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Counting Integer Flows in Networks

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Abstract. This paper discusses analytic algorithms and software for the enumeration of all integer flows inside a network. Concrete applications abound in graph theory, representation theory, and statistics. Our methods are based on the study of rational functions with poles on arrangements of hyperplanes; they surpass traditional exhaustive enumeration and can even yield formulas when the input data contains some parameters. We also discuss the calculation of chambers in detail because it is a necessary subroutine.

1. Introduction

A *network* is a graph with directed edges, with multiple copies of the edges allowed, and where each node v has an integer value specified, the so-called *excess* of v, and

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each arc has an assigned nonnegative integer value, or infinity, called its *capacity*. We can think of the excess and capacities as functions. A *feasible flow* is an assignment of real values to the arcs of the network so that for any node v the difference between the sum of values in outgoing arcs minus the sum of values in incoming arcs equals the prescribed excess of the node v and the capacities of the arcs are not surpassed. In this paper we study the problem of effectively counting the number of different *integral feasible flows* in a network. It is well-known that this problem is $\#P$-hard in the computational category of counting problems (see Section 7.3 in [**15**] and Chapter 18 of [**21**]) because the problem of counting perfect matchings in bipartite graphs reduces to it. Despite this bad complexity, concrete applications abound in graph theory [**16**], representation theory [**18**], and statistics [**14**] and thus finding good methods for attacking concrete examples is of importance. Our goal is to show that using the algebraic–analytic structure of the problem allows us to count flows in complicated instances very fast, surpassing traditional exhaustive enumeration.

Continuing the work started in [**2**] we present practical counting algorithms from which one can, in fact, derive counting formulas when the excess values at the vertices have parameters. This is not the only case where residues formulas are used for counting lattice points (see [**4**], [**19**] and references therein), but here we stress a special type of "partial fraction" decomposition that simplifies calculations. We only discuss the problem of exact counting and we do not discuss approximation or estimation. This is another very active area of research, mostly based on probabilistic methods (e.g., random walks). For more information, see [**14**], [**29**].

The set of all *feasible flows* with given excess vector b and capacity vector c is a convex polytope, the well-known *flow polytope*, which is defined by the constraints $\Phi_G x = b$, $0 \le x \le c$, where Φ_G denotes the node-arc incidence matrix of G (a *network matrix*). The incidence matrix Φ_G has one column per arc and one row per node. Each column of Φ_G has as many entries as nodes. For an arc going from i to j, its corresponding column has zeros everywhere except at the ith and jth entries. The jth entry, the *head* of the arrow, receives a -1 and the ith entry, the *tail* of the arrow, a 1. A famous instance is the *max-flow min-cut* problem [**24**]. This is the case when b has first entry v, last entry $-v$, and 0 elsewhere. In Figure 1(b) we list all possible flows with $v = 11$, the maximal possible from the network information specified in Figure 1(a).

An important feature of the network incidence matrix Φ_G is that it is unimodular. We say that the matrix Φ_G is *unimodular*, if the columns of Φ_G span a lattice, denoted by $\mathbb{Z}\Phi_G$, and whenever a is in this lattice $\mathbb{Z}\Phi_G$, the polytope $P(\Phi_G, a) = \{x \mid \Phi_G x = a,\ x \ge 0\}$ has vertices with integral coordinates. Even more strongly, network matrices are, in fact, *totally unimodular* matrices (see Chapter 19 in [**24**]), which means that the lattice generated by their columns is the standard integral lattice $\mathbb{Z}^n$. Note that the integral feasible flows are precisely the integer lattice points inside the flow polytope.

The algorithms and formulas used in this paper for counting lattice points are based on the notion of *total residue* (to be reviewed in Section 2), the main concept

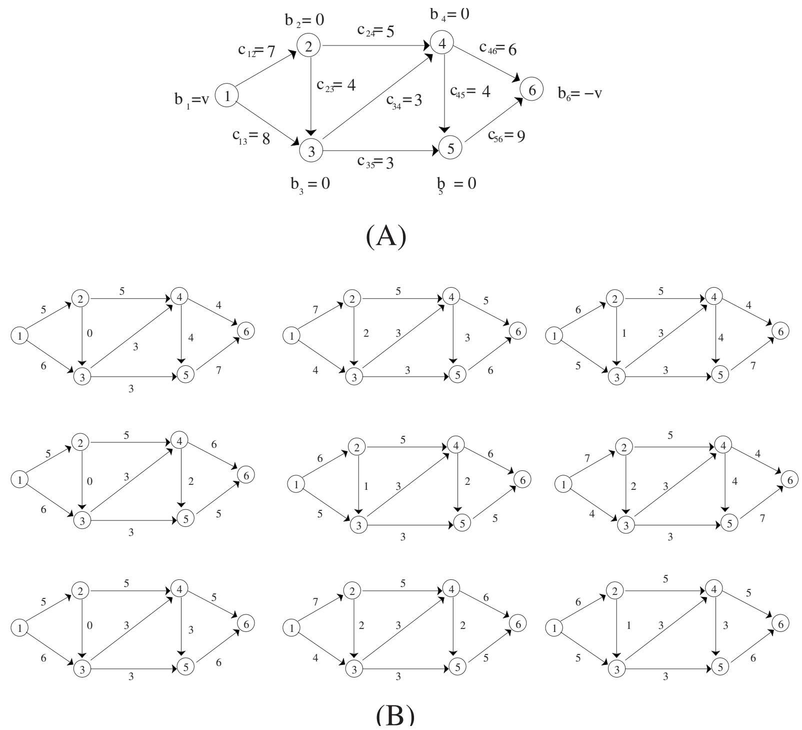
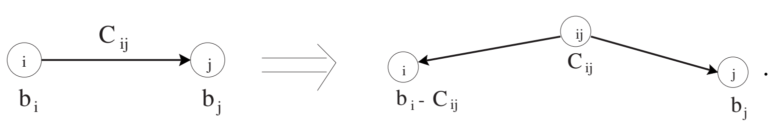


Fig. 1. (b) Counting all maximum flows of (a) a specific network.

involved being the study of rational functions with poles on an arrangement of hyperplanes. The enumeration theory we present was extended to arbitrary rational polyhedra in [**28**]. The particular description we do here is valid for all unimodular matrices. The following lemma implies that it is enough to describe our counting formulas and algorithms for networks without restricted capacities on the arcs and that have no directed cycles. These networks are called *acyclic uncapacitated networks*. The easy details of the proof are left to the reader.

Lemma 1. *Given a network G with n nodes and m arcs, with capacity c and excess function b, there is an acyclic uncapacitated network $\widehat{G}$ with $n+m$ nodes, $2m$ arcs, and excess function $\widehat{b}$ (a linear combination of b, c) such that the integral flows in both networks are in bijection. The network $\widehat{G}$ is obtained from G by replacing each arc by two new arcs and modifying the capacity and excess functions as illustrated in the figure below*:



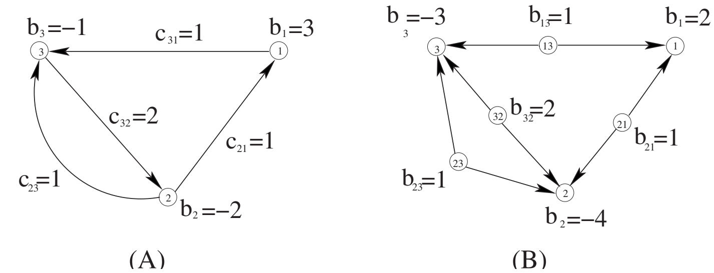


Fig. 2. (a) shows a network with specified capacity (c_i's) and excess functions (b_i's). By Lemma 1, we obtain from G a new network $\widehat{G}$ without restriction on capacities, shown in (b).

Here is a small example to illustrate some of the notions mentioned above: The node-arc incidence matrix for the graph G in Figure 2(a) is defined by:

$$\Phi_{G_1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix}.$$

The equation $\Phi_{G_1} x = b$ reads as the series of equations $x_1 - x_2 = b_1$, $x_2 - x_3 + x_4 = b_2$, $-x_1 + x_3 - x_4 = b_3$. Remark that to have solutions to these three equations, we must have $b_1 + b_2 + b_3 = 0$. The three equations express the fact that, at each node $v \in \{1, 2, 3\}$, the difference between the sum of values in outgoing arcs minus the sum of values in incoming arcs equals the prescribed excess b_i of the node v. Feasible flows are restricted furthermore by the conditions $0 \leq x_i \leq c_i$. In this particular example, the flow polytope is a two-dimensional polytope (a polygon).

Because of Lemma 1, and due to interesting applications in representation theory, it makes sense to focus our efforts on the special case of uncapacitated acyclic graphs, and we do so in Section 3. An important case is what representation theorists would call the *Kostant partition function* associated to the complete graph K_n with n nodes. There are many ways to induce an acyclic orientation to the complete graph, here we take the following convention of orientation: Whenever there is an edge of the graph G between i and j, with $i < j$, then we direct the arrow from i to j. Another example of flow polytope is the *Pitman–Stanley polytope* [**22**] that is a multiple edge graph with vertices $(1, \ldots, n)$ and edges $\{i, i+1\}$ and $\{i, n\}$ and the last edge $\{n-1, n\}$ of multiplicity two. Another interesting class of flow polytopes are the *transportation polytopes* [**24**]. These polytopes are usually described in terms of m by n real matrices (denoted here by $M_{m,n}(\mathbb{R})$): Fix $c = (c_1, \ldots, c_n) \in \mathbb{R}^n_{\geq 0}$ and $d = (d_1, \ldots, d_m) \in \mathbb{R}^m_{\geq 0}$ such that $\sum_{i=1}^m d_i = \sum_{i=1}^n c_i$ and define $T_{m,n}(d, c)$ as the set

$$\left\{ X = \{x_{i,j}\} \in M_{m,n}(\mathbb{R}); \quad \begin{array}{ll} x_{i,j} \geq 0, 1 \leq i \leq m, 1 \leq j \leq n, & \\ \sum_k x_{i,k} = d_i, & 1 \leq i \leq m, \\ \sum_k x_{k,j} = c_j, & 1 \leq j \leq n. \end{array} \right\}.$$

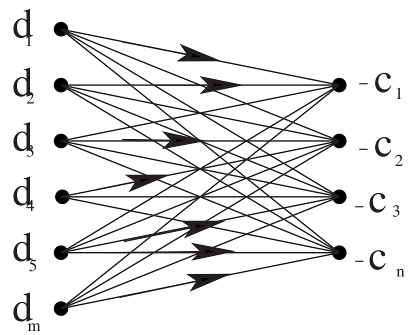


Fig. 3. The transportation polytopes are network polytopes of complete bipartite graphs.

Then $T_{m,n}(d, c)$ is a polytope called the *transportation polytope* associated for the vectors d, c. We can easily see that this is another flow polytope over a complete bipartite network $K_{m,n}$: See Figure 3 where the first m nodes receive excess values $(d_1, \ldots, d_m)$ and the n nodes in the second block receive the excess values $(-c_1, -c_2, \ldots, -c_n)$. The arcs are oriented from the first block to the second. In the family of transportation polytopes there is a distinguished member, the *Birkhoff polytope* that has been extensively studied (see, for instance, the references in the recent paper [**4**]).

It is well-known that the counting formulas of integer flows in a network come in piecewise polynomial functions (see [**8**], [**27**], [**28**]). It is therefore of interest to understand the regions of validity of each polynomial formula, the so-called *chambers*. In Section 4 we discuss the structure of the chambers and how to determine the number of chambers. This part of the theory applies to nonunimodular matrices too. The question of how many chambers are possible was first raised in [**18**]. The combinatorial investigation of the chambers partition functions was initiated by [**1**]. See also [**10**].

2. Formulas for the Volume and the Number of Integral Points in General Polytopes

In this section, we outline the principles used in the algorithms we implemented for counting integer flows. The ideas of this section are valid for general convex polytopes [**2**], [**28**], thus we describe things in a general setting when possible. Later, in Section 3, we will use particular properties of flow polytopes associated with graphs to explicitly compute counting formulas.

Let Φ be an integral r by N matrix with column vectors $\varphi_1, \ldots, \varphi_N$. Let b be an r-dimensional column vector and $\mathcal{P} = \{x \in \mathbb{R}^N_{\geq 0} \mid \Phi x = b\}$, the rational convex polytope associated to Φ and b. We assume that b is in the cone $C(\Phi)$ spanned by the nonnegative linear combinations of columns $\varphi_1, \varphi_2, \ldots, \varphi_N$ of Φ. Without loss of generality, we may assume that $\operatorname{rank}(\Phi) = r$. If this is not the case, take the subspace of $\mathbb{R}^r$ generated by the columns of our matrix and rewrite the polytope in terms of an appropriate rank k matrix of dimension k by N. For example, for the network polytopes with r nodes, the matrices are not full rank, as the sum of the rows is always equal to 0. Thus the last row is a linear combination of the first $r-1$ rows. In the examples we treat, deleting this last row will turn this network matrix into a matrix of full rank.

