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MULTIPLICITIES FORMULA FOR GEOMETRIC QUANTIZATION, PART II

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1. Introduction. Let P be a compact manifold. Let H be a compact Lie group acting on the right on P. We assume that the stabilizer of each element $y \in P$ is a finite subgroup of H. The space $M = P/H$ is an orbifold and every orbifold can be presented this way. If H acts freely, then M is a manifold. If $\mathscr{L}$ is an H-equivariant line bundle on P, the space $\mathscr{L}/H$ will be called an *orbifold line bundle* on M. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ acting on the compact orbifold $M = P/H$. We consider the case where M is a prequantized symplectic orbifold. Let $\mathscr{L}$ be a G-equivariant Kostant-Souriau orbifold line bundle on M. Then the quantized representation $Q(M, \mathscr{L})$ associated to $(M, \mathscr{L})$ is a virtual representation of G constructed as the $\mathbb{Z}/2\mathbb{Z}$-graded space of H-invariant solutions of the H-horizontal Dirac operator on P twisted by the line bundle $\mathscr{L}$.

Let $\mu: M \to \mathfrak{g}^*$ be the moment map for the G-action. Assume 0 is a regular value of μ. Let M_{red} be the reduced orbifold of M; that is, $M_{\text{red}} = \mu^{-1}(0)/G$. Consider the reduced orbifold line bundle $\mathscr{L}_{\text{red}} = \mathscr{L}|_{\mu^{-1}(0)}/G$ on M_{red}. In the case where both G and H are torus, we prove here the formula

$$Q(M, \mathscr{L})^G = Q(M_{\text{red}}, \mathscr{L}_{\text{red}}).$$

This formula was conjectured by Guillemin-Sternberg [4] and proved when M is a complex manifold and $\mathscr{L}$ a sufficiently positive G-equivariant holomorphic line bundle. Here, we do not assume the existence of complex structure on M. Initially, we obtained a proof [7] of the formula $Q(M, \mathscr{L})^G = Q(M_{\text{red}}, \mathscr{L}_{\text{red}})$ for the case where M is a symplectic manifold with Hamiltonian action of a torus G such that G acts freely on $\mu^{-1}(0)$. Let us recall that independently E. Meinrenken [6] had obtained a proof of the formula $Q(M, \mathscr{L})^G = Q(M_{\text{red}}, \mathscr{L}_{\text{red}})$ including the case where M_{red} is an orbifold. It is possible to generalise the method sketched in [7] to cover the case of orbifolds. Indeed, after writing a character formula [9] for $Q(P/H, \mathscr{L})$, similar arguments can be given. We give here an alternative approach that requires almost no calculations. This approach is the K-theoretical version of the deformation argument in equivariant cohomology employed in Part I of this article [8]. However, we have tried to write the present article in such a way that the reading of Part I (although reassuring) is not necessary to understand our arguments. In Part I, we wrote in detail the case of an S^1-action using a deformation formula for the character of $Q(M, \mathscr{L})$. The original inspiration of

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this deformation argument goes back to Witten [10]. We here replace this argument by a simpler argument based on the deformation of the symbol of the Dirac operator inside G-transversally elliptic symbols. Although this approach is more direct, it uses Atiyah's theory [1] of the index of transversally elliptic symbols: existence of the index as a trace-class virtual representation of G, homotopy invariance of the index, excision, multiplicativity, and the explicit calculation of the index for the Atiyah symbol m (see Formula 5).

Let us explain briefly our argument in the case of an S^1-Hamiltonian action on a compact symplectic manifold M. Let $G = S^1$. Let $E \in \mathfrak{g}$ be a basis of the Lie algebra $\mathfrak{g}$ and let $f = \mu(E)$. Let $(.,.)$ be a G-invariant metric. We introduce on M the G-invariant 1-form

$$\lambda(\cdot) = \mu(E)(E_M, \cdot),$$

where E_M is the vector field on M generated by the S^1-action. We denote by (x, ξ) a point of the cotangent bundle T^*M. With the help of λ, we deform the symbol $c(x, \xi) \otimes I_{\mathscr{L}_x}$ of the twisted Dirac operator in a G-transversally elliptic symbol $c_t(x, \xi) = c(x, \xi - t\lambda_x) \otimes I_{\mathscr{L}_x}$. As the index is invariant under homotopy, we obtain $Q(M, \mathscr{L}) = \text{index}(c_1)$. A model for the virtual representation index(c_1) can be realised in the space of solutions of an operator D_1 with symbol homotopic to c_1. The form λ vanishes on $M_0 = f^{-1}(0)$ and on M^G. We can choose such an operator D_1 so that its solutions are supported near the set of zeroes of λ. Using this, we need to analyse c_1 on a neighbourhood of M_0 and M^G. Assume zero is a regular value of f. Then M_0/G is an orbifold. It is easy to see that the restriction c_1^0 on a neighbourhood of M_0 isomorphic to $M_0 \times \mathbb{R}$ is just the tensor product of the symbol of the G-horizontal Dirac operator $D_{\mathscr{L}_0}$ on M_0 (twisted by $\mathscr{L}|_{M_0}$) and the Bott symbol on $\mathbb{R}$. We thus have

$$\text{index}(c_1^0)^G = Q(G\backslash M_0, G\backslash \mathscr{L}|_{M_0}).$$

We analyse the symbol c_1^a on a neighbourhood of a fixed-point component M_a of M^G. If M_a is a point p_a, the symbol c_1^a coincides with the Atiyah symbol m shifted by a line bundle. Thus, we obtain that the index of c_1^a is $e_{i\mu_a+\rho_a}\sum_k Tr_{S^k N_a}$, where N_a is the tangent space at p_a equipped with a particular complex structure. By calculations very similar to those in Part I [8], we see that the trivial representation of G does not occur in index(c_1^a).

