# Local Euler-Maclaurin expansion of Barvinok valuations and Ehrhart coefficients of a rational polytope

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## 1   Introduction

Let $\mathfrak{p}$ be a rational polytope in $V = \mathbb{R}^d$ and $h(x)$ a polynomial function on $V$. A classical problem in Integer Programming is to compute the sum of values of $h(x)$ over the set of integral points of $\mathfrak{p}$,

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

When $\mathfrak{p}$ is dilated by an integer $n \in \mathbb{N}$, we obtain a function of $n$ which is quasi-polynomial, the so-called Ehrhart quasi-polynomial of the pair $(\mathfrak{p}, h)$

$$S(n\mathfrak{p}, h) = \sum_{m=0}^{d+N} E_m(\mathfrak{p}, h, n) n^m$$

of degree $d + N$ where $N = \deg h$. The coefficients $E_m(\mathfrak{p}, h, n)$ are periodic functions of $n \in \mathbb{N}$, with period the smallest integer $q$ such that $q\mathfrak{p}$ is a lattice polytope.

Replacing $h(x)$ by an exponential, we are led to study the analytic function on $V^*$ defined by

$$S(\mathfrak{p})(\xi) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} e^{\langle \xi, x \rangle}.$$

If $\mathfrak{p}$ is any rational polyhedron, this sum still makes sense as a meromorphic function defined near 0 and the map $\mathfrak{p} \mapsto S(\mathfrak{p})(\xi)$ is a valuation.

In [4], we proved that the meromorphic function $S(\mathfrak{p})(\xi)$ has a local Euler-Maclaurin expansion

$$S(\mathfrak{p})(\xi) = \sum_{\mathfrak{f}} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) \int_{\mathfrak{f}} e^{\langle \xi, x \rangle} dx.$$

The sum is taken over the set of faces $\mathfrak{f}$ of the polyhedron $\mathfrak{p}$. For each face $\mathfrak{f}$, the function $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi)$ is holomorphic near 0, and it depends only on the transverse cone $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ of $\mathfrak{p}$ along $\mathfrak{f}$. More precisely, once a rational scalar product is chosen on $V$, we define canonically a map $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ from the set of rational affine cones $\mathfrak{a}$ in quotient spaces of $V$, with values in the space of functions on $V$ which are analytic near 0, then we prove that these functions satisfy the above formula. The map $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ is invariant under lattice translations, equivariant with respect to lattice preserving isometries, and it is a valuation on the set of affine cones with a fixed vertex ([4], Theorems 17 and 18).

It is easy to see that the Ehrhart quasi-polynomial can be computed in terms of the Taylor coefficients of the functions $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi)$. For example, if $\mathfrak{p}$ is a lattice polytope and $h(x) = 1$, we have ([4], Corollary 28)

$$(1) \qquad \mathrm{Card}(n\mathfrak{p} \cap \mathbb{Z}^d) = \sum_{\mathfrak{f}} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(0) \, \mathrm{vol}(\mathfrak{f}) n^{\dim \mathfrak{f}}.$$

Using the valuation property of $\mu(\mathfrak{a})$ and Barvinok's decomposition of a cone into unimodular cones, we thus obtained in [4] an algorithm for computing the Ehrhart quasi-polynomial. It has polynomial length with respect to the input $(\mathfrak{p}, h)$, when the dimension $d$ and the degree $N$ are fixed.

The valuation $S(\mathfrak{p}, h)$ has a natural generalization used by Barvinok in [2], the *mixed* valuation $S^L(\mathfrak{p}, h)$, where $L \subseteq V$ is a rational vector subspace. Denote the projected lattice on $V/L$ by $\Lambda_L$. For a polytope $\mathfrak{p} \subset V$ and a polynomial $h(x)$

$$S^L(\mathfrak{p}, h) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (y+L)} h(x) dx.$$

In other words, the polytope $\mathfrak{p}$ is sliced along lattice affine subspaces parallel to $L$ and the integrals of $h$ over the slices are added up. For $L = V$, there is only one term and $S^V(\mathfrak{p}, h)$ is just the integral of $h(x)$ over $\mathfrak{p}$, while, for $L = \{0\}$, we recover $S(\mathfrak{p}, h)$, the sum of values of $h(x)$ over the set of integral points of $\mathfrak{p}$.

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In the case $h(x) = 1$, we write $S(\mathfrak{p})$ and $S^L(\mathfrak{p})$ in place of $S(\mathfrak{p}, 1)$ and $S^L(\mathfrak{p}, 1)$.

Using these mixed valuations, Barvinok gave an algorithm which computes the $r+1$ highest degree Ehrhart coefficients of $S(n\mathfrak{p}) = \mathrm{Card}(n\mathfrak{p} \cap \mathbb{Z}^d)$, when $\mathfrak{p}$ is a simplex in $\mathbb{R}^d$. Barvinok's algorithm has polynomial length when $d$ is an input, provided $r$ is fixed. The method consists in reducing the problem to summations over lattice points in dimension $\leq r$.

Barvinok considers particular linear combinations

$$\sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p}),$$

where $\mathcal{L}$ is a finite set of rational vector subspaces of $V$ which is closed under sum, and the coefficients $\rho(L)$ are integers which satisfy the following relation between characteristic functions:

$$\chi(\cup_{L \in \mathcal{L}} L^\perp) = \sum_{L \in \mathcal{L}} \rho(L) \chi(L^\perp),$$

where $L^\perp \subseteq V^*$ is the orthogonal of $L$. We call a function $\mathcal{L} \to \mathbb{Z}$ with this property a patchwork function on $\mathcal{L}$.

When $\mathfrak{p}$ is dilated by an integer $n$, $S^L(n\mathfrak{p})$ is again given by a quasi-polynomial in $n$, as is a linear combination

$$\sum_{L \in \mathcal{L}} \rho(L) S^L(n\mathfrak{p}) = \sum_{m=0}^{d} \nu_m(\mathfrak{p}, n) n^m.$$

The main theoretical result of [2], Theorem (1.3), is the following: if $\mathcal{L}$ is a family of subspaces which is closed under sum and contains the vector subspace $\mathrm{lin}(\mathfrak{f})$ parallel to $\mathfrak{f}$, for every face $\mathfrak{f}$ of codimension $\leq r$ of $\mathfrak{p}$, and if $\rho$ is a patchwork function on $\mathcal{L}$, then the $r+1$ highest degree coefficients $\nu_m(\mathfrak{p}, n)$, for $m = d, \ldots, d - r$, are equal to the corresponding Ehrhart coefficients of $S(n\mathfrak{p}) = \mathrm{Card}(n\mathfrak{p} \cap \mathbb{Z}^d)$.

In the present article, we introduce the meromorphic functions which extend $S^L(\mathfrak{p})$. For any polyhedron $\mathfrak{p}$,

$$S^L(\mathfrak{p})(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (y+L)} e^{\langle \xi, x \rangle} dx$$

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is defined as a meromorphic function near $\xi = 0$. We show that $S^L(\mathfrak{p})(\xi)$ also enjoys a local Euler-Maclaurin expansion (Theorem 8),

$$S^L(\mathfrak{p})(\xi) = \sum_{\mathfrak{f}} \mu^L(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) \int_{\mathfrak{f}} e^{\langle \xi, x \rangle} dx.$$

Furthermore, for a linear combination of Barvinok type, if $\mathfrak{f}$ is a face of $\mathfrak{p}$ such that $\mathrm{lin}(\mathfrak{f}) \in \mathcal{L}$, we prove that the $\mathfrak{f}$-term in the Euler-Maclaurin expansions of

$$S^{\mathcal{L},\rho}(\mathfrak{p})(\xi) = \sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{p})(\xi)$$

and of the *usual* valuation $S(\mathfrak{p})(\xi)$ are equal (Theorem 17):

$$\sum_{L \in \mathcal{L}} \rho(L) \mu^L(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) = \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi).$$

This is the main result of the present article. From the relation between Ehrhart quasi-polynomials and Euler-Maclaurin expansions, it implies Barvinok's Theorem (1.3).

Actually, we derive from Theorem 17 another computation of the $r + 1$ highest coefficients of the Ehrhart quasi-polynomial for a pair $(\mathfrak{p}, h)$, based on Brion's decomposition of a polytope into cones, in the line of [1] and [6].

