THE INFINITESIMAL INDEX

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Abstract. In this note, we study an invariant associated to the zeros of the moment map generated by an action form, the *infinitesimal index*. This construction will be used to study the compactly supported equivariant cohomology of the zeros of the moment map and to give formulas for the multiplicity index map of a transversally elliptic operator.

Introduction

Let G be a compact Lie group acting on a manifold N. Then G acts on the cotangent bundle $M = T^*N$ in a Hamiltonian way. The set M^0 of zeroes of the moment map $\mu : M \to \mathfrak{g}^*$ is the union of the conormals to the G-orbits in N. An element S of the equivariant K theory $K_G(M^0)$ of M^0 is called a transversally elliptic symbol, and Atiyah-Singer (see [1]) associated to S a trace class representation index(S) of G. If $\hat{G}$ is the dual of G, the representation index(S) gives rise to a function $m(\tau)$ on $\hat{G}$: index$(S) = \sum_{\tau\in\hat{G}} m(\tau)\tau$ called the multiplicity index map.

The analog of the equivariant K-theory of M^0 is the equivariant cohomology with compact support $H^*_{G,c}(M^0)$. Here we construct a map infdex^μ_G, called the infinitesimal index, associating to an element $[\alpha] \in H^*_{G,c}(M^0)$ an invariant distribution on $\mathfrak{g}^*$. We prove a certain number of functorial properties of this map, mimicking the properties of the multiplicity index map.

More generally, we consider the case when M is a G-manifold provided with a G-invariant one form σ (and we do not assume that $d\sigma$ is non degenerate). This allows us to obtain a map $\mathrm{infdex}^\mu_G : H^*_{G,c}(M^0) \to \mathcal{D}'(\mathfrak{g}^*)^G$, where M^0 is the set of zeroes of the associated moment map $\mu : M \to \mathfrak{g}^*$ and $\mathcal{D}'(\mathfrak{g}^*)^G$ the space of G-invariant distributions on $\mathfrak{g}^*$. Our construction is strongly related to Paradan's localization on M^0 of the equivariant cohomology of M (see [21]).

Let us summarize the content of this article.

In the first section, we give a "de Rham" definition of the equivariant cohomology with compact support $H^*_{G,c}(Z)$ of a topological space Z which is a closed invariant subspace of a G-manifold M: a representative of a class $[\alpha]$ is an equivariant differential form $\alpha(x)$ on M with compact support and

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such that the equivariant differential $D\alpha$ of α vanishes in a neighborhood of Z. In the appendix, we show under mild assumptions on M and Z that our space $H^*_{G,c}(Z)$ is naturally isomorphic with the (topological) equivariant cohomology of Z with compact support.

In the second section, we define the infinitesimal index. Let M be a G-manifold provided with a G-invariant one form σ (we will say that σ is an action form). If v_x is the vector field on M associated to $x \in \mathfrak{g}$, the moment map $\mu : M \to \mathfrak{g}^*$ is defined by $\mu(x) = -\langle \sigma, v_x \rangle$. Then

$$\Omega(x) = \mu(x) + d\sigma = D\sigma(x)$$

is a closed (in fact exact) equivariant form on M. The symbol D denotes in this paper the equivariant differential as defined in the Cartan model (see Formula (2).)

Our main remark is that, if f is a smooth function on $\mathfrak{g}^*$ with compact support, the double integral

$$\int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$$

is independent of s for s sufficiently large. Here $\hat{f}(x)$ is the Fourier transform of f. Some comment is in order: if $\alpha(x)$ is closed (and compactly supported) on M, it is clear that the integral $\int_M e^{is\Omega(x)}\alpha(x)$ is independent of s as $\Omega(x) = D\sigma(x)$ is an exact equivariant form. In our context, $\alpha(x)$ is compactly supported, but $\alpha(x)$ is **not closed** on M: only its restriction to a neighborhood of M^0 is closed. This is however sufficient to prove that

$$\text{(1)} \qquad \langle \mathrm{infdex}^{\mu}_G([\alpha]), f \rangle = i^{-\dim M/2} \lim_{s \to \infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$$

is a well defined map from $H^*_{G,c}(M^0)$ to invariant distributions on $\mathfrak{g}^*$.

In the third and fourth sections, we prove a certain number of functorial properties of the infinitesimal index. One important property is the free action property. Consider the situation where the compact Lie group L acts freely on M and 0 is a regular value of μ. Then the infinitesimal index of a class $[\alpha]$ is a polynomial density on $\mathfrak{l}^*$. Its value at 0 is the integral of the cohomology class corresponding to $[\alpha]$ by the Kirwan map over the reduced space $\mu^{-1}(0)/L$. This is essentially Witten non abelian localization theorem [22]. We give also the double equivariant version, where a compact Lie group G acts on M commuting with the free action of L.

Let us comment on previous work around this theme.

As we said, the use of the form $e^{isD\sigma}$ to "localize" integrals is the main principle in Witten non abelian localization theorem [22], [11], and our definition of the infinitesimal index is strongly inspired by this principle.

P.-E. Paradan has studied systematically the situation of a manifold M provided with a G-invariant action form σ. Indeed, he constructed in [17] a closed equivariant form P_σ on M, congruent to 1 in cohomology and supported near M^0. Paradan's form P_σ is constructed using equivariant cohomology with $C^{-\infty}$-coefficients. Multiplying $\alpha(x)$ by Paradan's form $P_\sigma(x)$ leads to a closed compactly supported equivariant form on M and $i^{\dim M/2}I(x) := \int_M P_\sigma(x)\alpha(x)$ is a generalized function on $\mathfrak{g}$. As we explain in Remark 3.8, our infinitesimal index is the Fourier transform of $I(x)$. Properties of the infinitesimal index could thus be deduced by Fourier transform from the functorial properties of P_σ proven in [17], [19]. We choose here to prove directly our properties by using our limit definition.

In a future article, we will use the infinitesimal index map to describe the multiplicity index map of a transversally elliptic operator.

In the case where $M = N \oplus N^*$, where N is a representation space for a linear action of a torus G , we have used the infinitesimal index to identify $H^*_{G,c}(M^0)$ to a space of spline distributions on $\mathfrak{g}^*$, in a companion article [8] to the article [7], where we determined $K_G(M^0)$ as a space of functions on $\hat{G}$.

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1. Equivariant de Rham cohomology

Let M be a C^∞ manifold with a C^∞ action of a compact Lie group G, we are going to define its equivariant cohomology with compact support following Cartan definition (see [10]).

We define the space of compactly supported equivariant forms as

$$\mathcal{A}_{G,c}(M) = (S(\mathfrak{g}^*) \otimes \mathcal{A}_c(M))^G$$

with the grading given setting $\mathfrak{g}^*$ in degree 2. Here $\mathcal{A}_c(M)$ is the algebra of differential forms on M with compact support.

Each element $x \in \mathfrak{g}$ of the Lie algebra of G induces a vector field v_x on M, the *infinitesimal generator* of the action: here the sign convention is that $v_x = \frac{d}{d\epsilon}\exp(-\epsilon x)\cdot m$ in order that the map $x \to v_x$ is a Lie algebra homomorphism. A vector field V on M induces a derivation ι_V on forms, such that $\iota_V(df) = V(f)$ and for simplicity we denote by $\iota_x = \iota_{v_x}$.

One defines the differential as follows. Given $\alpha \in \mathcal{A}_{G,c}(M)$, we think of α as an equivariant polynomial map on $\mathfrak{g}$ with values in $\mathcal{A}_c(M)$, thus for any $x \in \mathfrak{g}$ we set

$$D\alpha(x) := d(\alpha(x)) - \iota_x(\alpha(x)) \tag{2}$$

where d is the usual de Rham differential.

It is easy to see that D increases the degree by one and that $D^2 = 0$. Thus we can take cohomology and we get the G-equivariant cohomology of M with compact support.

Now take a G-stable closed set Z in a manifold M. Consider the open set $U = M \setminus Z$. Then U is a manifold and we have an inclusion of complexes $\mathcal{A}_{G,c}(U) \subset \mathcal{A}_{G,c}(M)$ given by extension by zero. We set

$$\mathcal{A}_{G,c}(Z, M) := \mathcal{A}_{G,c}(M)/\mathcal{A}_{G,c}(U).$$

Definition 1.1. The equivariant de Rham cohomology with compact support $H^*_{G,c}(Z)$ is the cohomology of the complex $\mathcal{A}_{G,c}(Z,M)$.

Notice that $\mathcal{A}_{G,c}(U)$ is an ideal in $\mathcal{A}_{G,c}(M)$ so $\mathcal{A}_{G,c}(Z,M)$ is a differential graded algebra and $H^*_{G,c}(Z)$ is a graded algebra (without 1 if Z is not compact).

In this model, a representative of a class in $H^*_{G,c}(Z)$ is an equivariant form $\alpha(x)$ with **compact support** on M. The form α is not necessary equivariantly closed on M, but there exists a neighborhood of Z such that the restriction of $\alpha(x)$ to this neighborhood is equivariantly closed.

If Z is compact, the class 1 belongs to $H^*_{G,c}(Z)$: a representative of 1 is a G-invariant function χ on M with compact support and identically equal to 1 on a neighborhood of Z in M.

Remark 1.2. Our model for $H^*_{G,c}(Z)$ seems to depend of the ambient manifold M. However, in the appendix we are going to see that under mild assumptions on M and Z, $H^*_{G,c}(Z)$ is naturally isomorphic with the equivariant singular cohomology of Z with compact support.

By the very definition of $H^*_{G,c}(Z)$, we also deduce

Proposition 1.3. *Let M be a G-space, $Z \subset X$ a closed G-stable subset (denote by $j : Z \to M$ the inclusion). Set $U = M \setminus Z$ (denote by $i : U \to M$ the inclusion). We have a long exact sequence*

(3)

$$\cdots \to H^h_{G,c}(U) \xrightarrow{i_*} H^h_{G,c}(M) \xrightarrow{j^*} H^h_{G,c}(Z) \longrightarrow H^{h+1}_{G,c}(U) \to \cdots .$$

If $i : Z \to M$ is a closed G-submanifold of a manifold M, the restriction of forms gives rise to a well defined map $i^* : \mathcal{A}_{G,c}(Z,M) \to \mathcal{A}_{G,c}(Z)$.

Proposition 1.4. *If Z is a closed G-invariant submanifold of a manifold M admitting an equivariant tubular neighborhood, the map i^* induces an isomorphism in cohomology.*

Proof. We reduce to the case in which M is a vector bundle on Z by restriction to a tubular neighborhood. Put a G-invariant metric on this bundle and let $p : M \to Z$ be the projection. Choose a C^∞ function f on $\mathbb{R}$ with compact support and equal to 1 in a neighborhood of 0. We map an equivariant form $\omega \in \mathcal{A}_{G,c}(Z)$ to $f(\|m\|^2)p^*\omega(m)$ and then to its class modulo $\mathcal{A}_{G,c}(U)$. It is easily seen that this map is an inverse in cohomology of the map i^*. □

Assume that M is a $L \times G$ manifold and that L acts freely on M. Let Z be a $G \times L$ closed subset of M. Denote by $p : M \to M/L$ the projection. The pull back of forms on M/L induces a map from $p^* : H^*_{G,c}(Z/L) \to H^*_{L\times G,c}(Z)$. The following proposition is proven as in Cartan (see [10] or [9]).

Proposition 1.5. *The pull back*

$$p^* : H^*_{G,c}(Z/L) \to H^*_{L\times G,c}(Z)$$

is an isomorphism

Proof. The fact that the pull back of forms induces an isomorphism between $H^*_{G,c}(M/L)$ and $H^*_{L\times G,c}(M)$, and between $H^*_{G,c}((M\setminus Z)/L)$ and $H^*_{L\times G,c}(M\setminus Z)$, is proven as in Cartan (see [10] or [9]). Our statement then follows from Proposition 1.3. □

2. Basic definitions

2.1. **Action form and the moment map.** Let G be a Lie group and M a G-manifold.

Definition 2.2. An *action form* is a G-invariant real one form σ on M.

The prime examples of this setting are when M is even dimensional and $d\sigma$ is non degenerate. In this case $d\sigma$ defines a symplectic structure on M.

Example 2.3. For every manifold N, we may take its cotangent bundle $M := T^*N$ with projection $\pi : T^*N \to N$. The canonical action form σ on a tangent vector v at a point (n, ϕ), $n \in N, \phi \in T^*_nN$ is given by

$$\langle \sigma \,|\, v\rangle := \langle \phi \,|\, d\pi(v)\rangle.$$

In this setting, $d\sigma$ is a canonical symplectic structure on T^*N and, if $r = \dim(N)$, the form $\frac{d\sigma^r}{r!}$ determines an orientation and a measure, the *Liouville measure* on T^*N. If a group G acts on N, then it acts also on T^*N preserving the canonical action form and hence the symplectic structure and the Liouville measure.

Remark 2.4. If M is a manifold with a G–invariant Riemannian structure, we can consider an invariant vector field instead of a 1–form.

Definition 2.5. Given an action form σ we define the *moment map* $\mu_\sigma : M \to \mathfrak{g}^*$ associated to σ by:

$$\mu_\sigma(m)(x) := -\langle \sigma \,|\, v_x\rangle(m) = -\iota_x(\sigma)(m). \tag{4}$$

for $m \in M$, $x \in \mathfrak{g}$.

Remark 2.6. Due to our sign convention for v_x, we have

$$\mu(m)(x) := \langle \sigma \,|\, \frac{d}{d\epsilon}\exp(\epsilon x)\cdot m\rangle.$$

The moment map is a G–equivariant map, where on $\mathfrak{g}^*$ we have the *coadjoint action.*

The form $d\sigma$ is a closed 2-form on M. Then $D\sigma(x) = \mu(x) + d\sigma$ is a closed (in fact exact) equivariant form on M.

Let us present a few examples.

Example 2.7. In example 2.3 take $N = S^1 = \{e^{2\pi i\theta}\}$. The form $d\theta$ gives a trivialization $T^*S^1 = S^1 \times \mathbb{R}$. The vector field $\frac{\partial}{\partial\theta}$ gives a canonical generator of the Lie algebra of S^1 and $d\theta$, a generator for the dual. The circle group S^1 acts freely by rotations on itself. If $[e^{2\pi i\theta}, t]$ is a point of T^*S^1 with $t \in \mathbb{R}$, the action form σ is $\sigma = td\theta$. The function t is the moment map and $dt \wedge d\theta$ the symplectic form.

More generally, take $N = G$ a Lie group. Denote by

$$L_g : h \mapsto gh,\ \ R_g : h \mapsto hg^{-1}$$

the left and right actions of G on G and by extension also on T^*G. Let us now trivialize $T^*G = G \times \mathfrak{g}^*$ using left invariant forms.

