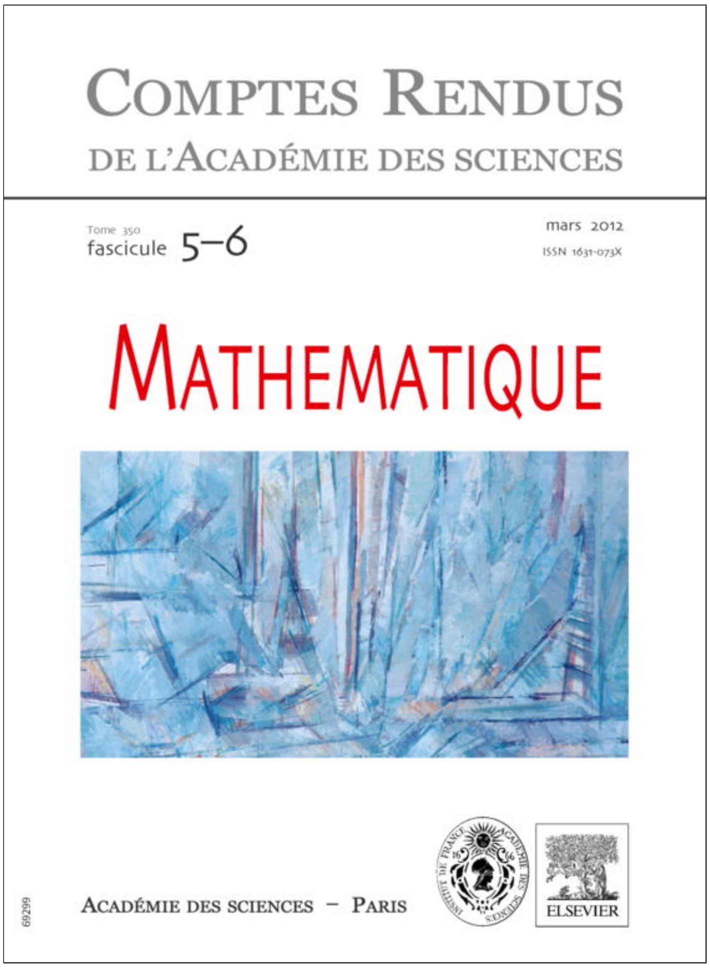
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Combinatorics

A relation between number of integral points, volumes of faces and degree of the discriminant of smooth lattice polytopes

Une relation entre nombre de points entiers, volumes des faces et degré du discriminant des polytopes entiers non singuliers

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ABSTRACT

We present a formula for the degree of the discriminant of a smooth projective toric variety associated to a lattice polytope P, in terms of the number of integral points in the interior of dilates of faces of dimension greater or equal than $\lceil \frac{\dim P}{2} \rceil$.

RÉSUMÉ

Nous donnons une formule pour le degré du discriminant d'une variété torique projective non singulière associée à un polytope entier P, en terme du nombre de points entiers des intérieurs de dilatations de faces de dimension supérieure ou égale à $\lceil \frac{\dim P}{2} \rceil$.

1. Introduction

The relation between the volume and the number of integral points of a lattice polytope $P \subset \mathbb{R}^n$ has a long history. Here, a lattice polytope is a polytope whose vertices are integral, i.e. in $\mathbb{Z}^n$. The Ehrhart polynomial [6] $\mathrm{ehr}_P(t)$ is the polynomial of degree $\dim(P)$ in one variable t such that the number of integral points in tP is equal to $\mathrm{ehr}_P(t)$ when t is a non-negative integer. On the other hand, by Ehrhart–Macdonald reciprocity, for t a positive integer, $\mathrm{ehr}_P(-t)$ equals $(-1)^{\dim(P)}$ times the number of integral points in the relative interior of tP. The leading coefficient of these polynomials equals $1/\dim(P)!$ times the lattice volume $\mathrm{Vol}_{\mathbb{Z}}(P)$ of P. (The volume $\mathrm{Vol}_{\mathbb{Z}}$ is normalized so that the volume of the fundamental parallelepiped is $\dim(P)!$.)