In what follows we assume that $\mathrm{kernel}(\Phi) \cap \mathbb{R}^N_{\geq 0} = \{0\}$. Then 0 is not in the convex hull of the vectors φ_k and the cone $C(\Phi)$ is a pointed polyhedral cone in $\mathbb{R}^r$ (pointed cones have no linear subspace contained inside). For $a \in \mathbb{R}^r$ we denote by

$$P(\Phi, a) = \left\{ (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N_{\geq 0} \;\middle|\; \sum_{j=1}^{N} x_j \varphi_j = a \right\}.$$

It is obvious that $P(\Phi, a)$ is a convex polytope determined by the matrix Φ. Define

$$v(\Phi, a) = \mathrm{volume}(P(\Phi, a)).$$

If Φ spans a lattice in $\mathbb{R}^r$ and a belongs to this lattice, then define

$$k(\Phi, a) = |P(\Phi, a) \cap \mathbb{Z}^N|.$$

Thus $k(\Phi, a)$ is the number of solutions $(x_1, x_2, \ldots, x_N)$, in nonnegative integers x_j, of the equation $\sum_{j=1}^{N} x_j \varphi_j = a$. The function $k(\Phi, a)$ is called the *vector partition function* associated to Φ. The name partition comes from the fact that if $\Phi = [1, 2, \ldots, N]$, then $P(\Phi, a) \cap \mathbb{Z}$ is the set of solutions of the equation $a = 1x_1 + 2x_2 + \cdots + Nx_N$, thus the solution $(x_1, x_2, \ldots, x_N)$ clearly gives a partition of the number a whose parts are the numbers 1 to N.

The function $a \to k(\Phi, a)$ depends strongly of the multiplicities in the system Φ, i.e., how many times a vector appears as a column of the matrix Φ. From now on we will refer to Φ both as a matrix or as a multiset of vectors.

Lemma 2. *Let Φ be an integral r by N matrix with column vectors $\varphi_1, \ldots, \varphi_N$. Let $z \in \mathbb{R}^r$ denote a vector in the dual cone $\{z \mid \langle \varphi_i, z\rangle > 0$ for all i=1,...,$N\}$. Then,*

$$\sum_{a \in C(\Phi) \cap \mathbb{Z}^r} k(\Phi, a) e^{-\langle a, z\rangle} = \frac{1}{\prod_{\varphi \in \Phi} 1 - e^{-\langle \varphi, z\rangle}},$$

$$\int_{C(\Phi)} v(\Phi, a) e^{-\langle a, z\rangle}\, da = \frac{1}{\prod_{\varphi \in \Phi} \langle \varphi, z\rangle}.$$

Proof. The first formula arises by calculating in two different ways the sum

$\sum_{x\in\mathbb{Z}^N_{\geq 0}} e^{-\langle\Phi x,z\rangle}$. On one hand, we have

$$\sum_{x\in\mathbb{Z}^N_{\geq 0}} e^{-\langle\Phi x,z\rangle} = \sum_{x\in\mathbb{Z}^N_{\geq 0}} e^{-\left\langle\sum_i x_i\varphi_i,z\right\rangle} = \prod_{i=1}^{N}\sum_{x_i\in\mathbb{Z}_{\geq 0}} e^{x_i(-\langle\varphi_i,z\rangle)} = \prod_{i=1}^{N}\frac{1}{1-e^{-\langle\varphi_i,z\rangle}}.$$

The last equality is a trivial consequence of the geometric series identity. On the other hand, we can reorder the infinite sum $\sum_{x\in\mathbb{Z}^N_{\geq 0}} e^{-\langle\Phi x,z\rangle}$ in terms of the values that Φx takes when x changes. We group together summands of x, x' if $\Phi x = \Phi x' = a$. The proof of the other formula is identical to it. □

As a consequence of the lemma our goal is simply to compute the inverses of these two equations. The point is that one can write efficient formulas for the inversion of Laplace transforms in terms of residues. In the sequel, we will write indifferently $\langle\varphi, z\rangle$ or $\varphi(z)$.

Let Δ^+ denote the set $\{\Phi\}$, this means the elements of Φ are present *without multiplicities*. We define $\Delta = \Delta^+ \cup -\Delta^+$. The *chamber complex* is the unique polyhedral subdivision of the cone $C(\Delta^+)$ which is obtained as the common refinement of the simplicial cones $C(\sigma)$ running over all possible basic subsets of Δ^+. A subset σ of Δ is called a *basic subset* of Δ, if the elements $\varphi \in \sigma$ form a vector space basis for $\mathbb{R}^r$. The pieces of the resulting subdivision are called *chambers*. We will discuss the chambers in detail, specially how to compute the chambers, in Section 4. The important fact to remember is that for each chamber there is a quasipolynomial formula for $k(\Phi, a)$ and we explain now how to derive the formula on a given chamber.

Each $\varphi \in \Delta$ determines a linear form on $\mathbb{C}^r$ and a complex hyperplane $\{z \in \mathbb{C}^r \mid \varphi(z) = 0\}$ in $\mathbb{C}^r$. Consider the hyperplane arrangement

$$\mathcal{H}_{\mathbb{C}} = \bigcup_{\varphi\in\Delta_+}\{z \in \mathbb{C}^r \mid \varphi(z) = 0\}$$

and let R_Δ denote the space of rational functions of $z \in \mathbb{C}^r$ with poles on $\mathcal{H}_{\mathbb{C}}$. A function in R_Δ can be written $P(z)/\prod_{\varphi\in\Delta}\varphi(z)^{n_\varphi}$ where P is a polynomial function on r complex variables and n_φ are nonnegative integers. For a basic set σ, set

$$f_\sigma(z) := \frac{1}{\prod\limits_{\varphi\in\sigma}\varphi(z)}.$$

After a linear change of coordinates, the function f_σ is simply $1/z_1z_2\cdots z_r$ and we denote by S_Δ the subspace of R_Δ spanned by such "simple" elements f_σ . Elements f_σ are, in general, *not* linearly independent, as we see in the example below.

Example 3. Let Δ^+ be the set $\Delta^+ = \{e_1, e_2, e_1 - e_2\}$. Then we have the linear relation

$$\frac{1}{xy} = \frac{1}{y(x-y)} - \frac{1}{x(x-y)}$$

between elements $f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3}$ with $\sigma_1 = \{e_1, e_2\}$, $\sigma_2 = \{e_1, e_1 - e_2\}$ and $\sigma_3 = \{e_2, e_1 - e_2\}$ basic subsets of Δ^+. Here we have identified e_1, e_2 to the coordinate function x, y of an element $xe^1 + ye^2$ of the dual space.

Partial differentiation ∂_i preserves the space R_Δ. The key result we need is that there is a well-defined decomposition of R_Δ under the action of partial differentiations, a free module part generated by the basic rational functions f_σ, and a torsion module part, which is unnecessary for calculations and can be neglected.

Theorem 4 (Brion–Vergne [**9**]). *The vector space S_Δ is contained in the homogeneous component of degree $-r$ of R_Δ and we have the direct sum decomposition*

$$R_\Delta = S_\Delta \oplus \left(\sum_{i=1}^{r} \partial_i R_\Delta \right).$$

We call the projection map

$$\mathrm{Tres}_\Delta \colon R_\Delta \to S_\Delta$$

according to this decomposition the **total residue** *map.*

The projection $\mathrm{Tres}_\Delta(f)$ of a function f with poles on the union of hyperplanes $\mathcal{H}_\mathbb{C}$ depends only on the smallest hyperplane arrangement $\mathcal{H}'_\mathbb{C}$ containing the poles of f. Therefore we just denote by $\mathrm{Tres}(f)$ the residue of a rational function f with denominator a product of linear forms.

Example 5. Observe that if we work in $\mathbb{R}^1$ and $\Delta = \{\pm e_1\}$, then R_Δ is the space of Laurent series

$$L = \left\{ f(z) = \sum_{k \geq -q} a_k z^k \right\}.$$

The total residue of a function $f(z) \in L$ is the function a_{-1}/z. The usual residue, denoted $\mathrm{Res}_{z=0}\ f$, is the constant a_{-1}.

We denote by $\hat{R}_\Delta$ the obvious extension of R_Δ, when we replace the space of polynomial functions on r variables by the space of formal power series on r variables. Let $F\colon \mathbb{C}^r \to \mathbb{C}^r$ be an analytic map, such that $F(0) = 0$ and F preserves each hyperplane $\varphi = 0$. If $f \in \hat{R}_\Delta$, the function $(F^* f)(z) = f(F(z))$ is again in $\hat{R}_\Delta$. Let $\mathrm{Jac}(F)$ be the Jacobian of the map F. The function $\mathrm{Jac}(F)$ is calculated as follows: Write $F(z) = (F_1(z_1, z_2, \ldots, z_r), \ldots, F_r(z_1, z_2, \ldots, z_r))$. Then $\mathrm{Jac}(F)(z) = \det(((\partial/\partial z_i)F_j)_{i,j})$. We assume that $\mathrm{Jac}(F)(z)$ does not vanish at $z = 0$. For any f in $\hat{R}_\Delta$, the following change of variable formula, which will be useful in our calculations later on, holds in S_Δ:

$$\mathrm{Tres}(f) = \mathrm{Tres}(\mathrm{Jac}(F)(F^* f)).$$

Note that the total residue of a rational function is again a rational function. By definition, this function can be expressed as a linear combination of the simple fractions $f_\sigma(z)$. If $f \in S_\Delta$, then $\mathrm{Tres}(f)$ is just equal to f. We also know that Tres vanishes on homogeneous rational functions of degree m, whenever $m \neq -r$ and that Tres vanishes on derivatives. If $f = P/\prod_k \langle \varphi_k, z\rangle$ (with P a polynomial in r variables) has as a denominator a product of linear forms $\langle \varphi_k, z\rangle$, where the associated normal vectors φ_i do not span $\mathbb{R}^r$, then it is easy to see that f is a derivative and the total residue of f is equal to 0. We are now ready to fix our notation and recall the key formulas.

Definition 6. For $a \in \mathbb{R}^r$, define

$$J_\Phi(a)(z) = \mathrm{Tres}\left(\frac{e^{\langle a,z\rangle}}{\prod_{k=1}^{N}\langle \varphi_k, z\rangle}\right) = \frac{1}{(N-r)!}\mathrm{Tres}\left(\frac{\langle a, z\rangle^{N-r}}{\prod_{k=1}^{N}\langle \varphi_k, z\rangle}\right)$$

and its "periodic" version

$$K_\Phi(a)(z) = \mathrm{Tres}\left(\frac{e^{\langle a,z\rangle}}{\prod_{k=1}^{N} 1 - e^{-\langle \varphi_k, z\rangle}}\right).$$

The equality

$$\mathrm{Tres}\left(\frac{e^{\langle a,z\rangle}}{\prod_{k=1}^{N}\langle \varphi_k, z\rangle}\right) = \frac{1}{(N-r)!}\mathrm{Tres}\left(\frac{\langle a, z\rangle^{N-r}}{\prod_{k=1}^{N}\langle \varphi_k, z\rangle}\right)$$

follows right away from the fact that the total residue vanishes on homogeneous rational functions of degree m, whenever $m \neq -r$.

By definition, $J_\Phi(a)(z)$ and $K_\Phi(a)(z)$ are rational functions of z homogeneous in z of degree $-r$. They are polynomial functions of a of degree $N-r$ and the homogeneous part in a of degree $(N-r)$ in $K_\Phi(a)(z)$ is $J_\Phi(a)(z)$.

Example 7. Let us compute $J_\Phi(a)(z)$ and $K_\Phi(a)(z)$ in the case of the Pitman–Stanley polytope associated to the graph G with three nodes and edges $\{1, 2\}$, $\{1, 3\}$ and the last edge $\{2, 3\}$ with multiplicity two. Then

$$\Phi_G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix}.$$

Deleting the last row leads to

$$\Phi = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

Then $J_\Phi(a_1, a_2)(z_1, z_2) = \mathrm{Tres}\left(e^{(a_1z_1+a_2z_2)}/(z_1 - z_2)z_1z_2^2\right)$ is

$$\begin{aligned}
&= \frac{1}{2!}\mathrm{Tres}\left(\frac{(a_1z_1 + a_2z_2)^2}{(z_1 - z_2)z_1z_2^2}\right)\\
&= \frac{a_1^2}{2}\mathrm{Tres}\left(\frac{z_1^2}{(z_1 - z_2)z_1z_2^2}\right) + a_1a_2\mathrm{Tres}\left(\frac{z_1z_2}{(z_1 - z_2)z_1z_2^2}\right)\\
&\quad + \frac{a_2^2}{2}\mathrm{Tres}\left(\frac{z_2^2}{(z_1 - z_2)z_1z_2^2}\right)\\
&= \frac{a_1^2}{2}\mathrm{Tres}\left(\frac{z_1}{(z_1 - z_2)z_2^2}\right) + a_1a_2\,\mathrm{Tres}\left(\frac{1}{(z_1 - z_2)z_2}\right)\\
&\quad + \frac{a_2^2}{2}\,\mathrm{Tres}\left(\frac{1}{(z_1 - z_2)z_1}\right).
\end{aligned}$$

Now $1/(z_1 - z_2)z_2$ and $1/(z_1 - z_2)z_1$ are simple elements so that they are equal to their respective total residue. To compute the total residue of $z_1/(z_1 - z_2)z_2^2$, we write z_1 as a linear combination of linear forms in the denominator, in order to reduce the degree of the denominator:

$$\frac{z_1}{(z_1 - z_2)z_2^2} = \frac{(z_1 - z_2) + z_2}{(z_1 - z_2)z_2^2} = \frac{1}{z_2^2} + \frac{1}{(z_1 - z_2)z_2}.$$