Call $\pi : T^*G \to G$ the canonical projection. Fix a basis $\psi_1, \dots, \psi_r$ of left invariant 1–forms on G so that a point of $T^*G = G \times \mathfrak{g}^*$ is a pair $(g, \zeta) = (g, \sum_i \zeta_i\psi_i)$. Clearly the action form is $\sigma = \sum_i \zeta_i\pi^*(\psi_i)$, the symplectic form is $\sum_i d\zeta_i \wedge \pi^*(\psi_i) + \sum_i \zeta_i\pi^*(d\psi_i)$. In the non commutative case, in general $d\psi_i \neq 0$, nevertheless when we compute the Liouville form we immediately see that these terms disappear and

$$\frac{d\sigma^r}{r!} = d\zeta_1 \wedge \pi^*(\psi_1) \wedge \cdots \wedge d\zeta_r \wedge \pi^*(\psi_r). \tag{5}$$

We can rewrite this as

$$\frac{d\sigma^r}{r!} = (-1)^{\frac{r(r+1)}{2}} d\zeta \wedge V_\psi \tag{6}$$

where we set $V_\psi := \psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_r$ and $d\zeta := d\zeta_1 \wedge \cdots \wedge d\zeta_r$. At this point it is clear that V_ψ gives a Haar measure on G while $d\zeta$ gives a translation invariant measure on $\mathfrak{g}^*$.

Let us call μ_ℓ, μ_r the moment maps for the left or right action of G respectively.

We denote by Ad_g the adjoint action and by Ad_g^* the coadjoint action. By definition

$$\langle \mathrm{Ad}_g^*(\phi) \,|\, x\rangle = \langle \phi \,|\, \mathrm{Ad}_{g^{-1}} x\rangle.$$

Then, we have

Proposition 2.8.

$$\mu_\ell(g, \psi) = \mathrm{Ad}_g^*(\psi), \quad \mu_r(g, \psi) = -\psi, \quad \textit{left trivialization.} \tag{7}$$

Proof. This follows from the formula $\sigma = \sum_i \zeta_i\pi^*(\psi_i)$. The vector fields generating the right action are the left invariant vector fields so, for this action, the formula follows from (4). On the other hand multiplication on the left by g on right invariant vector fields is the adjoint action. Thus the pairing between right invariant vector fields and left invariant forms is given by the previous formula. □

If we had used right invariant forms in order to trivialize the bundle, we would have

$$\mu_r(g, \psi) = -\mathrm{Ad}_g^*(\psi), \quad \mu_\ell(g, \psi) = \psi, \quad \text{right trivialization.} \tag{8}$$

Example 2.9. We get another example in the case of a symplectic vector space V with antisymmetric form B. Then $\sigma = \frac{1}{2}B(v, dv)$ is a 1- form on V invariant under the action of the symplectic group G so that $d\sigma = \frac{1}{2}B(dv, dv)$ is a symplectic two form on V. The moment map $\mu : V \to \mathfrak{g}^*$ is given by $\mu(v)(x) = \frac{1}{2}B(v, xv)$.

For example, let $M := \mathbb{R}^2$ with coordinates $v := [v_1, v_2]$, $B = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$. The action form σ is $\frac{1}{2}(v_1 dv_2 - v_2 dv_1)$ and $d\sigma = dv_1 \wedge dv_2$. The compact part of the symplectic group is the circle group S^1 acting by rotations. The moment map is $\frac{v_1^2+v_2^2}{2}$.

Remark 2.10. Given a vector space N, the space $N \oplus N^*$ has a canonical symplectic structure given by

$$\langle (u, \phi) \,|\, (v, \psi) \rangle := \langle \phi \,|\, v \rangle - \langle \psi \,|\, u \rangle. \tag{9}$$

The symplectic structure $d\sigma$ on the cotangent bundle to a vector space N gives a symplectic structure B to the vector space $T^*N = N \oplus N^*$.

The action form σ coming from the cotangent structure is not the same than the action form on $N \oplus N^*$ given by duality (9) (in case $V = \mathbb{R}$, ydx versus $\frac{1}{2}(ydx - xdy)$), but the moment map relative to the subgroup $GL(N)$ acting by $(gn, {}^t g^{-1}\phi)$ is the same, as well as $d\sigma$.

2.11. **The cohomology groups $\mathcal{H}^\infty_{G,c}(M)$.** We will need to extend the notion of equivariant cohomology groups. Consider the space $C^\infty(\mathfrak{g})$ of C^∞ functions on $\mathfrak{g}$. We may consider the $\mathbb{Z}/2\mathbb{Z}$-graded spaces $\mathcal{A}^\infty_G(M)$ (or $\mathcal{A}^\infty_{G,c}(M)$ consisting of the G-equivariant C^∞ maps from $\mathfrak{g}$ to $\mathcal{A}(M)$ (or to $\mathcal{A}_c(M)$). The equivariant differential D is well defined on $\mathcal{A}^\infty_G(M)$ (or on $\mathcal{A}^\infty_{G,c}(M)$) and takes even forms to odd forms and vice versa. Thus we get the cohomology groups $\mathcal{H}^\infty_G(M)$, $\mathcal{H}^\infty_{G,c}(M)$. The group $\mathcal{H}^\infty_{G,c}(M)$ is a module over $\mathcal{H}^\infty_G(M)$, and in particular on $C^\infty(\mathfrak{g})^G = \mathcal{H}^\infty_G(pt)$.

Proceeding as in the previous case, we may define for any G-stable closed subspace Z of M (this may depend on the embedding) the cohomology groups $\mathcal{H}^\infty_{G,c}(Z)$. An element in $\mathcal{H}^\infty_{G,c}(Z)$ is thus represented by an element in $\mathcal{A}^\infty_{G,c}(M)$ whose boundary has support in $M \setminus Z$. We have a natural map $H^*_{G,c}(Z) \to \mathcal{H}^\infty_{G,c}(Z)$.

In order to take Fourier transforms, we will need to use yet another space.

Consider the space $\mathcal{P}^\infty(\mathfrak{g})$ of C^∞ functions on $\mathfrak{g}$ with at most polynomial growth. We may consider the spaces $\mathcal{A}^{\infty,m}_{G,c}(M)$ consisting of the G-equivariant C^∞ maps with at most polynomial growth from $\mathfrak{g}$ to $\mathcal{A}_c(M)$. The index m indicates the moderate growth on $\mathfrak{g}$ of the coefficients. We get the cohomology groups $\mathcal{H}^{\infty,m}_{G,c}(M)$. This new cohomology has $\mathcal{H}^{\infty,m}_G(pt) = \mathcal{P}^\infty(\mathfrak{g})^G$ and is a module over $\mathcal{P}^\infty(\mathfrak{g})^G$. We may define in the same way the groups $\mathcal{H}^{\infty,m}_{G,c}(Z)$ of cohomology with compact support, and with coefficients of at most polynomial growth, for any G-stable closed subspace Z of M.

2.12. **Connection forms.** We shall use a fundamental notion in Cartan's theory of equivariant cohomology. Let us recall

Definition 2.13. Given a free action of a compact Lie group L on a manifold P, a *connection form* is a L-invariant one form $\omega \in \mathcal{A}^1(P)\otimes\mathfrak{l}$ with coefficients in the Lie algebra of L such that $-\iota_x\omega = x$ for all $x \in \mathfrak{l}$.

If on P with free L action we also have a commuting action of another group G, it is easy to see that there exists a $G \times L$ invariant connection form $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{l}$ on P for the free action of L.

Let $M = P/L$. Define the curvature R and the G-equivariant curvature R_y of the bundle $P \to M$ by

$$R := d\omega + \frac{1}{2}[\omega, \omega], \quad R_y := -i_y\omega + R. \tag{10}$$

Example 2.14. Consider $L = G$ and $P = G$ with left and right action. A connection form for the right action can be constructed as follows. Each element x of the Lie algebra of G defines the vector field v_x by right action. These are left invariant vector fields. Given a basis $e_1, \dots, e_r$ of $\mathfrak{g}$, set $v_i := v_{e_i}$. This determines a dual basis and correspondingly left invariant forms ω_i with $i_{v_j}(\omega_i) = \langle \omega_i \,|\, v_j \rangle = \delta^i_j$ so that $\sum_i \omega_i e_i$ is a connection form for the right action.

This form is also left invariant and $R = 0$, so by (10) the equivariant curvature is $-i_y\omega$ where now i_y is associated to the left action. We then have

$$R_y(g) = -\sum_i i_y(\omega_i)(g)e_i = -(Ad_g)^{-1}y. \tag{11}$$

The equivariant Chern-Weil homomorphism ([4],[6], see [5]) associates to any L invariant smooth function a on $\mathfrak{l}$ a closed G-equivariant form, with C^∞-coefficients as in §2.11, denoted by $y \to a(R_y)$, on $M = P/L$.

The formula for this form is obtained via the Taylor series of the function a as follows. Choose a basis $e_j,\ j = 1, \dots, r$ of $\mathfrak{l}$ and write $R = \sum_j R_j \otimes e_j$. For a multi–index $I := (i_1, \dots, i_r)$, denote by $R^I := \prod_{j=1}^r R_j^{i_j}$. Then, given a point $p \in P$, we set

Definition 2.15.

$$a(R_y)(p) := a(-i_y\omega + R) = a(-\iota_y\omega(p)) + \sum_I R^I(\partial_I a)(-\iota_y\omega(p)) \tag{12}$$

which is a finite sum since R is a nilpotent element.

One easily verifies that this is independent of the chosen basis. Moreover one can prove (as in the construction of ordinary characteristic classes) the following proposition.

Proposition 2.16. *([4],[6], see [5]) The differential form $a(R_y)$ is the pull back of a G-equivariant closed form (still denoted by $a(R_y)$) on $M = P/L$. Its cohomology class in $\mathcal{H}_G^\infty(M)$ is independent of the choice of the connection.*

3. Definition of the infinitesimal index

3.1. **Infinitesimal index.** As before, consider a compact Lie group G and a G-manifold M equipped with an action form σ. We assume M oriented. Let $\mu := \mu_\sigma : M \to \mathfrak{g}^*$ be the corresponding moment map given by (4).

Set
$$M_G^0 := \mu^{-1}(0), \qquad U := M \setminus M_G^0.$$
We simply denote M_G^0 by M^0 when the group G is fixed.

Consider the equivariant form
$$\Omega := d\sigma + \mu = D\sigma.$$

Let $\mathcal{D}'(\mathfrak{g}^*)$ be the space of distributions on $\mathfrak{g}^*$. It is a $S[\mathfrak{g}^*]$-module where $\mathfrak{g}^*$ acts as derivatives. When G is non commutative, we need to work with the space $\mathcal{D}'(\mathfrak{g}^*)^G$ of G-invariant distributions.

By Lemma 1.1, a representative of a class $[\alpha] \in H^*_{G,c}(M^0)$ is a form $\alpha \in [S(\mathfrak{g}^*) \otimes \mathcal{A}_c(M)]^G$ such that $D\alpha$ is compactly supported in U.

Let us define a map called the infinitesimal index
$$\mathrm{infdex}_G^\sigma : H^*_{G,c}(M^0) \to \mathcal{D}'(\mathfrak{g}^*)^G$$
as follows.

We fix a Euclidean structure on $\mathfrak{g}^*$ which induces a translation invariant Lebesgue measure $d\xi$. We choose a square root i of -1 and define the Fourier transform:
$$\hat{f}(x) := \int_{\mathfrak{g}^*} e^{-i\langle \xi \,|\, x\rangle} f(\xi) d\xi.$$
We normalize dx on $\mathfrak{g}$ so that the inverse Fourier transform is
$$f(\xi) = \int_{\mathfrak{g}} e^{i\langle \xi \,|\, x\rangle} \hat{f}(x) dx. \tag{13}$$

The measure $dx d\xi$ is independent of the choice of $d\xi$.

Let $f(\xi)$ be a C^∞ function on $\mathfrak{g}^*$ with compact support in a ball B_R of radius R in $\mathfrak{g}^*$ and $\hat{f}(x)$ its Fourier transform, a rapidly decreasing function on $\mathfrak{g}$.

Consider the differential form on M depending on a parameter s:
$$\Psi(s, \alpha, f) = \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx,$$
and define (choosing a square root of i)
$$\langle \mathrm{infdex}(s, \alpha, \sigma), f\rangle := i^{-\dim M/2} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx \tag{14}$$

$$= i^{-\dim M/2}\int_M \Psi(s,\alpha,f).$$

This double integral on $M \times \mathfrak{g}$ is absolutely convergent, since α is compactly supported on M and depends polynomially on x while $\hat{f}(x)$ is rapidly decreasing.

More precisely, write $\alpha(x) = \sum_a P_a(x)\alpha^a$ with α^a compactly supported forms on M and $P_a(x)$ polynomial functions of x. Then

$$\Psi(s,\alpha,f)(m) = \sum_a \Big[\int_{\mathfrak{g}} \hat{f}(x)P_a(x)e^{is\langle\mu(m),x\rangle}dx\Big]e^{isd\sigma}\alpha^a.$$

By Fourier inversion (as in (13))

$$\int_{\mathfrak{g}} \hat{f}(x)P_a(x)e^{is\langle\mu(m),x\rangle}dx = (P_a(-i\partial)f)(s\mu(m)), \tag{15}$$

thus

$$\Psi(s,\alpha,f) = \sum_a ((P_a(-i\partial)f)\circ(s\mu))e^{isd\sigma}\alpha^a. \tag{16}$$

In particular, remark that $\Psi(s,\alpha,f)$ does not depend of the choice of $d\xi$. Another consequence of this analysis is

Proposition 3.2. *Let $K \subset M$ be the support of α and $C \subset \mathfrak{g}^*$ the support of f. If $s\mu(K)\cap C = \emptyset$, then $\Psi(s,\alpha,f) = 0$.*

Given $s > 0$, set $V_s = \mu^{-1}(B_{R/s})$. We can then choose some $s_0 >> 0$ so large that the restriction of α to the small neighborhood V_{s_0} of M^0 is equivariantly closed. This is possible since $D\alpha$ has a compact support K in $U = M \setminus M^0$ so that $\rho := \min_{m\in K}\|\mu(m)\| > 0$ and it suffices to take $s_0 > R/\rho$.

We have $(P_a(-i\partial)f)(s\mu(m)) = 0$ if $\|s\mu(m)\| > R \iff \|\mu(m)\| > R/s$. Thus we see that, for $s \geq s_0$, if K is the support of α, $\Psi(s,\alpha,f)$ has compact support contained in $V_s \cap K$.

We have then the formula:

$$i^{\dim M/2}\langle \mathrm{infdex}(s,\alpha,\sigma), f\rangle = \int_M \Psi(s,\alpha,f) = \int_{V_s}\Psi(s,\alpha,f). \tag{17}$$

Note that from Formula (17) follows the

Lemma 3.3. *If α has support in U then, for s large, $\Psi(s,\alpha,f) = 0$.*

We will often make use of the following lemma.