For each vertex $s$ of $\mathfrak{p}$, let $\mathfrak{c}_s$ be the cone of feasible directions of $\mathfrak{p}$ at $s$. Instead of the full family $\mathcal{L}$ generated by taking sums of the subspaces $\mathrm{lin}(\mathfrak{f})$, when $\mathfrak{f}$ runs over the set of faces of codimension $\leq r$ of the polytope $\mathfrak{p}$, we consider, for each vertex $s$ of $\mathfrak{p}$, the family $\mathcal{L}_s$ generated by faces of $\mathfrak{c}_s$ of codimension $\leq r$. The point in taking a family which depends on $s$ lies in the case where $\mathfrak{p}$ is simplicial. Then $\mathcal{L}_s$ consists only of the spaces $\mathrm{lin}(\mathfrak{f})$ where $\mathfrak{f}$ is a face of $\mathfrak{c}_s$ of codimension $\leq r$, as this set is already closed under sum. Moreover the coefficients $\rho(L)$ are just signed binomial coefficients (Lemma 15), and the computation of $S^L(\mathfrak{c}_s)$ is easier when $L$ is parallel to a face of $\mathfrak{c}_s$ (Example 6).

Let us describe our method in the simpler case of a lattice polytope $\mathfrak{p}$ and polynomial $h(x) = 1$. By Brion's theorem, we have

$$S(\mathfrak{p})(\xi) = \sum_{s} e^{\langle \xi, s \rangle} S(\mathfrak{c}_s)(\xi).$$

For each vertex $s$, let $\rho_s : \mathcal{L}_s \to \mathbb{Z}$ be a patchwork function. We define

$$\mathcal{B}(\mathfrak{p})(\xi) = \sum_{s} e^{\langle \xi, s \rangle} S^{\mathcal{L}_s, \rho_s}(\mathfrak{c}_s)(\xi).$$

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For the dilated polytope $n\mathfrak{p}$, we have

$$S(n\mathfrak{p})(\xi) = \sum_s e^{n\langle\xi,s\rangle} S(\mathfrak{c}_s)(\xi) = \sum_{m\geq 0} \frac{n^m}{m!} \sum_s \langle\xi,s\rangle^m \; S(\mathfrak{c}_s)(\xi).$$

Hence, the meromorphic function $\frac{1}{m!}\sum_s \langle\xi,s\rangle^m \; S(\mathfrak{c}_s)(\xi)$ is actually regular at $\xi = 0$ and its value at $\xi = 0$ is the $m$th Ehrhart coefficient of $\mathfrak{p}$.

We have similarly

$$\mathcal{B}(n\mathfrak{p})(\xi) = \sum_{m\geq 0} \frac{n^m}{m!} \sum_s \langle\xi,s\rangle^m \; S^{\mathcal{L}_s,\rho_s}(\mathfrak{c}_s)(\xi).$$

The meromorphic functions $S^L(\mathfrak{c}_s)(\xi)$ and $S^{\mathcal{L}_s,\rho_s}(\mathfrak{c}_s)(\xi)$ have a special form: they can be written as the quotient of an analytic function by a product of $d' \leq d$ linear forms. Such a function $\phi$ has an expansion into rational functions $\phi = \sum_{j\geq -d} \phi_{[j]}$ where $\phi_{[j]}$ is homogeneous of total degree $j$.

Now it follows from our main theorem that, for $m \geq d - r$, we have

$$S(\mathfrak{c}_s)_{[-m]}(\xi) = S^{\mathcal{L}_s,\rho_s}(\mathfrak{c}_s)_{[-m]}(\xi),$$

hence the zero degree terms of $\sum_s \langle\xi,s\rangle^m \; S(\mathfrak{c}_s)(\xi)$ and $\sum_s \langle\xi,s\rangle^m \; S^{\mathcal{L}_s,\rho_s}(\mathfrak{c}_s)(\xi)$ are equal. Therefore the latter is also analytic at $\xi = 0$ and its value at $\xi = 0$ is the $m$th Ehrhart coefficient of $\mathfrak{p}$. This is the content of Theorem 20.

Thus, besides taking care of any polynomial $h(x)$, not only $h(x) = 1$, this method to compute the $r+1$ highest degree Ehrhart coefficients for the pair $(\mathfrak{p}, h)$ leads to a simpler algorithm than the one proposed in [2]. When $\mathfrak{p}$ is a rational simplex, the contributions of the terms of the form $S^L(\mathfrak{c}_s)(\xi)$ when $L \in \mathcal{L}_s$ are immediately reduced to the computation of a function $S(\mathfrak{a})$ with $\mathfrak{a}$ a simplicial cone of dimension smaller or equal to $r$.

There is also another possible implementation of an algorithm to compute the $r+1$ Ehrhart highest degree coefficients for the pair $(\mathfrak{p}, h)$ based on the results of [4]. As seen from Equation (1), this involves the computation of the analytic function $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$, also associated to simplicial cones in dimension smaller or equal to $r$. We plan to compare the implementation of both methods in the near future.

# 2 Local Euler-Maclaurin expansion of a mixed valuation $S^L$

We consider a rational vector space $V$ of dimension $d$, that is to say a finite dimensional real vector space with a lattice denoted by $\Lambda_V$ or simply $\Lambda$. We will need to consider subspaces and quotient spaces of $V$, this is why we cannot just let $V = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$. By lattice, we mean a discrete additive subgroup of $V$ which generates $V$ as a vector space. Hence, a lattice is generated by a basis of the vector space $V$. A basis of $V$ which is a $\mathbb{Z}$-basis of $\Lambda_V$ is called an integral basis. The elements of $\Lambda$ are called integral. An element $x \in V$ is called rational if $qx \in \Lambda$ for some integer $q \neq 0$. The space of rational points in $V$ is denoted by $V_{\mathbb{Q}}$. A subspace $L$ of $V$ is called rational if $L \cap \Lambda$ is a lattice in $L$. If $L$ is a rational subspace, the image of $\Lambda$ in $V/L$ is a lattice in $V/L$, so that $V/L$ is a rational vector space. We will call the image of $\Lambda$ in $V/L$ the projected lattice.

**Example 1** *Let $V = \mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$. Let $v_1, v_2$ be two primitive integral independent vectors. Using an integral basis with first basis vector $v_1$, a straightforward computation shows that the projected lattice on $\mathbb{R}^2/\mathbb{R}v_1$ is $\mathbb{Z}\frac{\overline{v}_2}{\det(v_1, v_2)}$, where $\overline{v}_2$ is the projection of $v_2$ on $\mathbb{R}^2/\mathbb{R}v_1$.*

A rational space $V$, with lattice $\Lambda$, has a canonical Lebesgue measure, for which $V/\Lambda$ has measure 1. An affine subspace $L$ of $V$ is called rational if it is a translate of a rational subspace by a rational element. It is similarly provided with a canonical Lebesgue measure. We will denote this measure by $dm_L$.

We will denote elements of $V$ by latin letters $x, y, v, \ldots$ and elements of $V^*$ by greek letters $\xi, \alpha, \ldots$. We denote the duality bracket by $\langle \xi, x \rangle$.

If $S$ is a subset of $V$, we denote by $< S >$ the affine subspace generated by $S$. If $S$ consists of rational points, then $< S >$ is rational. Remark that $< S >$ may contain no integral point. We denote by $\text{lin}(S)$ the vector subspace of $V$ parallel to $< S >$.

If $S$ is a subset of $V$, we denote by $S^\perp$ the subspace of $V^*$ orthogonal to $S$:

$$S^\perp = \{\xi \in V^* \; ; \; \langle \xi, x \rangle = 0 \text{ for all } x \in S\}.$$

If $L$ is a subspace of $V$, the dual space $(V/L)^*$ is canonically identified with the subspace $L^\perp \subset V^*$.

A convex rational polyhedron $\mathfrak{p}$ in $V$ (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces bounded by rational affine hyperplanes. We say that $\mathfrak{p}$ is solid (in V) if $< \mathfrak{p} >= V$. A polytope $\mathfrak{p}$ is a **compact** polyhedron.

The set of non negative real numbers is denoted by $\mathbb{R}_+$. A convex rational cone $\mathfrak{c}$ in $V$ is a closed convex cone $\sum_{i=1}^{k} \mathbb{R}_+ v_i$ which is generated by a finite number of elements $v_i$ of $V_{\mathbb{Q}}$. In this article, we simply say cone instead of convex rational cone.

An affine (rational) cone $\mathfrak{a}$ is, by definition, the translate of a cone in $V$ by a rational element $s \in V_{\mathbb{Q}}$. This cone is uniquely defined by $\mathfrak{a}$.

