Discrete series representations and K multiplicities for $U(p,q)$. User's guide

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Abstract

This document is a companion for the Maple program **Discrete series and K-types for** $U(p,q)$ available on

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We explain an algorithm to compute the multiplicities of an irreducible representation of $U(p) \times U(q)$ in a discrete series of $U(p,q)$. It is based on Blattner's formula. We recall the general mathematical background to compute Kostant partition functions via multidimensional residues, and we outline our algorithm. We also point out some properties of the piecewise polynomial functions describing multiplicities based on Paradan's results.

Contents

Introduction

The present article is a user's guide for the Maple program **Discrete series and K types for** $U(p, q)$, available at

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In the first part, we explain what our program does with simple examples. The second part sets the general mathematical background. We recall Blattner's formula, and we discuss the piecewise (quasi)-polynomial behavior of multiplicities for a discrete series of a reductive real group. In the third part, we outline the algorithm of computing partition functions for arbitrary set of vectors, based on

Jeffrey-Kirwan residue and maximally nested subsets. In the fourth part, we specialize the method to $U(p,q)$. We in particular explain how to compute maximally nested subsets of the set of non compact positive roots of $U(p,q)$.

In the last two sections, we give more examples and some details on the implementation of the algorithms for our present application.

Here G is the Lie group $G = U(p,q)$ and $K = U(p) \times U(q)$ is a maximal compact subgroup of G. Of course all these issues can be addressed for the other reductive real Lie groups following the same approach described here. To introduce the function we want to study, we briefly establish some notations (see Sec. 2.2). We denote by π^λ a discrete series representation with Harish Chandra parameter λ. The restriction of π^λ to the maximal compact subgroup K decomposes in irreducible finite dimensional K representations with finite multiplicities, in formula

$$\pi^\lambda\ _{|K} = \sum_{\tau_\mu} m_\mu^\lambda \tau_\mu$$

where we sum over $\hat{K}$: the classes τ_μ of irreducible finite dimensional K representations, μ being the Harish Chandra parameter of τ_μ.
m_μ^λ is a finite number called the **multiplicity** of the K-type τ_μ, or simply of μ, in π^λ and this paper addresses the question of computing m_μ^λ.
The algorithm described in this paper checks whether a certain K-type τ_μ appears in the K-spectrum of a discrete series π^λ by computing the multiplicity of such a K-type. The input μ can also be a symbolic variable as we will explain shortly.
By Blattner's formula, computing the K-multiplicity is equivalent to compute the number of integral points, that is a partition function, for specific polytopes. We thus use the formulae developed in [3] to write the algorithm.

One important aspect of these results is that the input datas (λ, μ) can also be treated as parameters. Thus, in principle, given (λ_0, μ_0), we can output a convex cone, containing (λ_0, μ_0), and a polynomial function of the parameters (λ, μ) whose value at (λ, μ) is the multiplicity of the K-type μ in the representation π^λ as long as we stay within the region described by the cone and (λ, μ) satisfy some integrality conditions to be defined later. We also plan to explicitly decompose our parameter space (λ, μ) in such regions of polynomiality in a future study, thus describing fully the piecewise polynomial function m_μ^λ, at least for some low rank cases.

In practise here, we only address a simpler question: we will fix λ, μ and a direction $\vec{v}$ and compute a piecewise polynomial function of t coinciding with $m_{\mu+t\vec{v}}^\lambda$ on integers t. This way, we can also check if a direction $\vec{v}$ is an asymptotic direction of the K-spectrum, in the sense explained in Sec. 3.7.3.

The Atlas of Lie Groups and Representations, [1], within the problem of classifying all of the irreducible unitary representations of a given reductive Lie group, addresses in particular the problem of computing K-types of discrete series. The multiplicities results needed for the general unitary problem is of different nature as we are going to explain.

Given as input λ and some height h, Atlas computes **the list with multiplicities** of all the representations occurring in π^λ **of height smaller** than h. But the efficiency, in this setting, is limited by the height. In contrast, the efficiency of our program is unsensitive to the height of λ, μ, but the output is **one number**: the multiplicity of μ in π^λ. It takes (almost) the same time to compute the multiplicity of the lowest K-type of π^λ (fortunately the answer is 1) than the multiplicity of a representation of very large height. Our calculation are also very sensitive to the rank: $p+q-1$.