Lemma 3.4. *We have*

$$-i\frac{d}{ds}\int_M\int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx = \int_M\int_{\mathfrak{g}} \sigma e^{is\Omega(x)}D(\alpha)(x)\hat{f}(x)dx.$$

Proof. Indeed, since $\Omega(x) = D\sigma(x)$,

$$-i\frac{d}{ds}\int_M\int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx = \int_M\int_{\mathfrak{g}} D\sigma(x)e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx$$
$$= \nu + r$$

with

$$\nu = \int_{\mathfrak{g}}\int_M D\left(\sigma e^{is\Omega(x)}\alpha(x)\right)\hat{f}(x)dx$$

and

$$r = \int_M\int_{\mathfrak{g}} \sigma e^{is\Omega(x)} D(\alpha)(x)\hat{f}(x)dx$$

since D is a derivation, $D(\Omega) = 0$ and hence $D(e^{is\Omega(x)}) = 0$.

As $\alpha(x)$ is compactly supported, $\nu = 0$, and we obtain the lemma. □

Let us see that

$$\langle \text{infdex}(s,\alpha,\sigma), f\rangle = i^{-\dim M/2}\int_M\int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx$$

does not depend of the choice of $s \geq s_0$.

We use Lemma 3.4 above to compute $\frac{d}{ds}\langle \text{infdex}(s,\alpha,\sigma), f\rangle$. By the hypotheses made the form $\sigma D\alpha$ has compact support in U, thus by Lemma 3.3 the differential form $\Psi(s,\sigma D\alpha, f) = \int_{\mathfrak{g}} \sigma e^{is\Omega(x)}D(\alpha)(x)\hat{f}(x)dx$ is identically equal to 0 for $s \geq s_0$. This implies that for $s \geq s_0$

$$\frac{d}{ds}\langle \text{infdex}(s,\alpha,\sigma), f\rangle = 0$$

hence the independence of the choice of $s \geq s_0$.

We now see the independence on the choice of the representative α. In fact, take a different representative $\alpha + \beta$ with β compactly supported on U, then

$$\lim_{s\to\infty} i^{\dim M/2}\langle \text{infdex}(s,\beta,\sigma), f\rangle = 0$$

by Lemma 3.3.

Next let us show that $\lim_{s\to\infty} i^{\dim M/2}\langle \text{infdex}(s,\alpha,\sigma), f\rangle$ depends only on the cohomology class of α. Take $\alpha = D\beta$, with β compactly supported on M,

$$i^{\dim M/2}\langle \text{infdex}(s,\alpha,\sigma), f\rangle = \int_M\int_{\mathfrak{g}} e^{is\Omega(x)}D\beta(x)\hat{f}(x)dx$$
$$= \int_{\mathfrak{g}}\int_M D\left(e^{is\Omega(x)}\beta(x)\right)\hat{f}(x)dx = 0.$$

Finally, let us consider two action forms σ_1, σ_0, with $\sigma_0 = \sigma$. Then the moment map for $\sigma_t = t\sigma_1 + (1-t)\sigma_0$ is $\mu_t = t\mu_1 + (1-t)\mu_0$, with $\mu_0 = \mu$. We assume that the closed set $\mu_t^{-1}(0)$ remains equal to M^0, for $t \in [0,1]$. Let us see that $\text{infdex}(s,\alpha,\sigma_1) = \text{infdex}(s,\alpha,\sigma_0)$, for s large.

Indeed, consider $\Omega(t) = D\sigma_t$. Let

$$I(t,s) = \int_M \int_{\mathfrak{g}} e^{is\Omega(t,x)}\alpha(x)\hat{f}(x)dx.$$

We obtain

$$-i\frac{d}{dt}I(t,s) = s\int_M \int_{\mathfrak{g}} D(\sigma_1 - \sigma_0)(x)e^{is\Omega(t,x)}\alpha(x)\hat{f}(x)dx$$
$$= \nu + r$$

with

$$\nu = s\int_{\mathfrak{g}} \int_M D\left((\sigma_1 - \sigma_0)e^{is\Omega(t,x)}\alpha(x)\right)\hat{f}(x)dx$$

and

$$r = s\int_M \int_{\mathfrak{g}} (\sigma_1 - \sigma_0)e^{is\Omega(t,x)}D(\alpha)(x)\hat{f}(x)dx.$$

As $\alpha(x)$ is compactly supported, $\nu = 0$.

As for r, we remark that $\Omega(t,x) = \langle \mu_t, x\rangle + q(t)$ where $q(t)$ is a two form. The integral r involves the value of f, and its derivatives, at the points $s\mu_t(m)$. As the compact support K of $D\alpha$ is disjoint from M^0, our assumption implies that $\mu_t(m)$ is never equal to 0 for $m \in K$ and $t \in [0,1]$. Thus $\rho := \min_{m\in K, t\in[0,1]} \|\mu_t(m)\| > 0$ and, arguing as for Formulas (15) and (16) we deduce that $r = 0$ if we take $s_0 > R/\rho$.

One has still to verify that this linear map satisfies the continuity properties that make it a distribution. We leave this to the reader.

In conclusion we have shown

Theorem 3.5. *Let σ be an action form with moment map μ. Let $M^0 = \mu^{-1}(0)$. Then we can define a map*

$$\mathrm{infdex}_G^\sigma : H^*_{G,c}(M^0) \to \mathcal{D}'(\mathfrak{g}^*)^G$$

setting for any $[\alpha] \in H^*_{G,c}(M^0)$ and for any smooth function with compact support f on $\mathfrak{g}^*$

$$\langle \mathrm{infdex}_G^\sigma([\alpha]), f\rangle := i^{-\dim M/2} \lim_{s\to\infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx.$$

The map infdex_G^σ *is a well defined homomorphism of $S[\mathfrak{g}^*]^G$ modules.*

If the one form σ moves along a smooth curve σ_t with moment map μ_t such that $\mu_t^{-1}(0)$ remains equal to M^0 , then

$$\mathrm{infdex}_G^{\sigma_t} = \mathrm{infdex}_G^\sigma.$$

In particular, if two action forms σ_1, σ_2 have same moment map μ, the two infinitesimal indices $\mathrm{infdex}_G^{\sigma_1}$ and $\mathrm{infdex}_G^{\sigma_2}$ coincide. Indeed, the moment map μ_t associated to $(1-t)\sigma_1 + t\sigma_2$ is constant. In view of this property, we denote by infdex_G^μ the infinitesimal index map associated to σ.

Remark 3.6. In general, the maps infdex_G^{μ} and $\mathrm{infdex}_G^{-\mu}$ are different (cf. Exaple 3.14), although the zeroes of the moment maps associated to σ and $-\sigma$ are the same. Thus the stability condition that the set $\mu_t^{-1}(0)$ remains constant, when moving along σ_t, is essential in order to insure the independence of the infinitesimal index.

Let us give another formula for infdex_G^{μ}. On this formula, it will be clear that infdex_G^{μ} belongs to the space $\mathcal{S}'(\mathfrak{g}^*)^G$ of invariant tempered distributions on $\mathfrak{g}^*$.

Let f be a Schwartz function on $\mathfrak{g}^*$. If α is a representative of $[\alpha] \in H^*_{G,c}(M^0)$, we see that $\int_{\mathfrak{g}} e^{is\Omega(x)}(D\alpha)(x)\hat{f}(x)dx$ is a rapidly decreasing function of s: $D\alpha$ being identically equal to 0 on a neighborhood of M^0, this is expressed in terms of the value of the function f, and its derivatives, at points $s\mu(m)$, where $\mu(m)$ is non zero. Thus we can define the compactly supported differential form $\Phi(\alpha, f)$ on M by

$$\Phi(\alpha, f) := \int_{\mathfrak{g}} \alpha(x)\hat{f}(x)dx + i\sigma \int_{s=0}^{\infty} \left(\int_{\mathfrak{g}} e^{is\Omega(x)} D\alpha(x)\hat{f}(x)dx \right) ds. \tag{18}$$

Proposition 3.7. *We have*

$$\langle \mathrm{infdex}_G^{\mu}(\alpha), f\rangle = i^{-\dim M/2} \int_M \Phi(\alpha, f).$$

Proof. Let f be a function with compact support on $\mathfrak{g}^*$. Then

$$\lim_{s\to\infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x)\hat{f}(x)dx$$

is equal to

$$\int_M \int_{\mathfrak{g}} \alpha(x)\hat{f}(x)dx + \int_0^{\infty} \frac{d}{ds} \left(\int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x)\hat{f}(x)dx \right) ds.$$

By Lemma 3.4, we obtain the proposition. □

Remark 3.8. It is possible to define equivariant forms on M with $C^{-\infty}$ coefficients [13]. Such a form is an equivariant map from test densities on $\mathfrak{g}$ to differential forms on M. The equivariant differential D extends and we obtain the group $\mathcal{H}_G^{-\infty}(M)$, and similarly the group $\mathcal{H}_{G,c}^{-\infty}(M)$. If $\alpha \in H^*_{G,c}(M^0)$, and g is a test function on $\mathfrak{g}$, we may define the differential form

$$(p(\alpha), g dx) = \int_{\mathfrak{g}} \alpha(x)g(x)dx + i\sigma \int_{s=0}^{\infty} (\int_{\mathfrak{g}} e^{is\Omega(x)} D\alpha(x)g(x)dx)ds.$$

It is easy to see that $p(\alpha)$ is a compactly supported equivariant form on M with $C^{-\infty}$ coefficients such that $D(p(\alpha)) = 0$. Indeed, we have $p(\alpha) = \alpha - \sigma \frac{D\alpha}{D\sigma}$, where $\frac{D\alpha}{D\sigma}$ is well defined in the distribution sense by $-i \int_{s=0}^{\infty} e^{isD\sigma} D\alpha \, ds$.

We see that $\alpha \mapsto p(\alpha)$ defines a map from $H^*_{G,c}(M^0)$ to $\mathcal{H}^{-\infty}_{G,c}(M)$. In this framework, our distribution $i^{\dim M/2}\mathrm{infdex}^\mu_G(\alpha)$ on $\mathfrak{g}^*$ is the Fourier transform of the generalized function $\int_M p(\alpha)$ on $\mathfrak{g}$.

Associated to an action form σ, Paradan defined a particular element $P_\sigma \in \mathcal{H}^{-\infty}_G(M)$ representing 1 and supported in a neighborhood of M^0 [16]. This element is the form $p(1)$ defined above (when M^0 is compact). Most of our subsequent theorems could be obtained by Fourier transforms of Theorems proven in [17], [19] where basic functorial properties of P_σ are proved. However, we will work on $\mathfrak{g}^*$ instead that on $\mathfrak{g}$ and we will give direct proofs.

3.9. **Extension of the definition of the infinitesimal index.** Let us see that the definition of the infinitesimal index extends to $\mathcal{H}^{\infty,m}_{G,c}(M^0)$.

If $\alpha \in \mathcal{A}^{\infty,m}_{G,c}(M)$ is such that $D\alpha = 0$ in a neighborhood of M^0, we see that Lemma 3.4 still holds, f being a Schwartz function on $\mathfrak{g}^*$:

$$-i\frac{d}{ds}\int_M\int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx = \int_M\int_{\mathfrak{g}} e^{is\Omega(x)}\sigma D\alpha(x)\hat{f}(x)dx.$$

Since α is of at most polynomial growth, the function of x given by $D\alpha(x)\hat{f}(x)$ is still a Schwartz function of x. Thus by Fourier inversion, we again see that $-i\frac{d}{ds}\int_M\int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx$ is a rapidly decreasing function of s and we may define

$$\langle \mathrm{infdex}^\mu_G(\alpha), f\rangle = i^{-\dim M/2}\lim_{s\to\infty}\int_M\int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx.$$

We have again the formula:

$$\langle \mathrm{infdex}^\mu_G(\alpha), f\rangle = i^{-\dim M/2}\int_M \Phi(\alpha, f)$$

where $\Phi(\alpha, f)$ is given by Equation (18).

This formula shows that $\mathrm{infdex}^\mu_G(\alpha)$ is a G-invariant tempered distribution on $\mathfrak{g}^*$. With similar arguments, we obtain the following theorem.

Theorem 3.10. *We can define a map*

$$\mathrm{infdex}^\mu_G : \mathcal{H}^{\infty,m}_{G,c}(M^0) \to \mathcal{S}'(\mathfrak{g}^*)^G$$

setting for any $[\alpha] \in \mathcal{H}^{\infty,m}_{G,c}(M^0)$ *and for any Schwartz function* f *on* $\mathfrak{g}^*$

$$\langle \mathrm{infdex}^\mu_G([\alpha]), f\rangle := i^{-\dim M/2}\lim_{s\to\infty}\int_M\int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx.$$

If σ *moves smoothly along a curve* σ_t *such that* $\mu_t^{-1}(0)$ *remains equal to* M^0*, the map* $\mathrm{infdex}^{\mu_t}_G$ *remains constant.*

Furthermore, using the Fourier transform $\mathcal{F}$ *of tempered distributions*

$$\mathcal{F}(i^{\dim M/2}\mathrm{infdex}^\mu_G([\alpha])) = \lim_{s\to\infty}\int_M e^{is\Omega(x)}\alpha(x). \tag{19}$$

Remark 3.11. If f is with compact support and the Fourier transform of $\alpha(x)$ is a distribution with compact support on $\mathfrak{g}^*$, the value $\int_M \int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\hat{f}(x)dx$ is independent of s when s is sufficiently large.

Let us state some immediate properties of the infinitesimal index.

Theorem 3.12. *Witten non abelian localization theorem principle* [22].

Let $[\alpha] \in H^*_{G,c}(M)$ *(eventually with coefficients in* $\mathcal{P}^\infty(\mathfrak{g})$*). Let* $I(x) = \int_M \alpha(x)$*. Let* σ *be an action form, and let* M^0 *be the zeroes of the moment map. Then* $[\alpha]$ *defines an element* $[\alpha_0]$ *in* $H^*_{G,c}(M^0)$ *and*

$$\mathcal{F}(i^{\dim M/2}\mathrm{infdex}^{\mu}_{G}([\alpha_0]))(x) = I(x). \tag{20}$$

Proof. This is clear from Formula (19) as $\int_M e^{is\Omega(x)}\alpha(x)$ does not depend on s, as $\Omega(x)$ is exact and α is closed with compact support. □

The interest of this theorem is that the left hand side of (20) depends only of the restriction of α on a small neighborhood of M^0.

Remark 3.13. Let M be a G-manifold equipped with a G invariant Riemannian metric. Take a G-invariant vector field V on M so that V_m at each point $m \in M$ is tangent to the orbit Gm and let σ be the one form associated to V using the metric. Then M^0 is the set of zeroes of the vector field V.

• If G is abelian, we may choose $V = v_x$ with x generic in $\mathfrak{g}$, and then $M^0 = M^G$, the set of fixed points of G on M. Theorem 3.12 leads to the "abelian localization theorem" of Atiyah-Bott-Berline-Vergne [2],[3].

• When G is non necessarily abelian and M is provided with an Hamiltonian structure with symplectic moment map $\nu : M \to \mathfrak{g}^*$, then the Kirwan vector field $V_m = \exp(\epsilon\nu(m))m$ is such that M^0 coincides with the critical points of the function $\|\nu\|^2$ (we used an identification $\mathfrak{g}^* = \mathfrak{g}$). Then one of the connected components of M^0 is the zeros of the symplectic moment map ν, and μ and ν coincide near this component. This is the situation considered by Witten (and extensively studied by Paradan, [16]) with applications to intersection numbers of reduced spaces $\nu^{-1}(0)/G$ (as in [12]).

