/H(v)$. These formulae have been proved in [7], and in the close setting of quantization with ρ -correction (also called *Spin_c*-quantization) in [9]. In this note we consider only quantization with ρ -correction.

When M is not compact, it is not clear how to define a quantization of M . In the case where M is a coadjoint admissible orbit, closed and of maximal dimension, of a real connected reductive Lie group G , the representations Π associated to M by Harish-Chandra are the natural candidates for the quantization of M . When H is a maximal compact subgroup of G , Paradan [8] has shown that the motto “quantization commutes with reduction” still holds for these non compact Hamiltonian manifolds. We give another proof using character formulae. It holds for any connected compact subgroup H , provided the moment map $p : M \rightarrow \mathfrak{h}^*$ is proper. However, our proof uses a special feature

of these manifolds M : their \hat{A} genus is trivial. So it does not extend to representations associated to coadjoint orbits of G which are not of maximal dimension.

2. BOX SPLINES AND DAHMEN-MICCHELLI DECONVOLUTION FORMULA

Let V be a finite dimensional real vector space, $\Lambda \subset V$ a lattice, and dv the associated Lebesgue measure. For $v \in V$, we denote by δ_v the δ measure at v , by ∂_v the differentiation in the direction v . Let $\Phi := [\alpha_1, \alpha_2, \dots, \alpha_N]$ be a list of elements in Λ and let $\rho_\Phi := \frac{1}{2} \sum_{\alpha \in \Phi} \alpha$. The centered Box spline $B_c(\Phi)$ is the measure on V such that, for a continuous function F on V ,

$$\langle B_c(\Phi), F \rangle := \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} F\left(\sum_{i=1}^N t_i \alpha_i\right) dt_1 \cdots dt_N.$$

The Fourier transform $\hat{B}_c(\Phi)(x)$ is the function $\prod_{\alpha \in \Phi} \frac{e^{i\langle \alpha, x \rangle / 2} - e^{-i\langle \alpha, x \rangle / 2}}{i\langle \alpha, x \rangle}$.

Define a series of differential operators on V by

$$\hat{A}(\Phi) := \prod_{\alpha \in \Phi} \frac{\partial_\alpha}{e^{\frac{1}{2}\partial_\alpha} - e^{-\frac{1}{2}\partial_\alpha}} = 1 - \frac{1}{24} \sum_{\alpha} (\partial_\alpha)^2 + \cdots$$

We assume now that Φ generates V . A point $v \in V$ is called Φ -regular if v does not lie on any affine hyperplane $\rho_\Phi + \lambda + U$ where $\lambda \in \Lambda$ and U is a hyperplane spanned by elements of Φ . A connected component \mathfrak{c} of the set of Φ -regular elements is called an alcove. A vector $\epsilon \in V$ is called generic if ϵ does not lie on any hyperplane U .

Denote by $\mathcal{C}(\Lambda)$ the space of complex valued functions on Λ . Let $\mathbf{h} = (h^\mathfrak{c})_\mathfrak{c}$ be a family of polynomial functions on V indexed by the alcoves, and ϵ generic. We define $\lim_\epsilon \mathbf{h} \in \mathcal{C}(\Lambda)$ by $\lim_\epsilon \mathbf{h}(\lambda) = h^\mathfrak{c}(\lambda)$, where \mathfrak{c} is the alcove such that $\lambda + t\epsilon \in \mathfrak{c}$ for small $t > 0$. If P is a differential operator (or a series of differential operators) with constant coefficients, we define $P(\mathbf{h}) = (Ph^\mathfrak{c})_\mathfrak{c}$.

Consider the Box spline $B_c(\Phi)$. For each alcove \mathfrak{c} , there exists a polynomial function $b^\mathfrak{c}$ on V such that the measure $B_c(\Phi)$ coincide with $b^\mathfrak{c}(v)dv$ on \mathfrak{c} . We thus obtain a family $\mathbf{b} = (b^\mathfrak{c})_\mathfrak{c}$. If $m \in \mathcal{C}(\Lambda)$, the convolution of the discrete measure $\sum_\lambda m(\lambda)\delta_\lambda$ with $B_c(\Phi)$ is given by a polynomial function $b(m)^\mathfrak{c}$ on each alcove \mathfrak{c} . We denote by $\mathbf{b}(m)$ the family so obtained.