The total residue of $1/z_2^2$ is 0, as $1/z_2^2 = -\partial/\partial_{z_2}1/z_2$ is a derivative, thus

$$\mathrm{Tres}\left(\frac{z_1}{(z_1 - z_2)z_2^2}\right) = \frac{1}{(z_1 - z_2)z_2}.$$

We finally obtain

$$J_\Phi(a_1, a_2)(z_1, z_2) = \frac{1}{2}\frac{a_1^2 + 2a_1a_2}{(z_1 - z_2)z_2} + \frac{1}{2}\frac{a_2^2}{(z_1 - z_2)\,z_1}.$$

We now compute

$$K_\Phi(a_1, a_2)(z_1, z_2) = \mathrm{Tres}\left(\frac{e^{(a_1z_1+a_2z_2)}}{(1 - e^{-(z_1-z_2)})(1 - e^{-z_1})(1 - e^{-z_2})^2}\right).$$

This is

$$\mathrm{Tres}\left(\frac{1}{(z_1 - z_2)z_1z_2^2}e^{(a_1z_1+a_2z_2)}\frac{z_1 - z_2}{(1 - e^{-(z_1-z_2)})}\frac{z_1}{(1 - e^{-z_1})}\frac{z_2^2}{(1 - e^{-z_2})^2}\right).$$

We replace the analytic function

$$e^{(a_1 z_1 + a_2 z_2)} \frac{z_1 - z_2}{(1 - e^{-(z_1 - z_2)})} \frac{z_1}{(1 - e^{-z_1})} \frac{z_2^2}{(1 - e^{-z_2})^2}$$

by its Taylor series at $z_1 = 0, z_2 = 0$, and keep only its term $N(a_1, a_2)(z_1, z_2)$ of homogeneous degree 2 in z_1, z_2 which is

$$(\tfrac{5}{12} + a_1 + \tfrac{1}{2} a_1{}^2)\, z_1{}^2 + (\tfrac{7}{12} + a_2 + \tfrac{1}{2} a_1 + a_1 a_2)\, z_1 z_2 + (\tfrac{1}{2} a_2 + \tfrac{1}{2} a_2{}^2)\, z_2{}^2.$$

Thus $K_\Phi(a_1, a_2)(z_1, z_2)$ is equal to

$$\mathrm{Tres}\left(\frac{N(a_1, a_2)(z_1, z_2)}{(z_1 - z_2) z_1 z_2^2}\right).$$

Arguing as for J_Φ, we finally obtain that $K_\Phi(a_1, a_2)(z_1, z_2)$ is equal to

$$\frac{1}{2} \frac{a_1^2 + 2a_1 a_2 + 3a_1 + 2a_2 + 2}{(z_1 - z_2) z_2} + \frac{1}{2} \frac{a_2^2 + a_2}{(z_1 - z_2)\, z_1}.$$

We are now ready to write the formulas to compute the volume and number of integral points. See [**2**, Section 2] for details. To each chamber $\mathfrak{c}$ of the subdivision of $C(\Delta^+)$ is associated a linear form $f \to \langle\langle \mathfrak{c}, f \rangle\rangle$ on S_Δ, the Jeffrey–Kirwan residue [**17**]. Here is the first time the theory is different for unimodular matrices: If the system Φ is unimodular, as is the case for networks, it takes value 1 or 0 on f_σ depending upon whether or not $\mathfrak{c}$ is contained in $C(\sigma)$.

Theorem 8 (Baldoni–Vergne [**2**]). *Let Φ be an integral r by N matrix with column vectors $\varphi_1, \ldots, \varphi_N$. Let $\mathfrak{c}$ be a chamber of the subdivision of $C(\Delta^+)$.*

(1) *The functions $\langle\langle \mathfrak{c}, J_\Phi(a) \rangle\rangle$ and $\langle\langle \mathfrak{c}, K_\Phi(a) \rangle\rangle$ are polynomial functions of degree $N - r$.*
(2) *For $a \in \overline{\mathfrak{c}}$, the volume of $P(\Phi, a)$ is given by*

$$v(\Phi, a) = \langle\langle \mathfrak{c}, J_\Phi(a) \rangle\rangle.$$

(3) *If in addition the matrix Φ is unimodular, as is the case for networks, then: for $a \in \overline{\mathfrak{c}} \cap \mathbb{Z}\Phi$, the number of integral points in $P(\Phi, a)$ is given by*

$$k(\Phi, a) = \langle\langle \mathfrak{c}, K_\Phi(a) \rangle\rangle.$$

Thus, in the case of unimodular matrices the function $a \mapsto k(\Phi, a)$ is polynomial on closure of chambers and is given by an explicit formula in function of the Jeffrey–Kirwan residue.

A more general formula for arbitrary Φ spanning a lattice $\mathbb{Z}\Phi$ in $\mathbb{R}^r$ is given in [**28**]. Now, the question is how to apply these two formulas for the computations with flow polytopes. The calculation of total residues will simplify considerably.

3. Counting Integer Flows in Networks

In this section we will focus on flow polytopes for acyclically directed graphs. We already justified in the Introduction that this makes sense, as other networks can be reduced to acyclic uncapacitated networks. Consider an $(r+1)$-dimensional real vector space with basis e_i. Let A_r^+ be defined by

$$A_r^+ = \{e_i - e_j \mid 1 \le i < j \le (r+1)\}.$$

(A_r^+ is a choice of positive roots for the root system of type A_r.)

Consider E_r the vector space spanned by the elements $(e_i - e_j)$, then

$$\begin{aligned} E_r \;=\; \{a \in \mathbb{R}^{r+1} \mid\, & a = a_1e_1 + \cdots + a_re_r + a_{r+1}e_{r+1} \\ \text{with} \quad & a_1 + a_2 + \cdots + a_r + a_{r+1} = 0\}. \end{aligned}$$

The vector space E_r is of dimension r and the map

$$f\colon \mathbb{R}^r \to E_r \tag{1}$$

defined by

$$a = (a_1, a_2, \ldots, a_r) \quad \mapsto \quad \mathbf{a} = a_1e_1 + \cdots + a_re_r - (a_1 + \cdots + a_r)e_{r+1}$$

explicitly provides an isomorphism of E_r with the Euclidean space $\mathbb{R}^r$. Let, as before, $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ denote a multiset of nonzero linear forms belonging to A_r^+. We assume that the vector space spanned by Φ is E_r. This multiset is completely specified by the multiplicity $m_{i,j}$ of the vector $e_i - e_j$ in Φ. Explicitly, for the transportation polytope $T_{m,n}(d,c)$, if we denote by $\Phi_{m,n} \subset A_{m+n-1}^+$ the roots associated to it, then we have

$$\Phi_{m,n} = \{e_i - e_j \mid 1 \le i < m,\ m+1 \le j \le m+n\}$$

and thus $m_{i,j} = 1$ if $1 \le i \le m,\ m+1 \le j \le m+n$, $m_{i,j} = 0$ otherwise.

It is clear that the polytope $P(\Phi, a)$ is the polytope associated to the uncapacitated network with $(r+1)$ nodes, where the arc $i \mapsto j$ $(i < j)$ appears $m_{i,j}$ times ($m_{i,j}$ can be 0 for some arcs), and with excess function a_i at each node $1, 2, \ldots, r$ and $-(a_1 + a_2 + \cdots + a_r)$ at the last node $r+1$. Indeed we have seen in Remark 3 that the columns of the matrix corresponding to $P(\Phi, a)$ are vectors of the form $e_i - e_j$ for some i and j.

The hyperplane arrangement (setting $z_{r+1} = 0$) generated by A_r^+ is given by the following set of hyperplanes:

$$\{z_i \mid 1 \le i \le r\} \cup \{z_i - z_j \mid 1 \le i < j \le r\}.$$

A function in R_{A_r} is thus a rational function $f(z_1, z_2, \ldots, z_r)$ on $\mathbb{C}^r$, with poles on the hyperplanes $z_i = z_j$ or $z_i = 0$. The following result is proved by induction in [**2**, Proposition 14]:

Lemma 9. *Let Σ_r be the set of permutations on $\{1, 2, \ldots, r\}$ and let $f_\pi, f_w, w \in \Sigma_r$ be defined by*

$$f_\pi(z_1, z_2, \ldots, z_r) = \frac{1}{(z_1 - z_2)(z_2 - z_3)\cdots(z_{r-1} - z_r)z_r}$$

and

$$f_w(z_1, \ldots, z_r) = w \cdot f_\pi(z_1, \ldots, z_r) = \frac{1}{\prod_{i=1}^{r-1}(z_{w(i)} - z_{w(i+1)})z_{w(r)}},$$

then

$$\dim S_{A_r} = r! \tag{2}$$

and

$$\{f_w(z_1, \ldots, z_r) = w \cdot f_\pi(z_1, \ldots, z_r), \quad w \in \Sigma_r\} \tag{3}$$

is a basis for S_{A_r}.

The cone $C(A_r^+)$ generated by positive-roots is the cone

$$a_1 \geq 0, \qquad a_1 + a_2 \geq 0, \ldots, \qquad a_1 + a_2 + \cdots + a_r \geq 0.$$

We denote by $\mathfrak{c}^+$ the open set of $C(A_r^+)$ defined by

$$\mathfrak{c}^+ = \{a \in C({A_r}^+) \mid a_i > 0, i = 1, \ldots, r\}.$$

It is a chamber of our subdivision, and will be called the *nice chamber*. The reason for this name will be clear later on.

If $\mathfrak{c}$ is a chamber for $C(A_r^+)$, then there exists a unique chamber of $C(\Phi)$ that contains $\mathfrak{c}$. In other words, the polyhedral subdivision for $C(A_r^+)$ is much finer than for an arbitrary acyclic subnetwork.

Definition 10 ([**2**]). Let $m_{i,j}$ $(i < j)$ be the multiplicity of the vector $e_i - e_j$ in Φ (i.e., this is the number of times the arc $\{i, j\}$ is present in the network). Let $N = \sum_{i,j} m_{i,j}$ be the total number of arcs. We explicitly write the functions $J_\Phi(a)$ and $K_\Phi(a)$ for our choice of Φ, a. Recalling that $z_{r+1} = 0$, we have that:

- $$J_\Phi(a)(z_1, \ldots, z_r) = \frac{1}{(N-r)!}\operatorname{Tres}\left(\frac{(a_1 z_1 + \cdots + a_r z_r)^{N-r}}{\prod_{i=1}^{r} z_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (z_i - z_j)^{m_{i,j}}}\right),$$
- $$K_\Phi(a)(z_1, \ldots, z_r) = \operatorname{Tres}\left(\frac{e^{a_1 z_1} e^{a_2 z_2} \cdots e^{a_r z_r}}{\prod_{i=1}^{r}(1 - e^{-z_i})^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (1 - e^{-(z_i - z_j)})^{m_{i,j}}}\right).$$

We now write these functions in two specific examples.

Example 11. We consider the polytope associated to a complete bipartite graph with three nodes on each side. Recall that in this case the matrix that determines the polytope is given by the vectors $\Phi = \{e_1 - e_4, e_1 - e_5, e_1 - e_6, e_2 - e_4, e_2 - e_5, e_2 - e_6, e_3 - e_4, e_3 - e_5, e_3 - e_6\}$. So

$$m_{i,j} \begin{cases} = 1 & \text{if } 1 \leq i \leq 3 \text{ and } 4 \leq j \leq 6, \\ 0 & \text{otherwise,} \end{cases}$$

and:

- $J_\Phi(a)(z_1, \ldots, z_5) = \dfrac{1}{4!}\mathrm{Tres}\left(\dfrac{(a_1z_1 + a_2z_2 + a_3z_3 + a_4z_4 + a_5z_5)^4}{z_1z_2z_3 \prod\limits_{\substack{1\leq i\leq 3\\ 4\leq j\leq 5}} (z_i - z_j)} \right)$;
- $K_\Phi(a)(z_1, \ldots, z_5) = \mathrm{Tres}\left(\dfrac{e^{a_1z_1}e^{a_2z_2}e^{a_3z_3}e^{a_4z_4}e^{a_5z_5}}{\prod\limits_{i=1}^{3}(1 - e^{-z_i}) \prod\limits_{\substack{1\leq i\leq 3\\ 4\leq j\leq 5}} (1 - e^{-(z_i - z_j)})} \right)$.

Example 12. We consider the polytope determined by the complete graph K_5, in other words, $\Phi = A_4^+$. We obtain:

- $J_\Phi(a)(z_1, \ldots, z_4) = \dfrac{1}{6!}\mathrm{Tres}\left(\dfrac{(a_1z_1 + a_2z_2 + a_3z_3 + a_4z_4)^6}{z_1z_2z_3z_4 \prod\limits_{1\leq i<j\leq 4} (z_i - z_j)} \right)$;
- $K_\Phi(a)(z_1, \ldots, z_4) = \mathrm{Tres}\left(\dfrac{e^{a_1z_1}e^{a_2z_2}e^{a_3z_3}e^{a_4z_4}}{\prod\limits_{i=1}^{4}(1 - e^{-z_i}) \prod\limits_{1\leq i<j\leq 4} (1 - e^{-(z_i - z_j)})} \right)$.