In our proof of the equality

$$Q(M, \mathscr{L})^G = Q(M_{\text{red}}, \mathscr{L}_{\text{red}}),$$

the number $Q(M_{\text{red}}, \mathscr{L}_{\text{red}})$ is by definition the index of an elliptic operator on the orbifold M_{red}. When M_{red} is a complex algebraic variety, the number $Q(M_{\text{red}}, \mathscr{L}_{\text{red}})$ is given by Riemann-Roch-Kawasaki's formula [5]. When (as is the case in our setting) M_{red} is a quotient of a manifold by a torus action, integral formula for the number $Q(M_{\mathrm{red}}, \mathcal{L}_{\mathrm{red}})$ is given by Atiyah [1] Corollary 9.12. It follows that our result coincides with Meinrenken's expression. In [9], we give similar integral expressions for the trace of the index of G-elliptic operators on orbifolds.

2. Quantization on orbifolds. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let M be a smooth G-manifold. Let T^*M be the cotangent bundle to M with projection $p: T^*M \to M$. We follow notations of [2] for transversally elliptic symbols. If $\mathscr{E}^\pm$ are G-equivariant Hermitian vector bundles over M, a morphism $\sigma \in \Gamma(T^*M, \mathrm{Hom}(p^*\mathscr{E}^+, p^*\mathscr{E}^-))$ of G-equivariant vector bundles will be called a symbol. We denote by $\sigma^*: p^*\mathscr{E}^- \to p^*\mathscr{E}^+$ the adjoint of the morphism σ. A point of T^*M will be denoted by (x, ξ), with $x \in M$ and $\xi \in T_x^*M$. Then $\sigma(x, \xi): \mathscr{E}_x^+ \to \mathscr{E}_x^-$. The subset of point $(x, \xi) \in T^*M$ where $\sigma(x, \xi)$ is not invertible will be called the characteristic set of σ. We do not assume any homogeneity assumption on σ. However, we assume that σ is defined and is C^∞ on all of T^*M. If σ is invertible outside a compact subset of T^*M, we will say that σ is elliptic. Then an elliptic symbol σ defines an element of $K_G(T^*M)$, and if M is compact, the index of σ is a virtual finite-dimensional representation of G, that is, a difference of two finite-dimensional representations of G. We denote by T_G^*M the closed subset of T^*M which is the union of conormal to the orbits of G in M; that is,

$$T_G^*M = \{(x, \xi); \langle \xi, X_M(x) \rangle = 0 \quad \text{for all } X \in \mathfrak{g}\}.$$

A symbol σ will be called transversally elliptic if the restriction of $\sigma(x, \xi)$ to T_G^*M is invertible outside a compact subset of T_G^*M. Then σ defines an element of $K_G(T_G^*M)$. If M is compact, then the index of σ is defined as in [1] and is a trace-class virtual representation of G. If U is a G-invariant open subset of a compact G-manifold M, and if σ is a G-transversally elliptic symbol on U, then the index of σ is also defined. If σ is a symbol on M invertible on all points of T^*M above $M - U$, then $\mathrm{index}(\sigma) = \mathrm{index}(\sigma|_U)$. This is the excision lemma.

Let us recall the definition of the external product of symbols. Let M_1 and M_2 be two manifolds with G-actions. Let $\mathscr{E}_i^\pm$ be G-equivariant Hermitian vector bundles over M_i. Let $p_i^*\mathscr{E}_i^+ \xrightarrow{\sigma_i} p_i^*\mathscr{E}_i^-$ be two transversally elliptic symbols. We denote by

$$\sigma_1 \odot \sigma_2: p_1^*\mathscr{E}_1^+ \otimes p_2^*\mathscr{E}_2^+ \oplus p_1^*\mathscr{E}_1^- \otimes p_2^*\mathscr{E}_2^- \to p_1^*\mathscr{E}_1^- \otimes p_2^*\mathscr{E}_2^+ \oplus p_1^*\mathscr{E}_1^+ \otimes p_2^*\mathscr{E}_2^-$$

the symbol defined by

$$\sigma_1 \odot \sigma_2 = \begin{pmatrix} \sigma_1 \otimes I & -I \otimes \sigma_2^* \\ I \otimes \sigma_2 & \sigma_1^* \otimes I \end{pmatrix}. \tag{1}$$

Under some conditions on characteristic sets of σ_1 and σ_2, the symbol $\sigma_1 \odot \sigma_2$ is transversally elliptic. Furthermore, the tensor product $\mathrm{index}(\sigma_1) \otimes \mathrm{index}(\sigma_2)$ of the virtual trace-class representations $\mathrm{index}(\sigma_i)$ is a virtual trace-class representation of G (for example, if one of the σ_i is elliptic, these conditions are satisfied), and

$$\text{index}(\sigma_1 \odot \sigma_2) = \text{index}(\sigma_1) \otimes \text{index}(\sigma_2). \tag{2}$$

This is the multiplicativity property of the index. We will use this equality for symbols verifying the hypothesis of [1, Theorem 3.5]. (The group G is a product $G_1 \times G_2$, and G_2 acts trivially on M_1.)

Let P be a compact smooth manifold. Let G and H be two compact Lie groups acting on P with commuting actions. We assume that the action of H is infinitesimally free; that is, the stabilizer of any point $y \in P$ is a finite subgroup of H. We write the action of H on the right and the action of G on the left. The quotient space P/H is provided with an action of G, and the space P/H will be called a G-orbifold. A tangent vector on P tangent at $y \in P$ to the orbit $H \cdot y$ will be called a vertical tangent vector. We will say that the space T_H^*P is the horizontal cotangent bundle. The bundle T_H^*P is a $(G \times H)$-equivariant vector bundle over P. Let us assume that T_H^*P is an even-dimensional $(G \times H)$-equivariant orientable vector bundle. For simplicity, we assume that T_H^*P admits a $(G \times H)$-invariant spin structure. Let $K_{G \times H}(P)$ be the Grothendieck group of $(G \times H)$-equivariant vector bundles on P. Choose a $(G \times H)$-invariant orientation o on T_H^*P. Then there is a well-defined quantization map

$$Q_{P,H}^o\colon K_{G\times H}(P) \to R(G).$$

We can construct this map as follows. Choose a $(G \times H)$-invariant metric on P. The bundle of vertical vectors is isomorphic to $P \times \mathfrak{h}$. Let