A cone $\mathfrak{c}$ is called simplicial if it is generated by independent elements of $V_{\mathbb{Q}}$. A simplicial cone $\mathfrak{c}$ is called unimodular if it is generated by independent integral vectors $v_1, \ldots, v_k$ such that $\{v_1, \ldots, v_k\}$ can be completed in an integral basis of $V$. An affine cone $\mathfrak{a}$ is called simplicial (resp. simplicial unimodular) if it is the translate of a simplicial (resp. simplicial unimodular) cone.

An affine cone $\mathfrak{a}$ is called pointed if it does not contain any straight line.

The set of faces of an affine cone $\mathfrak{a}$ is denoted by $\mathcal{F}(\mathfrak{a})$. If $\mathfrak{a}$ is pointed, then the vertex of $\mathfrak{a}$ is the unique face of dimension 0, while $\mathfrak{a}$ is the unique face of maximal dimension $\dim \mathfrak{a}$.

Let us recall the definition of the *transverse cone* $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ of a polyhedron $\mathfrak{p}$ along one of its faces $\mathfrak{f}$. Let $x$ be a point in the relative interior of $\mathfrak{f}$. The cone of feasible directions of $\mathfrak{p}$ at $x$ is the set $\mathfrak{c}(\mathfrak{p}, \mathfrak{f}) := \{v \in V \; ; x + \epsilon v \in \mathfrak{p}$ for $\epsilon > 0$ small enough$\}$. It does not depend on the choice of $x$. We denote the projection $V \to V/\operatorname{lin}(\mathfrak{f})$ by $\pi_{\mathfrak{f}}$. Then $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ is the image $\pi_{\mathfrak{f}}(\mathfrak{f} + \mathfrak{c}(\mathfrak{p}, \mathfrak{f}))$ of the affine cone $\mathfrak{f} + \mathfrak{c}(\mathfrak{p}, \mathfrak{f})$ in $V/\operatorname{lin}(\mathfrak{f})$. It is a solid pointed **affine** cone in the quotient space $V/\operatorname{lin}(\mathfrak{f})$ with vertex $\pi_{\mathfrak{f}}(< \mathfrak{f} >)$. In particular, if $v$ is a vertex of $\mathfrak{p}$, the transverse cone $\mathfrak{t}(\mathfrak{p}, v)$ coincides with the supporting cone $v + \mathfrak{c}(\mathfrak{p}, v) \subset V$.

If $\mathfrak{a}$ is an affine cone and $\mathfrak{f}$ is a face of $\mathfrak{a}$, then $\mathfrak{c}(\mathfrak{a}, \mathfrak{f}) = \mathfrak{a} + \operatorname{lin}(\mathfrak{f})$ and the transverse cone $\mathfrak{t}(\mathfrak{a}, \mathfrak{f})$ of $\mathfrak{a}$ along $\mathfrak{f}$ is just the projection $\pi_{\mathfrak{f}}(\mathfrak{a})$ of $\mathfrak{a}$ on $V/\operatorname{lin}(\mathfrak{f})$.

**Definition 2** *Denote by $\mathcal{H}(V^*)$ the ring of analytic functions around $0 \in V^*$. Denote by $\mathcal{M}(V^*)$ the ring of meromorphic functions defined around $0 \in V^*$ and by $\mathcal{M}_\ell(V^*) \subset \mathcal{M}(V^*)$ the subring consisting of those meromorphic functions $\phi(\xi)$ such that there exists a product of linear forms $D(\xi)$ with*

$$D(\xi)\phi(\xi) \in \mathcal{H}(V^*).$$

A function $\phi(\xi) \in \mathcal{M}_\ell(V^*)$ has a unique expansion into homogeneous rational functions

$$\phi(\xi) = \sum_{m >> -\infty} \phi_{[m]}(\xi)$$

where m is the total degree.

If $P$ is a homogeneous polynomial on $V^*$ of degree $p$, and $D$ a product of $r$ linear forms, then $\frac{P}{D}$ is an element in $\mathcal{M}_\ell(V^*)$ homogeneous of degree $m = p - r$.

Let us recall the definition of the function $I(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$ associated to a polyhedron $\mathfrak{p}$, (see for instance the survey [3]).

**Proposition 3** *There exists a map I which to every polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function with rational coefficients $I(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold:*

*(a) If $\mathfrak{p}$ contains a straight line, then $I(\mathfrak{p})=0$.*

*(b) If $\xi \in V^*$ is such that $|e^{\langle \xi, x \rangle}|$ is integrable over $\mathfrak{p}$, then*

$$I(\mathfrak{p})(\xi) = \int_{\mathfrak{p}} e^{\langle \xi, x \rangle} dm_{<\mathfrak{p}>}(x).$$

*(c) For every point $s \in V_{\mathbb{Q}}$, we have*

$$I(s + \mathfrak{p})(\xi) = e^{\langle \xi, s \rangle} I(\mathfrak{p})(\xi).$$

*(d) The map I is a* simple valuation*: if the characteristic functions $\chi(\mathfrak{p}_i)$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i \chi(\mathfrak{p}_i) = 0$, then the functions $I(\mathfrak{p}_i)$ satisfy the relation*

$$\sum_{\{i, <\mathfrak{p}_i>=V\}} r_i I(\mathfrak{p}_i) = 0.$$

In the following proposition, we define the *mixed valuation* $\mathfrak{p} \mapsto S^L(\mathfrak{p})$ associated to a rational vector subspace $L \subsetneq V$. To any polyhedron $\mathfrak{p}$, we associate a *meromorphic function* $S^L(\mathfrak{p})(\xi) \in \mathcal{M}(V^*)$. If $\mathfrak{p}$ is compact, this function is actually regular at 0, and its value for $\xi = 0$ is the valuation $E_{L^\perp}(\mathfrak{p})$ considered by Barvinok [2].

We denote by $\Lambda_{V/L}$ the projection on $V/L$ of the lattice $\Lambda$.

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**Proposition 4** *Let $L \subseteq V$ be a rational subspace. There exists a map $S^L$ which to every rational polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function with rational coefficients $S^L(\mathfrak{p}) \in \mathcal{M}(V^*)$ so that the following properties hold:*

*(a) If $\mathfrak{p}$ contains a line, then $S^L(\mathfrak{p})=0$.*

*(b)*

$$(2) \qquad S^L(\mathfrak{p})(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (y+L)} e^{\langle \xi, x \rangle} dm_L(x),$$

*for every $\xi \in V^*$ such that the above sum converges.*

*(c) For every point $s \in \Lambda$, we have*

$$S^L(s + \mathfrak{p})(\xi) = e^{\langle \xi, s \rangle} S^L(\mathfrak{p})(\xi).$$

*(d) The map $S^L$ is a valuation: if the characteristic functions $\chi(\mathfrak{p}_i)$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i \chi(\mathfrak{p}_i) = 0$, then the functions $S^L(\mathfrak{p}_i)$ satisfy the same relation*

$$\sum_i r_i S^L(\mathfrak{p}_i) = 0.$$

For $L = \{0\}$, we recover the valuation $S$ given by

$$S(\mathfrak{p})(\xi) = \sum_{x \in \mathfrak{p} \cap \Lambda} e^{\langle \xi, s \rangle},$$

provided this sum is convergent.

For $L = V$, we have $S^V(\mathfrak{p}) = I(\mathfrak{p})$, if $\mathfrak{p}$ is solid, and $S^V(\mathfrak{p}) = 0$ otherwise.

The proof is entirely analogous to the case $L = \{0\}$, see Theorem 3.1 in [3], and we omit it.