For other applications (weight multiplicities, tensor products multiplicities) based on computations of Kostant partition functions in the context of finite dimensional representations, see [4], [6], [2].

1 The algorithm for Blattner's formula: main commands and simple examples

Let p, q be integers. We consider the group $G = U(p,q)$. The maximal compact subgroup is $K := U(p) \times U(q)$. More details in parametrization are given in Section 4.

A discrete series representation π^λ is parametrized according to Harish-Chandra parameter λ, that we input as **discrete**:

$$discrete := [[\lambda_1, \ldots, \lambda_p], [\gamma_1, \ldots, \gamma_q]].$$

Here λ_i, γ_j are **integers** if $p+q$ is odd, or **half-integers** if $p+q$ is even. They are all distinct. Furthermore $\lambda_1 > \cdots > \lambda_p$ and $\gamma_1 > \cdots > \gamma_q$.

A unitary irreducible representation of K (that is a couple of unitary irreducible representations of $U(p)$ and of $U(q)$) is parametrized by its Harish-Chandra parameters μ that we input as **Krep**:

$$Krep := [[a_1, a_2, \ldots, a_p], [b_1, \ldots, b_q]]$$

with $a_1 > \cdots > a_p$ and $b_1 > \cdots > b_q$.

Here a_i are integers if p is odd, half-integers if p is even. Similarly b_j are integers if q is odd, half-integers if q is even.

As we said, our objective is to study the function m_μ^λ for $\mu \in \hat{K}$ where $\mu = Krep$ and $\lambda = discrete$.

The examples are runned on a MacBook Pro, Intel Core 2 Duo, with a Processor Speed of 2.4 GHz. The time of the examples is computed in seconds. They are recorded by

```
TT
```

.

Some of these examples are very simple and can be checked by hand (as we did, to reassure ourselves). Other examples are given at the end of this article.

To compute the multiplicity of the K-type given by **Krep** in the discrete series with parameter given by **discrete**, the **command** is

```
>discretemult(Krep,discrete,p,q)
```

Example 1

```
Krep:=[[207/2, -3/2], [3/2, -207/2]];
discrete:=[[5/2, -3/2], [3/2, -5/2]];

>discretemult(Krep,discrete,2,2);

101
```

Here is another example of m_μ^λ that our program can compute.

Example 2

We consider the discrete series indexed by

```
lambda33:=[[11/2, 7/2, 3/2], [9/2, 5/2, 1/2]];
```

Its lowest K-type is

```
lowlambda33:= [[7,4,1], [5,2,-1]];
```

Of course, the multiplicity of the lowest K-type is 1. Our programm fortunately returns the value 1 in 0.03 seconds.

Consider now the representation of K with parameter:

```
biglambda33:= [[10006, 4, -9998], [10004, 2, -10000]];
```

Then the multiplicity of this K-type is computed computed in 0.05 seconds as

```
> discretemult(biglambda33,lambda33,3,3);

  2500999925005000;
```

Here are two other examples that verify the known behavior of holomorphic discrete series. The notation *ab...* that we use to label the discrete series parameters is introduced in 4.1 and it very effective to picture the situation, but it is not relevant to understand the following computation.

Example 3

Consider a holomorphic discrete series of type "aaabbb" for $G = U(3,3)$ (see Sec.4.1) with lowest K-type of dimension 1. We verify that the multiplicity is 1 for μ in the cone spanned by strongly orthogonal non compact positive roots.

```
hol33:=[[11/2,9/2,7/2],[5/2,3/2,1/2]];
lowhol33:=[[7, 6, 5], [1, 0, -1]];
bighol33:=[[7+1000, 6+100, 5+10], [1-10, -100, -1-1000]];

>discretemult(lowhol33,hol33,3,3);

1
TT:=  0.052

>discretemult(bighol33,hol33,3,3);

1
TT:=  1.003
```