Example 3.14. • If $G := \{1\}$ is trivial, $H^*_{G,c}(M^0)$ is equal to $H^*_c(M)$ and the infdex maps to constants, by just integration of compactly supported cohomology classes.

• If $M = \{p\}$ is a point, the moment map and $\Omega(x)$ are both 0 while $M^0 = M = \{p\}$. Its equivariant cohomology is $S[\mathfrak{g}^*]^G$.

By Proposition 3.5 it is then enough to compute the infinitesimal index of the class 1. This is given by

$$f \mapsto \int_{\mathfrak{g}} \hat{f}(x)dx = f(0)$$

by Fourier inversion formula. So, in this case the infinitesimal index of 1 is the δ–function δ_0.

More generally, we have extended the definition of infdex_G^{μ} to $\mathcal{P}^{\infty}(\mathfrak{g})^G$. If $\alpha(x)$ is any G-invariant function on $\mathfrak{g}$ with polynomial growth and $\hat{\alpha}$ its Fourier transform (a distribution on $\mathfrak{g}^*$), we obtain

$$\mathrm{infdex}_G^{\mu^0}(\alpha) = \hat{\alpha}. \tag{21}$$

• Consider $M = T^*S^1$ with the canonical action form as in Example 2.7. Then $M^0 = S^1$. We compute the infinitesimal index of the class $1 \in H^*_{G,c}(M^0) = \mathbb{R}$. Let $\chi(t)$ be a function identically equal to 1 in a neighborhood of $t = 0$. Then $D\sigma(x) = xt + dt \wedge d\theta$, and by definition

$$\langle \mathrm{infdex}_G^{\mu}([\alpha]), f\rangle = -i \lim_{s\to\infty} \int_{T^*S^1} \Big(\int_{-\infty}^{\infty} \chi(t) e^{isxt + isdtd\theta} \hat{f}(x) dx \Big)$$

$$= -i \lim_{s\to\infty} \int_{T^*S^1} \chi(t) f(st) e^{isdtd\theta} = \lim_{s\to\infty} s \int_{\mathbb{R}} \chi(t) f(st) dt$$

$$= \lim_{s\to\infty} \int_{\mathbb{R}} \chi(t/s) f(t) dt.$$

Thus we see that *the limit distribution is just the integration against the Lebesgue measure dt.*

• Consider now $M = \mathbb{R}^2$ as in Example 2.9. As we have seen, the action form σ is $\frac{1}{2}(v_1 dv_2 - v_2 dv_1)$, so $D\sigma(x) = dv_1 \wedge dv_2 + x\|v\|^2/2$. Then $M^0 = 0$.

We compute the infinitesimal index of the class $1 \in H^*_{G,c}(M^0)$. Let $\chi(t)$ be a function on $\mathbb{R}$ with compact support and identically equal to 1 in a neighborhood of $t = 0$. Then by definition and the normalization of the Lebesque measure on the Lie algebra of S^1, we get

$$\langle \mathrm{infdex}_G^{\mu}([\alpha]), f\rangle = -i \lim_{s\to\infty} \int_{\mathbb{R}^2} \Big(\frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(\|v\|^2) e^{isx\frac{\|v\|^2}{2} + isdv_1dv_2} \hat{f}(x) dx \Big).$$

Passing in polar coordinates, we see *the limit distribution is the Heaviside distribution supported on $\mathbb{R}^+$.*

4. Properties of the infinitesimal index

There are several functorial properties of the infinitesimal index that we need to investigate: *locality, product, restriction, the map $i_!$, free action.*

4.1. **Locality.** The easiest property is *locality.*

Let M be a G–action manifold with moment map μ and $i : U \to M$ an invariant open set, then we have a mapping $i_* : \mathcal{A}_{G,c}(U) \to \mathcal{A}_{G,c}(M)$ which induces also a mapping

$$i_* : H^*_{G,c}(U^0) \to H^*_{G,c}(M^0).$$

Proposition 4.2. *The mapping i_* is compatible with the infinitesimal index.*

Proof. This is immediate from the definitions. ∎

4.3. **Product of manifolds.** If we have a product $M_1 \times M_2$ of two manifolds relative to two different groups $G_1 \times G_2$, we have

$$(M_1 \times M_2)^0 = M_1^0 \times M_2^0$$

and the cohomology is the product.

Proposition 4.4. *The infinitesimal index of the external product of two cohomology classes is the external product of the two distributions.*

Proof. This is immediate from the definitions. □

4.5. **Restriction to subgroups.** Let $L \subset G$ be a compact subgroup of G so that $\mathfrak{l}$, the Lie algebra of L, is a subalgebra of $\mathfrak{g}$. The moment map μ_L for L is just the composition of μ_G with the restriction $p : \mathfrak{g}^* \to \mathfrak{l}^*$. Thus $\mu_L^{-1}(0) \supset \mu_G^{-1}(0)$.

If f is a test function on $\mathfrak{l}^*$, then $p^* f$ is a smooth function on $\mathfrak{g}^*$ constant along the fibers of the projection.

Definition 4.6. We will say that a distribution Θ on $\mathfrak{g}^*$ is a distribution with compact support along the fibers, if for any test function f on $\mathfrak{l}^*$, the distribution $(p^* f)\Theta$ is with compact support on $\mathfrak{g}^*$.

If Θ is a distribution on $\mathfrak{g}^*$ with compact support along the fibers, we may define $p_*\Theta$ as a distribution on $\mathfrak{l}^*$ by

$$\langle p_*\Theta, f\rangle := \int_{\mathfrak{g}^*} (p^* f)\Theta.$$

The right hand side is computed as the limit when T tends to ∞ of $\langle \Theta, (p^* f)\chi_T\rangle$ when χ_T is a smooth function with compact support and equal to 1 on the ball B_T of $\mathfrak{g}^*$.

Let Z_G be a closed G-invariant subset of M containing $\mu_L^{-1}(0)$ (if G is abelian, we can take $Z_G = \mu_L^{-1}(0)$). Then we have two maps

$$j : H^*_{G,c}(Z_G) \to H^*_{G,c}(\mu_G^{-1}(0))$$

and

$$r : H^*_{G,c}(Z_G) \to H^*_{L,c}(\mu_L^{-1}(0)).$$

Theorem 4.7. *If* $[\alpha] \in H^*_{G,c}(Z_G)$ *then* $\operatorname{infdex}_G^{\mu_G}(j[\alpha])$ *is compactly supported along the fibers of the map* $p : \mathfrak{g}^* \to \mathfrak{l}^*$*, and*

$$p_*(\operatorname{infdex}_G^{\mu_G}(j[\alpha])) = \operatorname{infdex}_L^{\mu_L}(r[\alpha]). \tag{22}$$

Proof. Write $\mathcal{F}^{\mathfrak{g}^*}(h)$ for the Fourier transform $\hat{h}$ of a function h on $\mathfrak{g}^*$.

Let f be a test function on $\mathfrak{l}^*$ with support on a ball B_R. We have, for χ a test function on $\mathfrak{g}^*$,

$$i^{\dim M/2}\langle (p^* f)\operatorname{infdex}_G^{\mu_G}(j[\alpha]), \chi\rangle = \lim_{s\to\infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\mathcal{F}^{\mathfrak{g}^*}((p^* f)\chi)(x)dx.$$

By our assumption on α, there exists $\epsilon > 0$ such that $D\alpha$ is equal to 0 on the subset $\|\mu_L(m)\| < \epsilon$ of M. The argument used in Lemma (3.4) proves that the distribution

$$\chi \to \int_M \int_{\mathfrak{g}} e^{is\Omega(x)}\alpha(x)\mathcal{F}^{\mathfrak{g}^*}(\chi p^* f)(x)dx$$

stabilizes as soon as $s > R/\epsilon$.

Write for $s_0 > R/\epsilon$

$$i^{\dim M/2}\langle (p^* f)\mathrm{infdex}_G^{\mu_G}(j[\alpha]), \chi\rangle = \int_M \int_{\mathfrak{g}} e^{is_0\Omega(x)}\alpha(x)\mathcal{F}^{\mathfrak{g}^*}(\chi p^* f)(x)dx$$

$$= \int_M \Psi(s_0, \alpha, \chi p^* f)$$

where

$$\Psi(s_0, \alpha, \chi p^* f)(m) = \sum_a \Big[\int_{\mathfrak{g}} P_a(x)e^{is_0\langle \mu(m), x\rangle}\mathcal{F}^{\mathfrak{g}^*}(\chi p^* f)(x)dx\Big]e^{is_0 d\sigma}\alpha^a$$

$$= \sum_a ((P_a(-i\partial)(\chi p^* f) \circ (s_0\mu))e^{is_0 d\sigma}\alpha^a. \tag{23}$$

Applying Proposition 3.2 we have that, if K is the compact support of α, and as s_0 is greater than R/ϵ, the form $\Psi(s_0, \alpha, \chi p^* f)$ is supported on the compact subset $s_0\mu_G(K)$ in $\mathfrak{g}^*$. This shows the first statement that $\mathrm{infdex}_G^{\mu_G}(j[\alpha])$ is compactly supported along the fibers of p.

We pass next to Formula (22). We then have

$$i^{\dim M/2}\langle (p^* f)\mathrm{infdex}_G^{\mu_G}(j[\alpha]), \chi_T\rangle = \int_M \Psi(s_0, \alpha, \chi_T p^* f)$$

for any T large.

Using Formula (23), when T is sufficiently large, as χ_T is equal to 1 on a large ball, $\Psi(s_0, \alpha, \chi_T p^* f)$ is simply

$$\sum_a ((P_a(-i\partial)p^* f) \circ (s_0\mu))e^{is_0 d\sigma}\alpha^a.$$

As $p^* f$ is constant along the fibers, if we denote by α_0 the restriction of $\alpha(x)$ to $\mathfrak{l}$, we see that $\Psi(s_0, \alpha, \chi_T p^* f)$ is equal to the differential form $\Psi(s_0, \alpha_0, f)$ as all derivatives in the $\ker p$ direction annihilate $p^* f$. We thus obtain our theorem.

□

4.8. **Thom class and the map $i_!$.** Let Z be an oriented G manifold of dimension d and $i : M \hookrightarrow Z$ a G-stable oriented submanifold of dimension $n = d - k$. Assume that M is an action manifold with moment map μ and that Z is equipped with an action form σ_Z such that the associated moment map μ_Z extends μ. Thus $Z^0 \cap M = M^0$. Under these assumption, we will define a map

$$i_! : H^*_{G,c}(M^0) \to H^*_{G,c}(Z^0)$$

preserving the infdex.

Let us recall the existence of an equivariant Thom class ([15], see [10] pag. 158, [18]) . We assume first that M has a G-stable tubular neighborhood N in Z, with projection $p : N \to M$. Then there exists a unique class τ_M of equivariantly closed forms on N with compact support along the fibers so that the integral $p_*\tau_M$ is identically equal to 1 along each fiber. Thus for any equivariant form $\alpha(x)$ on M with compact support (but not necessarily closed), we have that

$$\int_M \alpha = \int_N p^*\alpha \wedge \tau_M.$$

In general, let us take a class $[\alpha] \in H^*_{G,c}(M^0)$ where $\alpha \in \mathcal{A}_{G,c}(M)$ and $D\alpha$ has support K in $M \setminus M^0$.

Consider a G-stable open set $U \subset M$ with the following properties.

i) The support of α is contained in U.
ii) The closure of U is compact and has an open neighborhood A in Z such that $M \cap A$ has a G-stable tubular neighborhood in A.

By locality, we can then substitute U to M and thus assume that the pair (Z, M) has all the properties which insure the existence of a Thom class τ_M.

Consider a G-invariant Riemannian metric on the normal bundle $\mathcal{N}$ to M in Z. Define S^ϵ as the (open) disk bundle of radius ϵ in $\mathcal{N}$. Then we can take our tubular neighborhood in such a way that it is diffeomorphic to S^ϵ for some ϵ.

We claim that we can take S^ϵ so close to M that $p^{-1}K \cap S^\epsilon \cap Z^0 = \emptyset$. Indeed, $p^{-1}K \cap \overline{S^\epsilon}$ is a compact set and, since K is disjoint from M^0 and hence from Z^0, for a sufficiently small ϵ, $p^{-1}K \cap \overline{S^\epsilon}$ is disjoint from Z^0.

Let us now fix the Thom form τ_M in $\mathcal{A}_{G,c}(\mathcal{N})$ with support in S^ϵ.

Consider then the form $p^*\alpha \wedge \tau_M$. We have that $D(p^*\alpha \wedge \tau_M) = p^*D\alpha \wedge \tau_M$ has support in $p^{-1}K \cap S^\epsilon \subset Z \setminus Z^0$. It follows that $p^*\alpha \wedge \tau_M$ defines an element in $H^*_{G,c}(Z^0)$.

We claim that this element depends only on the class $[\alpha]$. So first take another Thom form τ'_M with the same properties. Then there is a form $r_M \in \mathcal{A}_{G,c}(S^\epsilon)$ so that $\tau_M - \tau'_M = Dr_M$ and

$$p^*\alpha \wedge \tau_M - p^*\alpha \wedge \tau'_M = p^*\alpha \wedge Dr_M = D(p^*\alpha \wedge r_M) - p^*D\alpha \wedge r_M$$

where $p^*\alpha \wedge r_M$ has compact support and $p^*D\alpha \wedge r_M$ has support in $Z \setminus Z^0$.

Next assume that α is supported outside M^0, then again we may take τ_M so that $p^*\alpha \wedge \tau_M$ is supported outside Z^0.

Finally, if $\alpha = D\beta$, we have $p^*\alpha \wedge \tau_M = D(p^*\beta \wedge \tau_M)$.

Hence we can set

$$i_![\alpha] := i^{\dim Z/2 - \dim M/2}[p^*\alpha \wedge \tau_M]. \tag{24}$$

Theorem 4.9. *Assume that M is an action manifold with action form σ and moment map μ and that Z is equipped with an action form σ_Z such that* *the associated moment map μ_Z extends μ. Then the morphism*

$$i_! : H^*_{G,c}(M^0) \to H^*_{G,c}(Z^0)$$

preserves the infinitesimal index.

Remark 4.10. We do not need to assume that the restriction of σ_Z to M is the action form σ on M, only that the moment map μ_Z restricts to μ.

Proof. First let us see that $\mathrm{infdex}_G^{\mu_Z}(i_![\alpha])$ does not depend of the choice of the form σ_Z on Z, if the moment map μ_Z restricts to μ. We can assume $Z = N$. Let $\beta = p^*\alpha \wedge \tau_M$. The form β is compactly supported.