**Remark 5** *The function $S^L(\mathfrak{p})$ is actually an element of $\mathcal{M}_\ell(V^*)$, but we do not prove it at this point. Let $\mathfrak{a}$ be an affine cone and $\{v_i\}$ the generators of its edges. It will follow from the Euler-Maclaurin expansion of $S^L(\mathfrak{a})$ (Theorem 8) that $\prod_i \langle \xi, v_i \rangle S^L(\mathfrak{a})(\xi)$ is analytic near zero for any $L$. It would be interesting to prove it a priori. By Brion's theorem and the valuation property, it follows in particular that $S^L(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$.*

**Example 6** *Let $\mathfrak{a}$ be a simplicial affine cone in the space $V$, and assume that $L = \mathrm{lin}(\mathfrak{f}_1)$ for some face $\mathfrak{f}_1$ of $\mathfrak{a}$. In this case, $S^L(\mathfrak{a})(\xi)$ decomposes as product of an integral and a discrete sum. For simplicity, assume that $\mathfrak{a}$ is solid. Let $\mathfrak{f}_2$ be the face of $\mathfrak{a}$ such $V = \mathrm{lin}(\mathfrak{f}_1) \oplus \mathrm{lin}(\mathfrak{f}_2)$. We write $x = x_1 + x_2$ and $\xi = \xi_1 + \xi_2$ for the corresponding decompositions of $x \in V$ and $\xi \in V^*$. Then $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ where $\mathfrak{a}_i$ is a simplicial affine cone in $\mathrm{lin}(\mathfrak{f}_i)$. Let us denote by $\Lambda_2$ the projected lattice in $V/\mathrm{lin}(\mathfrak{f}_1) \sim \mathrm{lin}(\mathfrak{f}_2)$. From (2), we obtain immediately*

$$(3) \qquad S^L(\mathfrak{a})(\xi_1 + \xi_2) = I(\mathfrak{a}_1)(\xi_1) \sum_{x_2 \in \mathfrak{a}_2 \cap \Lambda_2} e^{\langle \xi_2, x_2 \rangle}.$$

*Notice that the lattice $\Lambda_2$ is usually bigger than $\Lambda \cap \mathrm{lin}(\mathfrak{f}_2)$.*

**Example 7** *Let $V = \mathbb{R}^2$ with the standard lattice. We compute $S^L(\mathfrak{a})$ when $\mathfrak{a}$ is a cone and $L$ is a line. Let $\mathfrak{a} = \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2$, where $v_1, v_2$ are two linearly independent primitive integral vectors.*

*(a) Assume that $L$ is the line supporting an edge of $\mathfrak{a}$, say $L = \mathbb{R}v_1$. We identify $V/L$ to $\mathbb{R}v_2$. The projected lattice is $\Lambda_2 = \mathbb{Z}\frac{v_2}{\det(v_1, v_2)}$, (Example 1), hence, by (3) in Example 6, we have*

$$(4) \qquad S^L(\mathfrak{a})(\xi) = -\frac{1}{\langle \xi, v_1 \rangle} \; \frac{1}{1 - e^{\frac{\langle \xi, v_2 \rangle}{|\det(v_1, v_2)|}}}.$$

*(b) Assume now that $L$ is transverse to both edges of $\mathfrak{a}$. Assume that $\det(v_1, v_2) > 0$. Let $L = \mathbb{R}u$ where $u$ is a primitive integral vector chosen so that $\det(u, v_2) > 0$. Let $\mathfrak{a}_i = \mathbb{R}_+ u + \mathbb{R}_+ v_i$ for $i = 1, 2$. We decompose the characteristic function of the cone $\mathfrak{a}$ as $\chi(\mathfrak{a}) = \chi(\mathfrak{a}_2) + \chi(\mathfrak{a}_1) - \chi(\mathbb{R}_+ u)$ or $\chi(\mathfrak{a}) = \chi(\mathfrak{a}_2) - \chi(\mathfrak{a}_1) + \chi(\mathbb{R}_+ v_1)$, depending on whether $u$ belongs to $\mathfrak{a}$ or not. Using the valuation property, case (a) and the relation*

$$\frac{1}{1 - e^x} + \frac{1}{1 - e^{-x}} = 1,$$

*we obtain in both cases*

$$(5) \qquad S^L(\mathfrak{a})(\xi) = -\frac{1}{\langle \xi, u \rangle} \left( \frac{1}{1 - e^{\frac{\langle \xi, v_2 \rangle}{\det(u, v_2)}}} - \frac{1}{1 - e^{\frac{\langle \xi, v_1 \rangle}{\det(u, v_1)}}} \right).$$

*In this example, we see that $\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle S^L(\mathfrak{a})(\xi)$ is indeed analytic near $\xi = 0$.*

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In the following theorem and its applications, we will consider the functions $S^L(\mathfrak{p})$ when the space $V$ is replaced with a quotient space $W$. We denote by $\mathcal{C}(W)$ the set of affine cones in $W$. Thus if $\mathfrak{a} \in \mathcal{C}(W)$, and $L$ a rational subspace of $W$, the function $S^L(\mathfrak{a})$ is a meromorphic function on $W^*$. We are going to show that the function $S^L(\mathfrak{a})$ has a local Euler-Maclaurin expansion, which generalizes the case $L = \{0\}$ of [4].

**Theorem 8** *Let $V$ be a rational space and $Q$ a rational scalar product on $V^*$. There exists a unique family of maps $\mu_W^L$, indexed by pairs $(W, L)$ where $W$ is a rational quotient space of $V$ and $L$ is a rational vector subspace of $W$ such that the family enjoys the following properties:*

*(a) $\mu_W^L$ maps $\mathcal{C}(W)$ to $\mathcal{H}(W^*)$, the space of analytic functions on $W^*$.*
*(b) If $W = \{0\}$, then $\mu_{\{0\}}^{\{0\}}(\{0\}) = 1$.*
*(c) For $\dim W > 0$ and $L = W$, then $\mu_W^W(\mathfrak{a}) = 0$.*
*(d) If the affine cone $\mathfrak{a} \in \mathcal{C}(W)$ contains a straight line, then $\mu_W^L(\mathfrak{a}) = 0$.*
*(e) For any affine cone $\mathfrak{a}$ in $W$, the following formula holds*

$$S^L(\mathfrak{a}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a})} \mu_{W/\operatorname{lin}(\mathfrak{f})}^{L+\operatorname{lin}(\mathfrak{f})/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))I(\mathfrak{f})$$

*where the sum is over all faces of the cone $\mathfrak{a}$.*

*In this last formula, the function $\mu_{W/\operatorname{lin}(\mathfrak{f})}^{L+\operatorname{lin}(\mathfrak{f})/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))$ is considered as a function on $W^*$ itself by means of the orthogonal projection $W^* \to (W/\operatorname{lin}(\mathfrak{f}))^* = (\operatorname{lin}(\mathfrak{f}))^\perp$ with respect to the scalar product on $W^* \subset V^*$.*

**Proof.** The proof is entirely similar to the case $L = \{0\}$ studied in [4]. Note that $\mu_W^{\{0\}}$ coincides with the map denoted by $\mu_W$ in [4]. The only new item is (c). It follows immediately from the relation $S^W(\mathfrak{a}) = I(\mathfrak{a})$.

**Remark 9** *Let $\mathfrak{a}$ be a solid cone in $W$, and let $\mathfrak{f}$ be a face of $\mathfrak{a}$ such that $\dim \mathfrak{f} < \dim W$. If $L$ is transverse to the face $\mathfrak{f}$, that is, if $L + \operatorname{lin}(\mathfrak{f}) = W$, then $\mu_{W/\operatorname{lin}(\mathfrak{f})}^{L+\operatorname{lin}(\mathfrak{f})/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) = 0$. This follows from (c).*

From now on we omit the subscript $W$, thus we write $\mu^L$ in place of $\mu_W^L$. The next theorem and its proof are also entirely similar to the case $L = \{0\}$ in [4].

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**Theorem 10** *The analytic functions defined in Theorem 8 have the follow-ing properties:*

*(a) For any $x \in \Lambda$, one has $\mu^L(x + \mathfrak{a}) = \mu^L(\mathfrak{a})$.*

*(b) The map $(\mathfrak{a}, L) \mapsto \mu^L(\mathfrak{a})$ is equivariant with respect to lattice-preserving isometries. In other words, let $g$ be an isometry of $W$ which preserves the lattice $\Lambda_W$. Then $\mu^{g(L)}(g(\mathfrak{a}))({}^tg^{-1}\xi) = \mu^L(\mathfrak{a})(\xi)$.*

*(c) If $W$ is an orthogonal sum $W = W_1 \oplus W_2$ of rational spaces, $L_i \subseteqq W_i$ and $\mathfrak{a}_i$ is an affine cone in $W_i$ for $i = 1, 2$, then*

$$\mu^{L_1 \oplus L_2}(\mathfrak{a}_1 + \mathfrak{a}_2) = \mu^{L_1}(\mathfrak{a}_1)\mu^{L_2}(\mathfrak{a}_2).$$

*(d) For a fixed $s \in W_{\mathbb{Q}}$, the map $\mathfrak{c} \to \mu^L(s + \mathfrak{c})$ is a valuation on the set of cones in $W$.*

*(e) Let $\mathfrak{p} \subset W$ be a rational polyhedron, then*

(6)
$$S^L(\mathfrak{p}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu^{L+\mathrm{lin}(\mathfrak{f})/\mathrm{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))I(\mathfrak{f}).$$

**Example 11** *Let us compute the function $\mu^L$ for the various transverse cones of Example 7. We define a function $B(u)$ on $\mathbb{C}$, holomorphic near $0$, by*

$$B(u) = \frac{1}{1 - e^u} + \frac{1}{u}.$$

*We have*

$$I(\mathfrak{a})(\xi) = \frac{|\det(v_1, v_2)|}{\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle}.$$