Example 4

Consider a holomorphic discrete series of type "aaabbb" for $G = U(3,3)$ (see Sec. 4.1) with lowest K-type of dimension d. We verify that the multiplicities are bounded by d.

```
Hol33:=[[27/2,9/2,7/2],[5/2,3/2,-5/2]];
lowHol33:=[[15, 6, 5], [1, 0, -4]];
bigHol33:=[[15+1000, 6+1000, 5+1000], [1-1000, -1000, -4-1000]];
verybigHol33:=[[15+100000, 6+10000, 5+10000], [1-10000, -10000, -4-100000]];

>discretemult(lowHol33,Hol33,3,3);

 1
TT:= 0.069

>discretemult(bigHol33,Hol33,3,3);

 4
TT:=0.971

>discretemult(verybigHol33,Hol33,3,3);

4
TT:= 0.873
```

Fix now a K-type μ, a direction $\vec{v}$ given by a dominant weight for K (more details in Sec. 3.7.3) and a discrete series parameter λ. The half-line $\mu + t\vec{v}$ stays

inside the dominant chamber for K. A very natural question is that of investigating the behavior of the multiplicity function as a function of $t \in \mathbb{Z}$, when we move from μ along the positive $\vec{v}$ direction, that is the function $t \to m^{\lambda}_{\mu+t\vec{v}},\ t \geq 0$. The answer to this question will be given by two sets of datas: a covering of $\mathbb{N}$ determined by a finite number of closed intervals $I_i \subset \mathbb{R}$ with integral end points, that is $\mathbb{N} = \cup_{1\leq i\leq s}(I_i \cap \mathbb{N})$, together with polynomial functions $P_i(t),\ 1 \leq i \leq s$, of degree bounded by $pq-(p+q-1)$, that compute the multiplicity on such intervals I_i: in formula $m^{\lambda}_{\mu+t\vec{v}} = P_i(t)$ for $t \in I_i \cap \mathbb{N}$.
We remark two aspects. First the $\{I_i \cap \mathbb{N},\ 1 \leq i \leq s\}$ constitutes a covering in the sense that we recover all of $\mathbb{N}$ but $(I_j \cap \mathbb{N}) \cap (I_{j+1} \cap \mathbb{N})$ can intersect in the extreme points and hence in this case P_j and P_{j+1} have to coincide on the intersection. Secondly the "intervals" $I_i \cap \mathbb{N}$ can be reduced just to a point, so that the polynomial P_i (if not constant) is not uniquely determined by its value on one point !. More generally, if the length of the interval I_i is smaller that the degree of P_i, the polynomial P_i is not uniquely determined.
Because of our focus on the polynomiality aspects we keep in the output of the algorithm the polynomials P_i even if the intervals I_i are reduced to a point (or with small numbers of integral points).
In our application, μ will be the lowest K-type and we will give some examples of the situations occurring, in particular we will examine the following cases:

- The first example outputs a covering of $\mathbb{N}$ given by a unique interval and a polynomial function on $\mathbb{N}$ that computes the multiplicity. Thus in this case, all the integer points on the half line $\mu + t\vec{v}, t \geq 0$ give rise to K-types that appear in the restriction of the discrete series, in particular $\vec{v}$ is an asymptotic direction, (see Sec. 2.4 and 3.7.3),
- In the other examples, the covering of $\mathbb{N}$ has at least two intervals and illustrate different situations.

To compute, in the sense we just explained, the multiplicity of the K-type $\mu + t\vec{v}$ in the discrete series with parameter **discrete**, μ being the lowest K-type moving in the positive direction $\vec{v}$, labeled by **direction** , the **command** is

```
>function_discrete_mul_direction_lowest(discrete,direction,p,q);
```

Example 5

```
discrete:=[[5/2, -3/2], [3/2, -5/2]];
direction:=[[1,0],[0,-1]];

>function_discrete_mul_direction_lowest(discrete,[[1,0],[0,-1]], 2,2);

 [[t+1, [0, inf]]]
```

(inf stands for ∞).