In handling the formulas that we have for computing the volume and the number of integral points, the first problem is that of computing the total residue. This is in general a very difficult task. On the other hand, as we have seen, there is a very nice basis in S_{A_r} and this will allow us to rewrite the formulas in terms of iterated residues, which are certainly more tractable. The point is that one needs to find some, but not all, simplicial cones that contain the chamber determined by a. This is a step that allows the complexity of the algorithm to be reduced. We are now going to introduce the iterated residue for A_r.

Recall that, via the identification (1) of E_r with $\mathbb{R}^r$, a function in R_{A_r} is a rational function $f(z_1, z_2, \ldots, z_r)$ on $\mathbb{C}^r$, with poles on the hyperplanes $z_i = z_j$ or $z_i = 0$.

For a permutation $\sigma \in \Sigma_r$ define the linear form on R_{A_r},

$$\begin{aligned} \operatorname{Ires}^{\sigma}_{z=0} f &= \operatorname{Res}_{z_{\sigma(1)}=0} \operatorname{Res}_{z_{\sigma(2)}=0} \cdots \operatorname{Res}_{z_{\sigma(r)}=0} f(z_1, z_2, \ldots, z_r) \\ &= \operatorname{Res}_{z_1=0} \operatorname{Res}_{z_2=0} \cdots \operatorname{Res}_{z_r=0} f(z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, \ldots, z_{\sigma^{-1}(r)}). \end{aligned}$$

In particular, for $\sigma = \mathrm{id}$ the linear form $f \mapsto \operatorname{Ires}_{z=0} f$ defined by

$$\operatorname{Ires}_{z=0} f = \operatorname{Res}_{z_1=0} \operatorname{Res}_{z_2=0} \cdots \operatorname{Res}_{z_r=0} f(z_1, z_2, \ldots, z_r)$$

is called the *iterated residue*.

Remark.

- The linear form $f \mapsto \operatorname{Ires}^{\sigma}_{z=0} f$ on R_{A_r} induces a linear form on S_{A_r}, since it vanishes on the vector space of derivatives $\sum_{i=1}^{r} \partial_i R_{A_r}$.
- $\operatorname{Ires}^{\sigma}_{z=0} f_w = \delta^{\sigma}_w$, where δ^{σ}_w is the Kronecker δ-function.
- The $r!$ linear forms $\operatorname{Ires}^{\sigma}_{z=0} f, \sigma \in \Sigma_r$, on S_{A_r} are dual to the basis f_w.

Iterated residues are easier to understand, and we will see shortly how to use them in connection to our formulas. Let $w \in \Sigma_r$ and let $n(w)$ be the number of elements i such that $w(i) > w(i+1)$ (this is called the number of *descents* of the permutation w in [**25**]). We denote by $C^+_w \subset C({A_r}^+)$ the simplicial cone generated by the basic subset σ^+_w of A^+_r defined by

$$\sigma^+_w = \{\varepsilon(i)(e_{w(i)} - e_{w(i+1)})\}_{i=1,\ldots,r-1} \cup \{e_{w(r)} - e_{r+1}\},$$

where $\varepsilon(i)$ is 1 or -1 depending whether $w(i) < w(i+1)$ or not. When $w = 1$, then $C_1 = C(A^+_r)$. The following lemma is easy to see.

Lemma 13. *Let $a = \sum_{j=1}^{r+1} a_j e_j$ in E_r. The cone $C^+_w \subset E_r$ is given by the following system of inequalities $\sum_{j=1}^{i} a_{w(j)} \geq 0$, for all i such that $w(i) < w(i+1)$, but $\sum_{j=1}^{i} a_{w(j)} \leq 0$ if $w(i) > w(i+1)$.*

Let $f \in S_{A_r}$. According to the definition, to compute $\langle\langle \mathfrak{c}, f \rangle\rangle$, we must rewrite f as a linear combination of simple elements f_σ where σ ranges over basic subsets of the set A^+_r and then check if $\mathfrak{c}$ is contained in $C(\sigma)$ or not. As, for $w \in \Sigma_r$, the linear forms $\operatorname{Ires}^{w}_{z=0} f$ on S_{A_r} are dual to the basis f_w, we have

$$f = \sum_{w \in \Sigma_r} (\operatorname{Ires}_{z=0} w^{-1} f) f_w.$$

Consider the function

$$f_w(z_1, \ldots, z_r) = \frac{1}{\prod_{i=1}^{r-1} (z_{w(i)} - z_{w(i+1)}) z_{w(r)}}.$$

Then $f_w = (-1)^{n(w)} f_{\sigma_w^+}$. By definition $\langle\langle \mathfrak{c}, f_{\sigma_w^+}\rangle\rangle$ is equal to 1 if $\mathfrak{c} \subset C_w^+$, and 0 otherwise.

Combining these remarks, we obtain

Theorem 14 ([**2**]). *Let $\mathfrak{c}$ be a chamber of $C(\Phi)$. Consider the set of elements $w \in \Sigma_r$ such that $\mathfrak{c} \subset C_w^+$. Then, for $f \in S_{A_r}$,*

$$\langle\langle \mathfrak{c}, f\rangle\rangle = \sum_{w\in\Sigma_r,\, \mathfrak{c}\subset C_w^+} (-1)^{n(w)} \operatorname{Ires}_{z=0} w^{-1} f.$$

In particular, for $f = J_\Phi(a)$, we obtain:

Formula 1: *For $a \in \bar{\mathfrak{c}}$, we have*

$$v(\Phi, a) = \langle\langle \mathfrak{c}, J_\Phi(a)\rangle\rangle = \sum_{w\in\Sigma_r,\, \mathfrak{c}\subset C_w^+} (-1)^{n(w)} \operatorname{Ires}^w_{z=0} J_\Phi(a).$$

We have seen that to compute the number of integral points of our polytope we need to compute $K_\Phi(a)$. Let $t_j = m_{j,j+1} + \cdots + m_{j,r+1} - 1$, where we recall that $m_{i,j}$ is the multiplicity of the root $e_i - e_j$ in Φ. After a change of variable for the total residue [**2**, Section 10], we obtain

Theorem 15. *Let $a = \sum_{i=1}^{r+1} a_i e_i$ in $E_r \cap \mathbb{Z}^{r+1}$. Let*

$$f_\Phi(a)(z) = \frac{\prod_{i=1}^{r} (1+z_i)^{a_i+t_i}}{\prod_{i=1}^{r} z_i^{m_{i,r+1}} \prod_{1\le i<j\le r} (z_i - z_j)^{m_{i,j}}}.$$

Then

Formula 2: *For $a \in \bar{\mathfrak{c}}$,*

$$k(\Phi, a) = \sum_{w\in\Sigma_r,\, \mathfrak{c}\subset C_w^+} (-1)^{n(w)} \operatorname{Ires}^w_{z=0} f_\Phi(a).$$

We now want to give an even more explicit formulation of the above result suited to be directly implemented. For this purpose we need to introduce some more notations. For $a \in E_r$, let $\operatorname{def}(a)$ be defined by

$$\operatorname{def}(a) = a + \varepsilon \sum_{\alpha\in\Phi} \alpha + \varepsilon^2 \left(\sum_{i=1}^{r} e_i - r e_{r+1} \right)$$

with $\varepsilon = 1/2mr^2$ and m the maximum of the multiplicities $m_{i,j}$.

A *wall* of A_r^+ is a hyperplane generated by $r-1$ linearly independent elements of A_r^+. The cells in $C(A_r^+)\backslash\mathcal{H}^*$ ($\mathcal{H}^*$ being the set of walls for A_r^+) are open cells, interior of polyhedral cones. We will call these open cells *topes*. We will say that

$a \in C(A_r^+)$ is *regular* if a is not on any wall for A_r^+. The walls of A_r^+ are easily characterized since they are the kernel of a linear form as $\sum_{i \in J} a_i$ where J is a subset of $\{1, 2, \ldots, r\}$. It is then easy to decide whether a vector a is regular or not.

If a is a regular element, we let $\mathfrak{c}$ denote the unique chamber of $C(A_r^+)$ containing it. Then the set $\mathrm{Sp}(a) = \{w \in \Sigma_r \mid \mathfrak{c} \subset C_w^+\}$ can be computed without explicit knowledge of the chamber. In fact, one can easily see that the set $\mathrm{Sp}(a)$ consists of those permutations $w \in \Sigma_r$ that satisfy the following conditions:

$$w(i) < w(i+1) \quad \text{if and only if} \quad a_{w(1)} + \cdots + a_{w(i)} \geq 0,$$

for $i = 1, 2, \ldots, r-1$.

An element of $\mathrm{Sp}(a)$ will be called a *special permutation*.

Remark. If $a_i \geq 0$ for all $i \leq r$, then $a = \sum_{i=1}^{r} a_i e_i - (\sum_{i=1}^{r} a_i) e_{r+1}$ belongs to the closure of the *nice chamber* $\mathfrak{c}^+$ and $\mathrm{Sp}(a) = \{\mathrm{id}\}$. This is the *nice* property of this chamber leading to its name.

Now we can state Theorem 15 as follows:

Theorem 16. *Let $\Phi \subset A_r^+$ be a system generating E_r. Let $a = \sum_{i=1}^{r+1} a_i e_i \in E_r,\ a_{r+1} = -(a_1 + \cdots + a_r),\ a_i \in \mathbb{Z}$, and assume that $a \in C(A_r^+)$.*

Write

$$f_\Phi(a_1, a_2, \ldots, a_r)(z) = \frac{\prod_{i=1}^{r} (1 + z_i)^{a_i + t_i}}{\prod_{i=1}^{r} z_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (z_i - z_j)^{m_{i,j}}}.$$

Then:

- **Formula 2A:** *If a is regular, then*
$$k(\Phi, a) = \sum_{w \in \mathrm{Sp}(a)} (-1)^{n(w)} \operatorname{Ires}_{z=0}^{w} f_\Phi(a).$$
- **Formula 2B:** *If a is not regular, then*
$$k(\Phi, a) = \sum_{w \in \mathrm{Sp}(\mathrm{def}(a))} (-1)^{n(w)} \operatorname{Ires}_{z=0}^{w} f_\Phi(a).$$

Remark. If a is in the nice chamber, the sum above is reduced to a single iterated residue.

Remark. Formula 2B in the theorem follows by observing that the chamber containing the regular element $\mathrm{def}(a)$ contains a in its closure. The deformation has to be done with care to deal with some *border* cases. The following lemma, that we state for completeness, shows that the deformation with a_i integers is small enough to take care of such cases.

Lemma 17. *Given* $a \in C(A_r^+) \cap \mathbb{Z}^{r+1}$, *define* $\mathrm{def}(a) := a + \varepsilon \sum_{\alpha\in\Phi} \alpha + \varepsilon^2(\sum_{i=1}^r e_i - r e_{r+1})$, $\varepsilon = 1/(2mr^2)$, *where m is the maximum of the multiplicities* $m_{i,j}$. *Then the following holds*:

- $\mathrm{def}(a)$ *is regular, i.e., it belongs to a chamber.*
- *If* τ *is a tope and* $a \in \tau$, *then* $\mathrm{def}(a) \in \tau$.
- $a \in C(A_r^+)$ *if and only if* $\mathrm{def}(a) \in C(A_r^+)$.
- *In general, if* Φ *is a subset of* A_r^+, $a \notin C(\Phi)$ *if and only if* $\mathrm{def}(a) \notin C(\Phi)$.

For example, we obtain the following formula for the complete network K_{r+1} on $r+1$ nodes, with excess vector $a_1, a_2, \ldots, a_r, a_{r+1} = -\sum_{i=1}^r a_i$. In this case, the function $k(A_r^+, a)$ is the so-called Kostant partition function and has special importance for the representation theory of the group $GL(r+1, \mathbb{C})$.

Corollary 18. *For* $a \in C(A_r^+) \cap \mathbb{Z}^{r+1}$, *the Kostant partition function is given by*

$$k(A_r^+, a) = \sum_{w\in \mathrm{Sp}(a')} (-1)^{n(w)} \operatorname{Ires}_{z=0}^w \frac{\prod_{i=1}^r (1+z_i)^{a_i+r-i}}{\prod_{i=1}^r z_i \prod_{1\le i<j\le r} (z_i - z_j)},$$

where

$$a' = \begin{cases} a & \text{if } a \text{ is regular,} \\ \mathrm{def}(a) & \text{otherwise.} \end{cases}$$

In particular, if $a_i \ge 0$ *for* $1 \le i \le r$, *we have*

$$k(A_r^+, a) = \operatorname{Res}_{z_1=0} \operatorname{Res}_{z_2=0} \cdots \operatorname{Res}_{z_r=0} \left(\frac{\prod_{i=1}^r (1+z_i)^{a_i+r-i}}{\prod_{i=1}^r z_i \prod_{1\le i<j\le r} (z_i - z_j)} \right).$$

Similarly we may write a formula for the transportation polytope $T_{m,n}(d, c)$.