*Consider case (b) where $L$ is transverse to both edges $\mathfrak{f}_i = \mathbb{R}_+ v_i$ of $\mathfrak{a}$.*

*Using the equation $\det(v_1, v_2)u = \det(u, v_2)v_1 - \det(u, v_1)v_2$, we have*

$$I(\mathfrak{a})(\xi) = \frac{1}{\langle \xi, u \rangle}\left(\frac{\det(u, v_2)}{\langle \xi, v_2 \rangle} - \frac{\det(u, v_1)}{\langle \xi, v_1 \rangle}\right).$$

*Thus we can rewrite (5) as*

$$S^L(\mathfrak{a}) = \mu^L(\mathfrak{a}) + \sum_{i=1,2} \mu^{L+\mathrm{lin}(\mathfrak{f}_i)/\mathrm{lin}(\mathfrak{f}_i)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_i))I(\mathfrak{f}_i) + I(\mathfrak{a}),$$

*with*

(7)
$$\mu^L(\mathfrak{a})(\xi) = \frac{1}{\langle \xi, u \rangle}\left[B\left(\frac{\langle \xi, v_1 \rangle}{\det(u, v_1)}\right) - B\left(\frac{\langle \xi, v_2 \rangle}{\det(u, v_2)}\right)\right],$$
$$\mu^{L+\mathrm{lin}(\mathfrak{f}_i)/\mathrm{lin}(\mathfrak{f}_i)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_i)) = 0 \ \text{ for } i = 1, 2,$$

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*Observe that (7) is indeed regular at $\xi = 0$.*

*In case (a) where $L = \mathbb{R}v_1$, we have $L + \operatorname{lin}(\mathfrak{f}_1)/\operatorname{lin}(\mathfrak{f}_1) = \{0\}$. Let us assume that $\det(v_1, v_2) > 0$. Then we have, by [4],*

$$\mu^{\{0\}}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1))(\xi) = B\left(\frac{-C_1\langle\xi, v_1\rangle + \langle\xi, v_2\rangle}{\det(v_1, v_2)}\right)$$

*with $C_1 = \frac{Q(v_1, v_2)}{Q(v_1, v_1)}$.*

*As $I(\mathfrak{f}_1)(\xi) = -\frac{1}{\langle\xi, v_1\rangle}$, we can rewrite (4) as*

$$S^L(\mathfrak{a}) = \mu^L(\mathfrak{a}) + \mu^{\{0\}}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1))I(\mathfrak{f}_1) + \mu^{L+\operatorname{lin}(\mathfrak{f}_2)/\operatorname{lin}(\mathfrak{f}_2)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_2))I(\mathfrak{f}_2) + I(\mathfrak{a}),$$

*with*

$$\mu^L(\mathfrak{a})(\xi) = \frac{1}{\langle\xi, v_1\rangle}\left[B\left(\frac{-C_1\langle\xi, v_1\rangle + \langle\xi, v_2\rangle}{\det(v_1, v_2)}\right) - B\left(\frac{\langle\xi, v_2\rangle}{\det(v_1, v_2)}\right)\right],$$

*which is indeed regular at $\xi = 0$, and, again,*

$$\mu^{L+\operatorname{lin}(\mathfrak{f}_2)/\operatorname{lin}(\mathfrak{f}_2)}((\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_2)) = 0.$$

# 3  Barvinok valuations

Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, and let $\rho(L), L \in \mathcal{L}$, be a set of rational coefficients. The linear combination $\sum_{L\in\mathcal{L}} \rho(L)S^L(\mathfrak{p})$ is again a valuation on the set of polyhedra $\mathfrak{p} \subset V$, with values in $\mathcal{M}_\ell(V^*)$. By taking linear combinations, we obtain a local Euler-Maclaurin expansion for the function $\sum_{L\in\mathcal{L}} \rho(L)S^L(\mathfrak{p})(\xi)$.

**Definition 12** *Let $\mathfrak{p} \subset V$ be a polyhedron.*

*(a) We denote*

$$S^{(\mathcal{L},\rho)}(\mathfrak{p}) = \sum_{L\in\mathcal{L}} \rho(L)S^L(\mathfrak{p}).$$

*(b) We define the $\mathfrak{f}$-term in the local Euler-Maclaurin expansion of $S^{(\mathcal{L},\rho)}(\mathfrak{p})$ to be*

$$\mu^{(\mathcal{L},\rho)}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})) = \sum_{L\in\mathcal{L}} \rho_\mathcal{L}(L)\mu^{L+\operatorname{lin}(\mathfrak{f})/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})).$$

Thus we have

$$S^{(\mathcal{L},\rho)}(\mathfrak{p}) = \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{p})} \mu^{(\mathcal{L},\rho)}(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))I(\mathfrak{f}).$$

We are going to compute the $\mathfrak{f}$-term in the case of the following particular linear combinations introduced by Barvinok [2].

**Definition 13** *The valuation $S^{(\mathcal{L},\rho)}$ is called a Barvinok valuation if*
  *(a) the family of subspaces $\mathcal{L}$ is stable under sum,*
  *(b) $\rho$ is an integer valued function on the set $\mathcal{L}$ such that the characteristic function of the union of the subspaces $L^{\perp} \subseteq V^*$ can be written as the linear combination*

(8)
$$\chi(\cup_{L\in\mathcal{L}}L^{\perp}) = \sum_{L\in\mathcal{L}}\rho(L)\chi(L^{\perp}).$$

**Definition 14** *We call a function $\mathcal{L} \to \mathbb{Z}$ which satisfies (8) a patchwork function on $\mathcal{L}$.*

As the set of orthogonal subspaces $L^{\perp} \subseteq V^*$ is stable under intersection, a particular function $\rho_{\mathcal{L}}$ with this property can be computed in terms of the Moebius function of the partially ordered set $\mathcal{L}$ ([7], vol I, section 3.7), as explained in [2].

Let us compute a patchwork function $\rho$ in the following case. $V = \mathbb{R}^d$ with standard basis $e_i, i = 1, \ldots, d$, and $\mathcal{L}_{d,q}$ is the set of subspaces $L_I = \oplus_{i\in I}\mathbb{R}e_i$ with cardinal $|I| \geq q$. The function $\rho_{d,q}$ defined below is actually the one associated to the Moebius function, but we will not need this fact.

We denote the binomial coefficient $\frac{m!}{k!(m-k)!}$ by $C_m^k$.

**Lemma 15** *The function $\rho_{d,q}$ on $\mathcal{L}_{d,q}$ defined by*

$$\rho_{d,q}(L_I) = (-1)^{n-q}C_{n-1}^{q-1} \quad \textit{if } |I| = n,$$

*satisfies Equation (8).*

**Proof.** If $e^i$ is the dual basis, the orthogonal space $L_I^{\perp}$ is equal to $\sum_{i\notin I}\mathbb{R}e^i$. Let $\xi = \sum_{i=1}^d \xi_i e^i \in \cup_{L\in\mathcal{L}_q}L^{\perp}$. Let $I_0$ be the set of indices $i \in [1,\ldots,d]$ such that $\xi_i = 0$. Then $|I_0| \geq q$, and $\xi \in L_I^{\perp}$ if and only $I \subseteq I_0$. Let $|I_0| = N$. The value at $\xi$ of the right-hand side of (8) is equal to

$$E(N,q) = \sum_{I\subseteq I_0}\rho_{d,q}(L_I) = \sum_{n=q}^{N}(-1)^{n-q}C_N^n C_{n-1}^{q-1}.$$

We want to prove that $E(N, q) = 1$. Writing $n = q + i$, we have

$$E(N, q) = \sum_{i=0}^{N-q} (-1)^i \frac{N!}{(q+i)(N-q-i)!i!(q-1)!}.$$

Let us compute $(q-1)!\Big(E(N+1, q) - E(N, q)\Big)$. This is equal to

$$(-1)^{N+1-q} \frac{(N+1)!}{(N+1)(N+1-q)!} + \sum_{i=0}^{N-q} (-1)^i \frac{1}{(q+i)i!} \left( \frac{(N+1)!}{(N+1-q-i)!} - \frac{N!}{(N-q-i)!} \right)$$

$$= N! \sum_{i=0}^{N+1-q} (-1)^i \frac{N!}{i!(N+1-q-i)!} = 0.$$

We obtain $E(N, q) = E(q, q) = 1$. $\square$

In the case of a Barvinok valuation, it turns out that the $\mathfrak{f}$-term in the Euler-Maclaurin expansion of $S^{(\mathcal{L}, \rho)}(\mathfrak{p})$ coincides with that of $S(\mathfrak{p})$, if the vector subspace $\mathrm{lin}(\mathfrak{f})$ belongs to the set $\mathcal{L}$. This is the crucial result of the present article. It is an easy consequence of the following combinatorial lemma.