An n-dimensional lattice polytope P with $N+1$ integral points defines an embedded projective toric variety $M_P \subset \mathbb{P}^N$. The polytope P is called smooth (or Delzant) when M_P is nonsingular, which implies that P is simple and the primitive vectors on the n edges emanating from each vertex form a basis of $\mathbb{Z}^n$. The dual projective variety $M_P^\vee \subset (\mathbb{P}^N)^\vee$ consisting of the closure of the locus of those hyperplanes that intersect M_P non-transversally is generically a hypersurface. When this is the case, its degree equals

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$$c(P) = \sum_{p=0}^{n} (-1)^{n-p}(p+1) \sum_{F \in \mathcal{F}_p(P)} \mathrm{Vol}_{\mathbb{Z}}(F), \tag{1}$$

where $\mathcal{F}_p(P)$ denotes the subset of p-dimensional faces of P [7, Th. 28, Ch. 9.2]. In fact $c(P)$ equals the top Chern class of the first jet bundle of the embedding, and the lattice volumes in (1) occur as combinatorial translations of intersection products of line bundles on M_P.

In this Note, our main result, Theorem 3.1, gives a new representation of $c(P)$ for any simple n-dimensional lattice polytope P in terms of the number of integral points in the interior of dilates of faces of dimension greater or equal than $\lceil \frac{n}{2} \rceil$.

The search for this representation was motivated by the question by Batyrev and Nill raised in [1], whether a lattice polytope with sufficiently large dilates without interior lattice points necessarily has a Cayley structure. While this question was answered affirmatively in [8], it is in general still open what 'sufficiently large' precisely means. In the smooth case, this has recently been clarified in [5, Th. 2.1(i)]: if the $\frac{n}{2}+1$ dilate of a smooth polytope P does not have interior lattice points, then $c(P)=0$, so $M_P^\vee$ is not a hypersurface (M_P is *dual defective*), and P admits a (strict) Cayley structure (cf. [4]). The Ehrhart-theoretic proof in [5] relied heavily on non-trivial binomial identities and lacked any general insight in why $\frac{n}{2}+1$ works. Moreover, the method of proof did not employ Ehrhart reciprocity and did not allow one to deal with lattice polytopes with interior lattice points.

In Section 2, we provide a general result, Theorem 2.2, on involutions of 'Dehn–Sommerville type'. The proof of this formal statement is elementary and short. As our main application we deduce from Ehrhart reciprocity our new formula for $c(P)$ valid for any simple lattice polytope (not necessarily smooth). This shows that the surprising fact that $c(P)$ depends only on the lattice points of dilates of faces of high dimension holds actually for *any* simple polytope.

In Section 3, we explain how applying Theorem 2.2 yields the desired direct and conceptual proof of [5, Th. 2.1(i)].

We show in the last section an application of Theorem 2.2 in the realm of symmetric functions. We thank Alain Lascoux and Jean-Yves Thibon for their explanations about the ubiquitous occurrence of the transformation S in Definition 2.1, see for example [10].

Finally, let us remark that the method of proof of Theorem 2.2 can also be used to show an expression of the volume of P in terms of the number of integral points and integral boundary points on $\leqslant \lceil \frac{n}{2} \rceil$-dilates of P, similar to [9, Th. 2.5].

2. An involution

Definition 2.1. Define the transformation $S : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by

$$S[E_0, \ldots, E_n] = [F_0, \ldots, F_n] \quad \text{with} \quad F_p = \sum_{j=0}^{p} (-1)^j \binom{n-j}{n-p} E_j.$$

Then $S^2 = 1$, and we have the identity

$$\sum_{p=0}^{n} (-z)^p (z+1)^{(n-p)} E_p = \sum_{p=0}^{n} z^p F_p. \tag{2}$$

Later on, the elements E_i will be themselves functions of other variables.

The definition of S is motivated by the following examples:

Example 1. Let $[f_0, \ldots, f_n]$ be the f-vector of a simple n-dimensional polytope. Then $S([f_0, \ldots, f_n]) = [f_0, \ldots, f_n]$ by the Dehn–Sommerville equations, see e.g., [2, Th. 5.1]. On the other hand, in the dual situation when P is a simplicial n-dimensional polytope (i.e., all the facets of P are simplices), S is (up to a sign) precisely the transformation between the f- and h-vectors, see [12, 8.3].