Here the covering of $\mathbb{N}$ is $\mathbb{N} = [0, \infty] \cap \mathbb{N}$ and the polynomial is $P(t) = t + 1$. The output explicitly compute:

$$m^{\lambda}_{\mu+t\vec{v}} = t + 1,\ t \in \mathbb{N},\ t \geq 0$$

with $\mu = [[7/2, -3/2], [3/2, -7/2]]$ the lowest K-type of the representation π^{λ} (See Ex.8 for computation of the lowest K-type). Thus we compute the multiplicity, starting from the lowest K-type when we are moving off it in the direction of $\vec{v}$. In particular for $t = 100$ we get $m^{\lambda}_{\mu+100\vec{v}} = 101$ as predicted in Ex.1, since

```
discrete:=[[5/2, -3/2], [3/2, -5/2]]
```

and $\mu + 100\vec{v}$ is equal to

```
 Krep:=[[207/2, -3/2], [3/2, -207/2]];
```

Example 6

```
discrete:=[[9, 7], [-1, -2, -13]];
direction:= [[1, 0], [0, 0, -1]];

>function_discrete_mul_direction_lowest(discrete, direction,2,3);

[[1+(1/2)*t-(1/2)*t^2, [0, 0]], [1, [1, inf]]]
```

Here the covering is $\mathbb{N} = ([0, 0] \cap \mathbb{N}) \cup ([1, \infty] \cap \mathbb{N}$ and the polynomials are $P_1(t) = 1 + \frac{1}{2}t - \frac{1}{2}t^2$ on $[0, 0] \cap \mathbb{N}$ and $P_2(t) = 1$ on $([1, \infty] \cap \mathbb{N}$. Observe that $[0, 0] \cap \mathbb{N} = [0]$ is just a point and that $P_1(0) = 1$ as it should be, since μ is the lowest K-type. Explicitly we simply compute

$$m^{\lambda}_{\mu+t\vec{v}} = 1,\ t \in \mathbb{N},\ t \geq 0.$$

We conclude with one example in which the multiplicity grows: the last polynomial is not zero and has degree two. We give more examples at the end of this article.

Example 7

```
discrete:= [[57/2, 39/2, 3/2], [51/2, 5/2, -155/2]];
direction:=[[1, 0, 0], [0, 0, -1]];

>function_discrete_mul_direction_lowest(discrete,direction,3,3);

[(1/24)*t^4+(5/12)*t^3+(35/24)*t^2+(25/12)*t+1, [0, 16]], [-3059+(2242/3)*t-(133/2)*t^2+(19/6)*t^3, [17, 21]],
[-11914+(9597/4)*t-(4367/24)*t^2+(27/4)*t^3-(1/24)*t^4, [22, 40]], [100016-8664*t+228*t^2, [41, inf]]
```

That is for λ =discrete and $\mu = [[30, 20, 1], [26, 2, -79]]$ the lowest K-type

$$m^{\lambda}_{\mu+t\vec{v}} = \begin{cases} (1/24) * t^4 + (5/12) * t^3 + (35/24) * t^2 + (25/12) * t + 1 & 0 \leq t \leq 16 \\ -3059 + (2242/3) * t - (133/2) * t^2 + (19/6) * t^3 & 17 \leq t \leq 21 \\ -11914 + (9597/4) * t - (4367/24) * t^2 + (27/4) * t^3 - (1/24) * t^4 & 22 \leq t \leq 40 \\ 100016 - 8664 * t + 228 * t^2 & t \geq 41 \end{cases}$$

The time to compute the example is $TT := 0.835$ and the formula says for instance that $m^{\lambda}_{\mu+2000000\vec{v}} = 911982672100016$

To compare with other parametrizations of the discrete series representations, it may be useful also to give here the command for the lowest K-type of the discrete series with parameter $\lambda = discrete$ of the group $U(p, q)$. The command is:

```
>Inf_lowestKtype(p,q, discrete)
```

Example 8

```
> Inf_lowestKtype([[5/2,-3/2],[3/2,-5/2]],2,2);

 [[7/2,-3/2], [3/2, -7/2]]
```

Similarly, we may want to parametrize a representation of K by its highest weight. Then the **command** is:

```
> voganlowestKtype(discrete,p,q)
```

Example 9

```
> voganlowestKtype([[5/2,-3/2],[3/2,-5/2]],2,2);

 [[3,-1], [1, -3]]
```

Let us finally recall the simple case of the multiplicity function for $G = U(2,1)$ (see Sec. 4.1 for the notations).