Recall that the list Φ is called unimodular if any basis of V contained in Φ is a basis of the lattice Λ .

Theorem 2.1. [2]. *If Φ is unimodular, then for any ϵ generic,*

$$\lim_\epsilon (\hat{A}(\Phi)\mathbf{b}(m)) = m.$$

To explain what happens when Φ is not unimodular, we need more notations. We consider Λ as the group of characters of a torus T , and use the notation s^λ for the value of $\lambda \in \Lambda$ at $s \in T$. For $s \in T$, we denote by $\hat{s} \in \mathcal{C}(\Lambda)$ the corresponding character of Λ . Let $\mathcal{V}(\Phi)$ be the set of $s \in T$ such that the list $\Phi_s := [\alpha, s^\alpha = 1]$ still generates V (it is called the vertex set).

Consider a vertex $s \in \mathcal{V}(\Phi)$ and the convolution product

$$(1) \quad B_c(s, \Phi) := \left(\prod_{\alpha \in \Phi \setminus \Phi_s} \frac{\delta_{\alpha/2} - s^{-\alpha} \delta_{-\alpha/2}}{1 - s^{-\alpha}} \right) * B_c(\Phi_s).$$

If $m \in \mathcal{C}(\Lambda)$, Theorem 2.2 (basically due to Dahmen-Micchelli) below says that we can recover the value of m at a point λ_0 from the knowledge, in a neighborhood of λ_0 , of the locally polynomial measures (for all $s \in \mathcal{V}(\Phi)$)

$$(2) \quad b(s, m, \Phi) := \left(\sum_{\lambda} (\hat{s}m)(\lambda) \delta_{\lambda} \right) * B_c(s, \Phi)$$

and moreover a precise formula is given in terms of differential operators. The measure $b(s, m, \Phi)$ is given on each alcove \mathfrak{c} by a polynomial function. We denote by $\mathbf{b}(s, m)$ the family of polynomials so obtained.

Define the series of differential operators

$$E(s, \Phi) := \prod_{\alpha \in \Phi \setminus \Phi_s} \frac{1 - s^{-\alpha}}{e^{\partial_{-\alpha/2}} - s^{-\alpha} e^{\partial_{\alpha/2}}} = 1 + \frac{1}{2} \sum_{\alpha \in \Phi \setminus \Phi_s} \frac{1 + s^{-\alpha}}{1 - s^{-\alpha}} \partial_{\alpha} + \dots$$

and

$$\hat{A}(s, \Phi) := E(s, \Phi) \hat{A}(\Phi_s).$$

Theorem 2.2. (see [3]). *Let $m \in \mathcal{C}(\Lambda)$. For any ϵ generic, we have*

$$\sum_{s \in \mathcal{V}(\Phi)} \hat{s}^{-1} \lim_{\epsilon} \hat{A}(s, \Phi)(\mathbf{b}(s, m)) = m.$$

3. KIRILLOV CHARACTER FORMULAE

Let G be a connected reductive real Lie group with Lie algebra \mathfrak{g} . The function $j_{\mathfrak{g}}(X) := \det_{\mathfrak{g}} \left(\frac{e^{adX/2} - e^{-adX/2}}{adX} \right)$ admits a square root $j_{\mathfrak{g}}^{1/2}(X)$, an analytic function on \mathfrak{g} with $j_{\mathfrak{g}}^{1/2}(0) = 1$. Let s be a semi-simple element of G , and $\mathfrak{g}(s)$ its centralizer. The function $\det_{\mathfrak{g}/\mathfrak{g}(s)} \left(\frac{1 - se^{adX}}{1 - s} \right)$ admits a square root $D^{1/2}(s, X)$, an analytic function on $\mathfrak{g}(s)$ with $D^{1/2}(s, 0) = 1$.