Example 2. Let $P \subset \mathbb{R}^n$ be a polytope with integral vertices. Denote by $\mathcal{F}_k(P)$ the set of k-dimensional faces of P and define $E_k^P(t) = \sum_{F \in \mathcal{F}_k(P)} \mathrm{ehr}_F(t)$. We obtain an element $\mathbf{E}^P = [E_0^P(t), E_1^P(t), \ldots, E_n^P(t)]$ of $\mathbb{C}^{n+1}$ depending of t. We write $S(\mathbf{E}^P) = [F_1^P(t), F_2^P(t), \ldots, F_n^P(t)]$.

Assume P is simple. In this case, any face F of P of dimension $j \leqslant p$ is contained in precisely $\binom{n-j}{n-p}$ faces G of P of dimension p. Using the inclusion-exclusion formula, we get, for every $p = 0, \ldots, n$, the extended Dehn–Sommerville equations

$$E_p^P(-t) = \sum_{j=0}^{p} (-1)^j \binom{n-j}{n-p} E_j^P(t) = F_p^P(t).$$

Thus, for any non-negative integer t, $(-1)^p F_p^P(t)$ equals the sum of the number of integral points in the relative interior of the p-dimensional faces of tP. This example is Theorem 5.3 in [2].

Example 3. The elementary symmetric functions $\sigma_i(\mathbf{x})$, $\mathbf{x} = (x_1, \dots, x_n)$, are defined by $\prod_{a=1}^n (x_a z + 1) = \sum_{i=0}^n \sigma_i(\mathbf{x}) z^i$. If we start with the sequence $E = [\sigma_0(\mathbf{x}), \dots, \sigma_n(\mathbf{x})]$, then $S(E)$ gives the sequence of elementary functions on $(1 - x_1, \dots, 1 - x_n)$. We clearly see in this case that S is an involution. More generally, let v, w be meromorphic functions of one variable with $v(x) + w(-x) = 1$, write $\prod_{a=1}^n (v(x_a)z + 1) = \sum_{i=0}^n V_i(\mathbf{x}) z^i$, $\prod_{a=1}^n (w(x_a)z + 1) = \sum_{i=0}^n W_i(\mathbf{x}) z^i$. If we start with the sequence $E = [V_0(\mathbf{x}), \dots, V_n(\mathbf{x})]$, then $S(E) = [W_0(-\mathbf{x}), \dots, W_n(-\mathbf{x})]$.

Let $\mathcal{P}_n$ be the space of families $[E_0(t), \dots, E_n(t)]$ of $n+1$ polynomials where $E_j(t)$ is a polynomial in t of degree less or equal to j. The transformation S induces a transformation of $\mathcal{P}_n$. We write each element $E_j(t) = \frac{t^j}{j!} v_j +$ lower terms. Define

$$c(\mathbf{E}) = \sum_{p=0}^n (-1)^{n-p} (p+1) v_p. \tag{3}$$

Theorem 2.2. *Let $\mathbf{E} = [E_0, E_1, \dots, E_n] \in \mathcal{P}_n$ and $\mathbf{SE} = [F_0, F_1, \dots, F_n]$. For n odd, and $m = (n+1)/2$, then*

$$c(\mathbf{E}) = \sum_{p=m}^n \sum_{i=1}^{p+1-m} (-1)^{p+m-i} \binom{p+1}{m+i} i \big(E_p(-i) + F_p(i)\big). \tag{4}$$

For n even, and $m = n/2$, then

$$c(\mathbf{E}) = \sum_{p=m}^n \sum_{i=1}^{p+1-m} (-1)^{p+1+m-i} \left(\binom{p+1}{m+i} - \binom{p+1}{m+i+1} \right) \frac{i}{2} \big(E_p(-i) + F_p(i)\big). \tag{5}$$

Let us give the proof of this identity for n odd, the case n even being similar.