Choose as positive compact root the root $e_1 - e_2$ and denote by w the corresponding simple reflection. Fix $\lambda = [[\lambda_1, \lambda_2], [\lambda_3]]$ the Harish Chandra parameter for a discrete series representation. We assume λ regular and $\lambda_1 > \lambda_2$. There are three chambers $\mathfrak{c}_1$, $\mathfrak{c}_2$, $\mathfrak{c}_3$ and hence three systems of positive roots containing $e_1 - e_2$. Precisely $\mathfrak{c}_1$ corresponds to the positive system $\{e_1 - e_2, e_1 - e_3, e_2 - e_3\}$, $\mathfrak{c}_2$ corresponds to the positive system $\{e_1 - e_2, e_1 - e_3, e_3 - e_2\}$ and $\mathfrak{c}_3$ to $\{e_1 - e_2, e_3 - e_1, e_3 - e_2\}$.

We examine the situation in which the discrete series parameter belongs to one of these chambers. Fig.1 and Fig.2, picture the two chambers $\mathfrak{c}_1$, $\mathfrak{c}_2$ and evidentiate the values for m^λ_μ when λ is in the chamber. The black lines mark the chambers containing the compact root $e_1 - e_2$ and the red ones the system of positive roots for the given chamber.

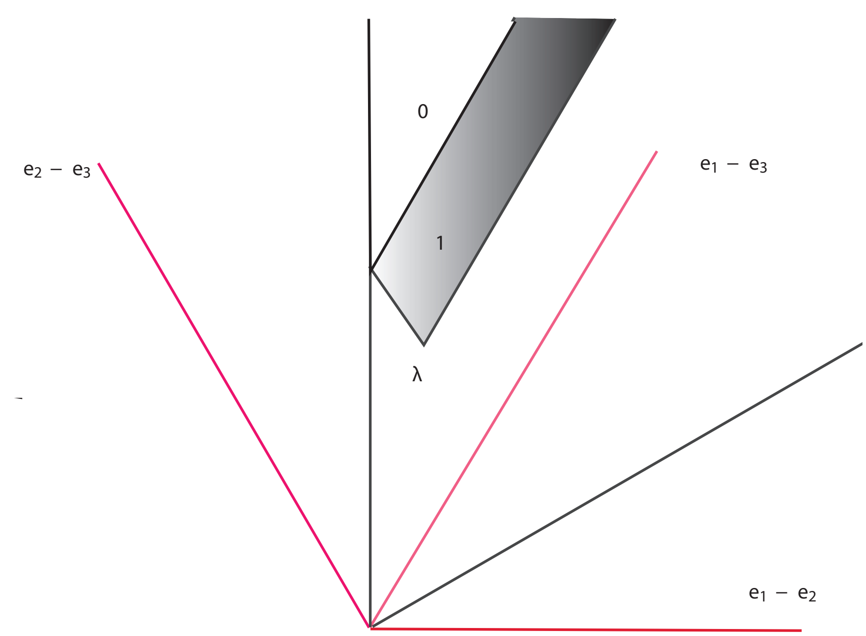

Figure 1: m^λ_μ for the chamber $\mathfrak{c}_1$ of U(2,1).