Let σ_1, σ_0 be two one forms on Z and consider $\sigma_t = t\sigma_1 + (1-t)\sigma_0$ and μ_t the corresponding moment map. Set $\Omega(t) = D\sigma_t$. We assume that the map μ_t coincides with μ on M for all t. Thus, provided we choose τ_M with support sufficiently close to M, there exists an $h > 0$ such that on the support of $D\beta$, we have $\|\mu_t\| > h > 0$.

Define

$$I(t,s) := \int_N \int_{\mathfrak{g}} \beta(x) e^{is\Omega(t,x)} \beta(x) \hat{f}(x) dx.$$

We can prove that $\frac{d}{dt} I(t,s) = 0$ in the same way that the invariance of the infinitesimal index $\mathrm{infdex}_G^{\mu_t}$ along a smooth curve μ_t (proof of Theorem 3.5), thus we skip the proof.

Having established the independence from σ, we choose for the final computation $\sigma_Z := p^*\sigma$. In this case, since $\beta = p^*\alpha \wedge \tau_M$,

(25)

$$i^{\dim Z/2} \langle \mathrm{infdex}_G^{\mu_Z}([\beta]), f\rangle = \lim_{s\to\infty} \int_{\mathfrak{g}} \int_N p^* \left(e^{is\Omega(x)}\alpha(x)\right) \wedge \tau_M(x) \hat{f}(x) dx.$$

As τ_M has integral 1 over each fiber of the projection $p : N \to M$, we obtain that (25) is equal to

$$\lim_{s\to\infty} \int_{\mathfrak{g}} \int_M e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$$

which is our statement. □

4.11. **Free action.** Let G and L be two compact groups. Consider now an oriented manifold N under $G \times L$ action, with action 1-form σ_N and moment map $\mu_{G\times L} = (\mu_G, \mu_L) : N \to \mathfrak{g}^* \oplus \mathfrak{l}^*$. We set $N^0 = \mu_{G\times L}^{-1}(0)$.

Assume that

- the group L acts freely on N.
- 0 is a regular value of μ_L.

Define $P = \mu_L^{-1}(0)$. By assumption P is a manifold with a free L-action so

$$M := \mu_L^{-1}(0)/L$$

is a G-manifold.

We denote by π the projection $P \to M$. The invariance of σ_N under L action then implies

Proposition 4.12. *The restriction $\overline{\sigma}$ of σ_N to P verifies $\iota_x\overline{\sigma} = 0$ for any $x \in \mathfrak{l}$ and descends to a G-invariant action form σ_M on M, thus M is an action manifold.*

We denote by μ the moment map on M associated to σ_M. The map μ is obtained factoring the restriction of μ_G to P which is L invariant, that is $\mu_G = \mu \circ \pi$ on P. Since N^0 is the subset of P where μ_G equals 0, we see that M^0, the fiber at zero of μ, is N^0/L.

Recall (Proposition 1.5) that since the action of L is free, we have an isomorphism $\pi^* : H^*_{G,c}(M^0) \to H^*_{G\times L,c}(N^0)$.

Our goal in this section is, given a class $[\gamma] \in H^*_{G,c}(M^0)$, to compare $\operatorname{infdex}^{\mu}_G([\gamma])$ and $\operatorname{infdex}^{\mu_{G\times L}}_{G\times L}(\pi^*([\gamma]))$.

As 0 is a regular value of μ_L, any L-stable compact subset K in P has an L-stable neighborhood in N isomorphic to $K \times \mathfrak{l}^*$ with moment map μ_L being the projection on the second factor. Since the computations of the infinitesimal index of a given class with compact support are local around N^0 (by Proposition 4.2), we may assume that $N = P \times \mathfrak{l}^*$ and that the moment map μ_L is the projection on the second factor. We write an element of N as (p, ζ) with $p \in P$, $\zeta \in \mathfrak{l}^*$.

The composition of the projection $\eta : N = P \times \mathfrak{l}^* \to P$ and of $\pi : P \to M$ is a fibration with fiber $L \times \mathfrak{l}^* = T^*L$. Since the symplectic structure gives a natural orientation on T^*L, the orientation of N induces an orientation on M.

Let us choose now a connection form $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{l}$ for the free action of L on P. We want to apply Definition 2.15 to the following functions. For ζ a point in $\mathfrak{l}^*$, define $\theta_\zeta \in C^\infty(\mathfrak{l})^L$ by

$$\theta_\zeta(x) := \int_L e^{i\langle x, \mathrm{Ad}^*(l)\zeta\rangle} dl$$

where dl is a Haar measure on L.

Thus for any $\zeta \in \mathfrak{l}^*$, we may consider the G-equivariant closed form $\theta_\zeta(R_y)$ on M given by

$$\theta_\zeta(R_y) = \int_L e^{i\langle R_y, \mathrm{Ad}^*(l)\zeta\rangle} dl = \int_L e^{i\langle -\iota_y\omega, \mathrm{Ad}^*(l)\zeta\rangle} e^{i\langle R, \mathrm{Ad}^*(l)\zeta\rangle} dl. \tag{26}$$

We need some growth properties of the function $y \to \theta_\zeta(R_y)$.

We write $\theta_\zeta(x) = \int_{\mathfrak{l}^*} e^{i\langle f, x\rangle} d\beta_\zeta(f)$ where $d\beta_\zeta(f)$ is a L-invariant measure on the orbit $L\zeta \subset \mathfrak{l}^*$.

If we fix $p \in P$ and $\zeta \in \mathfrak{l}^*$, let us see that the function $y \to \theta_\zeta(R_y)(p)$ is the Fourier transform of a compactly supported measure $d\mu_{p,\zeta}$ on $\mathfrak{g}^*$ (with values in $\Lambda T^*_p P$). Indeed, let $f \in \mathfrak{l}^*$. The function $\langle -\iota_y\omega(p), f\rangle$ is linear in $y \in \mathfrak{g}$, so we write $\langle -\iota_y\omega(p), f\rangle = \langle y, h(p, f)\rangle$ with $h(p, f) \in \mathfrak{g}^*$ depending smoothly on p, f. We see that

$$\theta_\zeta(R_y)(p) = \int_{\mathfrak{l}^*} e^{i\langle y, h(p,f)\rangle} e^{i\langle f, R\rangle} d\beta_\zeta(f).$$

Let us integrate over the fiber of the map $h_p : \mathfrak{l}^* \to \mathfrak{g}^*$ given by $f \to h(p,f) = \xi$. We obtain that

$$\theta_\zeta(R_y)(p) = \int_{\mathfrak{g}^*} e^{i\langle y,\xi\rangle}(h_p)_*(e^{i\langle f,R\rangle}d\beta_\zeta(f)). \tag{27}$$

In this formula, $(h_p)_*(e^{i\langle f,R\rangle}d\beta_\zeta(f))$ is a measure supported on the compact set $h_p(L\zeta)$ as $d\beta_\zeta(f)$ is supported in the compact set $L\zeta$. In particular, we see that, over a compact subset of P, $y \to \theta_\zeta(R_y)(p)$ is a bounded function of y as well as all its derivatives in y and estimates are uniforms in ζ if ζ varies in a compact set of $\mathfrak{l}^*$.

If $[\gamma] \in H^*_{G,c}(M^0)$, we choose a representative $\gamma(y)$ which is a form with compact support on M and depending of y in a polynomial way. Set

$$\tilde{\gamma}_\zeta(y) := \gamma(y)\theta_\zeta(R_y). \tag{28}$$

Proposition 4.13. *The equivariant form $\tilde{\gamma}_\zeta(y)$ is of at most polynomial growth in y. It represents a class in $\mathcal{H}^{\infty,m}_{G,c}(M^0)$ which does not depend of the choice of the connection ω but only on the choice of the Haar measure dl.*

Proof. The fact that $\tilde{\gamma}_\zeta(y)$ is of at most polynomial growth follows from the preceding discussion. The second statement is proved as in ([4],[6], see [5]). □

Remark that $\theta_0(R_y) = \mathrm{vol}(L, dl)$ where $\mathrm{vol}(L, dl)$ is the volume of the compact Lie group L for the Haar measure dl.

Given $[\gamma] \in H^*_{G,c}(M^0)$, we may apply the infdex construction (Theorem 3.10) to the cohomology class $[\tilde{\gamma}_\zeta] \in \mathcal{H}^{\infty,m}_{G,c}(M^0)$ of the equivariant form $\tilde{\gamma}_\zeta(y) = \gamma(y)\theta_\zeta(R_y)$.

With this notation, we can state:

Theorem 4.14. *Let f_1 be a test function on $\mathfrak{l}^*$ and f_2 be a test function on $\mathfrak{g}^*$. Then $\langle \mathrm{infdex}^\mu_G([\tilde{\gamma}_\zeta]), f_2\rangle$ is a smooth function of ζ and*

$$\langle \mathrm{infdex}^{\mu_{G\times L}}_{G\times L}(\pi^*([\gamma])), f_1 f_2\rangle = \int_{\mathfrak{l}^*} \langle \mathrm{infdex}^\mu_G([\tilde{\gamma}_\zeta]), f_2\rangle f_1(\zeta)d\zeta. \tag{29}$$

In this formula, $\tilde{\gamma}_\zeta$ is constructed using the Haar measure dl such that $dld\zeta$ is the canonical measure on $T^*L = L\times\mathfrak{l}^*$ (by right or left trivialization).

Let us first write a corollary of this theorem.

Corollary 4.15. *Let f_2 be a test function on $\mathfrak{g}^*$. Then the distribution $f_1 \to \langle \mathrm{infdex}^{\mu_{G\times L}}_{L\times G}(\pi^*([\gamma])), f_1 f_2\rangle$ on $\mathfrak{l}^*$ is a smooth density $D(\zeta)d\zeta$. The value of D at 0 is equal to $\mathrm{vol}(L, dl)\langle \mathrm{infdex}^\mu_G([\gamma]), f_2\rangle$.*

We now prove Theorem 4.14.

Proof. Let $\gamma(y)$ be a compactly supported equivariant form on M representative of $[\gamma]$. Any $G \times L$–equivariant form ψ with compact support on $N = P \times \mathfrak{l}^*$ which restricted to P coincides with $\pi^*\gamma$ can be taken as a representative for the cohomology class $\pi^*[\gamma] \in H^*_{G\times L,c}(N^0)$. Thus take an L-invariant function $\rho : \mathfrak{l}^* \to \mathbb{R}$ supported near zero and such that ρ equals 1 on a neighborhood of 0 and take the form ψ which is still L-invariant and G-equivariant:

$$\psi(y)(p,\zeta) = \rho(\zeta)\xi^*\gamma(y). \tag{30}$$

Recall that $\overline{\sigma}$ is the restriction of σ_N on P and consider the one form $\eta^*(\overline{\sigma})$ on $N = P \times \mathfrak{l}^*$, the pull back of $\overline{\sigma}$ under the projection $\eta : P \times \mathfrak{l}^* \to P$. Let $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{l}$ be our connection form. Then $\langle \omega, \zeta\rangle$ is an action form on N, with moment map for L the second projection. Its moment map for G vanishes on P.

Consider $\sigma_0 = \sigma_N$ and $\sigma_1 = \eta^*(\overline{\sigma}) + \langle \omega, \zeta\rangle$ with moment maps μ_0, μ_1.

Lemma 4.16. *The moment map $\mu_t = t\mu_1 + (1-t)\mu_0$ associated to $t\sigma_1 + (1-t)\sigma_0$ is such that $\mu_t^{-1}(0) = N^0$ for all $t \in [0,1]$.*

Proof. This follows from the fact that the component under L of these maps is the second projection, so that $\mu_t^{-1}(0) \subset P$ for all t and moreover μ_1, μ_0 coincide on P. Thus $\mu_t^{-1}(0) = P^0 = N^0$. □

According to Theorem 3.5, we may thus assume that $\sigma_N = \eta^*(\overline{\sigma}) + \langle\omega,\zeta\rangle$ and compute with this "normal form" the values of $\mathrm{infdex}^{\mu_{G\times L}}_{G\times L}$.

Recall that $\mu : M \to \mathfrak{g}^*$ is the moment map relative to G associated to σ_M. By abuse of notations, we still denote by μ its pull back by $\pi\eta$ to N. This is the moment map associated to $\eta^*(\overline{\sigma})$.

Lemma 4.17. *Let $\Omega := D\sigma_N$, for $(x,y) \in \mathfrak{l} \oplus \mathfrak{g}$. At a point $(p,\zeta) \in P \times \mathfrak{l}^*$, we have:*

$$\Omega(x,y) = \langle x,\zeta\rangle - \langle \iota_y\omega,\zeta\rangle + \langle y,\mu\rangle + d\eta^*(\overline{\sigma}) + d\langle\omega,\zeta\rangle.$$

Proof. By the definition of a connection form (for the action of L), we have $\langle x,\zeta\rangle = -\langle\iota_x\omega,\zeta\rangle$ so $\langle x,\zeta\rangle - \langle\iota_y\omega,\zeta\rangle$ is the value of the moment map at (x,y) of $\langle\omega,\zeta\rangle$. As for $\eta^*(\overline{\sigma})$, by definition of $P = \mu_L^{-1}(0)$, the part relative to L of its moment map equals to 0. □

We write $\Omega(x,y) = \langle x,\zeta\rangle + \Omega'(y)$ with

$$\Omega'(y) = -\langle\iota_y\omega,\zeta\rangle + \langle y,\mu\rangle + \eta^* d(\overline{\sigma}) + d\langle\omega,\zeta\rangle$$

independent of x. We have

$$\Omega'(y) = \eta^*(D\overline{\sigma}) - \langle\iota_y\omega,\zeta\rangle + d\langle\omega,\zeta\rangle. \tag{31}$$

For s sufficiently large,

$$i^{\dim N/2}\langle \mathrm{infdex}^{\mu_{G\times L}}_{G\times L}(\pi^*([\gamma])), f_1 f_2\rangle = I(s)$$

with

$$I(s) = \int_N \int_{\mathfrak{g}\times\mathfrak{l}} e^{is\Omega(x,y)} \psi(y) \hat{f}_1(x) \hat{f}_2(y) dx dy.$$

Applying Fourier inversion

$$\int_{\mathfrak{l}} e^{is\langle x,\zeta\rangle} \hat{f}_1(x) dx = f_1(s\zeta),$$

we obtain that

$$I(s) = \int_N \int_{\mathfrak{g}} e^{is\Omega'(y)} \psi(y) f_1(s\zeta) \hat{f}_2(y) dy$$

where $\psi(y)(p,\zeta) = \rho(\zeta)\xi^*\gamma(y)$ is defined by Formula (30).

Write $\omega = \sum_{i=1}^r \omega_i e_i$ on a basis $\{e_1, \dots, e_r\}$ of $\mathfrak{l}$, and set $\zeta_i = \langle e_i, \zeta\rangle$ for $i = 1, \dots, r$.