Corollary 19. *Let* $a = \sum_{i=1}^m d_i e_i - \sum_{j=1}^n c_j e_{m+j}$, *with* d_i *and* c_j *nonnegative integers. Then the number of integral points in* $T_{m,n}(d,c)$ *is equal to*

$$\sum_{w\in \mathrm{Sp}(a')} (-1)^{n(w)} \operatorname{Ires}_{z=0}^w \frac{\prod_{i=1}^m (1+z_i)^{d_i+n-1} \prod_{j=1}^{n-1} (1+z_{m+j})^{-c_j-1}}{\prod_{i=1}^m z_i \prod_{\substack{1\le i\le m \\ 1\le j\le n-1}} (z_i - z_{m+j})}$$

where

$$a' = \begin{cases} a & \text{if } a \text{ is regular,} \\ \mathrm{def}(a) & \text{otherwise.} \end{cases}$$

3.1. *Total Residue Software for Counting Integral Flows*

The scope of this section is a brief description of the various algorithmic procedures that were implemented with the symbolic language `Maple` and that achieve the formula for the number of integral points described in Theorem 15. This software is available at `www.math.ucdavis.edu/~totalresidue`. The initial data are an r by N matrix A whose columns are the elements of Φ and an element $a = \{a_1, \ldots, a_r\} \in \mathbb{Z}^r$ that determines the polytope. The ingredients that we need to compute are:

(1) The element $a' = \mathrm{def}(a)$ obtained by deforming the initial parameter a.
(2) The set of permutations that appear in the formula, that is, the set of special permutations $\mathrm{Sp}(a')$.
(3) The residues that appear in Formula 2.

We will discuss the ingredients for each one of these steps listing the various algorithms that are related to the part we are describing.

First of all we want to check if our vector is in $C(A_r^+)$, that is, in the cone generated by $\{e_1 - e_2, e_2 - e_3, \ldots, e_{r-1} - e_r, e_r\}$ because otherwise the polytope is empty and there is nothing to do. To be in the cone, a must satisfy $a_1 \geq 0$, $a_1 + a_2 \geq 0, \ldots, \ a_1 + a_2 + \cdots + a_r \geq 0$. The procedure **check-vector** verifies whether this is true or not. In fact, because of Lemma 17, we may use $\mathrm{def}(a)$ instead of a and we do this to simplify the procedures. We compute the element $\mathrm{def}(a)$ via the `Maple` procedure **def-vector**. The vector $\mathrm{def}(a)$ is used in all the formulas defining $\mathrm{Sp}(a)$ instead of a, whether or not a is regular. This takes care of the first part.

To find the subset $\mathrm{Sp}(a)$ of Σ_r, we use the procedure **special-permutations**. We stress that using the `Maple` function $combinat[permute]$ is impractical and does not go very far because of memory limitations. Our approach constructs recursively the permutations subject to our conditions; thus we save much memory in listing only those permutations. The set $\mathrm{Sp}(a)$ depends strongly on the element a. We do not have upper bound estimates on the subset $\mathrm{Sp}(a) \subset \Sigma_r$, but it seems that this set is small compared to Σ_r. One of the worst experimental cases for the complete graph K_{10} on 10 nodes (the case of A_9^+) is the case of the vector $a = [30201, 59791, 70017, 41731, 58270, -81016, -68993, -47000, -43001, -20000]$ where the number $\mathrm{Sp}(a) \subset \Sigma_9$ is 9572, certainly much smaller that 9! Experiments show that the time spent to compute this set is rather small.

Every permutation $w \in \mathrm{Sp}(a)$ gives rise to the simplicial cone C_w^+ containing a. This simplicial cone corresponds to selecting a unique vertex of the polytope $P(A_r^+, a)$. Note that the cardinality of $\mathrm{Sp}(a)$ is much smaller than the number of vertices of the polytope $P(A_r^+, a)$. For example, for

$$a = \left[a_1, a_2, \ldots, a_r, -\left(\sum_{i=1}^{r} a_i\right)\right]$$

with $a_i > 0$, we have already remarked that the cardinality of $\mathrm{Sp}(a)$ is 1, as $\mathrm{Sp}(a)$ is reduced to the identity permutation. Finally, for the last step we need to compute the residue. Recall that we need to compute

$$\mathrm{Ires}^w_{z=0} \frac{\prod_{i=1}^{r}(1+z_i)^{a_i+t_i}}{\prod_{i=1}^{r} z_i^{m_{i,r+1}} \prod_{1\le i<j\le r}(z_i - z_j)^{m_{i,j}}}$$

with w one of the special permutations. Let us denote by F the function appearing in the formula above. The function F is a product of a certain number of functions. This allows us to take the residues by introducing, little by little, the part of the function F containing the needed variable. To make things simpler we assume that w is the identity permutation. We start by taking the residue at $z_r = 0$ of the function $g := (1+z_r)^{(a_r+t_r)}/z_r^{m_{r,r+1}} \prod_{j=1}^{r-1}(z_j - z_r)^{m_{j,r}}$. Suppose $g_r(z_1, z_2, \ldots, z_{r-1})$ is the result. We continue by taking the residue in z_{r-1} of the function g_r multiplied by all the factors of the original function F that involve the variables z_{r-1} and so on. The way we compute the residue in one variable z of a function $g(z) = F(z)/z^u$, where F is analytic, is by computing the Taylor expansion of F up to the estimate we have for the order u of the pole of the function g and then taking the coefficient of $1/z$. The argument just described is implemented via different procedures: **coeex**, **invi**, **trunc-next-function** and **RRK**. Finally the procedure **number-kostant** adds up, with a sign (the appropriate sign is computed using **segnop**), all residues coming from the different special permutations, thus getting Formula 2. The procedure **polynomial-kostant** computes the polynomial $a \mapsto k(\Phi, a)$ on the chamber determined by a.

As we pointed out we need a uniform estimate for the order of poles appearing. The result for the order of poles is the content of the subsection that follows and it is implemented in procedure **E**.

3.2. *Estimates for the Order of Poles*

Let G_r be a Laurent polynomial in the r variables $z = (z_1, z_2, \ldots, z_r)$ and let $D_r = \prod_{1\le i<j\le r}(z_i - z_j)$. We have seen that we need to compute iterated residues of the form

$$\mathrm{Res}_{z_1=0}\,\mathrm{Res}_{z_2=0}\cdots\mathrm{Res}_{z_r=0}\;\; G_r/D_r^m.$$

The following key lemma will handle the situations that will appear in computing the estimate we are looking for:

Lemma 20. *Assume that* $G_r = [F(z_1,\ldots,z_r)/(z_1 z_2\cdots z_r)^g]H_r(1/z_1,\ldots,1/z_r)$ *where* F *is analytic and* H_r *is a homogeneous polynomial of degree* h *, then*

$$\mathrm{Res}_{z_r=0}\, G_r/D_r^m$$

is a linear combination of functions of the form G_{r-1}/D_{r-1}^m with

$$G_{r-1} = \frac{F(z_1, \ldots, z_{r-1})}{(z_1 z_2 \cdots z_{r-1})^{(g+m)}} H_{r-1}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_{r-1}}\right),$$

where H_{r-1} is a homogeneous polynomial of degree at most $g + h - 1$ and $F(z_1, \ldots, z_{r-1})$ is analytic.

Proof. Let us prove the lemma for a monomial $H_r = z_1^{i_1} \cdots z_{r-1}^{i_{r-1}} z_r^{i_r}$ where $i_1, i_2, \ldots, i_r$ are nonnegative integers such that $i_1 + i_2 + \cdots + i_r = h$. We write $\prod_{1 \le i \le r-1}(z_i - z_r)^m = (z_1 z_2 \cdots z_{r-1})^m \prod_{1 \le i \le r-1}(1 - z_r/z_i)^m$.

The Taylor expansion of $1/\prod_{1 \le i \le r-1}(1 - z_r/z_i)^m$ at $z_r = 0$ is

$$\sum_{U_1, \ldots, U_{r-1}} z_1^{-|U_1|} z_2^{-|U_2|} \cdots z_{r-1}^{-|U_{r-1}|} z_r^{|U_1|+|U_2|+\cdots+|U_{r-1}|},$$

where $U_s = \{j_1^s, j_2^s, \ldots, j_m^s\}$ varies over the m tuples of nonnegative integers. Write also $F(z_1, \ldots, z_r) = \sum_k F_k(z_1, \ldots, z_{r-1}) z_r^k$. Thus we obtain

$$\begin{aligned} \mathrm{Res}_{z_r=0} \frac{G_r}{D_r^m} &= \frac{z_1^{-i_1} \cdots z_{r-1}^{-i_{r-1}}}{(z_1 z_2 \cdots z_{r-1})^{g+m}} \frac{1}{D_{r-1}^m} \mathrm{Res}_{z_r=0} \frac{F(z_1, \ldots, z_r)}{z_r^{g+i_r} \prod_{i=1}^{r-1} \left(1 - \dfrac{z_r}{z_i}\right)^m} \\ &= \left(\frac{z_1^{-i_1} \cdots z_{r-1}^{-i_{r-1}}}{(z_1 z_2 \cdots z_{r-1})^{g+m}} \frac{1}{D_{r-1}^m}\right) \\ &\quad \times \sum_{k=0}^{g-1+i_r} \left(F_k(z_1, \ldots, z_{r-1}) \sum_{\substack{(U_1, \ldots, U_{r-1}) \\ \sum_{i=1}^{r-1} |U_i| = g-1+i_r-k}} z_1^{-|U_1|} \cdots z_{r-1}^{-|U_{r-1}|} \right). \end{aligned}$$

For $0 \le k \le i_r + g - 1$, the monomial

$$z_1^{-i_1} \cdots z_{r-1}^{-i_{r-1}} z_1^{-|U_1|} z_2^{-|U_2|} \cdots z_{r-1}^{-|U_{r-1}|}$$

is such that

$$i_1 + \cdots + i_{r-1} + |U_1| + \cdots + |U_{r-1}| = i_1 + \cdots + i_{r-1} + i_r + g - 1 - k \le h + g - 1$$

and we obtain the lemma. □

Observe that if $F = 1$, then the same proof shows that H_r is homogeneous of degree precisely $h + g - 1$. Now starting from $G_r = F(z_1, \ldots, z_r)/(z_1 \cdots z_r)^m$ we want to compute

$$\mathrm{Res}_{z_{k+1}=0}\, \mathrm{Res}_{z_{k+2}=0} \cdots \mathrm{Res}_{z_{r-1}=0}\, \mathrm{Res}_{z_r=0}\, G_r/D_r^m.$$

Applying the lemma with $h = 0$, we obtain that

$$\mathrm{Res}_{z_r=0}\, G_r/D_r^m$$

is a linear combination of functions of the form G_{r-1}/D_{r-1}^m where

$$G_{r-1} = \frac{F(z_1, \ldots, z_{r-1})}{(z_1 \cdots z_{r-1})^{2m}} H\left(\frac{1}{z_1}, \ldots, \frac{1}{z_{r-1}}\right)$$

and H is homogeneous of degree at most $m - 1$. Thus at the next residue we get again a linear combination of functions of the form G_{r-2}/D_{r-2}^m where

$$G_{r-2} = \frac{F(z_1, \ldots, z_{r-2})}{(z_1 \cdots z_{r-2})^{3m}} H\left(\frac{1}{z_1}, \ldots, \frac{1}{z_{r-2}}\right)$$

with H homogeneous of degree at most $2m + m - 1 - 1 = 3m - 2$. Finally, the last residue in $z_{k+1} = 0$ leaves a linear combination of functions of the form

$$\frac{G_k}{D_k^m}$$

with

$$G_k = \frac{F(z_1, \ldots, z_k)}{(z_1 \cdots z_k)^{(r-k+1)m}} H\left(\frac{1}{z_1}, \ldots, \frac{1}{z_k}\right).$$

Here H is homogeneous of degree at most $(r-k)(r-k+1)m/2-(r-k)$. In particular, considering $H(1/z_1, \ldots, 1/z_k)$ we have the estimate on poles we were looking for.

Corollary 21.

1. *Let $G_r = F(z_1, \ldots, z_r)/(z_1 \cdots z_r)^m$, with F analytic. Then the function*

$$\mathrm{Res}_{z_{k+1}=0}\,\mathrm{Res}_{z_{k+2}=0} \cdots \mathrm{Res}_{z_{r-1}=0}\,\mathrm{Res}_{z_r=0}\, G_r/D_r^m$$

has a pole in z_k of order at most $m(r-k)(r-k+1)/2 - (r-k)$.

2. *In particular, with the notation as in Theorem* 15, *if $m = \max_{\mathrm{ij}}(m_{i,j})$, then the pole in $\sigma(z_k)$ of the function*

$$\mathrm{Res}_{z_{\sigma(k+1)}=0} \cdots \mathrm{Res}_{z_{\sigma(r)}=0}\, f_\Phi(a_1, a_2, \ldots, a_r)(z)$$
$$= \mathrm{Res}_{z_{\sigma(k+1)}=0} \cdots \mathrm{Res}_{z_{\sigma(r)}=0} \frac{\prod_{i=1}^{r}(1+z_i)^{a_i+t_i}}{\prod_{i=1}^{r} z_i^{m_{i,r+1}} \prod_{1\le i<j\le r}(z_i - z_j)^{m_{i,j}}}$$

has at most order $m(r-k)(r-k+1)/2 - (r-k)$ independently from $\sigma \in \Sigma_r$.