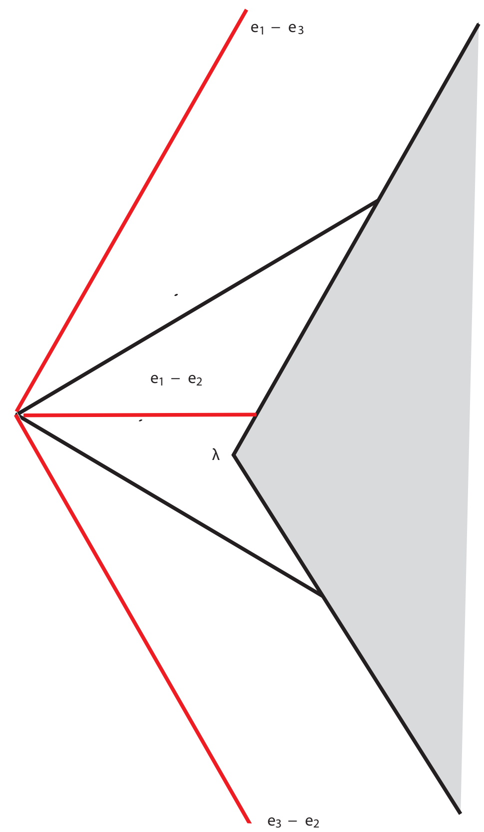

Figure 2: m_μ^λ for the chamber $\mathfrak{c}_2$ of U(2,1)

We give some examples concerning the two situations. The parameter λ in Fig.1 is

$$hol21 := [[2,1],[-3]]$$

In Figure 2 the parameter λ is

$$aba := [[2,-3],[1]].$$

2 Mathematical background

2.1 Notations

$G, \mathfrak{g}$	semi-simple real Lie group, Lie algebra of G.
$K, \mathfrak{k}$	maximal compact subgroup of G, Lie algebra of K.
$T, \mathfrak{t}$	maximal torus of K, Lie algebra of T.
$P \subset \mathfrak{t}^*$	lattice of weights of T.
$\Delta^+, \Delta^+(\lambda) \subset \mathfrak{t}^*$	system of positive roots for G.
$\Delta_c^+, \Delta_c^+(\lambda)$	system of positive compact roots.
$\Delta_n^+, \Delta_n^+(\lambda)$	system of positive non compact roots.
$\Delta^+(A, B)$	system of positive roots of parabolic type determined by (A, B).
$\mathfrak{a}, \mathfrak{a}_c$	positive chamber for G and K respectively
ρ, ρ_c^+, ρ_n^+	half the sum of positive, positive compact, positive non compact roots.
$P_{\mathfrak{g}}, P_{\mathfrak{k}}$	set of G admissible, K admissible parameters.
$P_{\mathfrak{g}}^r, P_{\mathfrak{k}}^r$	set of G admissible and regular , K admissible and regular parameters.
U	r-dimensional real vector space; $x \in U$.
V	dual vector space of U, $h \in V$.
$\langle\ ,\ \rangle$	the pairing between U and V.
$V_{\mathbb{Z}}$	lattice of V.
$U_{\mathbb{Z}}$	dual lattice in U.
$\mathcal{A}^+$	a sequence of vectors in $V_{\mathbb{Z}}$; $\alpha \in \mathcal{A}^+$.
τ	a tope.
T	torus $U/U_{\mathbb{Z}}$; $t \in T$.
F	finite subset of T.
$\Pi_{\mathcal{A}^+}(h)$	polytope defined by $\mathcal{A}^+$.
$N_{\mathcal{A}^+}(h)$	number of integral points for $\Pi_{\mathcal{A}^+}(h)$.
$\mathfrak{c}$	chamber.
$\mathrm{JK}_{\mathfrak{c}}$	Jeffrey-Kirwan residue.
Ires	iterated residue.

2.2 Blattner formula and multiplicities

Let G be a reductive connected linear Lie group with Lie algebra $\mathfrak{g}$ and denote by K a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$.

We assume that the ranks of G and K are equal. Under this hypothesis the group G has discrete series representations. Recall Harish-Chandra's parametrization of discrete series representations. We choose a compact Cartan subgroup $T \subset K$ with Lie algebra $\mathfrak{t}$. Let $P \subset \mathfrak{t}^*$ be the lattice of weights of T. They correspond to characters of T. Here if $\lambda \in P$, the corresponding character of T