We have $\langle \omega, \zeta\rangle = \sum_{i=1}^r \zeta_i \omega_i$ and thus

$$d\langle \omega, \zeta\rangle = \sum_{i=1}^r \zeta_i d\omega_i + \sum_{i=1}^r d\zeta_i \wedge \omega_i. \tag{32}$$

Let us now integrate along the fiber $\mathfrak{l}^*$ of the projection $\eta : N = P \times \mathfrak{l}^* \to P$. We thus need to identify the highest term of $e^{is\Omega'(y)}$ in the $d\zeta_i$. By (32),(31) this highest term equals

$$(is)^r d\zeta_1 \wedge \omega_1 \wedge \cdots \wedge d\zeta_r \wedge \omega_r = (-1)^{\frac{r(r+1)}{2}} (is)^r d\zeta \wedge V_\omega$$

where we set $V_\omega := \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_r$ and $d\zeta := d\zeta_1 \wedge \cdots \wedge d\zeta_r$. We obtain

$$I(s) = \int_N \int_{\mathfrak{g}} e^{is\Omega'(y)} \psi(y) f_1(s\zeta) \hat{f}_2(y) dy$$

$$= (-1)^{\frac{r(r+1)}{2}} i^r \int_{P\times\mathfrak{g}} e^{isD\overline{\sigma}} \gamma(y) \hat{f}_2(y) \left(\int_{\mathfrak{l}^*} s^r e^{-is\langle \iota_y \omega, \zeta\rangle} e^{is\langle d\omega, \zeta\rangle} \rho(\zeta) f_1(s\zeta) d\zeta \wedge V_\omega \right) dy.$$

In the integral on $\mathfrak{l}^*$, we change ζ to $s\zeta$ and obtain

$$(-1)^{\frac{r(r+1)}{2}} i^r \int_{P\times\mathfrak{g}} e^{isD\overline{\sigma}} \gamma(y) \hat{f}_2(y) \left(\int_{\mathfrak{l}^*} e^{-i\langle \iota_y \omega, \zeta\rangle} e^{i\langle d\omega, \zeta\rangle} \rho(\zeta/s) f_1(\zeta) d\zeta \wedge V_\omega \right) dy.$$

On the compact support of $f_1(\zeta)$, if s is sufficiently large, $\rho(\zeta/s) = 1$. Also we may replace $d\omega$ by R as $R - d\omega = \frac{1}{2}[\omega, \omega]$ is annihilated by wedge product with $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_r$ and obtain (for s sufficiently large):

$$i^{\frac{\dim M}{2}} \langle \text{infdex}_{G\times L}^{\mu_{G\times L}} \psi, f_1 f_2\rangle$$

$$= (-1)^{\frac{r(r+1)}{2}} \int_N \int_{\mathfrak{g}} e^{is\eta^* D\overline{\sigma}} \gamma(y) \hat{f}_2(y) e^{i\langle R_y, \zeta\rangle} f_1(\zeta) d\zeta \wedge V_\omega dy.$$

Now consider the fibration $N \to M \times \mathfrak{l}^*$ with fiber L. On each fiber, the form $V_\omega = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_r$ induces an orientation and restricts to a Haar measure dl on L. Let us now integrate over the fiber. Recall that σ_M denotes the action form on M. Let $\Omega_M := D\sigma_M$, we have $\pi^*\sigma_M =$

$\bar{\sigma}, \eta^* D\overline{\sigma} = \eta^*\pi^*\Omega_M$. So, keeping track of orientations, and using Formula (26), we finally obtain that $I(s)$ is equal to

$$\int_{\mathfrak{l}^*}\left(\int_M\int_{\mathfrak{g}} e^{is\Omega_M(y)}\gamma(y)\theta_\zeta(R_y)\hat{f}_2(y)dy\right)f_1(\zeta)d\zeta.$$

Remark that when ζ varies in the compact support of f_1, and over a compact subset K of M, the Fourier transform (in y) of $\theta_\zeta(R_y)$ stays supported on a fixed compact subset of $\mathfrak{g}^*$. Indeed, using Formula (27), we see that the Fourier transform of $\theta_\zeta(R_y)$ is supported on the compact subset $h(\pi^{-1}K, \mathrm{Ad}^* L\zeta)$. By Remark 3.11, for $s >> s_0$

$$\int_{M\times\mathfrak{g}} e^{is\Omega_M(y)}\gamma(y)\theta_\zeta(R_y)\hat{f}_2(y)dy = i^{\dim M/2}\mathrm{infdex}_G^\mu([\tilde{\gamma}_\zeta], f_2\rangle$$

for any ζ in the support of f_1.

Thus we obtain our claim. □

Example 4.18. Consider $N = T^*G$, with left and right action of G and moment map $\mu = \mu_\ell + \mu_r$.

Consider the $G\times G$ equivariant form $1 \in H^*_{G,c}(G)$ on $(T^*G)^0 = G$.

Theorem 4.19.

$$\langle \mathrm{infdex}^\mu_{G\times G}(1), f_1 f_2\rangle = \int_{\mathfrak{g}^*}\int_G f_1(g\zeta)f_2(-\zeta)dgd\zeta.$$

Here the measure $dgd\zeta$ is the canonical measure on T^*G.

Proof. We apply the previous construction to T^*G with free right action of G. Then $P = G$ and M is a point. Thus 1 is the lift of 1 and we apply Theorem 4.14. By Example 2.14, the equivariant curvature R_y at the point $p \in G$ is simply $R_y = -Ad_p^{-1}y$. Thus $\theta_\zeta(R_y) = \int_G e^{-i\langle p^{-1}y, g\zeta\rangle}dg = \int_G e^{-i\langle y, g\zeta\rangle}dg$. Using Formula (29), we obtain

$$\langle \mathrm{infdex}^\mu_{G\times G}1, f_1 f_2\rangle = \int_{\mathfrak{g}^*}\int_G f_2(-g\zeta)f_1(\zeta)dgd\zeta.$$

□

Another important particular case of the free action property is when G is trivial. We then have $y = 0$ in all the steps of the proof of Theorem 4.14. We summarize the result that we obtain in this particular case of Theorem 4.14.

Let N be an oriented L-manifold with action form, and assume that the group L acts freely on N and that 0 is a regular value of μ_L. Let $M = N^0/L$ and let $[\gamma] \in H^*_{G,c}(N^0) = H^*_c(M)$.

Let R be the curvature of the fibration $N^0 \to M$. For any $\zeta \in \mathfrak{l}^*$, we consider the closed differential form on M given by

$$\theta_\zeta(R) = \int_L e^{i\langle R, \mathrm{Ad}^*(l)\zeta\rangle}dl. \tag{33}$$

Here, as R is a $\mathfrak{l}$ valued two form, $\theta_\zeta(R)$ is a polynomial function of ζ.

Then we obtain

Proposition 4.20. *The distribution* $\mathrm{infdex}_L^{\mu_L}([\gamma]$ *is a polynomial density on* $\mathfrak{l}^*$. *More precisely*

$$\langle \mathrm{infdex}_L^{\mu_L}([\gamma]), f_1\rangle = \int_{\mathfrak{l}^*}\left(\int_M \gamma\theta_\zeta(R)\right) f_1(\zeta)d\zeta.$$

In particular the value of $\mathrm{infdex}_L^{\mu_L}([\gamma])$ at 0 is well defined and computes the integral on the reduced space $\mu_L^{-1}(0)/L$ of the compactly supported cohomology class associated to $[\gamma]$. This is the essentially Witten localization formula [22],[11].

4.21. **Extension of the properties of the infinitesimal index.** We have extended the definition of the infinitesimal index to $\mathcal{H}_{G,c}^{\infty,m}(M^0)$. Analyzing the proofs of the properties *locality, product, the map* $i_!$, we see that these properties hold for the infinitesimal index map on $\mathcal{H}_{G,c}^{\infty,m}(M^0)$. The proofs for the restriction property and the free action extend, provided we are in the situation of Remark 3.11: we consider the infinitesimal index on classes $[\alpha]\in H_{G,c}^{\infty,m}(M^0)$ such that the Fourier transform of $\alpha(x)$ is a distribution with compact support on $\mathfrak{g}^*$, so that the infinitesimal index stabilizes for s large. This will be always the situation in the applications to index formulae.

5. Some consequences of the functorial properties of the infinitesimal index

We list here some corollaries of the functorial properties: excision, product, restriction, push-forward, free action proved in the preceding section.

5.1. **Diagonal action and convolution.** Consider two G action manifolds M_1, M_2 with moment maps μ_1,μ_2 with zeroes M_1^0, M_2^0. Let Δ be the diagonal subgroup. The moment map for Δ is $\mu_1+\mu_2$.

Let us assume that $(M_1\times M_2)_\Delta^0 = M_1^0\times M_2^0$. If $\alpha\in H_{G,c}^*(M_1^0)$ and $\beta\in H_{G,c}^*(M_2^0)$, we may apply the product property (Proposition 4.4) and the restriction property (Theorem 4.7). As the restriction map is such that $r^*f(\xi_1,\xi_2)=f(\xi_1+\xi_2)$ $(\xi_1,\xi_2\in\mathfrak{g}^*)$, we obtain the following proposition.

Proposition 5.2. *Under the hypothesis* $(M_1\times M_2)_\Delta^0 = M_1^0\times M_2^0$, *the infinitesimal index* $\mathrm{infdex}_\Delta^{\mu_1+\mu_2}(\alpha_1\wedge\alpha_2)$ *is the convolution product* $\mathrm{infdex}_G^{\mu_1}(\alpha_1) * \mathrm{infdex}_G^{\mu_2}(\alpha_2)$ *of the distributions* $\mathrm{infdex}_G^{\mu_1}(\alpha_1)$ *and* $\mathrm{infdex}_G^{\mu_2}(\alpha_2)$.

Let us give an important example of this situation.

Let M_X be a complex representation space for the action of a torus G, where $X=[a_1,a_2,\dots,a_m]$ is a list of non zero weights $a_i\in\mathfrak{g}^*$. We assume that X spans a pointed cone in $\mathfrak{g}^*$. Recall the definition of the *multivariate spline* T_X, it is a tempered distribution defined by:

$$\langle T_X \mid f\rangle = \int_0^\infty \dots \int_0^\infty f(\sum_{i=1}^m t_i a_i)dt_1\cdots dt_m. \tag{34}$$

Let us consider on $M_X = \mathbb{C}^m$ the action form such that $\mu(z_1, \dots, z_m) = \sum_i \frac{|z_i|^2}{2} a_i$. Then $M_X^0 = \{0\}$ and the class 1 is a class in $H^*_{G,c}(M_X^0)$. Using our computation in Example 3.14 of $\mathrm{infdex}^\mu_G(1)$ in the case of $\mathbb{R}^2 = \mathbb{C}$, we obtain the following formula.

Proposition 5.3.

$$\mathrm{infdex}^\mu_G(1) = T_X.$$

We will use this calculation in [8] to identify $H^*_{G,c}((T^*M_X)^0)$ to a space of spline distributions on $\mathfrak{g}^*$.

Another example that we will use in Subsection 5.15 is the case were one of the action forms, say σ_1, is equal to 0, so that $\mu_1 = 0$ and μ is the pullback of μ_2. Then

$$(M_1 \times M_2)^0_\Delta = M_1 \times M_2^0.$$

In this case, the space $H^*_{G,c}(M_1^0)$ is simply $H^*_{G,c}(M_1)$ and $\int_{M_1} \alpha_1(x)$ is a polynomial function of $x \in \mathfrak{g}$. Thus $\mathrm{infdex}^0_G(\alpha_1)$, the Fourier transform, is a distribution of support 0 on $\mathfrak{g}^*$.

Corollary 5.4.

$$\mathrm{infdex}^\mu_\Delta[\alpha_1 \times \alpha_2] = \mathrm{infdex}^0_G(\alpha_1) * \mathrm{infdex}^{\mu_2}_G(\alpha_2).$$

5.5. **Induction of distributions.** Choose Lebesgue measures on $\mathfrak{g}$, and $\mathfrak{l}$ by fixing translation invariant top differential forms. This determines dual measures and forms on $\mathfrak{g}^*$, $\mathfrak{l}^*$ and a Haar measure dg on G. If p is the restriction map $\mathfrak{g}^* \to \mathfrak{l}^*$, we let p_* be the integration over the fiber (with respect to the chosen forms and orientations). It sends a test function on $\mathfrak{g}^*$ to a test function on $\mathfrak{l}^*$. Let

$$A(f)(\xi) = \int_G f(g\xi)dg. \tag{35}$$

The operator A transform a test function on $\mathfrak{g}^*$ to an invariant test function on $\mathfrak{g}^*$.

Definition 5.6. For a distribution V on $\mathfrak{l}^*$, we define the G-invariant distribution $\mathrm{Ind}^{\mathfrak{g}^*}_{\mathfrak{l}^*} V$ on $\mathfrak{g}^*$ by

$$\langle \mathrm{Ind}^{\mathfrak{g}^*}_{\mathfrak{l}^*} V, f\rangle = \mathrm{vol}(L, dl)^{-1} \langle V, p_*(A(f))\rangle,$$

f being a test function on $\mathfrak{g}^*$.

It is easy that $\mathrm{Ind}^{\mathfrak{g}^*}_{\mathfrak{l}^*} V$ is independent of the choices of measures.

5.7. **Induction of action manifolds.** Assume that $L \subset G$ is a subgroup. Take M a L manifold with action form σ and moment map μ_L.

Consider T^*G as a $G \times L$ action manifold where G acts on the left and L on the right, and the action form ω is the canonical one-form on T^*G.

Set $N := T^*G \times M$ and p_1, p_2 be the first and second projection of this product manifold. We consider the action form $\psi = p_1^*\omega + p_2^*\sigma$ on N, and denote by $\tilde{\mu}_{G\times L} = \tilde{\mu}_G \oplus \tilde{\mu}_L$ the corresponding moment map.

Let us trivialize $T^*G = G \times \mathfrak{g}^*$ using left trivialization (7), so that we identify $N = G \times \mathfrak{g}^* \times M$. According to Formula (7), if $(g, \xi, m) \in N$ we have:

$$\tilde{\mu}_G(g,\xi,m) = \mathrm{Ad}_g^*(\xi) := g\xi, \quad \tilde{\mu}_L(g,\xi,m) = -\xi|_{\mathfrak{l}} + \mu_L(m). \tag{36}$$

We denote by N^0 the zero fiber of the moment map $\tilde{\mu}_{G\times L}$ for $G \times L$, by M^0 the zero fiber of the moment map μ_L on M for L.

Lemma 5.8. *We have $N^0 = G \times M^0$.*

Proof. From Formula (36) the set of points of N where $\tilde{\mu}_G = 0$ is $G \times M$, and on these points we have $\tilde{\mu}_L(g,m) = \mu_L(m)$. □

Lemma 5.9. *i) If we take the zero fiber of $\tilde{\mu}_L$, we obtain the manifold*

$$P := \{(g,\xi,m); g \in G, \xi \in \mathfrak{g}^*, m \in M; \xi|_{\mathfrak{l}} = \mu_L(m)\}. \tag{37}$$

ii) 0 is a regular value for the moment map $\tilde{\mu}_L$.