4. The Chamber Complex

In this section we discuss the chambers and how to compute them. It is important to emphasize that everything that we present in this section is valid for general matrices, not necessarily unimodular. There is an implementation of these ideas in the `Maple` program **chambers** available at `www.math.ucdavis.edu/~totalresidue`. Let Δ^+ denote the set of distinct vectors $\{\Phi\}$. Recall that the *chamber complex* is a polyhedral subdivision of the cone $C(\Delta^+)$ of nonnegative linear combinations of Δ^+. Recall it is defined as the common refinement of the simplicial cones $C(\sigma)$ running over all possible basic subsets σ of Δ^+. To be more precise we now introduce notation and the key definitions. In what follows, when we consider a subset $I = \{s_1, s_2, \ldots, s_k\}$, where the elements s_i of I are subsets of a set X, we assume there is a partial order on I by containment. Thus the set of minimal elements of I is denoted by **minimalize**(I). We adopt the convention that the intersection of an empty family of subsets of X is X itself.

Let Δ^+ be the set $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ of vectors in $\mathbb{R}^r$. Recall that a *wall* is a hyperplane in $\mathbb{R}^r$ spanned by $(r-1)$ vectors of Δ^+. Each wall W partitions the set of indices $\{1, 2, \ldots, N\}$ into three sets: $\text{zeros}(W) = \{i \mid \varphi_i \in W\}$, and two disjoint subsets, $\text{pos}(W)$, $\text{neg}(W)$, whose union, $\text{pos}(W) \cup \text{neg}(W)$, is precisely the subset of $\{1, 2, \ldots, N\} \backslash \text{zeros}(W)$. Remark that the two half-spaces determined by W are arbitrarily thought of as positive or negative. We denote by $\mathcal{B}$ the set of subsets σ of $\{1, 2, \ldots, N\}$ such that σ is of cardinality r and the set of vectors $\{\varphi_i \mid i \in \sigma\}$ is linearly independent. For convenience, we continue to call such σ a *basic subset* of Δ^+, thinking of σ as a subset of integers or as a subset of elements of Φ labeled by indices.

For $\sigma \in \mathcal{B}$, we consider the closed cone $C(\sigma)$ generated by σ. If I is a subset of $\mathcal{B}$, let $F(I) = \bigcap_{\sigma \in I} C(\sigma)$ be the intersection of the cones $C(\sigma)$, when σ runs in I. We will say that I is a *feasible subset* of $\mathcal{B}$ if the interior of $F(I)$ is nonempty. A *combinatorial chamber* I is a maximal feasible subset of $\mathcal{B}$. The polyhedral cone $F(I)$ will be called a *geometric chamber*. The actual chamber **Chamber**(I) is the interior of $F(I)$. Reciprocally, the collection I is entirely determined by $F(I)$. We have $I = \{\sigma \in \mathcal{B} \mid F(I) \subset C(\sigma)\}$. The collection of all geometric chambers and their faces form a polyhedral complex that partition the cone $C(\Delta^+)$, the so-called *chamber complex* [**1**], [**5**], [**10**].

Figure 4 shows an example, the chamber complex for the cone associated to the acyclic complete graph K_4 we discussed in the previous section. The picture represents a two-dimensional slice of the cone decomposition (the cone is three-dimensional and pointed at the origin). The six dots labeled $(e_i - e_j)$ on the drawing are the intersections of the rays $\mathbb{R}_{\geq 0}(e_i - e_j)$ with the hyperplane $(3x_1 + x_2 - x_3 - 3x_4) = 2$. Seven chambers, numbered from 1 to 7, are present. In the configuration of vectors of Figure 4 there are seven walls, one for each of the distinct lines obtained from the vectors in the configuration.

Let $\mathcal{H}^*$ denote the hyperplane arrangement spanned by all possible subsets of d independent vectors on the input matrix (see, e.g., the left side of Figure 5). The

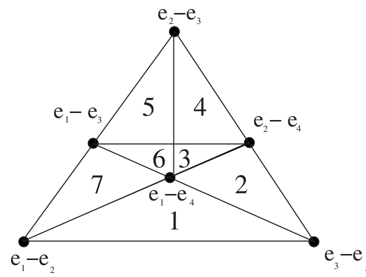


Fig. 4. A slice of the chamber complex for K_4.

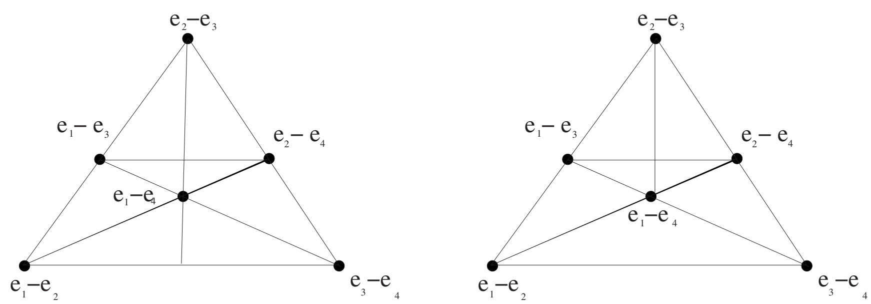


Fig. 5. (a) Eight topes (left) versus (b) seven chambers (right).

cells in $C(\Delta^+)\backslash\mathcal{H}^*$ are open cells, interior of polyhedral cones. We will call these open cells *topes* (following the oriented matroid terminology [**7**]). It is important for the reader to observe that the set of topes is (typically) a much finer subdivision of $C(\Delta^+)$ than its chambers. See Figure 5 for a comparison between the chamber complex and the tope complex of the hyperplane arrangement $\mathcal{H}^*$ associated with the example in Figure 4.

For each wall W and each tope τ, we denote by $\text{pos}(W, \tau)$ the set of elements $i \in \{1, 2, \ldots, N\}$ such that $\varphi_i \in \Delta^+$ lies on the same open half-spaces determined by W as the tope τ. Note that this makes sense since a tope τ is an open set disjoint from the walls. We say that $\text{pos}(W, \tau)$ is a *nonface* (this terminology is justified because these are the nonfaces of a certain simplicial complex in the sense of Chapter 2 of [**26**]). We denote by $\mathbf{Chamber}(\tau)$ the chamber containing the tope τ.

To each tope τ, we associate the family of positive nonfaces determined by the tope τ (we have a nonface for each wall). Let us call this full family $\mathbf{Polarized}(\tau)$. Consider the family $\mathbf{MNF}(\tau)$ of minimal elements of $\mathbf{Polarized}(\tau)$. This is the family $\mathbf{MNF}(\tau) = \mathbf{minimalize}(\mathbf{Polarized}(\tau))$. The first main observation is that we can reconstruct the chamber $\mathbf{Chamber}(\tau)$ containing the tope τ from the set $\mathbf{MNF}(\tau)$. This is very useful to construct one initial chamber. Later all others will be found from it.

The set $\mathbf{MNF}(\tau)$ is a set of nonfaces. Let f be the cardinality of the set $\mathbf{MNF}(\tau)$. Let us list $\mathbf{MNF}(\tau) := \{p_1, p_2, \ldots, p_f\}$. Each p_i is a nonface. We construct the family $\mathcal{P}(\tau)$ of sets ν of the form $\nu := \{i_1, i_2, \ldots, i_f\}$ with $i_1 \in p_1, i_2 \in p_2, \ldots,$ $i_f \in p_f$. These we call *transversals* of a family of sets. This family is denoted by **transversal**(**MNF**(τ)) in the computer program we present. Again $\mathcal{P}(\tau)$ is a set whose elements are sets of indexes, its elements being subsets of $\{1, 2, \ldots, N\}$. The cardinality of a set $\nu \in \mathcal{P}(\tau)$ may be smaller than f, as the family **MNF**(τ) does not consist of disjoint sets. It is important to observe that if ν is in $\mathcal{P}(\tau)$, then for any wall W, the intersection $\nu \cap \text{pos}(W, \tau)$ is not empty. Now we prove the following theorem:

Theorem 22. *The minimal elements of the family*

$$\mathcal{P}(\tau) := \textbf{transversal}(\textbf{MNF}(\tau))$$

are exactly the basic subsets σ of Δ^+ such that $\tau \subset C(\sigma)$.

In other words, given the set **MNF**(τ) associated to a tope τ, the family of basic subsets σ of Δ^+ such that τ is contained in the simplicial cone $C(\sigma)$ is precisely the set **minimalize**(**transversal**(**MNF**(τ))). We start by a lemma:

Lemma 23. *Every $\nu \in \mathcal{P}(\tau)$ is such that the set of vectors $\{\varphi_i \mid i \in \nu\}$ spans $\mathbb{R}^r$.*

Proof. Let us see that a set $\nu \in \mathcal{P}(\tau)$ spans $\mathbb{R}^r$. Indeed, if not, the set of vectors $\{\varphi_i \mid i \in \nu\}$ would be contained in a wall W. Consider the set $\text{pos}(W, \tau)$ and a minimal element p of the family

$$\textbf{MNF}(\tau) := \textbf{minimalize}(\textbf{Polarized}(\tau))$$

contained in $\text{pos}(W, \tau)$. Then p (meaning the set of elements φ_i indexed by p) is contained in one of the open half-spaces determined by W. Thus, contrary to our hypothesis, we would have $\nu \cap p = \emptyset$. $\square$

We go on proving Theorem 22.

Proof. Let σ be a basic subset of Δ^+ (σ (elements indexed by σ) generates a simplicial cone). We now prove that if $\tau \subset C(\sigma)$, then $\sigma \in \mathcal{P}(\tau)$ and is a minimal element in the family of tranversal sets $\mathcal{P}(\tau)$.

For each wall W, the set $\sigma \cap \text{pos}(W, \tau)$ is nonempty. Otherwise, σ would be contained in the closed half-space determined by W, but would be on the opposite to τ with respect to W, and the cone $C(\sigma)$ will not contain τ. Let us pick, for each $p \in \textbf{MNF}(\tau)$, an element $\varphi_p \in \sigma \cap p$. It follows that σ necessarily contains the set $\nu := \{\varphi_p \mid \varphi_p \in \sigma \cap p;\ p \in \textbf{MNF}(\tau)\}$, belonging to the family $\mathcal{P}(\tau)$. But then $\sigma = \nu$, as σ is a basic subset of Δ^+ and ν indexes a set of generators of $\mathbb{R}^r$ by Lemma 23. Furthermore, σ is minimal, as all sets belonging to the family $\mathcal{P}(\tau)$ have cardinality at least equal to r.

We now prove the converse. Let ν be a minimal set of $\mathcal{P}(\tau)$. We claim that τ is contained in the cone $C(\nu)$. Otherwise, there would be a wall W separating τ and $C(\nu)$. But by construction of ν there is an element $p \in \nu$ contained in $\mathrm{pos}(W, \tau)$; a contradiction with W separating $C(\nu)$ and τ. Now all we have to prove is that ν has cardinality r.

Let x be a point in τ. By Carathéodory's theorem, there is a basic subset σ contained in ν such that $x \in C(\sigma)$. Then the tope τ is entirely contained in $C(\sigma)$ because a tope is, by definition, not separated in two by any wall. The set σ belongs to $\mathcal{P}(\tau)$ by the preceding discussion. But $\sigma \subset \nu$ and ν is minimal, thus $\nu = \sigma$.

So we conclude that the set **Chamber**(τ) of basic subsets σ of Δ^+ such that $\tau \subset C(\sigma)$ is the set **minimalize**$(\mathcal{P}(\tau))$ of minimal elements of $\mathcal{P}(\tau) =$ **transversal**(**MNF**(τ)). $\square$

The *lexicographic tope* is the tope containing the vector $\xi = \varphi_1+\varepsilon\varphi_2+\varepsilon^2\varphi_3+\cdots$ where ε is a small number. The lexicographic chamber is the chamber that contains the lexicographic tope.

Corollary 24. *The following algorithm determines the r-simplicial cones $C(\sigma)$ that contain the lexicographic chamber associated with a particular labeling of the elements of Δ^+, by finding the basic sets σ that define them.*

(1) *Create the list L of lexicographic nonfaces* $\mathrm{pos}(W, \tau)$ *where τ is the lexicographic tope, and W runs over all possible walls of Δ^+.*
(2) *Let $F = \{A_1, A_2, \ldots, A_m\}$ be the minimal nonfaces from L.*
(3) *Find the transversal sets to the family F, then minimalize the set of transversals. The result is $\sigma_1, \ldots, \sigma_k$, the desired basic sets.*

Now we are concerned with producing all other chambers from one initial chamber, such as the lexicographic chamber. For this we need to understand the polyhedron $F(I)$. This is a pointed polyhedral cone. We recall, say from Chapter 8 in the book [**24**], that for a polyhedron P (e.g., $F(I)$) given by a finite set of inequalities $Ax \leq b$, a *supporting hyperplane* is an affine hyperplane $\{x \mid cx = d\}$ such that $d = \max\{cx \mid Ax \leq b\}$. A subset of P is a *face* if $F = P$ or F is the intersection of P with a supporting hyperplane of P. A *facet* of P is a maximal face distinct from P. We say a wall W is an *essential wall* of the geometric chamber $F(I)$, if $F(I) \cap W$ is a facet of the pointed polyhedral cone $F(I)$. This is equivalent to W being a supporting hyperplane of $F(I)$ and $\dim(F(I) \cap W) = r - 1$. We say that two geometric chambers $F(I)$ and $F(I')$ are *W-adjacent* if they share a common essential wall W and $\dim(F(I) \cap F(I') \cap W) = r - 1$. In particular, the wall W is an interior wall. In what follows we may sometimes say that two chambers are "adjacent chambers" without specifying the wall they share. We present now an operation that allows us to move, under certain conditions, from a geometric chamber to another adjacent geometric chamber. Since the geometric chambers form a connected polyhedral complex, we can then apply some standard search procedure, such as depth-first search, to enumerate and list all chambers.