$$TP = T_{\text{hor}}P \oplus P \times \mathfrak{h}$$

be the orthogonal decomposition of the tangent bundle. The bundles $T_{\text{hor}}P$ and T_H^*P are isomorphic. Let $\mathcal{S}_{\text{hor}}$ be the spin bundle for $T_{\text{hor}}P$. The orientation o determines a $\mathbb{Z}/2\mathbb{Z}$-gradation $\mathcal{S}_{\text{hor}} = \mathcal{S}_{\text{hor}}^+ \oplus \mathcal{S}_{\text{hor}}^-$. If $v \in (T_{\text{hor}}P)_y$, then the Clifford multiplication $c(v)$ is an odd operator on $(\mathcal{S}_{\text{hor}})_y$. Let $\mathcal{E}$ be a $(G \times H)$-equivariant Hermitian vector bundle on P. Let $\mathcal{S}_{\text{hor}} \otimes \mathcal{E}$ be the twisted horizontal spin bundle. With the help of a choice of a $(G \times H)$-invariant unitary connection $\nabla = \nabla^+ \oplus \nabla^-$ on $\mathcal{S}_{\text{hor}} \otimes \mathcal{E} = \mathcal{S}_{\text{hor}}^+ \otimes \mathcal{E} \oplus \mathcal{S}_{\text{hor}}^- \otimes \mathcal{E}$, we may define the formally self-adjoint "horizontal" Dirac operator $D_{\text{hor},\mathcal{E}}$ by

$$D_{\text{hor},\mathcal{E}} = \sum_i c(e_i)\nabla_{e_i},$$

where e_i runs over an orthonormal basis of $T_H^*P = T_{\text{hor}}P$. We have $D_{\text{hor},\mathcal{E}} = D_{\text{hor},\mathcal{E}}^+ \oplus D_{\text{hor},\mathcal{E}}^-$ with

$$D_{\text{hor},\mathcal{E}}^+\colon \Gamma(P, \mathcal{S}_{\text{hor}}^+ \otimes \mathcal{E}) \to \Gamma(P, \mathcal{S}_{\text{hor}}^- \otimes \mathcal{E})$$

and

$$D^-_{\mathrm{hor},\mathscr{E}}\colon \Gamma(P, \mathscr{S}^-_{\mathrm{hor}} \otimes \mathscr{E}) \to \Gamma(P, \mathscr{S}^+_{\mathrm{hor}} \otimes \mathscr{E}).$$

Clearly, the operators $D^{\pm}_{\mathrm{hor},\mathscr{E}}$ are H-transversally elliptic operators and commute with the natural action of G. The index $\mathrm{index}(D^+_{\mathrm{hor},\mathscr{E}})$ is defined as in [1] and is a trace-class virtual representation of H. It follows in particular that the spaces $(\mathrm{Ker}\ D^{\pm}_{\mathrm{hor},\mathscr{E}})^H$ of H-invariant solutions of $D^{\pm}_{\mathrm{hor},\mathscr{E}}$ are finite-dimensional representation spaces of G. We define the virtual representation $Q^o_{P,H}(\mathscr{E}) \in R(G)$ by

$$Q^o_{P,H}(\mathscr{E}) = (-1)^{(\dim P - \dim H)/2}[(\mathrm{Ker}\ D^+_{\mathrm{hor},\mathscr{E}})^H] - [(\mathrm{Ker}\ D^-_{\mathrm{hor},\mathscr{E}})^H].$$

When H acts freely on P, then the manifold $M = P/H$ admits a G-invariant spin structure, the space $\mathscr{E}/H$ is a vector bundle on P/H, and we have indeed

$$Q^o_{P,H}(\mathscr{E}) = Q^o(P/H, \mathscr{E}/H)$$

where the map $Q^o(M, \cdot)$ was defined in Part I for a spin even-dimensional compact manifold M. Thus, we will write $Q^o_{P,H}(\mathscr{E}) = Q^o(P/H, \mathscr{E})$.

If $G = \{e\}$, then $Q^o(P/H, \mathscr{E})$ is an integer called the Riemann-Roch number of $\mathscr{E}$. If P is a complex manifold and $\mathscr{E}$ a holomorphic H-equivariant bundle on H, the number $Q^o(P/H, \mathscr{E})$ is computed by Kawasaki's formula [5]. In general, it is given by a similar formula [1].

We note for further use the following trivial result. Let T be a torus acting trivially on P. If $\mathscr{E}$ is a $(T \times H)$-equivariant vector bundle on P, we write $\mathscr{E} = \Sigma_\xi \mathscr{E}_\xi$, where T acts on $\mathscr{E}_\xi$ by e_ξ. Each vector bundle $\mathscr{E}_\xi$ is an H-equivariant vector bundle on P, and $Q^o(P/H, \mathscr{E}_\xi)$ is a number. We have

$$Q^o(P/H, \mathscr{E}) = \oplus_{\xi \in \hat{T}}\, Q^o(P/H, \mathscr{E}_\xi) e_\xi; \tag{3}$$

in particular, the set of weights ξ occurring in $Q^o(P/H, \mathscr{E})$ is contained in the set of weights ξ such that $\mathscr{E}_\xi$ is nonzero.

We can extend without difficulty all the results of [8] to the case of an S^1-Hamiltonian action on an orbifold P/H. We first define what is a Hamiltonian action in this case. A differential form α on P will be called *horizontal* if $\imath(Y)\cdot\alpha = 0$ for all $Y \in \mathfrak{h}$. The differential form α is called *basic* if it is horizontal and H-invariant.

Definition 1. A symplectic form on P/H is a closed, basic 2-form σ on P such that, for each $y \in P$, the form σ_y is nondegenerate on $(T_{\mathrm{hor}}P)_y$.

If P/H is a symplectic orbifold, then the bundle $T^*_H P$ has a canonical orientation. We will choose o as being the symplectic orientation, and we may omit it in the notation.

We say that the action of G on P/H is Hamiltonian, if there exists an H-

invariant map $\mu: P \to \mathfrak{g}^*$ such that

$$d\mu(X) = \iota(X_P)\sigma$$

for all $X \in \mathfrak{g}$.

The following lemma is easily proved.

LEMMA 2. *Assume that zero is a regular value of* μ. *Then the action of* $G \times H$ *on* $P_0 = \mu^{-1}(0)$ *is infinitesimally free. Furthermore, the restriction* σ_0 *of* σ *to the manifold* P_0 *is a symplectic structure on the orbifold* $P_0/(G \times H)$.