Denote by τ be the translation operator $(\tau h)(t) = h(t+1)$. Define $e_p(t) = t E_p(t)$, a polynomial function of degree $p+1$. Then $(\tau - 1)^{p+1} e_p$ is just the constant function $(p+1)v_p$, as can be checked on binomials. Hence, $(\tau^{-1} - 1)^{p+1} e_p = (-1)^{p+1}(p+1)v_p$. Since translating by τ^m doesn't change a constant function, we obtain that

$$c(\mathbf{E}) = \sum_{p=0}^n (-1)^{n+1} \tau^m \big(\tau^{-1} - 1\big)^{p+1} e_p(0).$$

Since $\tau^m(\tau^{-1} - 1)^{p+1}$ equals

$$\sum_{j=0}^{n+1} \binom{p+1}{j} \tau^{-j+m} (-1)^{p+1-j} = \sum_{i=-m}^{m} \binom{p+1}{m-i} \tau^i (-1)^{p+1+i-m},$$

we get

$$c(\mathbf{E}) = \sum_{p=0}^n \sum_{i=-m}^{m} (-1)^{p+1+i-m} \binom{p+1}{m-i} i E_p(i). \tag{6}$$

Now we write the right-hand side of (4) as

$$\mathrm{RHS} := \sum_{p=m}^n \sum_{i=0}^{p+1-m} (-1)^p i \operatorname{coeff}\big((1-z)^{p+1}, z^{m+i}\big)\big(E_p(-i) + F_p(i)\big).$$

If $i > p+1-m$, or $p < m$, the number $i(\operatorname{coeff}(1-z)^{p+1}, z^{m+i})$ is equal to 0. Thus, since $p+1-m \leqslant n+1-m \leqslant m$,

$$\mathrm{RHS} = -\sum_{i=0}^m i \sum_{p=0}^n \operatorname{coeff}\big((z-1)^{p+1}, z^{m+i}\big)\big(E_p(-i) + F_p(i)\big).$$

By relation (2) we get $\sum_p (z-1)^p F_p(i) = \sum_p (1-z)^p z^{(n-p)} E_p(i)$; hence, $\sum_p (z-1)^{p+1} F_p(i) = -\sum_p (1-z)^{p+1} z^{(n-p)} E_p(i)$. We deduce that RHS equals

$$\sum_{p=0}^n \sum_{i=0}^m \Big(\operatorname{coeff}\big((z-1)^{p+1}, z^{m-(-i)}\big)(-i) E_p(-i) + \operatorname{coeff}\big((1-z)^{p+1}, z^{m+i-(n-p)}\big) i E_p(i)\Big).$$

As $\binom{p+1}{m+i-n+p} = \binom{p+1}{m-i}$, this is equal to the expression of $c(\mathbf{E})$ given in Eq. (6).

3. An application to lattice polytopes

We return to Example 2. Let us denote by $I_p(i)$ the number of integral points in the relative interior of i-th multiples of p-dimensional faces of P.

Theorem 3.1. *Let P be an n-dimensional simple lattice polytope. Let*

$$c(P)=\sum_{p=0}^{n}(-1)^{n-p}(p+1)\sum_{F\in\mathcal{F}_p(P)}\mathrm{Vol}_{\mathbb{Z}}(F).$$

For n odd, and $m=(n+1)/2$, then

$$c(P)=\sum_{p=m}^{n}\sum_{i=1}^{p+1-m}(-1)^{m-i}\binom{p+1}{m+i}2iI_p(i).$$

For n even, and $m=n/2$, then

$$c(P)=\sum_{p=m}^{n}\sum_{i=1}^{p+1-m}(-1)^{m+1-i}\left(\binom{p+1}{m+i}-\binom{p+1}{m+i+1}\right)iI_p(i).$$

This result follows from Theorem 2.2. Indeed, let $\mathbf{E}^P$ be the sequence of polynomials described in Example 2. Then, the coefficient v_p equals the normalized volume of the skeleton of p-dimensional faces of P, and $F_p^P(i)=E_p^P(-i)=(-1)^pI_p(i)$.

In particular, we get the following alternative proof of [5, Th. 2.1(i)]. Assume $I_n(i)=\mathrm{ehr}_P(-i)=0$ for any positive integer $i\leqslant\frac{n}{2}+1$. For simplicity, take n odd. So, the polytope iP has no integral interior points for $i=1,2,\dots,m=\frac{n+1}{2}$. The monotonicity theorem of Stanley [11] implies that any face of codimension k of P has no integral interior points for $i=1,\dots,m-k$. Thus we obtain from (4) that $c(P)=c(\mathbf{E})=0$. If P is smooth, then this implies that M_P is dual defective.