Proof. The first statement is immediate from Formula (36). As for the second, by the same formula, we see easily that the differential of $\tilde{\mu}_L$ is surjective everywhere. □

We are thus in the situation of Subsection 4.11. The manifold N is a $G \times L$ manifold, L acts freely on N and 0 is a regular value of the moment map $\tilde{\mu}_L$ for L. Consider the manifold $\mathcal{M} = P/L$. Applying Proposition 4.12, we see

Lemma 5.10. *The quotient $\mathcal{M} = P/L$ is a G-manifold. The action form on N restricted to P descends to $\mathcal{M}$.*

The induced moment map $\mu_G : P/L \to \mathfrak{g}^*$ is obtained by quotient from the moment map $\tilde{\mu}_G : (g,\xi,m) \to g\xi$ on P.

Definition 5.11. We will say that $\mathcal{M}$ is the *induced action manifold*.

By Lemma 5.8, the closed set N^0, the zero fiber of the moment map $\tilde{\mu}_{G\times L}$, equals $G \times M^0$ and it is contained in P. Since, by definition, on $P = \tilde{\mu}_L^{-1}(0)$ the moment map $\tilde{\mu}_L$ equals 0, we have that on P the moment map $\tilde{\mu}_{G\times L}$ equals $\tilde{\mu}_G$. Therefore we obtain the

Lemma 5.12. *Under the inclusions $N^0 \subset P$, $N^0/L \subset P/L$, the zero fiber $\mathcal{M}_G^0 \subset \mathcal{M}$ of the moment map μ_G is identified with $N^0/L = G \times_L M^0$.*

We denote by $\pi : G \times M^0 = N^0 \to N^0/L = G \times_L M^0$ the quotient map. Thus we get isomorphisms

$$H^*_{L,c}(M^0) \xrightarrow{p_2^*} H^*_{G\times L,c}(N^0) \xleftarrow{\pi^*} H^*_{G,c}(G \times_L M^0).$$

We set $j = \pi^{*-1}p_2^*$:

$$j : H^*_{L,c}(M^0) \xrightarrow{p_2^*} H^*_{G\times L,c}(N^0) \xrightarrow{(\pi^*)^{-1}} H^*_{G,c}(\mathcal{M}^0_G). \tag{38}$$

Remark 5.13. As in the usual case (see [9], page 33), the isomorphism j^{-1} can be described as follows. Let $\gamma(y)$, with $y \in \mathfrak{g}$, be an equivariant form on $P/L = \mathcal{M}$ representing $[\gamma] \in H^*_{G,c}(\mathcal{M}^0_G) = H^*_{G,c}(G \times_L M^0)$. We restrict γ to the L invariant submanifold M embedded in $\mathcal{M}$ by $m \mapsto (e, \mu_L(m), m)$ and obtain an L-equivariant form on M. We can represent $j^{-1}[\gamma]$ by $\gamma(x)|_M$ with $x \in \mathfrak{l}$.

Given a class $[\alpha] \in H^*_{L,c}(M^0)$, our goal is to compare $\mathrm{infdex}_L^{\mu_L}([\alpha])$ and $\mathrm{infdex}_G^{\mu_G}(j([\alpha]))$, the first being a distribution on $\mathfrak{l}^*$ and the second one on $\mathfrak{g}^*$. We shall show that $\mathrm{infdex}_G^{\mu_G}(j([\alpha]))$ *is induced* by $\mathrm{infdex}_L^{\mu_L}([\alpha])$, according to Definition 5.6.

Theorem 5.14. *Let* $[\alpha] \in H^*_{L,c}(M^0)$*, then*

$$\mathrm{infdex}_G^{\mu_G}(j[\alpha]) = \mathrm{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*}(\mathrm{infdex}_L^{\mu_L}([\alpha])). \tag{39}$$

Proof. Consider the form $\gamma := 1 \wedge \alpha$ on $G \times M$, where α is a representative of $[\alpha]$. Take the map $\pi : N^0 = G \times M^0 \to G \times_L M^0$. By the definition of j we see that $[\gamma] = \pi^* j[\alpha]$.

Consider the $G\times L$ manifold $N = T^*G\times M$. To this manifold we can apply Corollary 4.15. Let f_1 be a variable test function on $\mathfrak{l}^*$ and f_2 be a given test function on $\mathfrak{g}^*$. The distribution $f_1 \to \langle \mathrm{infdex}_{G\times L}^{\tilde{\mu}_{G\times L}}([\gamma]), f_2 f_1\rangle$ is given by a smooth density $D(\zeta)d\zeta$ on $\mathfrak{l}^*$, and the value $D(0)$ is $\mathrm{vol}(L, dl)\langle \mathrm{infdex}_G^{\mu_G}(j[\alpha]), f_2\rangle$.

Let us compute $\langle \mathrm{infdex}_{G\times L}^{\tilde{\mu}_{G\times L}}([\gamma]), f_2 f_1\rangle$ using the fact that γ is the exterior product $1 \wedge \alpha$. We consider the product manifold $T^*G \times M$ provided with the action of $G\times G\times L$ where $G\times G$ acts by left and right action on T^*G and $G_2 = L$ acts on M. Consider next the embedding of $G \times L$ as the subgroup $\{((g, l), l), g \in G, l \in L\}$ of $G \times G \times L$. Remark that our given action form on N is is $G \times G \times L$ invariant and that $N^0 = G \times M^0$ is also the set of zeroes of the moment map μ for the group $G \times G \times L$. We may thus apply first the exterior product property (Proposition 4.4) and then the restriction property (Proposition 4.7) to compute $\mathrm{infdex}_{G\times L}^{\tilde{\mu}_{G\times L}}([\gamma])$.

Denote by $p : \mathfrak{g}^* \to \mathfrak{l}^*$ the restriction map. Then for $\zeta \in \mathfrak{l}^*$ and $(\xi_1, \xi_2) \in \mathfrak{g}^* \oplus \mathfrak{g}^*$ the restriction map $R :\ \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{l}^* \to \mathfrak{g}^* \oplus \mathfrak{l}^*$ is given by

$$R : \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{l}^* \to \mathfrak{g}^* \oplus \mathfrak{l}^*,\ (\xi_1, \xi_2, \zeta) \mapsto (\xi_1, \zeta - p(\xi_2))$$

and we have

$$\mathrm{infdex}_{G\times L}^{\tilde{\mu}_{G\times L}}(1 \wedge \alpha) = R_*(\mathrm{infdex}_{G\times G}^{\mu}(1) \otimes \mathrm{infdex}_L^{\mu_L}([\alpha])).$$

Let f_1 be a test function on $\mathfrak{l}^*$ and let f_2 be a test function on $\mathfrak{g}^*$. The function $R^*(f_1 f_2)(\xi_1, \xi_2, \zeta)$ is the function $f_1(\zeta + p(\xi_2))f_2(\xi_1)$.

Using the formula for $\mathrm{infdex}_{G\times G}^{\mu}(1)$ for T^*G of Proposition 4.19, we obtain

$$\langle \mathrm{infdex}_{G\times L}^{\tilde{\mu}_{G\times L}}(1 \wedge \alpha), f_1 f_2\rangle = \langle \mathrm{infdex}_{G\times G}^{\mu}(1) \otimes \mathrm{infdex}_L^{\mu_L}([\alpha]), R^*(f_1 f_2)\rangle$$

$= \langle \mathrm{infdex}^{\mu}_{G\times G}(1)\otimes \mathrm{infdex}^{\mu_L}_{L}([\alpha]), f_1(\zeta+p(\xi_2))f_2(\xi_1)\rangle = \langle \mathrm{infdex}^{\mu_L}_{L}([\alpha]), q(f_1,f_2)\rangle$
with (A is defined in (35)):

$$q(f_1,f_2)(\zeta) = \int_{\mathfrak{g}^*}\int_G f_1(\zeta+p(\xi))f_2(-g\xi)dgd\xi = \int_{\mathfrak{g}^*} f_1(\zeta+p(\xi))Af_2(-\xi)d\xi.$$

Integrating first on the fiber $p:\mathfrak{g}^*\to\mathfrak{l}^*$, then on $\mathfrak{l}^*$, we see that

$$q(f_1,f_2)(\zeta) = f_1 * (p_*(Af_2))(\zeta)$$

where $u*v$ is the convolution product of test functions on $\mathfrak{l}^*$.

Then we obtain

$$\langle \mathrm{infdex}^{\tilde\mu_{G\times L}}_{G\times L}([\gamma]), f_1f_2\rangle = \langle \mathrm{infdex}^{\mu_L}_{L}([\alpha]), f_1*(p_*(Af_2))(\zeta)\rangle.$$

This is a smooth density with respect to $\zeta\in\mathfrak{l}^*$, and if f_1 tends to $\delta_0(\zeta)$, then $\langle \mathrm{infdex}_{G\times L}([\gamma]), f_1f_2\rangle$ tends to

$$\langle \mathrm{infdex}^{\mu_L}_{L}([\alpha]), p_*(Af_2)(\zeta)\rangle = \mathrm{vol}(L,dl)\langle \mathrm{Ind}^{\mathfrak{g}^*}_{\mathfrak{l}^*}\mathrm{infdex}_L([\alpha]), f_2(\zeta)\rangle.$$

We thus obtain the wanted formula. □

5.15. **Maximal tori.** As usual, let M be a G-manifold with a G-invariant action form σ. Let $T\subset G$ be a maximal torus. We show next how to reduce the calculation of the infdex map for G to the calculation of the infdex map for T.

Associated to σ, we have the moment maps $\nu_G : M\to\mathfrak{g}^*$ and $\nu_T = p\circ\nu_G : M\to\mathfrak{t}^*$, with $p:\mathfrak{g}^*\to\mathfrak{t}^*$ the restriction map.

Consider M as a T-manifold, and consider $N = T^*G\times M$, provided, as in Subsection 5.7 (here the group L is T), with action form $\psi = p_1^*\omega + p_2^*\sigma$ and the action of $G\times T$: the group G acts on T^*G by left action, and trivially on M, the group T acts on G by right action and acts on M. We denote by $\tilde\mu_{G\times T} = \tilde\mu_G\oplus\tilde\mu_T$ the corresponding moment map.

Recall, by Formula (37), that

$$P = \tilde\mu_T^{-1}(0) = \{(g,\xi,m); g\in G, \xi\in\mathfrak{g}^*, m\in M; \xi|_{\mathfrak{t}} = \nu_T(m)\}$$

is a $G\times T$ manifold on which G acts by $g_0\cdot(g,\xi,m) = (g_0g,\xi,m)$, for $g_0\in G$, $(g,\xi,m)\in P$ and T acts by $t\cdot(g,\xi,m) = (gt^{-1}, t\xi, tm)$.

We then consider $\mathcal{M} := P/T$, with moment map $\mu_G([g,\xi,m]) = g\xi$ (36). Recall that $\mathcal{M}^0_G$ is isomorphic to $G\times_T M^0_T$ embedded in P/T by $[g,0,m]$.

For $[\alpha]\in H^*_{G,c}(M^0_G)$, we want to produce an element $r([\alpha])\in H^*_{G,c}(\mathcal{M}^0_G) = H^*_{G,c}(G\times_T M^0_T)$ which has the same infinitesimal index as $[\alpha]$.

Proposition 5.16. *We can embed $G\times M$ in P by the map*

$$\gamma(g,m) = (g,\nu_G(g^{-1}m), g^{-1}m).$$

The map γ is $G\times T$ equivariant, where G acts on $G\times M$ by diagonal action (left on G) while T acts by the right action on G and not on M.

Proof. First $(g, \nu_G(g^{-1}m), g^{-1}m) \in P$ since $\nu_G(g^{-1}m)|_{\mathfrak{t}^*} = \nu_T(g^{-1}m)$. Next $\gamma(hg, hm) = (hg, \nu_G(g^{-1}m), g^{-1}m)$ and

$$\gamma(gt^{-1}, m) = (gt^{-1}, \nu_G(tg^{-1}m), tg^{-1}m) = (gt^{-1}, t\nu_G(g^{-1}m), tg^{-1}m).$$

□

Corollary 5.17. *The map γ induces, modulo the action of T, an embedding still denoted by $\gamma : G/T \times M \hookrightarrow \mathcal{M} = P/T$. Thus the manifold $G/T \times M$, with diagonal G-action is identified to a G-invariant submanifold of $\mathcal{M}$.*

In fact more is true. Let $q : \mathcal{M} \to G/T \times M$ be the projection given by $q(g, \xi, m) = (gT, gm)$. Let $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{t}^{\perp}$ be the canonical T-invariant decomposition of $\mathfrak{g}^*$. Then we claim that

Proposition 5.18. *$q\gamma$ is the identity and $q : \mathcal{M} \to G/T \times M$ is a vector bundle with fiber $\mathfrak{t}^{\perp}$.*

Proof. The first claim comes from the definitions. As for the second, we may identify P with $G \times M \times \mathfrak{t}^{\perp}$ by the map

$$P \to G \times M \times \mathfrak{t}^{\perp}, \quad (g, \xi, m) \mapsto (g, m, \xi - \nu_T(m)).$$

□

Lemma 5.19. *The restriction of the moment map μ_G on $\mathcal{M}$ to $G/T \times M$ is just $(gT, m) \mapsto \nu_G(m)$ with zeroes $G/T \times M_G^0$.*

Proof. We have $\mu_G(g, \xi, m) = g\xi$ by the previous discussion. An element (g, m) corresponds to the triple $(g, \nu_G(g^{-1}m), g^{-1}m)$, so the claim follows since ν_G is G–equivariant. □

We now apply the construction $\gamma_!$ of Subsection 4.8 to the manifold $G/T \times M$ embedded by γ in $\mathcal{M}$.

Take an equivariant form β on G/T with class

$$[\beta] = \frac{i^{(\dim G/T)/2}}{|W|} e(G/T) \tag{40}$$

where W is the Weyl group and $e(G/T)$ is the equivariant Euler class. Notice that since $|W|$ equals the Euler characteristic of G/T, the polynomial function $\int_{G/T}[\beta]$ is the constant

$$\int_{G/T} [\beta] = i^{(\dim G/T)/2}.$$

Thus, by Theorem 3.12, the infinitesimal index of $[\beta]$ is just the δ–function on $\mathfrak{g}^*$. Let $[\alpha] \in H^*_{G,c}(M^0_G)$. We then construct the element $[\beta \wedge \alpha]$ in the compactly supported equivariant cohomology

$$[\beta \wedge \alpha] \in H^*_{G,c}((G/T \times M)^0_G) = H^*_{G,c}(G/T \times M^0_G).$$

Lemma 5.20. *The infinitesimal index of $[\beta \wedge \alpha]$ is equal to the infinitesimal index of $[\alpha]$.*

Proof. Apply Corollary 5.4. □

Under the embedding $\gamma : G/T \times M \hookrightarrow \mathcal{M}$ of action manifolds (cf. 5.17), by Theorem 4.9, we have now a homomorphism

$$\gamma_! : H^*_{G,c}(G/T \times M^0_G) \to H^*_{G,c}(\mathcal{M}^0_G)$$

preserving infdex.