For $X \in \mathfrak{g}$, we define $\sigma_{\mathfrak{g}}(X) = \mu(X) + \sigma$. Let $\mathscr{L}$ be a $(G \times H)$-invariant line bundle on P with $G \times H$-invariant connection $\mathbb{A}$. We say that $\mathscr{L}$ is a Kostant-Souriau line bundle on the G-Hamiltonian orbifold P/H, if the equivariant curvature of $\mathscr{L}$ is equal to $i\mu(X) + i\sigma$. In this case, we say that the orbifold P/H is prequantized. The space $Q(P/H, \mathscr{L})$ is a virtual representation of G.

Let G and K be two compact Lie groups. We assume that the group $G \times K$ acts in a Hamiltonian way on $(P/H, \sigma)$. Let $\mu: P \to \mathfrak{g}^*$ be the moment map for the G-action on P/H. Assume that zero is a regular value of μ. Let $P_0 = \mu^{-1}(0)$. Then $(P_0/(G \times H), \sigma_0)$ is a K-Hamiltonian orbifold.

THEOREM 3. *Let* $(P/H, \sigma)$ *be a symplectic orbifold for the action of a torus* H. *We consider a Hamiltonian action of* $G \times K$ *on* P/H, *where* G *and* K *are two compact Lie groups. Assume that* G *is a torus. Let* $\mu: P \to \mathfrak{g}^*$ *be the moment map for the* G*-action on* P/H. *Assume that zero is a regular value of* μ. *Let* $P_0 = \mu^{-1}(0)$. *Assume that the* $(G \times K)$*-Hamiltonian orbifold* P *is prequantized. Let* $\mathscr{L}$ *be a Kostant-Souriau line bundle for the Hamiltonian action of* $(G \times K)$ *on* P. *Then we have the equality in* $R(K)$:

$$Q(P/H, \mathscr{L})^G = Q(P_0/(G \times H), \mathscr{L}|_{P_0}).$$

Proof. In the setting of orbifolds, we can proceed by induction on G. We assume the theorem proven for an S^1-action. Let $\exp: \mathfrak{g} \to G$ be the exponential map and Γ its kernel. If $X \in \Gamma$, the 1-parameter subgroup $\theta \mapsto \exp \theta X/2\pi$ is closed and isomorphic to $S^1 = \{e^{i\theta}\}$. We can choose $X \in \Gamma$ such that zero is a regular value of $f = \mu(X)$. Indeed, let us denote by $P(G_{\text{vert}})$ the set of points $y \in P$ such that $Gy \in Hy$ (if $H = \{1\}$, this is the fixed point set of G). This is a closed submanifold of P. Our hypothesis that zero is a regular value of μ implies that $\mu(P(G_{\text{vert}}))$ is a finite set of nonzero elements ξ_i of $\mathfrak{g}^*$. The critical set of the map $f = \mu(X)$ is the set $P(X_{\text{vert}})$ where the vector field X_P is vertical. If X is generic in $\mathfrak{g}$, the set $P(X_{\text{vert}})$ coincides with $P(G_{\text{vert}})$. Thus, if X is generic in $\mathfrak{g}$, then $\xi_i(X) \neq 0$, and the critical set $P(G_{\text{vert}})$ of f does not intersect $f^{-1}(0)$.

Let $X \in \Gamma$ such that zero is a regular value of f. Consider the compact 1-parameter group $\exp \theta X$ isomorphic to S^1. There is a subgroup G' of G such that the map $S^1 \times G' \to G$ given by $(e^{i\theta}, g') \mapsto \exp(\theta X)g'$ is a finite cover. Applying the theorem to S^1, with symmetry group G', we obtain $Q(P/H, \mathscr{L})^{S_1 \times G'} = Q(P_0/(S^1 \times H), \mathscr{L})^{G'}$ with $P_0 = f^{-1}(0)$, and we conclude by induction.

Thus, it is sufficient to prove this theorem for an action of $G = S^1$.

Let $G = \{e^{i\theta}; \theta \in \mathbb{R}\}$. We choose a basis E of $\mathfrak{g}$ such that $\exp(\theta E) = e^{i\theta}$ and denote by $f: P \to \mathbb{R}$ the map $f = \mu(E)$.

We consider the vector field $W = fE_P$ on P. Denote by C the set of points $y \in P$ of P where the vector W_y is vertical. Morally, the space C/H is the set of zeroes of the vector field W acting on P/H. It is clear that $C = P_0 \cup P(E_{\text{vert}})$ where $P_0 = f^{-1}(0)$, and where $P(E_{\text{vert}})$ is the set of points $y \in P$ where E_y is vertical. Let $\mathscr{F}$ be the set of connected components of $P(E_{\text{vert}})$. For $a \in \mathscr{F}$, we write P_a for the corresponding connected component. (If $H = \{1\}$, then $P(E_{\text{vert}})$ is simply the set of fixed points of $G = \{\exp \theta E\}$). Using the vector field W, we first deform the Dirac operator $D^+_{\text{hor}, \mathscr{L}}$ in a $(G \times H)$-transversally elliptic operator "trivial" outside a neighbourhood of C. We denote by (y, ξ) an element of T^*P with $y \in P$ and $\xi \in T_yP$. Consider the symbol σ of $D^+_{\text{hor}, \mathscr{L}}$. We have

$$\sigma(y, \xi) = c^+(h(\xi)) \otimes I: (\mathscr{S}^+_{\text{hor}} \otimes \mathscr{L})_y \to (\mathscr{S}^-_{\text{hor}} \otimes \mathscr{L})_y,$$

where h is the horizontal projection of T^*P on T^*_HP. We identify TP with T^*P.

LEMMA 4. *For all $t \in \mathbb{R}$, the symbol*

$$\sigma_t(y, \xi) = \sigma(y, \xi - tW_y)$$

is a $(G \times H)$-transversally elliptic symbol.

Proof. We see that the vanishing of $\sigma_t(y, \xi)$ implies that $h(\xi)$ is proportional to $h(W_y)$. As W_y at each point is proportional to E_y, we see that the symbol σ_t is never 0 on $T^*_{G \times H}P - \{0\}$. ■

Consider the $(G \times H)$-transversally elliptic symbol σ_1. Its index is a trace-class virtual representation of $(G \times H)$. By homotopy invariance of the index, we have then $\operatorname{index}(D^+_{\text{hor}, \mathscr{L}}) = \operatorname{index}(\sigma_1)$.