4. An identity of symmetric functions

Let

$$V(s,\mathbf{x})(t)=\frac{e^{ts}}{\prod_{a=1}^{n}(-tx_a)},\qquad B(s,\mathbf{x})(t)=\frac{e^{ts}}{\prod_{a=1}^{n}(1-e^{tx_a})}$$

be meromorphic functions of t depending of the $(n+1)$ variables s and $\mathbf{x}$. The constant term $CTV(s,\mathbf{x})$ of the Laurent series (in t) of $V(s,\mathbf{x})(t)$ is $(-1)^n\frac{s^n}{n!x_1x_2\cdots x_n}$. The constant term $CTB(s,\mathbf{x})$ of the Laurent series of $B(s,\mathbf{x})(t)$ is a meromorphic function of $(s,x_1,x_2,\dots,x_n)$, symmetric in the x_i.

Let $P\subset\mathbb{R}^n$ be an n-dimensional smooth polytope, and let $\mathcal{V}(P)$ be the set of vertices of P. For v a vertex, let $g_a, a=1,\dots,n$, be the primitive generators on the edges of P starting at v. If ξ is generic in the dual space to $\mathbb{R}^n$, we can specialize the variables s,x_a to $\langle v,\xi\rangle$, $\langle g_a,\xi\rangle$ in $V(s,\mathbf{x})(t)$ and $B(s,\mathbf{x})(t)$ and we obtain meromorphic functions $V_v(\xi,t)$, $B_v(\xi,t)$ depending of the vertex v. Then the sum over $v\in\mathcal{V}(P)$ of the meromorphic function $V_v(\xi,t)$ or $B_v(\xi,t)$ is actually regular at $t=0$, and $n!\sum_v V_v(\xi,t)_{(t=0)}$ is the normalized volume of P while $\sum_v B_v(\xi,t)_{(t=0)}$ is the number of integral points in P by Brion's formulas [3].

Let $1\leqslant p\leqslant n$, and let

$$V_p(s,\mathbf{x})(t)=\sum_{J,|J|=p}\frac{e^{ts}}{\prod_{a\in J}(-tx_a)},\qquad B_p(s,\mathbf{x})(t)=\sum_{J,|J|=p}\frac{e^{ts}}{\prod_{a\in J}(1-e^{tx_a})},$$

where J runs over subsets of cardinal p of $\{1,2,\dots,n\}$. Similarly, the specialization s,x_a to $\langle v,\xi\rangle$, $\langle g_a,\xi\rangle$ gives us meromorphic functions of (t,ξ), and the sum over the vertices v of the polytope P is regular at $t=0$. If we dilate P by i, the vertices are changed in iv, while the generators g_a stay the same. Then the identities of Theorem 3.1 imply in particular that for n odd,

$$\sum_{v\in\mathcal{S}(P)}\left(\sum_{p=0}^{n}(-1)^{n-p}(p+1)!V_{v,p}(\xi,t)\right)_{(t=0)}=\sum_{v\in\mathcal{S}(P)}\left(\sum_{p=m}^{n}(-1)^p\sum_{i=1}^{p+1-m}(-1)^{m-i}\binom{p+1}{m+i}2iB_{iv,p}(-\xi,t)\right)_{(t=0)}.$$

A similar identity holds for n odd. Actually, this identity holds before summing over the vertices v and before specializing, as an identity for the symmetric functions in x_a obtained as the constant term $CTB_p(s,\mathbf{x})$ of the Laurent series in t of $B_p(s,\mathbf{x})(t)$.

Theorem 4.1. *For n odd, and $m = (n+1)/2$,*

$$\sum_{p=0}^{n} (-1)^{n-p} (p+1)! CTV_p(s, \mathbf{x}) = \sum_{p=m}^{n} (-1)^p \sum_{i=1}^{p+1-m} (-1)^{m-i} \binom{p+1}{m+i} 2i CTB_p(-is, \mathbf{x}).$$

For n even, and $m = n/2$,

$$\sum_{p=0}^{n} (-1)^{n-p} (p+1)! CTV_p(s, \mathbf{x}) = \sum_{p=m}^{n} (-1)^p \sum_{i=1}^{p+1-m} (-1)^{m+1-i} \left(\binom{p+1}{m+i} - \binom{p+1}{m+i+1} \right) i CTB_p(-is, \mathbf{x}).$$

We prove this again as a consequence of Theorem 2.2 using Example 3, and the identity $\frac{1}{1-\exp(x)} + \frac{1}{1-\exp(-x)} = 1$.

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