Let H be a compact connected group, with Lie algebra \mathfrak{h} . Let T be a Cartan subgroup of H with Lie algebra \mathfrak{t} . We will apply the results of the previous paragraph to the vector space $V = \mathfrak{t}^*$ equipped with the lattice $\Lambda \subset \mathfrak{t}^*$ of weights of T (thus $e^{i\lambda}$ is a character of T). Let

W be the Weyl group of (H, T) . Choose a positive system $\Delta^+ \subset \mathfrak{t}^*$ for the non zero weights of the adjoint action of T in $\mathfrak{h}_{\mathbb{C}}$. For $X \in \mathfrak{t}$, $j_{\mathfrak{h}}^{1/2}(X) = \prod_{\alpha \in \Delta^+} \frac{e^{i\langle \alpha, X \rangle/2} - e^{-i\langle \alpha, X \rangle/2}}{i\langle \alpha, X \rangle}$.

Let $\rho_H := \rho_{\Delta^+}$, and \mathfrak{t}_+^* be the open Weyl chamber. Thus \mathfrak{t}_+^* intersect every orbit of H in \mathfrak{h}^* of maximal dimension in one point. Consider the set $P_{\mathfrak{h}} := (\rho_H + \Lambda) \subset \mathfrak{t}^*$ and $P_{\mathfrak{h}}^+ := (\rho_H + \Lambda) \cap \mathfrak{t}_+^*$. A function *mult* on $P_{\mathfrak{h}}^+$ will be extended to a W -anti-invariant function m on $P_{\mathfrak{h}}$.

The set $P_{\mathfrak{h}}^+$ is in one-to-one correspondence $\mu \rightarrow \Pi^H(\mu)$ with the dual \hat{H} of H . The identity

$$\mathrm{Tr} \Pi^H(\mu)(\exp X) = \sum_{w \in W} \frac{\epsilon(w) e^{i w \mu, X}}{\prod_{\alpha \in \Delta^+} e^{i\langle \alpha, X \rangle/2} - e^{-i\langle \alpha, X \rangle/2}}$$

holds on \mathfrak{t} . This is the Atiyah-Bott fixed-point formula for the index of a twisted Dirac operator on $H\mu$, so that $\Pi^H(\mu)$ is the quantization $Q(H\mu)$ of the symplectic manifold $H\mu$.

Let $p : M \rightarrow \mathfrak{h}^*$ be the moment map of a connected H -hamiltonian manifold M . Let β_M be the Liouville measure. The *slice* S of M is the locally closed subset $p^{-1}(\mathfrak{t}_+^*)$ of M . It is a symplectic submanifold of M with associated Liouville measure β_S . If p is proper, the restriction p^0 of p to S defines a proper map $S \rightarrow \mathfrak{t}_+^*$. We extend the push-forward measure $p_*^0(\beta_S)$ on \mathfrak{t}_+^* to a W -anti-invariant signed measure on \mathfrak{t}^* denoted by $DH(M, p)$ (the Duistermaat-Heckman measure). If S is non empty (that is, if $p(M)$ contains an H -orbit of maximal dimension), the support of $DH(M, p)$ is equal to $p(M) \cap \mathfrak{t}^*$. Suppose moreover that there exists regular values $v \in \mathfrak{t}_+^*$ of p^0 . At such v , the reduced space $M_v := p^{-1}(v)/H(v)$ is an orbifold with symplectic form denoted by Ω_v , and corresponding Liouville measure β_{M_v} . By [6], the measure $DH(M, p)$ has a polynomial density with respect to dv in a neighborhood of v , and the value at v is the symplectic volume $\int_{M_v} e^{\Omega_v/2\pi} = \int_{M_v} \beta_{M_v}$.