Remark 2.1. In the identification of TP with T^*P, the vector field W becomes equal to the 1-form λ of [10]. The homotopy argument for the equality $\operatorname{index}(D^+_{\text{hor}, \mathscr{L}}) = \operatorname{index}(\sigma_1)$ is similar to the fact that the form $e^{itd_{\mathfrak{g}}\lambda}$ is congruent to 1 in equivariant cohomology.

The argument which follows corresponds to the localisation argument employed in [8], when t becomes large.

Outside a neighbourhood of C, the horizontal component of W is nonzero; we then can deform the symbol

$$\sigma_1(y, \xi) = c^+(h(\xi) - h(W_y)) \otimes I \tag{4}$$

to a symbol independent of ξ and given by the bundle isomorphism $c^+(-h(W_y))\otimes I$. Indeed, we may choose a $(G\times K\times H)$-invariant partition of unity, where ϕ_0 is identically 1 in a neighbourhood of C and with support in a small neighbourhood of C and deform $\sigma_1(y,\xi)$ in $\sigma_1'(y,\xi)=c^+(\phi_0(y)h(\xi)-\phi_1(y)h(W_y))\otimes I$. We denote by σ_1^0 the restriction of σ_1' to a tubular neighbourhood of P_0 and by σ_1^a the restriction of σ_1' to a tubular neighbourhood of P_a. By the excision lemma, we then have

$$\operatorname{index}(\sigma_1)=\operatorname{index}(\sigma_1^0)\oplus(\oplus_{a\in\mathscr{F}}\operatorname{index}(\sigma_1^a)).$$

To prove Theorem 3, it remains to prove the following proposition.

PROPOSITION 5. *Let $G=S^1$ acting on the symplectic orbifold P/H. Then*
(1) *Let a be a component of the set $P(G_{\text{vert}})$; then*

$$\operatorname{index}(\sigma_1^a)^{G\times H}=0;$$

(2) $\operatorname{index}(\sigma_1^0)^{G\times H}=Q(P_0/(G\times H),\mathscr{L}_0)$.

We will prove the equality (1) of Proposition 5 in Section 3 and the equality (2) in Section 4.

3. The local index near a fixed-point component. We first describe a particular transversally elliptic symbol which will be needed for the description of σ_1^a.

Let V be a complex Hermitian vector space. Let $U(V)$ be the unitary group of V. Let $S^1=\{e^{i\theta}\}$. Consider the exterior bundle $\Lambda V=\Lambda^+V\oplus\Lambda^-V$ graded in even and odd elements. For $v\in V$, we denote by $\varepsilon(v)$ the left exterior multiplication by v and by $\imath(v)$ the contraction by v. Let $\ell(v)=\varepsilon(v)+\imath(v)$ so that $\ell(v)^2=\|v\|^2I$. We write $\ell(v)=\ell^+(v)\oplus\ell^-(v)$ with $\ell^+(v)\colon\Lambda^+V\to\Lambda^-V$.

We identify T^*V with $V\oplus V$ with the help of the scalar product on V. Define

$$m(v,\xi)=\ell^+(v-i\xi). \tag{5}$$

It is a S^1-transversally elliptic symbol on V. Furthermore, m is $U(V)$-invariant. Thus, m is a fortiori a $U(V)$-transversally elliptic symbol. The index of m is a trace-class virtual representation of $U(V)$. We denote by $\det_V$ the 1-dimensional representation $u\to\det_V(u)$ of $U(V)$.

LEMMA 6. *We have the equality of virtual representations of $U(V)$*

$$\operatorname{index}(m)=(-1)^{\dim_{\mathbb{C}}V}\det_V\otimes\oplus_{k=0}^{\infty}S^k(V).$$

Proof. It is sufficient to prove this formula on the torus of $U(V)$ composed of diagonal matrices. Then, by multiplicativity of the index, it is sufficient to prove this formula for $V=\mathbb{C}$. This results from [1, Lemma 6.4] (see also [2, Appendix 2]). ∎

We also recall that if $V = V_0 \oplus V_1$ is an orthogonal decomposition of an even-dimensional Euclidean vector space V in two even-dimensional Euclidean spaces, then the spinor space S for V is the tensor product of the spinor spaces S_i for V_i:

$$S = S_0 \otimes S_1 .$$

If $\xi = \xi_0 + \xi_1$ with $\xi_i \in V_i$, then

$$c(\xi) = c(\xi_0) \otimes c(\xi_1).$$

Let us analyze $P(E_{\text{vert}})$. At a point $y \in P(E_{\text{vert}})$, there exists a unique vector $H_y \in \mathfrak{h}$, such that $(E + H_y)_y = 0$. Let I be the set of elements $Y \in \mathfrak{h}$ such that $P(E + Y)$ is nonempty. This set is finite. Then it is clear that $P(E_{\text{vert}}) = \bigcup_{i \in I} P(E + Y_i)$. For each $a \in \mathscr{F}$ there exists a unique $Y_a \in \mathfrak{h}$ such that P_a is a connected component of the manifold $P(E + Y_a)$. We write $S_a = E + Y_a$. The manifold P_a is an H-manifold with infinitesimally free action of H. In particular, the subgroup of H acting trivially on P_a is a finite group. Let us remark that the 1-parameter group generated by S_a in $G \times H$ is compact. Indeed, the action of $\exp 2\pi Y_a$ on P_a is trivial as $Y_a = (E + Y_a) - E$. Thus, $\gamma_a = \exp 2\pi Y_a$ belongs to the finite subgroup of H acting trivially on P_a. We thus see that there exists an integer k such that $\exp 2\pi k S_a = 1$ in $G \times H$. The map $\mathbb{R} \to G \times H$ given by $\theta \mapsto \exp \theta S_a$ gives a map from $\mathbb{R}/2\pi k\mathbb{Z}$ into $G \times H$ and the 1-parameter subgroup $G_a \subset G \times H$ generated by $S_a = E + Y_a$ is compact. The projection $G \times H \to G$ realizes G_a as a finite cover of G. The manifold P_a is a connected component of the manifold P^{G_a}. The function $\mu(E)$ is constant on P_a. Let μ_a be its constant value. The action of G_a on $\mathscr{L}|_{P_a}$ is thus given by $\exp(\theta S_a) \cdot v = e^{i\mu_a\theta} v$. In particular, μ_a is a rational number.