**Lemma 16** *Let $\mathcal{L}$ be a finite family of vector subspaces of $V$, stable under sum and let $\rho$ be a patchwork function on $\mathcal{L}$.*
*(a) Let $L_0 \in \mathcal{L}$. Then*

$$\sum_{\{L \in \mathcal{L}, \, L \subseteq L_0\}} \rho(L) = 1.$$

*(b) Let $L_0 \subsetneq L_1$ be two subspaces in the family $\mathcal{L}$. Then*

$$\sum_{\{L \in \mathcal{L}, \, L + L_0 = L_1\}} \rho(L) = 0.$$

**Proof.** There exists a $\xi_0 \in L_0^{\perp}$ such that, for $L \in \mathcal{L}$, $\xi_0 \in L^{\perp}$ if and only if $L \subseteq L_0$. We obtain *(a)* by evaluating both sides of (8) at this particular element $\xi_0$. Next, we deduce *(b)* from *(a)*, by induction on $\dim L_1 - \dim L_0$.
For two subspaces $M \subseteq M'$ in the family $\mathcal{L}$, let us denote

$$f(M, M') = \sum_{\{L \in \mathcal{L}, \, L + M = M'\}} \rho(L).$$

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If $M = M'$, we have $f(M, M) = \sum_{\{L \in \mathcal{L}, L \subseteq M\}} \rho(L) = 1$ by *(a)*.
We apply this with $M = L_1$. Thus

$$\sum_{\{L \in \mathcal{L}, L \subseteq L_1\}} \rho(L) = 1.$$

In this sum, we group the $L \in \mathcal{L}$ such that $L + L_0$ is equal to a given $M_1 \in \mathcal{L}$ together.

First we consider the case when $\dim L_1 - \dim L_0 = 1$. Then $M_1$ is either $L_0$ or $L_1$, hence we obtain

$$f(L_0, L_0) + f(L_0, L_1) = 1.$$

Since $f(L_0, L_0) = 1$ by *(a)*, we obtain $f(L_0, L_1) = 0$ as required.

Next we consider the case when $\dim L_1 - \dim L_0 > 1$. We obtain

$$\sum_{\{M_1 \in \mathcal{L}, \ L_0 \subseteq M_1 \subseteq L_1\}} f(L_0, M_1) = 1.$$

For $M_1 = L_0$, we have $f(L_0, L_0) = 1$ by *(a)*. For $M_1 \subsetneq L_1$, we have $f(L_0, M_1) = 0$ by induction hypothesis. Hence there remains only the term $f(L_0, L_1)$ which must be equal to 0. $\square$

We study now the Euler-Maclaurin expansion of a Barvinok valuation.

**Theorem 17** *Let $\mathfrak{p} \subset V$ be a rational polyhedron and let $\mathfrak{f}$ be a face of $\mathfrak{p}$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, stable under sum. Let $\rho$ be a patchwork function on $\mathcal{L}$, and let $S^{(\mathcal{L},\rho)} = \sum_{L \in \mathcal{L}} \rho(L) S^L$.*
*Assume that $\mathrm{lin}(\mathfrak{f})$ belongs to $\mathcal{L}$. Then*

$$\mu^{(\mathcal{L},\rho)}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})) = \mu^{\{0\}}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})).$$

*In other words, the $\mathfrak{f}$-term in the local Euler-Maclaurin expansion of $S^{(\mathcal{L},\rho)}(\mathfrak{p})$ coincides with that of $S(\mathfrak{p})$.*

**Proof.** In the sum $\sum_{L \in \mathcal{L}} \rho(L) \mu^{L + \mathrm{lin}(\mathfrak{f})/\mathrm{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$, we group the terms for which $L + \mathrm{lin}(\mathfrak{f})$ is equal to a given $L_1$ together. By Lemma 16 we obtain $\mu^{\{0\}}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$ for $L_1 = \mathrm{lin}(\mathfrak{f})$ and 0 otherwise. $\square$

**Corollary 18** *Let $\mathfrak{p} \subset V$ be a rational polyhedron. Let $0 \leq k \leq d$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, stable under sum, such that $\mathrm{lin}(\mathfrak{f}) \in \mathcal{L}$ for every $k$-dimensional face $\mathfrak{f}$ of $\mathfrak{p}$ and let $\rho$ be a patchwork function on $\mathcal{L}$.*

*   *Let $0 < k \leq d$. Then the meromorphic function*

$$S(\mathfrak{p})(\xi) - S^{(\mathcal{L},\rho)}(\mathfrak{p})(\xi)$$

*has lowest degree $\geq -k + 1$.*
*   *Let $k = 0$. Then*
$$S(\mathfrak{p})(\xi) = S^{(\mathcal{L},\rho)}(\mathfrak{p})(\xi).$$

**Proof.** By Theorem 17, the local Euler-Maclaurin expansion of the difference involves only faces of dimension $< k$.

$$S(\mathfrak{p})(\xi) - S^{(\mathcal{L},\rho)}(\mathfrak{p})(\xi) = \sum_{\{\mathfrak{f} \in \mathcal{F}(\mathfrak{p}),\, \dim \mathfrak{f} < k\}} \left( \mu^{\{0\}}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) - \mu^{(\mathcal{L},\rho)}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) \right) I(\mathfrak{f})(\xi).$$

For a face of dimension $j$, the function $I(\mathfrak{f})(\xi)$ is homogeneous of degree $-j$. Multiplied by the *holomorphic* function $\mu^{\{0\}}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) - \mu^{\mathcal{L}}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi)$, its lowest degree can only increase. $\square$

Remark that the statement of Corollary 18 above does not involve any scalar product. In the next section, we will show that this corollary implies our main Theorem (Theorem 20).

# 4   Ehrhart quasi-polynomial

Let $\mathfrak{p}$ be a rational polytope and let $h(x)$ be a polynomial function on $V$. Let

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

When $\mathfrak{p}$ is dilated by a non negative integer $n$, we obtain the Ehrhart quasi-polynomial of the pair $(\mathfrak{p}, h)$

$$S(n\mathfrak{p}, h) = \sum_{m=0}^{d+N} E_m(\mathfrak{p}, h, n)n^m,$$

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where $N = \deg h$. The coefficients $E_m(\mathfrak{p}, h, n)$ are periodic functions of $n \in \mathbb{N}$, with period the smallest integer $q$ such that $q\mathfrak{p}$ is a lattice polytope.

If an integer $r \leq d$ is fixed, and $h = 1$, Barvinok [2] proved that the $r + 1$ highest Ehrhart coefficients $E_d(\mathfrak{p}, 1, n), \ldots, E_{d-r}(\mathfrak{p}, 1, n)$ of $S(n\mathfrak{p}, 1)$ can be computed in polynomial time with respect to $d$, when $\mathfrak{p}$ is a rational simplex.

Let $L \subseteq V$ be a rational vector subspace. Denote the projected lattice on $V/L$ by $\Lambda_L$. Consider the mixed valuation

$$S^L(\mathfrak{p}, h) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (y+L)} h(x) dx.$$

As shown by Barvinok, and as we will see here, we can use linear combination of these mixed valuations to approximate $S(n\mathfrak{p}, h)$ when $n$ is big.