Some unitary irreducible representations of G can similarly be associated to closed admissible orbits of maximal dimension of the coadjoint representation of G . Recall Harish-Chandra parametrization. To simplify, we assume G linear. Let $f_0 \in \mathfrak{g}^*$ such that $\mathfrak{g}(f_0)$ (its centralizer in \mathfrak{g}) is a Cartan subalgebra of \mathfrak{g} . Denote by $\tilde{G}(f_0)$ the metaplectic two fold cover of the stabilizer $G(f_0)$ of f_0 . Let τ be a character of $\tilde{G}(f_0)$ such that $\tau(\exp X) = e^{i\langle f, X \rangle}$ and $\tau(\epsilon) = -1$ if $\epsilon \in \tilde{G}(f_0)$ projects on 1 and $\epsilon \neq 1$ (if such a character τ exists, f_0 is called admissible). As explained in [4], Harish-Chandra associated to this data an irreducible unitary representation $\Pi^G(f_0, \tau)$ of G . We consider it as a quantization $Q(M, \tau)$ of M . If f_0 is admissible and $G(f_0)$ is connected (as is the case when $G(f_0)$ is compact), the character τ is unique, and we simply write $Q(M)$ for $Q(M, \tau)$.

Irreducible unitary representations of G have a character, which, by Harish-Chandra theory, is a locally L^1 function on G . We denote by $\Theta(M, \tau)$ the character of $Q(M, \tau)$. Similarly, the measure β_M , considered as a tempered measure on \mathfrak{g}^* , has a Fourier transform which is a locally L^1 function on \mathfrak{g} . Kirillov formula (proven in this case by Rossmann [10]) is the equality of locally L^1 functions on \mathfrak{g} :

$$j_{\mathfrak{g}}^{1/2}(X) \Theta(M, \tau)(\exp X) = \int_M e^{i\langle f, X \rangle} d\beta_M(f).$$

We suppose that the connected compact group H is a subgroup of G , and we assume that the projection map $p : M \rightarrow \mathfrak{h}^*$ is proper. It implies that the restriction $Q(M, \tau)|_H = \sum_{\mu \in P_{\mathfrak{h}}^+} \text{mult}(\mu) \Pi^H(\mu)$ is a sum of irreducible representations of H with finite multiplicities $\text{mult}(\mu)$. We associated to $\text{mult}(\mu)$ an anti-invariant function $m(\mu)$ on $P_{\mathfrak{h}}$, and to the projection p an anti-invariant measure $DH(M, p)$ on \mathfrak{t}^* . Let $\Delta(\mathfrak{g}/\mathfrak{h}) \subset \mathfrak{t}^*$ be the list of weights for the action of T in $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$. Choose a sublist Φ so that $\Delta(\mathfrak{g}/\mathfrak{h})$ is the disjoint union of Φ , $-\Phi$ and the zero weights. The subsequent definitions do not depend of this choice. On \mathfrak{t} , we have $j_{\mathfrak{g}}^{1/2}(X) = j_{\mathfrak{h}}^{1/2}(X) \prod_{\alpha \in \Phi} \frac{e^{i\langle \alpha, X \rangle/2} - e^{-i\langle \alpha, X \rangle/2}}{i\langle \alpha, X \rangle}$.

We assume (and we can easily restrict to this case) that \mathfrak{h} does not contain any ideal of \mathfrak{g} . Then the set Φ generates \mathfrak{t}^* .

Kirillov formula, written for the characters of $Q(M, \tau)$ and $Q(H\mu)$, implies the equality of measures on \mathfrak{t}^* :

$$\left(\sum_{\nu \in P_{\mathfrak{h}}} m(\nu) \delta_{\nu} \right) * B_c(\Phi) = DH(M, p).$$

The measure $DH(M, p)$ is a polynomial $d^{\mathfrak{a}}$ on each alcove $\mathfrak{a} = \mathfrak{c} + \rho_H$, where \mathfrak{c} is an alcove for the system Φ . For $v \in \mathfrak{a}$, $r(v) := \hat{A}(\Phi) d^{\mathfrak{a}}(v) = \int_{M_v} e^{\Omega_v/2\pi} \hat{A}(M_v)$ where $\hat{A}(M_v)$ is the \hat{A} genus of M_v . This follows from expressing the linear variation of Ω_v in function of the curvature of the principal bundle $(p^0)^{-1}(v)/T$ ([6]).