Consider the normal bundle $\mathscr{N}_a$ to P_a in P. We then have an orthogonal decomposition

$$T_{\text{hor}} P|_{P_a} = T_{\text{hor}} P_a \oplus \mathscr{N}_a .$$

The action of S_a on $\mathscr{N}_a$ allows us to decompose $\mathscr{N}_a \otimes_{\mathbb{R}} \mathbb{C}$ as a sum of complex vector bundles $\mathscr{N}_a(\alpha)$ where G^a acts on $\mathscr{N}_a(\alpha)$ by the weight e_α.

If μ_a is positive, we write

$$\Delta_a^{\text{out}} = \{\alpha, i(\alpha, S_a) < 0\} .$$

At the opposite, if μ_a is negative, we write

$$\Delta_a^{\text{out}} = \{\alpha, i(\alpha, S_a) > 0\} .$$

We write

$$\mathscr{N}_a^{+} = \oplus_{\alpha \in \Delta_a^{\text{out}}} \mathscr{N}_a(\alpha) .$$

The vector bundle $T_{\mathrm{hor}}P_a$ has a spin structure, with spinor bundle $\mathcal{S}_{\mathrm{hor},a}$. Furthermore, we can construct a line bundle ρ_a over P_a such that $(\rho_a)^2 = \Lambda^{n_a}\mathcal{N}_a^+$ where n_a is the rank of $\mathcal{N}_a^+$.

The index of the $(G \times H)$-transversally elliptic symbol σ_1^a is a trace-class virtual representation of $G \times H$. The action of G in $\mathrm{index}(\sigma_1^a)^H$ is a trace-class virtual representation of G. As the group K acts as a group of symmetries, this virtual representation is indeed a virtual representation of $G \times K$. Choose an orientation o_a on $T_{\mathrm{hor}}P_a$. The sign $\varepsilon(o, o_a, \Delta_a^{\mathrm{out}})$ is defined in [8, Formula (23)].

THEOREM 7. *We have the equality of trace-class virtual representations of* $G \times K$

$$\mathrm{index}(\sigma_1^a)^H = \varepsilon(o, o_a, \Delta_a^{\mathrm{out}}) \sum_{k=0}^{\infty} Q^{o_a}(P_a/H, \mathcal{L}|_{P_a} \otimes \rho_a \otimes S^k(\mathcal{N}_a^+)).$$

Let us first note that we obtain from this theorem the corollary.

COROLLARY 8. *Let μ_a be the constant value of $\mu(E)$ on P_a.*

If $\mu_a > 0$, the virtual representation $\mathrm{index}(\sigma_1^a)^H$ *of $G = S^1$ is of the form $\oplus_n a_n e^{in\theta}$ with $n \geqslant \mu_a$.*

If $\mu_a < 0$, the virtual representation $\mathrm{index}(\sigma_1^a)^H$ *of $G = S^1$ is of the form $\oplus_n a_n e^{in\theta}$ with $n \leqslant \mu_a$.*

In particular, for all $a \in \mathcal{F}$, we have $(\mathrm{index}(\sigma_1^a))^{G\times H} = 0$.

Proof. To compute the action of G on $\mathrm{index}(\sigma_1^a)^H$, it is sufficient to compute the action of the covering group $G_a \subset G \times H$ of G. As G_a acts trivially on P_a, we can employ Formula (3) to obtain the result. Indeed, by our choice of Δ_a^{out}, then for $\mu_a > 0$, all weights of $\exp \theta S_a$ on $\mathcal{L}|_{P_a} \otimes \rho_a \otimes S^k(\mathcal{N}_a^+)$ are of the form $e^{i(\mu_a+k_a)\theta}$ with $k_a > 0$. ∎

Let us prove Theorem 7. The group K acts as a group of symmetries, and all of our construction will be K-invariant. Thus, we leave implicit the action of K.

We consider the noncompact manifold equal to the total space of $\mathcal{N}_a$. It is a $(G \times H)$-manifold. The manifold $\mathcal{N}_a$ is fibered over P_a with projection p_a. Consider the infinitesimally free action of H on $\mathcal{N}_a$. Using a connection on $\mathcal{N}_a$, we can write the tangent bundle to the total space $\mathcal{N}_a$ as

$$T\mathcal{N}_a = (\mathcal{N}_a \times \mathfrak{h}) \oplus p_a^*(T_{\mathrm{hor}}P|_{P_a}).$$

Thus, we have $T_{\mathrm{hor}}\mathcal{N}_a = p_a^*(T_{\mathrm{hor}}P|_{P_a})$ and $p_a^*(\mathcal{S}_{\mathrm{hor}}|_{P_a})$ is a spin bundle for $T_{\mathrm{hor}}\mathcal{N}_a$. We still denote by $\mathcal{L}$ the line bundle $p_a^*(\mathcal{L}|_{P_a})$ on $\mathcal{N}_a$. The horizontal $\mathcal{L}$-twisted Dirac symbol on $T^*\mathcal{N}_a$ is defined for $z \in \mathcal{N}_a$ above $y \in P_a$ and $\xi \in (T^*\mathcal{N}_a)_z$ by

$$d^a(z, \xi) = c^+(h(\xi))_z \otimes I: \mathcal{S}_y^+ \otimes \mathcal{L}_y \to \mathcal{S}_y^- \otimes \mathcal{L}_y,$$

where we still denote by h the projection of $(T^*\mathcal{N}_a)_z$ on $(T_{\mathrm{hor}}^*P)_y$.