For any polyhedron $\mathfrak{a}$, we define the meromorphic function $S^L(\mathfrak{a}, h)(\xi) \in \mathcal{M}_\ell(V^*)$ similarly to $S^L(\mathfrak{a}, h)$. For $\xi \in V^*$ such that the sum converges, we have

$$S^L(\mathfrak{a}, h)(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{a} \cap (y+L)} h(x) e^{\langle \xi, x \rangle} dm_L(x).$$

**Remark 19** *It is clear that $S^L(\mathfrak{a}, h)(\xi) = h(\partial_\xi) \cdot S^L(\mathfrak{a})(\xi)$.*

*If $\mathfrak{p}$ is a polytope, then $S^L(\mathfrak{p}, h)(\xi)$ is regular at $\xi = 0$ and $S^L(\mathfrak{p}, h)(0) = S^L(\mathfrak{p}, h)$.*

For a family $\mathcal{L}$ and a function $L \mapsto \rho(L), L \in \mathcal{L}$, we define

$$S^{(\mathcal{L}, \rho)}(\mathfrak{a}, h)(\xi) = \sum_{L \in \mathcal{L}} \rho(L) S^L(\mathfrak{a}, h)(\xi).$$

If $\mathfrak{p}$ is a polytope, and we dilate by $n \in N$, we have again a quasi-polynomial

$$S^{(\mathcal{L}, \rho)}(n\mathfrak{p}, h)(0) = \sum_{m=0}^{d+N} E_m(\mathcal{L}, \rho, \mathfrak{p}, h, n) n^m.$$

We can replace the quasi-polynomial $S^{(\mathcal{L}, \rho)}(n\mathfrak{p}, h)(0)$ by $q$ legal polynomials in the variable $u$, by splitting $\mathbb{N}$ into classes modulo $q$. Writing $n = qu + k$, for $k = 0, \ldots, q - 1$, we obtain the polynomial function of $u$:

$$S^{(\mathcal{L}, \rho)}((qu + k)\mathfrak{p}, h)(0) = \sum_{m=0}^{d+N} E_m^{(k)}(\mathcal{L}, \rho, \mathfrak{p}, h) u^m.$$

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We briefly recall how the usual Ehrhart quasi-polynomial of a polytope can be computed using Brion's theorem. We will then use a similar method in order to compute efficiently the $r+1$ highest coefficients only, using Barvinok valuations.

Let $\mathcal{V}(\mathfrak{p})$ be the set of vertices of $\mathfrak{p}$. For each vertex $s$, let $\mathfrak{c}_s$ be the cone of feasible directions of $\mathfrak{p}$ at $s$, so that the supporting cone at $s$ is $s + \mathfrak{c}_s$. By Brion's theorem [5], we have

$$S(\mathfrak{p}, h)(\xi) = \sum_{s \in \mathcal{V}(\mathfrak{p})} S(s + \mathfrak{c}_s, h)(\xi).$$

Let $n \in \mathbb{N}$ and consider the dilated polytope $n\mathfrak{p}$. The supporting cone at the vertex $ns$ is $ns + \mathfrak{c}_s$. Let $q \in \mathbb{N}$ such that $q\mathfrak{p}$ is a lattice polytope and fix $k \in \mathbb{N}$, $0 \le k \le q - 1$. Let $n = qu + k$. As $qus$ is an integral point, we have

$$S((qu + k)s + \mathfrak{c}_s, h)(\xi) = e^{qu\langle \xi, s \rangle} S^{\mathcal{L}_s, \rho_s}(ks + \mathfrak{c}_s, h)(\xi).$$

Expanding in powers of $u$, we obtain

$$S((qu + k)\mathfrak{p}, h)(\xi) = \sum_{m \ge 0} u^m \frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathfrak{p})} \langle \xi, s \rangle^m \, S(ks + \mathfrak{c}_s, h)(\xi).$$

It follows that for each $m$, the sum of meromorphic functions

$$\frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathfrak{p})} \langle \xi, s \rangle^m \, S(ks + \mathfrak{c}_s, h)(\xi)$$

is actually analytic. Its value at $\xi = 0$ is obtained by taking the zero degree term. We obtain

$$S((qu + k)\mathfrak{p}, h)(0) = \sum_{m \ge 0} E_m^{(k)}(\mathfrak{p}, h) \, u^m,$$

with

$$E_m^{(k)}(\mathfrak{p}, h) = \frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathfrak{p})} \langle \xi, s \rangle^m \, S(ks + \mathfrak{c}_s, h)_{[-m]}(\xi).$$

The right-hand side of this relation, a priori a meromorphic function of $\xi$, is actually constant. Moreover, we have $S(ks + \mathfrak{c}_s, h)_{[-m]}(\xi) = 0$ if $m > d + N$, hence the Ehrhart quasi-polynomial has degree $\le d + N$.

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We apply now Brion's theorem to $S^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi)$. We obtain

$$S^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi) = \sum_{s\in\mathcal{V}(\mathfrak{p})} S^{\mathcal{L},\rho}(s+\mathfrak{c}_s,h)(\xi).$$

For reasons to be explained later on, instead of one family $\mathcal{L}$, we take a family of subspaces $\mathcal{L}_s$ for each vertex $s$. Let $\rho_s : \mathcal{L}_s \to \mathbb{Z}$ be a function on $\mathcal{L}_s$. We denote now by $(\mathcal{L},\rho)$ the map $s \mapsto (\mathcal{L}_s,\rho_s)$.

We define:

(9) $$\mathcal{B}^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi) = \sum_{s\in\mathcal{V}(\mathfrak{p})} S^{\mathcal{L}_s,\rho_s}(s+\mathfrak{c}_s,h)(\xi).$$

If the family does not depend on $s$, $(\mathcal{L}_s,\rho_s) = (\mathcal{L}_0,\rho_0)$ for every vertex $s$, then, by Brion's theorem, we have

$$\mathcal{B}^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi) = S^{\mathcal{L}_0,\rho_0}(\mathfrak{p},h)(\xi).$$

We dilate (9). Let $n = qu + k$. We obtain

$$\mathcal{B}^{\mathcal{L},\rho}((qu+k)\mathfrak{p},h)(\xi) = \sum_{s\in\mathcal{V}(\mathfrak{p})} e^{qu\langle\xi,s\rangle} S^{\mathcal{L}_s,\rho_s}(ks+\mathfrak{c}_s,h)(\xi).$$

Expanding in powers of $u$, we obtain

(10) $$\mathcal{B}^{\mathcal{L},\rho}((qu+k)\mathfrak{p},h)(\xi) = \sum_{m\geq 0} u^m E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)(\xi)$$

with

(11) $$E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)(\xi) = \frac{q^m}{m!} \sum_{s\in\mathcal{V}(\mathfrak{p})} \langle\xi,s\rangle^m \, S^{\mathcal{L}_s,\rho_s}(ks+\mathfrak{c}_s,h)(\xi).$$

If the family does not depend on $s$, $(\mathcal{L}_s,\rho_s) = (\mathcal{L}_0,\rho_0)$ for all vertices, then $\mathcal{B}^{\mathcal{L},\rho}(\mathfrak{p},h)(\xi) = S^{\mathcal{L}_0,\rho_0}(\mathfrak{p},h)(\xi)$ is analytic near $\xi = 0$, and so are the coefficients (11).

On the contrary, if we take a different family $\mathcal{L}_s$ for each vertex $s$, the coefficient $E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)(\xi)$ of $u^m$ in (10) is no longer analytic near $\xi = 0$, in general. However, the meromorphic function $\xi \mapsto E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)(\xi)$ belongs to $\mathcal{M}_\ell(V^*)$, thus it has a term of degree 0 with respect to $\xi$, given by

(12) $$E_m^{(k)}(\mathcal{L},\rho,\mathfrak{p},h)_{[0]}(\xi) = \frac{q^m}{m!} \sum_{s\in\mathcal{V}(\mathfrak{p})} \langle\xi,s\rangle^m \, S^{\mathcal{L}_s,\rho_s}(ks+\mathfrak{c}_s,h)_{[-m]}(\xi).$$

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For a family $(\mathcal{L}, \rho)$ as described in the next theorem, it turns out that, for large $m$, this zero-degree part $E_m^{(k)}(\mathcal{L}, \rho, \mathfrak{p}, h)_{[0]}(\xi)$ is actually analytic, hence constant, and its value is equal to the $m$-th Ehrhart coefficient $E_m^{(k)}(\mathfrak{p}, h)$ of $S((k+qu)\mathfrak{p}, h)(0)$.