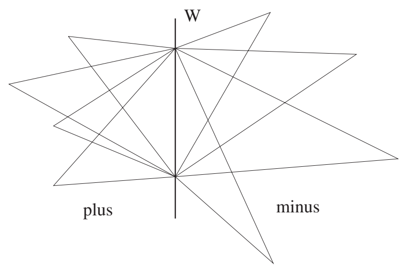


Fig. 6. A flip exchanges the simplicial cones supported on opposite sides of a wall.

We denote by $\mathcal{W}$ the set of subsets ν of $\{1, 2, \ldots, N\}$ such that ν is of cardinality $r - 1$ and the set of vectors $\{\varphi_i \mid i \in \nu\}$ is linearly independent. In other words, if ν is in $\mathcal{W}$, the vector space $\mathcal{L}(\nu)$ spanned by the vectors $\{\varphi_i \mid i \in \nu\}$ is a wall W. If W is a wall we denote by $\mathcal{W}(W)$ the subset of $\mathcal{W}$ with elements whose ν are such that $\mathcal{L}(\nu) = W$.

If ν is in $\mathcal{W}$, we consider the subsets $\mathrm{zeros}(\nu)$, $\mathrm{pos}(\nu)$, and $\mathrm{neg}(\nu)$. If i is not in $\mathrm{zeros}(\nu)$, then $\nu \cup \{i\}$ is an element of $\mathcal{B}$. We denote by $\delta^+(\nu)$ the subset of $\mathcal{B}$ consisting of elements $\sigma = \nu \cup \{i\}$ where i runs in $\mathrm{pos}(\nu)$; denote by $\delta^-(\nu)$ the subset of $\mathcal{B}$ consisting of elements $\sigma = \nu \cup \{i\}$ where i runs in $\mathrm{neg}(\nu)$.

If W is a wall and σ a subset of $\{1, 2, \ldots, N\}$ we denote by $\sigma \cap W = \sigma \cap \mathrm{zeros}(W)$. We denote by $\mathcal{B}(W \mid \text{facet})$ the subset of $\mathcal{B}$ consisting of those elements σ such that $\sigma \cap W$ is of cardinality $(r - 1)$. In other words, W is spanned by a facet of the cone $C(\sigma)$. We denote by $\mathcal{B}(W \mid \text{cut})$ the subset of $\mathcal{B}$ consisting of elements σ such that both sets $\sigma \cap \mathrm{pos}(W)$ and $\sigma \cap \mathrm{neg}(W)$ are nonempty. For any subset I of $\mathcal{B}$, we denote by $I(W \mid \text{facet}) = I \cap \mathcal{B}(W \mid \text{facet})$ and by $I(W \mid \text{cut}) = I \cap \mathcal{B}(W \mid \text{cut})$.

Let I be a combinatorial chamber which is a maximal feasible subset of $\mathcal{B}$. Let W be a wall, we define $B(W, I) = \{\sigma \cap W \mid \sigma \in I(W \mid \text{facet})\}$. This is a subset of $\mathcal{W}(W) = \{\nu \in \mathcal{W} \mid \mathcal{L}(\nu) = W\}$. If W is an essential wall of $F(I)$, then (as we will see later) for each subset $\nu \in B(W, I)$ either $\delta^+(\nu)$ is contained in I or $\delta^-(\nu)$ is contained in I, but not both.

If W is an interior wall and I a subset of $\mathcal{B}$, then define the *flip operation* $\mathbf{flip}(I, W)$. This is also a subset of $\mathcal{B}$ constructed in this way: We keep in $\mathbf{flip}(I, W)$ all elements $\sigma \in I(W \mid \text{cut})$, while we replace each subset $\delta^+(\nu) \subset I(W \mid \text{facet})$ by its opposite $\delta^-(\nu)$.

Applying a flip to a combinatorial chamber over any wall may not yield an adjacent chamber, as we now see in the example of Figure 7.

Indeed, consider the shaded geometric chamber presented in Figure 7. Then the corresponding combinatorial chamber is

$$I := \{\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\}, \{1, 4, 6\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 6\}\}.$$

If W is the wall $\{1, 4\}$, then $I(W \mid \text{facet}) := \{\{1, 4, 5\}, \{1, 4, 6\}\}$. To perform the flip, we replace this subset of I by $\{\{1, 4, 2\}, \{1, 4, 3\}\}$. We obtain that $\mathbf{flip}(I, W)$

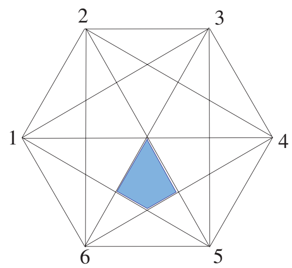


Fig. 7. A flip using the wall 1, 4 does not give a chamber, only a point is the intersection of all cones. On the other hand, a flip using wall 1, 5 yields a triangular chamber.

is equal to

$$\{\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 2\}, \{1, 4, 3\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 6\}\}.$$

But this subset of $\mathcal{B}$ is not feasible. For example, the intersection of the simplicial cones generated by $\{1, 2, 5\}$ and $\{3, 4, 6\}$ lies on the bottom side of the line picturing W, while the intersection of the simplicial cones generated by $\{1, 4, 2\}$ and $\{1, 4, 3\}\}$ lies on the top side.

In contrast the flip of the combinatorial chamber I across the wall W' spanned by the vectors $\{1, 5\}$ is a chamber. Indeed, in this case, we replace in I the subset $\{\{1, 5, 2\}, \{1, 5, 3\}, \{1, 5, 4\}\}$ just by $\{\{1, 5, 6\}\}$. Thus $\mathbf{flip}(I, W')$ is equal to (again see Figure 7),

$$\{\{1, 5, 6\}, \{1, 4, 6\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 6\}\}$$

which is again a maximal feasible subset. The important fact, illustrated by the previous example, is that if one performs the flips over essential walls the result is the desired one.

Lemma 25. *Let W be an essential interior wall of $F(I)$, and let $\mathbf{flip}(I, W)$ be the geometric chamber obtained by the flip of I along the essential wall W. Then the set $\mathbf{flip}(I, W)$ is the combinatorial chamber associated to the W-adjacent chamber sharing W with $F(I)$.*

Clearly all elements $\sigma \in I(W \mid \text{cut})$ and elements in $\delta^-(\nu)$, when ν runs over $B(W, I)$, give rise to simplicial cones containing the W-adjacent chamber. Conversely, any σ in $\mathcal{B}$, such that the cone $C(\sigma)$ contains the W-adjacent chamber, is either in $I(W \mid \text{cut})$ or in a set of the form $\delta^-(\nu)$, with $\nu \in B(W, I)$.

The above lemma stresses the importance of determining the essential walls and that is what we describe next. Each essential wall W is described by a linear inequality that reaches equality at $F(I) \cap W$. The chamber is contained in the corresponding half-space. The presentation we have of the chamber is as the intersection of simplicial cones, their facets provide us with a system of inequalities whose solution is precisely the chamber. The trouble is that this system contains *redundant* inequalities. An inequality is redundant if it is implied by the other constraints in the system, so redundant inequalities can be removed.

Our algorithm for finding the essential walls is based on the following statement, which is essentially Theorem 8.1 on page 101 of [**24**]. Here we state it for full-dimensional polyhedra (thus no equality constraints are present):

Theorem 26. *If no inequality in the system $Ax \leq b$ defining the full-dimensional polyhedron P is redundant, then there exists a one-to-one correspondence between the facets of a polyhedron and the inequalities in $Ax \leq b$ given by $F = \{x \in P \mid a_i x = \beta_i\}$, for any facet F of P and any inequality $a_i x \leq \beta_i$ from the system $Ax \leq b$.*

So if we manage to remove redundant inequalities from the original system of inequalities associated to $F(I)$ we would have found the essential facets of the pointed polyhedral cone $F(I)$. To do this, let us describe a direct method. Let $Ax \leq b, s^T x \leq t$ be a given system of $(m+1)$-inequalities in d-variables $x = (x_1, x_2, \ldots, x_d)^T$. We want to test whether the subsystem of first m inequalities $Ax \leq b$ implies the last inequality $s^T x \leq t$. If so, the inequality $s^T x \leq t$ is redundant and can be removed from the system. A linear programming formulation of this is rather simple:

$$f^* = \text{maximize} s^T x, \text{ subject to } Ax \leq b, s^T x \leq t + 1.$$

Then the inequality $s^T x \leq t$ is redundant if and only if the optimal value f^* is less than or equal to t. By successively solving this linear programming problem for each untested inequality against the remaining inequalities, one would finally obtain an equivalent nonredundant system. Thus the algorithm to recover all the essential walls is as follows:

(1) Find the inequalities of each of the simplicial cones in $F(I)$.
(2) Remove redundant inequalities using linear programming until there is no redundant inequality left. By the previous theorem the wall is uniquely determined by setting the inequalities to equality.

Thus to find all the chambers, we have:

Corollary 27. *The following algorithm finds all the chambers of the vector set Δ_+:*

(1) *Find the lexicographic chamber I_{initial}. Put that as the first element of a list of chambers L.*
(2) *Pick an element I of L for which we have not yet found its adjacent chambers. Determine its essential walls W using the method above.*

(3) *Perform the flips* $\mathbf{flip}(I, W) = I(W)$ *for each essential interior wall* W.
(4) *Add the* $I(W)$ *to the list* L *of existing chambers if not already there, and continue until we have found adjacent chambers for all elements in* L.

Although we now have a concrete algorithm to generate all chambers, for practical reasons it is highly desirable to improve the speed on recognizing the essential walls. For this, the following intuitive proposition states some necessary conditions of the essential walls of a chamber. The proof is left to the reader.

Proposition 28. *Let* I *be a combinatorial chamber (a maximal feasible subset of* $\mathcal{B}$*). Let* W *be a wall of* Δ^+*. If* W *is an essential wall of* $F(I)$*, then the following conditions hold true:*

(1) $I = I(W \mid \text{facet}) \cup I(W \mid \text{cut})$.
(2) $I(W \mid \text{facet}) \neq \emptyset$.
(3) *For each* $\nu \in \mathcal{W}$*, either:* $\delta^+(\nu) \cap I \neq \emptyset$*. Then* $\delta^+(\nu) \subset I$ *and* $\delta^-(\nu) \cap I = \emptyset$*; or* $\delta^-(\nu) \cap I \neq \emptyset$*. Then* $\delta^-(\nu) \subset I$ *and* $\delta^+(\nu) \cap I = \emptyset$.
(4) *Assume* $I(W \mid \text{cut})$ *is not empty. Then* $\bigcap_{\sigma \in I(W|\text{cut})} \overset{\circ}{C}(\sigma)$ *intersects* W *in an* $(r-1)$*-dimensional set.*

Remark. The wall $W = \{1, 4\}$ in Figure 7 satisfies conditions (1), (2), and (3) of Proposition 28, but not condition (4). Applying a flip over a wall satisfying conditions (1), (2), and (3) may not yield an adjacent chamber, as we see in the example of Figure 7. In fact, we have:

Corollary 29. *If* W *is a wall of* $F(I)$ *satisfying conditions* (1), (2), (3), *and not* (4)*; then* $\mathbf{flip}(I, W)$ *is not a feasible subset of* $\mathcal{B}$.

Proof. Assume W verifies (1), (2), and (3). Let $I' = \mathbf{flip}(I, W)$. If W does not satisfy (4), the set $F(\text{cut}) = \bigcap_{\sigma \in I(W|\text{cut})} C(\sigma)$ does not cut W in an open set. Thus $F(\text{cut})$ is contained in one side of the hyperplane W. The set $I(W \mid \text{cut})$ is left stable under the procedure **flip**. Clearly, the other cone $F'(\text{facet}) = \bigcap_{\sigma \in I'(W|\text{facet})} C(\sigma)$ is on the other side of the hyperplane W. Thus the set I' is not feasible. □

The following result justifies why it is so difficult finding the combinatorial chamber that contains a given input vector:

Proposition 30. *Let* A *be an integral matrix. Let* b *be a vector in the cone* $C(A)$ *generated by the columns of* A *and a list* F *of simplicial cones with rays in the columns of* A *such that all elements of* F *contain* b*. Deciding whether* F *includes all simplices that contain* b*, i.e., whether* F *determines the combinatorial chamber that contains* b*, is* NP*-hard.*

Proof. The following decision question is a well-known NP-complete problem: Given a complete graph with positive integral weights on the edges decide whether there is a Hamiltonian tour of cost less than β. We will explain why this problem can be transformed to the problem of "decide whether a list of a simplicial of cones is already enough to determine a chamber."