We consider the complex structure J_a on $\mathcal{N}_a$ such that the i-eigenspace of J_a on $\mathcal{N}_a \otimes_{\mathbb{R}} \mathbb{C}$ is $\mathcal{N}_a^+$. Thus, the complex vector bundle $(\mathcal{N}_a, J_a)$ is isomorphic with the vector bundle $\mathcal{N}_a^+$. The vector bundle $\mathcal{N}_a$ is a $(G \times H)$-equivariant Hermitian bundle over P_a. We consider the action of S^1 on $\mathcal{N}_a$ given by $e^{i\theta}$ on each fiber. This action of S^1 commutes with the action of $G \times H$. Consider the vector field W_a generated by this action of $e^{i\theta}$. Thus, $(W_a)_z$ is the vector $(y, -J_a \cdot v)$, if $z = (y, v)$ with $y \in P_a$ and $v \in (\mathcal{N}_a)_y$. We may use the vector field W_a to define the following $(G \times H)$-transversally elliptic symbol on $\mathcal{N}_a$:

$$d_1^a(z, \xi) = d^a(z, \xi - (W_a)_z).$$

Near P_a, the horizontal component of $W = fE$ is equal to the horizontal component of fS_a. By our choice of Δ_a^{out}, for all $t \in [0, 1]$ the map $tJ_a + (1 - t)\mu_a S_a$ has no zero eigenvalues. Thus, the action of $\mu_a S_a$ is homotopic to the transformation J_a. By homotopy arguments, we obtain

$$\operatorname{index}(\sigma_1^a) = \operatorname{index}(d_1^a).$$

Furthermore, if we consider the action of $S^1 = \{e^{i\theta}\}$ on $\mathcal{N}_a$, the symbol d_1^a is $(S^1 \times H)$-transversally elliptic. We have

$$\operatorname{index}(d_1^a) = \oplus_k \operatorname{index}(d_1^a)^k,$$

where $\operatorname{index}(d_1^a)^k$ is the isotypic component of type $e^{ik\theta}$ for the action of S^1. This series of representations of $G_a \times H$ defines a trace-class virtual representation of $G_a \times H$. We will prove the following lemma.

LEMMA 9. *The isotypic component* $\operatorname{index}(d_1^a)^k$ *is* 0 *if* $k < n_a$. *For* $k \geqslant 0$, *we have the equality of virtual representations of G:*

$$(\operatorname{index}(d_1^a)^{k+n_a})^H = \varepsilon(o, o_a, \Delta_a^{\text{out}}) Q^{o_a}(P_a/H, \mathcal{L}|_{P_a} \otimes \rho_a \otimes S^k(\mathcal{N}_a^+)).$$

Proof. We analyse the symbol d_1^a. Consider the decomposition of the H-horizontal tangent bundle

$$(T_H^* P)|_{P_a} = T_H^* P_a \oplus \mathcal{N}_a.$$

We denote by $\mathcal{S}_0$ the corresponding spin bundle for $T_H^* P_a$. Let $\mathcal{S}_1 \to P_a$ be the spin bundle for the Clifford algebra of the vector bundle $\mathcal{N}_a \to P_a$. Then

$$\mathcal{S}_{\text{hor}}|_{P_a} = \mathcal{S}_0 \otimes \mathcal{S}_1.$$

We can take as spinor space $\mathcal{S}_1$ the bundle $\rho_a^{-1} \otimes \Lambda \mathcal{N}_a$. Thus,

$$\mathcal{S}_{\text{hor}}|_{P_a} = \mathcal{S}_0 \otimes \rho_a^{-1} \otimes \Lambda \mathcal{N}_a.$$

Consider the Hermitian vector bundle $\mathcal{N}_a$. Let N_a be a Hermitian vector space of dimension n_a. We denote by U_a the unitary group of N_a. Let R_a be the principal bundle of Hermitian frames of $(\mathcal{N}_a, J_a)$ framed on N_a. Then R_a is provided with an infinitesimally free action of $H \times U_a$. The manifold $\mathcal{N}_a$ is isomorphic to $R_a \times_{U_a} N_a$. In this isomorphism, the group U_a acts on the left on N_a and on the right on R_a.

Using a connection on $R_a \to P_a$, the $(H \times U_a)$-horizontal tangent bundle on R_a is the lift of the bundle $T_H^* P_a$. Choose an orientation o_a on $T_H^* P_a$. This defines a $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{S}_0$. Denote $\mathcal{L}|_{P_a}$ by $\mathcal{L}_a$. We define the $\mathcal{L}_a \otimes \rho_a^{-1}$-twisted $(U_a \times H)$-horizontal Dirac symbol d_0^a on R_a by

$$d_0^a(r, \eta) = c_0^+(h_0(\eta)) \otimes I\colon (\mathcal{S}^0)_y^+ \otimes (\mathcal{L}_a \otimes \rho_a^{-1})_y \to (\mathcal{S}^0)_y^- \otimes (\mathcal{L}_a \otimes \rho_a^{-1})_y,$$

where r is a frame at $y \in P_a$ and $\eta \in T_r^* R_a$ projects on the H-horizontal cotangent vector $h_0(\eta) \in (T_H^* P_a)_y$.

The symbol d_0^a is a $(U_a \times H)$-transversally elliptic symbol on R_a. Thus, its index is a trace-class virtual representation of $U_a \times H$. Let τ be a finite-dimensional representation of U_a in a vector space V_τ. Let $\mathcal{V}_\tau$ be the associated vector bundle $R_a \times_{U_a} V_\tau$ on P_a. This is an H-equivariant vector bundle on P_a. By Frobenius reciprocity [1], the isotypic component of type τ^* of $\mathrm{index}(d_0^a)$ is a virtual representation of $G \times H$ equal to

$$\mathrm{index}(D^+_{\mathrm{hor}, \mathcal{L}_a \otimes \rho_a^{-1} \otimes \mathcal{V}_\tau}) \otimes V_{\tau^*}.$$

We identify T^*N_a to $N_a \oplus N_a$ and we write (z_1, ξ_1) an element of T^*N_a. Let ΛN_a be the exterior space of the Hermitian space N_a. Consider the symbol m_a on N_a given by Formula (5). Let us lift horizontally the $S^1 \times H$-transversally elliptic symbol d_1^a to $R_a \times N_a$. We then see that $\mathrm{hor}(d_1^a)$ is the external product of the symbols d_0^a and m_a up to signs. A check of orientations leads to

$$\mathrm{hor}(d_1^a) = (-1)^{n_a} \varepsilon(o, o_a, \Delta_a^{\mathrm{out}}) d_0^a \odot m_a.$$