**Theorem 20** *Let $\mathfrak{p}$ be a rational polytope in a rational vector space of dimension $d$. For each vertex $s$ of the polytope $\mathfrak{p}$, let $\mathfrak{c}_s$ be the cone of feasible directions of $\mathfrak{p}$ at $s$, so that the supporting cone at $s$ is $s + \mathfrak{c}_s$. For each vertex $s$, let $\mathcal{L}_s$ be a finite family of rational vector subspaces of $V$, stable under sum, such that $\mathrm{lin}(\mathfrak{f})$ belongs to $\mathcal{L}_s$ for every face $\mathfrak{f}$ of codimension $r$ of the cone $\mathfrak{c}_s$, and let $\rho_s$ be a patchwork function on $\mathcal{L}_s$. Let $q \in \mathbb{N}$ such that $q\mathfrak{p}$ is a lattice polytope and fix $k \in \mathbb{N}$, $0 \le k \le q - 1$. Let $h(x)$ be a homogeneous polynomial of total degree $N$.*

*Then, for $m \ge d + N - r$, the zero-degree term $E_m^{(k)}(\mathcal{L}, \rho, \mathfrak{p}, h)_{[0]}(\xi)$ defined by (12) is regular near $\xi = 0$, hence constant. Its value is the coefficient $E_m^{(k)}(\mathfrak{p}, h)$ of $u^m$ in the Ehrhart quasi-polynomial*

$$S((k+qu)\mathfrak{p}, h)(0) = \sum_{x \in ((k+qu)\mathfrak{p}) \cap \Lambda} h(x) = \sum_{m=0}^{d+N} u^m E_m^{(k)}(\mathfrak{p}, h).$$

**Proof.** We first consider the case $h(x) = 1$. We have, for every $m \ge 0$,

$$E_m^{(k)}(\mathfrak{p}, 1) = \frac{q^m}{m!} \sum_{s \in \mathcal{V}(\mathfrak{p})} \langle \xi, s \rangle^m \, S(ks + \mathfrak{c}_s)_{[-m]}(\xi)$$

where the right-hand side is actually a constant function of $\xi$. For $m > d - r - 1$, we have, by Corollary 18,

$$S(ks + \mathfrak{c}_s)(\xi)_{[-m]} = S^{\mathcal{L}_s, \rho_s}(ks + \mathfrak{c}_s)_{[-m]}(\xi).$$

This proves the theorem when $h(x) = 1$. The case of a non constant polynomial $h(x)$ is quite similar. If $h(x) = x_1^{N_1} \ldots x_d^{N_d}$, we just have to replace the meromorphic functions $S(ks + \mathfrak{c}_s)(\xi)$ and $S^{\mathcal{L}_s, \rho_s}(ks + \mathfrak{c}_s)(\xi)$ by their derivatives under $\partial_{\xi_1}^{N_1} \ldots \partial_{\xi_d}^{N_d}$. $\square$

If for each vertex $s$, we take $\mathcal{L}_s = \mathcal{L}$, the full collection generated by all $r$ codimensional faces of $\mathfrak{p}$, we obtain Corollary 21 below, that is Barvinok's theorem [2], with an extension to the sum of values of any polynomial $h(x)$ over the set of integral points of a rational polytope (Barvinok considers only the case $h(x) = 1$).

**Corollary 21** *Let $\mathfrak{p} \subset V$ be a rational polytope and let $h(x)$ be a polynomial function on $V$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, stable under sum. Assume that $\mathrm{lin}(\mathfrak{f})$ belongs to $\mathcal{L}$ for every face $\mathfrak{f}$ of codimension $r$ of $\mathfrak{p}$. Let $\rho$ be a patchwork function on $\mathcal{L}$ and let $S^{\mathcal{L},\rho} = \sum_{L \in \mathcal{L}} \rho(L) S^L$. Then the $r+1$ highest Ehrhart coefficients of $S(t\mathfrak{p},h)(0)$ and $S^{\mathcal{L},\rho}(t\mathfrak{p},h)(0)$ are equal.*

The point in taking a family $\mathcal{L}_s$ which depends on the vertex $s$ lies in the case where $\mathfrak{p}$ is *simplicial*. In this case, we can take $\mathcal{L}_s$ to be just the set of subspaces $\mathrm{lin}(\mathfrak{f})$, for all faces $\mathfrak{f}$ of codimension $\leq r$ of the supporting cone $\mathfrak{c}_s$ at vertex $s$. This family is stable under sum. Moreover the patchwork function on $\mathcal{L}_s$ is simple, (Lemma 15) and the computation of the function $S^L(ks + \mathfrak{c}_s)(\xi)$, when $L \in \mathcal{L}_s$, is immediately reduced (Example 6) to the computation of a function $S(\mathfrak{a})(\xi)$ for a simplicial cone $\mathfrak{a}$ in a rational vector space of dimension **smaller or equal** than $r$. When $\mathfrak{p}$ is a simplex, we obtain in this way a method for computing the $r+1$ highest Ehrhart coefficients for the pair $(\mathfrak{p},h)$.

# 5 Local Euler-Maclaurin formula for mixed sums

Finally in this last section, we discuss an application of the existence of the coefficients $\mu^L$ (Theorem 8) in the line of [4].

Let $\mathfrak{p}$ be a rational polytope in a rational vector space $V$ of dimension $d$ and let $h(x)$ be a polynomial function on $V$. Let $L$ be a rational subspace of $V$. Consider the mixed sum

$$S^L(\mathfrak{p}, h) = \sum_{y \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (y+L)} h(x) dm_L(x).$$

As in [4], we associate to the analytic function $\mu^L(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))$ a constant coefficients differential operator (of infinite order) on $V$.

**Definition 22** *Let $\mathfrak{f}$ be a face of $\mathfrak{p}$. We denote by $D^L(\mathfrak{p},\mathfrak{f})$ the differential operator on $V$ associated to analytic function $\mu^L(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))$:*

$$D^L(\mathfrak{p}, \mathfrak{f})(\partial_\xi) \cdot e^{\langle \xi, x \rangle} = \mu^L(\mathfrak{t}(\mathfrak{p},\mathfrak{f})))(\xi) e^{\langle \xi, x \rangle}.$$

The operators $D^L(\mathfrak{p}, \mathfrak{f})$ are **local**, that is they depend only of the transverse cone $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ of $\mathfrak{p}$ along $\mathfrak{f}$, and they involve only derivatives in directions orthogonal to the face $\mathfrak{f}$. We can state the following theorem with the same proof as in [4].

**Theorem 23** *(Local Euler-Maclaurin formula) Let $\mathfrak{p}$ be a polytope in $V$. For any polynomial function $h(x)$ on $V$, we have*

$$(13) \qquad S^L(\mathfrak{p}, h) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \int_{\mathfrak{f}} D^L(\mathfrak{p}, \mathfrak{f}) \cdot h$$

*where the integral on the face $\mathfrak{f}$ is taken with respect to the Lebesgue measure on $<\mathfrak{f}>$ defined by the lattice $\Lambda \cap \mathrm{lin}(\mathfrak{f})$.*

*In particular, for $h = 1$, we obtain*

$$(14) \qquad S^L(\mathfrak{p}, 1) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu^L(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(0) \, \mathrm{vol}(\mathfrak{f}).$$

Let us dilate the polytope $\mathfrak{p}$ by a non negative integer $n$. If $\mathfrak{f}$ is a face of $\mathfrak{p}$, let $q_{\mathfrak{f}}$ be the smallest positive integer such that $q_{\mathfrak{f}} <\mathfrak{f}>$ contains integral points. Define $D(\mathfrak{p}, \mathfrak{f}, n) = D(n\mathfrak{p}, n\mathfrak{f})$, if $n > 0$, and $D(\mathfrak{p}, \mathfrak{f}, 0) = D(q_{\mathfrak{f}}\mathfrak{p}, q_{\mathfrak{f}}\mathfrak{f})$. The function $n \mapsto D(\mathfrak{p}, \mathfrak{f}, n)$ is periodic of period $q_{\mathfrak{f}}$.

**Proposition 24** *Let $\mathfrak{p}$ be a rational polytope and $h$ a polynomial function of degree $N$ on $V$. Then, for any integer $n \geq 0$, we have*

$$(15) \qquad S^L(n\mathfrak{p}, h) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \int_{n\mathfrak{f}} D^L(\mathfrak{p}, \mathfrak{f}, n) \cdot h.$$

*Furthermore, if $\mathfrak{f} \in \mathcal{F}(\mathfrak{p})$, we have*

$$\int_{n\mathfrak{f}} D^L(\mathfrak{p}, \mathfrak{f}, n) \cdot h = \sum_{i=\dim \mathfrak{f}}^{\dim \mathfrak{f}+N} E_i(\mathfrak{p}, h, \mathfrak{f}, n) \, n^i$$

*where the coefficients $E_i(\mathfrak{p}, h, \mathfrak{f}, n)$ are periodic with period $q_{\mathfrak{f}}$.*

*Hence the Ehrhart coefficients are given by*

$$E_m^L(\mathfrak{p}, h, n) = \sum_{\mathfrak{f}, \dim \mathfrak{f} \leq m} E_m(\mathfrak{p}, h, \mathfrak{f}, n).$$

When we apply the last proposition to the function $h(x) = 1$, we obtain

$$(16) \qquad S^L(n\mathfrak{p}, 1) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu^L(n\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(0)\, \mathrm{vol}(\mathfrak{f})n^{\dim \mathfrak{f}}.$$

As $\mu^L(n\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$ is invariant by integral translations, the function $\mu^L(n\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(0)$ is of period $q_{\mathfrak{f}}$.

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