We define

$$r([\alpha]) := \gamma_!([\beta \wedge \alpha]) \in H^*_{G,c}(\mathcal{M}^0_G). \tag{41}$$

We then have, combining Lemma 5.15 with Theorem 4.9

$$\mathrm{infdex}^{\nu_G}_G([\alpha]) = \mathrm{infdex}^{\mu_G}_G(r[\alpha]). \tag{42}$$

On the other hand, we have the isomorphism

$$j : H^*_{T,c}(M^0_T) \to H^*_{G,c}(G \times_T M^0_T)$$

and we have shown in Theorem 39 that

$$\mathrm{infdex}^{\mu_G}_G(j[\theta]) = \mathrm{Ind}^{\mathfrak{g}^*}_{\mathfrak{t}^*}\mathrm{infdex}^{\nu_T}_T([\theta])$$

for any $[\theta] \in H^*_{T,c}(M^0_T)$.

We deduce

Theorem 5.21. *Take the commutative diagram*

$$\begin{array}{ccccc} H^*_{G,c}(M^0_G) & \xrightarrow{r} & H^*_{G,c}(G \times_T M^0_T) & \xrightarrow{j^{-1}} & H^*_{T,c}(M^0_T) \\ \mathrm{infdex}\downarrow & & \mathrm{infdex}\downarrow & & \mathrm{infdex}\downarrow \\ \mathcal{D}'(\mathfrak{g}^*)^G & \xrightarrow{id} & \mathcal{D}'(\mathfrak{g}^*)^G & \xleftarrow{\mathrm{Ind}^{\mathfrak{g}^*}_{\mathfrak{t}^*}} & \mathcal{D}'(\mathfrak{t}^*). \end{array} \tag{43}$$

The element $[\lambda] := j^{-1}r([\alpha]) \in H^*_{T,c}(M^0_T)$ *is such that*

$$\mathrm{infdex}^{\nu_G}_G([\alpha]) = \mathrm{Ind}^{\mathfrak{g}^*}_{\mathfrak{t}^*}\mathrm{infdex}^{\nu_T}_T(j^{-1}r([\alpha])) \tag{44}$$

Let us finally give an explicit formula for the element $[\lambda] = j^{-1}r([\alpha]) \in H^*_{T,c}(M^0_T)$ corresponding to $[\alpha] \in H^*_{G,c}(M^0_G)$.

Let $\mathrm{Pf}(x) = \det^{1/2}_{\mathfrak{t}^\perp}(x)$ be the Pfaffian associated to the action of $x \in \mathfrak{t}$ in the oriented orthogonal space $\mathfrak{t}^\perp$.

We need the

Proposition 5.22. *The restriction of the form $\beta(x)$ at the point $e \in G/T$ is the polynomial $|W|^{-1}(2i\pi)^{-(\dim G/T)/2}\mathrm{Pf}(x)$.*

Proof. By construction, the equivariant Euler class is the restriction to G/T of the Thom class of the tangent bundle. The fiber of the tangent bundle at the T fixed point e is isomorphic to $\mathfrak{t}^\perp$. Thus this class restricts at the fixed point e as $(-2\pi)^{-(\dim G/T)/2}\mathrm{Pf}(x)$ ([15], see [5], Theorem 7.41, [18]). □

Recall the decomposition $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{t}^\perp$. Let us consider the map $\nu_\perp : M \to \mathfrak{t}^\perp$ which is uniquely defined by the identity $\nu_G = \nu_T \oplus \nu_\perp$. Then $\nu_T^{-1}(0) \cap \nu_\perp^{-1}(0) = \nu_G^{-1}(0)$.

Denote by τ_0 the T-equivariant Thom class of the embedding $0 \to \mathfrak{t}^\perp$, a compactly supported equivariant class on $\mathfrak{t}^\perp$. Then $\tau_\perp := \nu_\perp^* \tau_0$ is a closed equivariant class on M supported on a small neighborhood A of $\nu_\perp^{-1}(0)$. It follows that

Lemma 5.23. *If $[\alpha] \in H^*_{G,c}(M^0_G)$, we can choose τ_0 so that the class $\tau_\perp \wedge \alpha$ defines a class in $H^*_{T,c}(M^0_T)$.*

Proof. Let $K \subset M \setminus M^0_G$ be the support of $D\alpha$, then $D(\tau_\perp \wedge \alpha) = \tau_\perp \wedge D\alpha$ is supported in $A \cap K$. Since $\emptyset = K \cap M^0_G = K \cap M^0_T \cap \nu_\perp^{-1}(0)$, we can choose τ_0 so that $A \cap K \cap M^0_T = \emptyset$. □

By Remark 5.13, an equivariant form representing $j^{-1} r([\alpha])$ is the restriction to $M = \{(e, 0, m), m \in M\}$ of $r(\alpha)(x)$, when $x \in \mathfrak{t}$. We still denote it by $j^{-1}(r(\alpha))(x)$.

Theorem 5.24. *We can choose the Thom classes so that*

$$j^{-1}(r(\alpha))(x) = |W|^{-1}(2i\pi)^{-(\dim G/T)/2}\mathrm{Pf}(x)\alpha(x) \wedge \tau_\perp(x).$$

Proof. Let $\tau_{G/T\times M}$ be a Thom class of the bundle $q : \mathcal{M} \to G/T \times M$ (Proposition 5.18). Then, by the $\gamma_!$ construction, the associated equivariant form on $\mathcal{M}$ which we denoted by $r(\alpha)$, is $q^*(\beta \wedge \alpha) \wedge \tau_{G/T\times M}$.

Now the bundle $q : \mathcal{M} \to G/T \times M$ is trivial over $e \times M$ and isomorphic to $\mathfrak{t}^\perp \times M$ by $(\xi, m) \mapsto (e, \xi + \nu_G(m), m)$.

The restriction of the Thom class $\tau_{G/T\times M}$ gives a Thom class for this trivial bundle. We can then assume that the restriction of $\tau_{G/T\times M}$ is $\tau_0(\xi)$.

As (e, M) is embedded by $\xi = \nu_\perp(m)$, we obtain our Theorem from Proposition 5.22.

□

Appendix A. Equivariant cohomology with compact support

A.1. **Compact supports.** We are going to assume in this appendix that all our spaces are locally compact and paracompact and we are going to work with Alexander-Spanier cohomology groups both ordinary and with compact support, and with real coefficients. We shall denote them by H^* or, if we take compact support, by H^*_c. H^* is a cohomology theory on spaces or pairs of spaces deduced from a functorial cochain complex $\mathcal{C}(X, Z)$ and H^*_c, the theory with compact supports, is associated to a natural subcomplex $\mathcal{C}_c(X, Z)$, (see [20] ch.6).

Let us now recall a few properties. The first is (see [20] ch.6, p.321, Lemma 11.)

Proposition A.2. *Let (X, Z) be a pair with X compact $Z \neq \emptyset$ closed. Set $U := X \setminus Z$. Then there are natural isomorphism $H^q_c(U) \simeq H^q(X, Z)$.*

In fact this is induced by the map of cochains complexes $\mathcal{C}_c(U) \to \mathcal{C}(X, Z)$ composition of the inclusions $\mathcal{C}_c(U) \to \mathcal{C}_c(X) \to \mathcal{C}(X)$ and of the quotient $\mathcal{C}(X) \to \mathcal{C}(X, Z)$.

In particular, if we take an open set U in a compact space X (for example we could take the one point compactification U^+ of a locally compact space U), we get that $H_c^*(U) = H^*(X, X \setminus U)$.

As an application of this, assume $Z \subset U$ is closed and U is open in a compact space X. Set $Y = X \setminus U$ and take the triple $(X, \tilde{Z}, Y)$ with $\tilde{Z} = Z \cup Y$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}_c^*(U \setminus Z) & \longrightarrow & \mathcal{C}_c^*(U) & \longrightarrow & \mathcal{C}_c^*(Z) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{C}^*(X, \tilde{Z}) & \longrightarrow & \mathcal{C}^*(X, Y) & \longrightarrow & \mathcal{C}^*(\tilde{Z}, Y) & \longrightarrow & 0 \end{array}$$

Using the exactness of the bottom line we deduce the long exact sequence

$$\cdots \to H_c^h(U \setminus Z) \xrightarrow{i_*} H_c^h(U) \xrightarrow{j^*} H_c^h(Z) \longrightarrow H_c^{h+1}(X \setminus Z) \to \cdots .$$

On the other hand, the top line induces a homomorphism of chain complexes

$$\mu : \mathcal{C}_c^*(U)/\mathcal{C}_c^*(U \setminus Z) \to \mathcal{C}_c^*(Z)$$

and since the vertical arrows induce isomorphism in cohomology, using the five lemma we easily deduce

Proposition A.3. *The homomorphism μ induces an isomorphism in cohomology.*

In order to compare the Alexander–Spanier and singular cohomology, one needs to pass to the associated sheaves (see [20] ch.6, p.324). Thus, under suitable topological conditions, we obtain a natural isomorphism between Alexander–Spanier and singular cohomology.

In particular consider a C^∞-manifold M and a closed subset $Z \subset M$. Further assume that Z is locally contractible. We then have (see [20] ch.6, p.341 Corollary 7) that, under these assumptions, we can use singular cochains and in fact, in the case of a manifold, singular C^∞ cochains to compute cohomology since Alexander Spanier and singular cohomology are naturally isomorphic in this case.

Integrating on singular C^∞-simplexes we get a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{A}_c^*(M \setminus Z) & \longrightarrow & \mathcal{A}_c^*(M) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & {}_\infty\mathcal{C}_c^*(M \setminus Z) & \longrightarrow & {}_\infty\mathcal{C}_c^*(M) \end{array}$$

$\mathcal{A}_c^*$ being the complex of differential forms with compact support. We deduce a homomorphism of cochain complexes

$$\nu : \mathcal{A}_c^*(M)/\mathcal{A}_c^*(M \setminus Z) \to {}_\infty\mathcal{C}_c^*(M)/{}_\infty\mathcal{C}_c^*(M \setminus Z)$$

Since the vertical arrows induce isomorphism in cohomology, we get a de Rham model for $H_c^*(Z)$.

Proposition A.4. *The homomorphism ν induces isomorphism in cohomology. In particular $H_c^*(Z)$ is naturally isomorphic to the cohomology of the complex $\mathcal{A}_c^*(M)/\mathcal{A}_c^*(M \setminus Z)$.*

A.5. **Classifying spaces.** We now take a compact Lie group G and denote by B_G its classifying space (which is not locally compact). Recall that B_G is a polyhedron with finitely many cells in each dimension and it has a filtration $(B_G)_0 \subset \cdots \subset (B_G)_n \subset (B_G)_{n+1} \subset \cdots \subset B_G$ by compact manifolds with the property that the inclusion $(B_G)_n \subset B_G$ induces isomorphism in cohomology up to degree n. For example, if G is a s-dimensional torus, $B_G = \mathbb{C}P(\infty)^s$ and we may take $(B_G)_n = \mathbb{C}P(n)^s$ (indeed in this case the inclusion induces an isomorphism up to degree $2n-1$).

We denote by $\pi : E_G \to B_G$ the universal fibration and set $(E_G)_n = \pi^{-1}((B_G)_n)$. Thus $(E_G)_n$ is also a compact C^∞ manifold and a principal bundle over $(B_G)_n$.

Recall now that for any G-space Y, $H_G^*(Y) = H^*(Y \times_G E_G)$.

We can define the equivariant cohomology with compact support of a G-space as follows. Take U locally compact. Embed U in his one point compactification U^+. The action of G extends to U^+ and we set

Definition A.6. $H_{G,c}^*(U) = H_G^*(U^+, \infty)$.

Some remarks are in order.

- If U is compact, then U^+ is the disjoint union $U \cup \{\infty\}$ so $H_{G,c}^*(U) = H_G^*(U)$.
- If U is non compact, then $H_{G,c}^*(U) = H^*(U^+ \times_G E_G, B_G)$ where $B_G = \{\infty\} \times_G E_G$.
- All the equivariant cohomologies are modules over $H_G^*(pt)$ and all the homomorphisms are module homomorphisms.

Recall that by the properties of $(B_G)_m$ for any $h \geq 0$, and for all m large enough, $H^r(B_G, R) = H^r((B_G)_m, R)$ for $0 \leq r \leq 2h$. So given a G-space X, the spectral sequences of the fibrations $X \times_G E_G \to B_G$ and $X \times_G (E_G)_m \to (B_G)_m$ have the same $E_r^{p,q}$ for all r and $p + q \leq h$. In particular we get for any pair (X, Z) of G-spaces that for large m, $H_G^h(X, Z) = H^h(X \times_G (E_G)_m, Z \times_G (E_G)_m)$. From Proposition A.2, we then deduce

Proposition A.7. *Let X be a G-space with X compact Hausdorff and $Z \neq \emptyset$ a closed G-stable subspace. Set $U := X \setminus Z$. Then there is a natural isomorphism $H_{G,c}^q(U) \simeq H_G^q(X, Z)$.*

Furthermore for m large with respect to h, $H_{G,c}^h(U) \simeq H_c^h(U \times_G (E_G)_m)$.

Take now a C^∞ manifold M with a C^∞ action of G and a closed G-stable subset Z in M which we assume to be locally contractible. For instance if Z is locally triangular as for instance when Z is semi–analytic [14]. The same is true for $Z \times_G (E_G)_m$ for any m so we can apply Proposition A.4 and deduce that for m large with respect to h, $H^h_{G,c}(Z)$ is the h-th cohomology group of the complex $\mathcal{A}^*_c(M \times_G (E_G)_m)/\mathcal{A}^*_c((M \setminus Z) \times_G (E_G)_m)$.

But one knows (see [10]) that for any m we have a natural morphism of complexes $\mathcal{A}_{G,c}(M) \to \mathcal{A}^*_c(M \times_G (E_G)_m)$ which induces isomorphism in cohomology in small degree. The same holds also for the open set $M \setminus Z$ so that we get a commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{A}_{G,c}(M \setminus Z) & \longrightarrow & \mathcal{A}_{G,c}(M) \\ & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathcal{A}^*_c((M \setminus Z) \times_G (E_G)_m)) & \longrightarrow & \mathcal{A}^*_c(M \times_G (E_G)_m)) \end{array}$$

which induces a morphism of complexes

$$\rho : \mathcal{A}_{G,c}(M)/\mathcal{A}_{G,c}(M \setminus Z) \to \mathcal{A}^*_c(M \times_G (E_G)_m)/\mathcal{A}^*_c((M \setminus Z) \times_G (E_G)_m)$$

From this we immediately deduce

Proposition A.8. *$H^*_{G,c}(Z)$ equals the cohomology of the complex $\mathcal{A}_{G,c}(Z, M)$.*

Proof. From the above considerations we have, if m is large with respect to h, ρ induces an isomorphism in cohomology in degree h. Since we have seen that in degree h the cohomology of the complex $\mathcal{A}^*_c(M \times_G (E_G)_m)/\mathcal{A}^*_c((M \setminus Z) \times_G (E_G)_m)$ is $H^h_{G,c}(Z)$, everything follows. □

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