We will use a theorem by K. Murty (see Theorem 2.1 in [**20**]): Consider the complete bipartite graph $K_{n,n}$. Orient the edges all in the same directions and assign excess 1 to the tail nodes and -1 to the head nodes of each arc. It is well-known that the associated flow polytope is the Birkhoff–Von Neumann polytope of doubly stochastic matrices. This polytope is embedded in $\mathbb{R}^{n^2}$ and the coordinates are in correspondence with the arcs of the bipartite network. The associated network matrix has rank $2n-1$, $2n$ rows, and n^2 columns, one per arc in the network, and we label them $(1,1), (1,2), \ldots, (n-1,n), (n,n)$.

Extend the above network matrix by adding a row of costs, where $c_{i,j}$, $i \neq j$, is the cost to go from i to j, except for the entry associated to the arc i, i where one can put a huge integer value M, much larger than the sum of the n largest $c_{i,j}$'s. On the right-hand side of the matrix equation we add an entry of value β. Written in terms of equations we have

$$\begin{aligned}
\sum_{i=1}^{n} x_{i,j} &= 1, \qquad j = 1, \ldots, n,\\
\sum_{j=1}^{n} -x_{i,j} &= -1, \qquad i = 1, \ldots, n,\\
\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}x_{i,j} &= \beta,\\
x_{ij} &\geq 0 \qquad \text{for all } i, j.
\end{aligned}$$

This system now has rank $2n$.

The important point is: If the set of columns $\{(1,1), (2,2), \ldots, (n,n), (i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}$ defines a simplicial cone containing the vector $b = (1, 1, 1, \ldots, -1, -1, -1, \beta)$, then $(i_1, j_1), \ldots, (i_n, j_n)$ must be a traveling salesman tour with cost less than or equal to β. Thus if we take as F the set of all simplicial cones of bases that do not use all the columns $\{(1,1), (2,2), \ldots, (n,n)\}$ and contain $b = (1, 1, \ldots, -1, -1, \beta)$, the remaining job of deciding whether any other cone contains the vector b is at least as hard as the solution of the traveling salesman problem. □

To conclude this section it is worth mentioning that one can abstractly apply flips to the nonessential walls satisfying (1), (2), and (3). The interior of the resulting "chamber" may actually have an empty interior in that case and thus is not useful for us here. Nevertheless, this phenomenon plays an important role in the theory under the name of *virtual chambers*. In fact, there is another characterization of the chambers using the triangulations of the Gale diagram of the original vectors. (See [**31**] for an introduction to Gale diagrams and triangulations.)

Lemma 31 (See [**5**], [**10**]). *The face lattice of the chamber complex of a vector configuration A is anti-isomorphic to the face lattice of the secondary polyhedron of the Gale transform $\hat{A}$ of A. The vertices of the polyhedron are the regular triangulations of $\hat{A}$.*

Thus generating the chambers of a network cone is the same as generating the distinct regular triangulations of the Gale diagram of an extended network matrix. Such calculations can also be done using the software *topcom* [**23**].

5. Computational Experiments

Now we present some computational experiments. All experiments were done in a 1 GHz pentium computer running Linux using `Maple 7`. All our software is available at `www.math.ucdavis.edu/~totalresidue`. We present our experiments in three tables. We begin with Tables 1 and 2 that deal with Kostant's partition function, this is the case of acyclic complete graphs. As we saw in Lemma 1, all other networks can be embedded into this case. We did examples in the cases of K_4 (A_3^+), K_5 (A_4^+) in the first table and in the second table we have bigger examples for the cases A_6^+, A_7^+, A_8^+, A_9^+, and A_{10}^+. We show computation times in both tables and Table 2 also shows the cardinality of the special permutation sets. The computations show that the total residue method is faster than brute force enumeration and the current implementation of software `LattE` [**11**], [**12**] by one or two orders of magnitude. `LattE`, on the other hand, is the only software that can deal with arbitrary rational convex polyhedra.

As is clear in the two first tables, the computation time does not increase significantly when the weights on nodes are very large. On the other hand, computation time quickly becomes very large when the number of nodes of the graph is growing. This is in agreement with Barvinok's result (see [**3**]): computing $k(\Phi, a)$ can be done in polynomial time, if the size of Φ is fixed. Let us stress here that our method for network polytopes is different from Barvinok's algorithm, recently implemented by `LattE`.

In the second table it is evident that, for a fixed number of nodes, the time of computation depends strongly on the cardinality of the set $\mathrm{Sp}(a)$, i.e., the signs of weights on the nodes (when all weights are positive, except the last, the cardinality of $\mathrm{Sp}(a)$ is 1).

Let us stress that one of the features of our method is that it can directly compute the polynomial $k(\Phi, a)$ giving the number of lattice points in the polytope $P(\Phi, a)$ in the chamber determined by a. In particular, the Ehrhart polynomial of the polytope $P(\Phi, a)$, i.e., the function $t \mapsto k(\Phi, ta)$, is also computed easily from

Table 1. Testing for the complete graphs K_4 and K_5. Time is given in seconds. Excess vectors are in the first column.

Weights on nodes	# of flows	Secs
[6, 8, −5, −9]	223	0.1
[9, 11, −12, −8]	330	0.1
[1000, 1, −1000, −1]	3002	0.009
[4383, −886, −2777, −720]	785528058	0.1
[4907, 2218, −3812, −3313]	20673947895	0.1
[47896, 30744, −46242, −32398]	19470466783680	0.01
[69295, 62008, −28678, −102625]	179777378508547	0.1
[3125352, 6257694, −926385, −8456661]	34441480172695101274	0.01
[6860556, 1727289, −934435, −7653410]	91608082255943644656	0.1
[12, 8, −9, −7, −4]	14805	0.081
[125, 50, −75, −33, −67]	6950747024	0.020
[763, 41, −227, −89, −488]	222850218035543	0.019
[11675, 88765, −25610, −64072, −10758]	563408416219655157542748	0.011
[78301, 24083, −22274, −19326, −60784]	1108629405144880240444547243	0.029
[52541, 88985, −1112, −55665, −84749]	3997121684242603301444265332	0.010
[71799, 80011, −86060, −39543, −26207]	160949617742851302259767600	0.010
[45617, 46855, −24133, −54922, −13417]	15711217216898158096466094	0.21
[54915, 97874, −64165, −86807, −1817]	102815492358112722152328	0.060
[69295, 62008, −28678, −88725, −13900]	65348330279808617817420057	0.010
[8959393, 2901013, −85873, −533630, −11240903]	6817997013081449330251623043931489475270	0.010
[2738090, 6701290, −190120, −347397, −8901863]	277145720781272784955528774814729345461	0.010
[6860556, 1727289, −934435, −818368, −6835042]	710305971948234346520365668331191134724	0.060

Table 2. Testing for complete graphs K_n with $n = 6, 7, 8, 9, 10, 11$. Time is given in seconds.

Weights on nodes	# of flows	Secs	$\|\mathrm{Sp}(a)\|$
[1, 2, 3, 4, 5, −15]	5880	0.02	1
[21128, 45716, 79394, −76028, −31176, 66462, −105496]	58733548560911702671 16780821466940568432 553474831987566395925	0.22	8
[82275, 33212, 91868, −57457, 47254, −64616, 94854, −227390]	22604049468113537772 228176193404009135 6424181	2.14	26
[31994, −12275, 55541, 72295, 26697, −3212, −38225, 6916, −139731]	11446847479255704222 87042245223206779226 01568734727431018393 069006356672309031382 51984519069399479632 6644137066000	7.94	24
[12275, 55541, 72295, 26697, −3212, −38225, 6916, 92409, 9528, −234224]	12970047729476531166 58326881685949118367 16319862924094634125 27856414458487356258 66474206451882923253 41990044115208492747 58896993761880000897 382293730	21.31	16
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, −55]	38883505145515430400	5	1
[46398, 36794, 92409, −16156, 29524, −68385, 93335, 50738, 75167, −54015,−285809]	20889867895116832060 28578373441423712122 50684806890637191792 33590765780756053509 92237184823590262176 29560725791309259479 21077842421668832691 54404688022155977982 34585056426719876125 028873152	2193.23	322

our algorithm. For example, corresponding to the first line of Table 2:

$$k(A_r^+, (t, 2t, 3t, 4t, 5t, -15t)) = \frac{1}{120960}(6t+1)(t+4)(t+3)(t+2)(t+1) \times (64921t^5 + 233897t^4 + 307649t^3 + 184639t^2 + 50574t + 5040)$$

which was computed in 0.55 seconds.

The polynomial function $k(\Phi, (a_1, a_2, a_3, a_4, a_5))$ (with $a_5 = -(a_1+a_2+a_3+a_4)$) in the chamber $\{a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0\}$ is computed in 0.48 seconds.

The Ehrhart polynomials for the second, third, and fourth examples in Table 2, i.e.,

$$k(A_r^+, (21128t, 45716t, 79394t, -76028t, -31176t, 66462t)),$$

$$k(A_r^+, (82275t, 33212t, 91868t, -57457t, 47254t, -64616t, 94854t)),$$

and

$$k(A_r^+, (31994t, -12275t, 55541t, 72295t, 26697t, -3212t, -38225t, 6916t)),$$

were computed in 1.36 seconds, 18.54 seconds, and 93.36 seconds, respectively.

It is also interesting to check the program on the value of the Kostant partition for A_r^+ on the vector $a = [1, 2, 3, 4, \dots, r, -r(r+1)/2]$. As proven by Zeilberger [**30**], this value is $\prod_{i=1}^r (2i)!/i!(i+1)!$.

The last table is dedicated to 4×4 transportation matrices. In the case of transportation polytopes, i.e., complete bipartite graphs, we were able to compare our speed to the special purpose C^{++} program written by Beck and Pixton [**4**]. Both `LattE` and Beck–Pixton's software are faster than our `Maple` implementation, with Beck–Pixton's significantly so, but it must still be emphasized that our calculations for transportation polytopes make use of the fact that they are embedded inside the complete graph for a large enough number of nodes. For example, the case of 4×4 transportation polytopes is treated via the complete graph K_8. The same kind of embedding can be done for other networks.

If we consider the case of 4×5 matrices with weights on nodes [3046, 5173, 6116, 10928], [182, 778, 3635, 9558, 11110], the number of lattice points is 23196436596128897574829611531938753 calculated in 11.15 seconds. The number of special permutations for this vector is 540 while the number of vertices of the corresponding polytope is 912. This same example takes 7.8 seconds in `LattE` and 0.1 seconds in the Beck–Pixton program.

The Ehrhart polynomial

$$k(\Phi_{4,5}, (3046t, 5173t, 6116t, 10928t, -182t, -778t, -3635t, -9558t, -11110t))$$

is computed in 30.72 seconds.

Table 3. Testing for 4×4 transportation polytopes.

Margins	# of lattice points	Secs
[220, 215, 93, 64], [108, 286, 71, 127]	1225914276768514	5.04
[109, 127, 69, 109], [119, 86, 108, 101]	993810896945891	10.43
[72, 67, 47, 96], [70, 70, 51, 91]	25387360604030	6.5
[179909, 258827, 224919, 61909], [190019, 90636, 276208, 168701]	13571026063401838164668296635065899923152079	5.87
[229623, 259723, 132135, 310952], [279858, 170568, 297181, 184826]	646911395459296645200004000804003243371154862	16.1
[249961, 232006, 150459, 200438], [222515, 130701, 278288, 201360]	319720249690111437887229255487847845310463475	16.1
[140648, 296472, 130724, 309173], [240223, 223149, 218763, 194882]	322773560821008856417270275950599107061263625	11.7
[65205, 189726, 233525, 170004], [137007, 87762, 274082, 159609]	6977523720740024241056075121611021139576919	9.0
[251746, 282451, 184389, 194442], [146933, 239421, 267665, 259009]	861316343280649049593236132155039190682027614	15
[138498, 166344, 187928, 186942], [228834, 138788, 189477, 122613]	63313191414342827754566531364533378588986467	19.4
[20812723, 17301709, 21133745, 27679151], [28343568, 18410455, 19751834, 20421471]	665711555567792389878908993624629379187969880179721169068827951	15.6
[15663004, 19519372, 14722354, 22325971], [17617837, 25267522, 20146447, 9198895]	63292704423941655080293971395348848807454253204720526472462015	27.4
[13070380, 18156451, 13365203, 20567424], [12268303, 20733257, 17743591, 14414307]	43075357146173570492117291685601604830544643769252831337342557	14.8

If we consider the case of 5×5 matrices with weights on nodes [30201, 59791, 70017, 41731, 58270], [81016, 68993, 47000, 43001, 20000], then the number of lattice points is

$$24640538268151981086397018033422264050757251133401758112509495633028,$$

which we computed in 23 minutes. The number of special permutations needed is 9572 while the number of vertices of the corresponding polytope is 13150. This example took 20 minutes with `LattE` and just 4 seconds with the Beck–Pixton program.

Transportation polytopes were treated by Beck and Pixton [**4**] in a special purpose C^{++} program dedicated to this particular family of flow polytopes. Their computation is also via residues and is the fastest at the moment. It is important to remark that their use of residues is quite different from ours; our main theorem can be thought of as a multidimensional analogue of the fact that sums of the residues of a rational function on $P_1(\mathbb{C})$ are zero. It is to be expected that in a forthcoming C^{++} implementation the timings discussed here will be considerably faster than those from this preliminary `Maple` implementation. Besides the obvious implementation speed-ups, the ideas presented in this paper could still be improved when the total residue method is applied directly to the bipartite graph, not as a subnetwork of K_n.

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