Consider the action of $U_a \times H \times S^1$ on $R_a \times N_a$. The group H acts only on R_a, the group S^1 acts only on N_a, while the group U_a acts both on R_a and U_a. By multiplicativity property of the index, we have the equality of virtual representations of $U_a \times H \times S^1$

$$\mathrm{index}(d_0^a \odot m_a) = \mathrm{index}(d_0^a) \otimes \mathrm{index}(m_a).$$

We thus have (using Lemma 6) the following equality of virtual representations of $H \times U_a \times S^1$:

$$\mathrm{index}(\mathrm{hor}(d_1^a)) = \varepsilon(o, o_a, \Delta_a^{\mathrm{out}})(\oplus_{k=0}^{\infty} \mathrm{index}(d_0^a) \otimes (\det_{N_a} \otimes S^k(N_a)) \otimes e^{i(n_a+k)\theta}).$$

By Frobenius reciprocity, $\text{index}(d_1^a) = (\text{index}(\text{hor}(d_1^a))^{U_a}$. To compute the space of U_a-invariants in $\text{index}(\text{hor}(d_1^a))$, we must consider the isotypic component of $\text{index}(d_0^a)$ of type $\det_{V_a} \otimes S^k(N_a)$. The associated vector bundle on P_a is the vector bundle $\rho_a^2 \otimes S^k(\mathcal{N}_a^+)$. We thus obtain Lemma 9 and thus Theorem 7. ■

Corollary 3 of Theorem 7 implies the equality (1) of Proposition 5.

4. The local index and reduction. We now analyse the symbol σ_1^0 near P_0. The action of $G \times H$ is infinitesimally free on P_0. We can assume our metric on P chosen such that $\|E_P\| = 1$ on P_0. We have

$$T_{\text{hor}} P|_{P_0} = T_{\text{hor}} P_0 \oplus P_0 \times \mathfrak{g}^*.$$

Furthermore, using a connection for the infinitesimally free action of $G \times H$ on P_0, we may write

$$T_{\text{hor}} P_0 = T_0 \oplus P_0 \times \mathfrak{g},$$

where T_0 is the $(G \times H)$-horizontal space of P_0. Thus,

$$T_{\text{hor}} P|_{P_0} = T_0 \oplus P_0 \times (\mathfrak{g} \oplus \mathfrak{g}^*).$$

The spinor space for the direct orthogonal sum $\mathfrak{g} \oplus \mathfrak{g}^*$ is a two-dimensional graded vector space $S_1 = S_1^+ \oplus S_1^-$ where $S_1^\pm = \mathbb{C}$. Let $E \in \mathfrak{g}$ and $E^* \in \mathfrak{g}^*$ be the dual element. If λ, $\eta \in \mathbb{R}$, the Clifford multiplication $c_1(\lambda E + \eta E^*)$ is given by the (2×2)-matrix

$$c_1(\lambda E + \eta E^*) = \begin{pmatrix} 0 & \lambda + i\eta \\ \lambda - i\eta & 0 \end{pmatrix}.$$

Let U be the noncompact manifold $P_0 \times \mathfrak{g}^*$. The horizontal tangent bundle for the H-action on U is

$$T_{\text{hor}} U = T_0 \oplus (U \times (\mathfrak{g} \oplus \mathfrak{g}^*)).$$

Let $\mathcal{S}_1$ be the trivial bundle on U with 2-dimensional fiber S^1. Let $\mathcal{S}_0$ be the spinor bundle for T_0. The spinor space $\mathcal{S}$ for $T_{\text{hor}} U$ is thus

$$\mathcal{S} = \mathcal{S}_0 \otimes \mathcal{S}_1 .$$

We denote by c the Clifford action of an element of $T_{\text{hor}} U$ on $\mathcal{S}$.

Let $(y, f) \in P_0 \times \mathfrak{g}^*$. If $\xi \in (T_{\text{hor}} U)_{(y,f)}$, we write $\xi = \xi_0 \oplus \lambda E_U \oplus \eta E^*$ with λ, $\eta \in \mathbb{R}$ and $\xi_0 \in (T_0)_y$. Consider the symbol c on U given by

$$c((y, f), \xi) = c(\xi_0 + (\lambda E + \eta E^*)) = c_0(\xi_0) \otimes c_1(\lambda E + \eta E^*).$$

Both bundles $\mathscr{S}_0$ and $\mathscr{S}_1$ have canonical $\mathbb{Z}/2\mathbb{Z}$-gradations inherited from the symplectic structures. Consider the vector field $W = fE$ on $P_0 \times \mathfrak{g}^*$. Let $z = (y, f) \in P_0 \times \mathfrak{g}^*$. Denote by $d_1^0(z, \xi)$ the symbol on $P_0 \times \mathfrak{g}^*$ defined by

$$d_1^0(z, \xi) = c^+(z, \xi - fE_y).$$

Then we see that d_1^0 is a G-transversally elliptic symbol on $P_0 \times \mathfrak{g}^*$. A neighbourhood of P_0 in P is diffeomorphic to $P_0 \times \mathfrak{g}^*$ and by homotopy, we have $\operatorname{index}(\sigma_1^0) = \operatorname{index}(d_1^0)$.

It is clear that inside $H \times G$-transversally elliptic operator (trivial outside $f = 0$), we can deform

$$d_1^0((y, f), \xi) = c^+(\xi_0 + (-f + \lambda)E + \eta E^*) \otimes I_{\mathscr{L}_y}$$

to

$$v_1^0((y, f), \xi) = c^+(\xi_0 - fE + \eta E^*) \otimes I_{\mathscr{L}_y}.$$

Let d_0 be the $(G \times H)$-horizontal Dirac symbol twisted by the line bundle $\mathscr{L}_0$. Consider the Bott elliptic symbol $b(f, \eta) = f + i\eta$ on $\mathbb{R}$. Checking orientations, we thus see that

$$v_1^0 = d_0 \odot b.$$

As the index of b is identically equal to 1, we obtain that $\operatorname{index}(v_1^0) = \operatorname{index}(d_1^0)$. Taking the H-invariants, we obtain the equality (2) of Proposition 5. Hence, Theorem 3 is proved.

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