In the (rare) case where the system Φ is unimodular (for example for G the adjoint group of $U(p, q)$, and H the maximal compact subgroup), the orbifold M_v is smooth. The value $r(v)$ can be defined at any $\nu \in P_{\mathfrak{h}}$, by taking a limit of $r(\nu + t\epsilon)$ where $\nu + t\epsilon$ stay in an alcove \mathfrak{a} , and any ϵ generic, and coincide with the number $Q(M_{\nu}) \in \mathbb{Z}$ defined as the quantization of the (possibly singular) reduction M_{ν} ([8]). Thus we obtain

$$Q(M)|_H = \sum_{\nu \in P_{\mathfrak{h}}^+ \cap p(M)} Q(M_{\nu}) Q(H\nu).$$

We now consider the general case. Consider a vertex $s \in T$ for Φ . Let $M(s) \subset M$ be the submanifold of fixed points of s . It may have several connected components. It is a symplectic submanifold, and we denote by β_s its Liouville measure. We can define the generalized

function $\Theta(f, \tau)(sg)$ where $g \in G$ commutes with s . The identity

$$(3) \quad j_{\mathfrak{g}(s)}^{1/2}(X)D^{1/2}(s, X)\Theta(f, \tau)(s \exp X) = \int_{M(s)} \epsilon(s, \tau)e^{i\langle f, X \rangle} \beta_s$$

holds as an identity of locally L^1 - functions on $\mathfrak{g}(s)$ ([1]). Here $\epsilon(s, \tau)$ is a locally constant function on $M(s)$ ([5]).

Recall (1) the measure $B_c(s, \Phi)$ on \mathfrak{t}^* associated to s . Denote by $p_s : M(s) \rightarrow \mathfrak{h}(s)^*$ the restriction of p to $M(s)$. We define $DH(M, s, \tau)$ as the sum of the measures $\epsilon(s, \tau)_i DH(M_s^i, p_s^i)$, where M_s^i are the connected components of M_s , and $\epsilon(s, \tau)_i$ the constant value of $\epsilon(s, \tau)$ on M_s^i . The support of $DH(M, s, \tau)$ is contained in the image $p(M)$ of M for any s . Formula (3) implies the identity of measures on \mathfrak{t}^* :

$$\left(\sum_{\nu \in P_{\mathfrak{h}}} (\hat{sm})(\nu) \delta_{\nu} \right) * B_c(s, \Phi) = DH(M, s, \tau).$$

Comparing with formula (2), we see that we can compute $m(\nu)$ from the knowledge, in a neighborhood of ν , of Duistermaat-Heckmann measures $DH(M, s, \tau)$ associated to all vertices s . In particular $m(\nu) = 0$, if ν is not in the interior of $p(M)^0$ of $p(M)$. More precisely, Theorem 2.2 and the definition of the quantization of (possibly singular) reduced spaces gives us

$$Q(M, \tau)|_H = \sum_{\nu \in P_{\mathfrak{h}}^+ \cap p(M)^0} Q(M_{\nu}, \tau)Q(H\nu).$$

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UNIVERSITÉ PARIS DIDEROT-PARIS 7, INSTITUT DE MATHÉMATIQUES DE JUSSIEU,
C.P. 7012, 2 PLACE JUSSIEU, F-75251 PARIS CEDEX 05
E-mail address: `duflo@math.jussieu.fr`

UNIVERSITÉ PARIS DIDEROT-PARIS 7, INSTITUT DE MATHÉMATIQUES DE JUSSIEU,
C.P. 7012, 2 PLACE JUSSIEU, F-75251 PARIS CEDEX 05
E-mail address: `vergne@math.jussieu.fr`