is $e^{i\lambda}$. Let $\Delta^+ \subset P$ be a positive system of roots and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Then the subset $\rho + P \subset \mathfrak{t}^*$ does not depend of the choice of the positive system Δ^+. We denote it by $P_{\mathfrak{g}}$. We denote by $P^r_{\mathfrak{g}} \subset P_{\mathfrak{g}}$ the subset of $\mathfrak{g}$-regular elements. We can similarly define $P^r_{\mathfrak{k}}$. We denote by $\mathcal{W}_c$ the Weyl group of K. For any $\lambda \in P^r_{\mathfrak{g}}$, Harish-Chandra defined a discrete series representation π^λ. Elements of $P^r_{\mathfrak{g}}$ are called Harish-Chandra parameters for G. Two representations, π^λ and $\pi^{\lambda'}$, coincide precisely when their parameters λ, λ' are related by an an element of $\mathcal{W}_c$. Thus the set of discrete series representations is parametrized by $P^r_{\mathfrak{g}}/\mathcal{W}_c$.

In the same way we can parametrize the set $\hat{K}$ of classes of irreducible finite dimensional representations of K by their Harish-Chandra parameter $\mu \in P^r_{\mathfrak{k}}/\mathcal{W}_c$. Once a positive system of compact roots is chosen, an element $\mu \in P^r_{\mathfrak{k}}$ can be conjugated to a unique regular element in the corresponding positive chamber $\mathfrak{a}_c \subset \mathfrak{t}^*$ for the compact roots. We denote by $\tau_\mu \in \hat{K}$, or simply by $\mu \in \hat{K}$, the corresponding representation.

A discrete series representation π^λ is K-finite:

$$\pi^\lambda\ _{|K} = \sum_{\tau_\mu\in\hat{K}} m^\lambda_\mu \tau_\mu.$$

To determine m^λ_μ, that is the multiplicity of the K-type τ_μ in π^λ, is a basic problem in representation theory.

Blattner's formula, [14], gives an answer to this problem. We need to introduce a little more notations before stating it.

We let Δ be the root system for $\mathfrak{g}$ with respect to $\mathfrak{t}$, Δ_c the system of compact roots , that is the roots of $\mathfrak{k}$ with respect to $\mathfrak{t}$, and Δ_n the system of noncompact roots. We let Δ^+ be the unique positive system for Δ with respect to which λ is dominant. We write $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, $\rho_c = \frac{1}{2}\sum_{\alpha\in\Delta^+_c}\alpha$ and $\rho_n = \frac{1}{2}\sum_{\alpha\in\Delta^+_n}\alpha$ where $\Delta^+_c = \Delta^+ \cap \Delta_c$ and $\Delta^+_n = \Delta^+ \cap \Delta_n$. Therefore $P_{\mathfrak{g}} = \rho + P$, $P_{\mathfrak{k}} = \rho_c + P$ and if $\xi \in P_{\mathfrak{g}}$, then $\xi + \rho_n \in P_{\mathfrak{k}}$.

Write $\mathfrak{a}, \mathfrak{a}_c \subset \mathfrak{t}^*$ for the (closed) positive chambers corresponding to Δ^+ and Δ^+_c. We will also write $\Delta^+(\lambda), \Delta^+_c(\lambda)$ and $\Delta^+_n(\lambda)$ for $\Delta^+, \Delta^+_c, \Delta^+_n$ if necessary to stress that these systems depend on λ.
Then for $\mu \in P^r_{\mathfrak{k}} \cap \mathfrak{a}_c$, **Blattner's formula** says:

$$m^\lambda_\mu = \sum_{w\in\mathcal{W}_c} \epsilon(w)\mathcal{P}_n(w\mu - \lambda - \rho_n) \tag{1}$$

where, given $\gamma \in \mathfrak{t}^*$, we define $\mathcal{P}_n(\gamma)$ to be the number of distinct ways in which γ can be written as a sum of **positive noncompact roots** (recall our identification for which $\Delta, \Delta_c \subset \mathfrak{t}^*$). The number $\mathcal{P}_n(\gamma)$ is a well-defined integer, since the elements of Δ^+_n span a cone which contains no straight lines. As usual, $\epsilon(w)$ will

stand for the sign of w. Remark that, as $\mu + \rho_c$ and $\lambda + \rho_n + \rho_c$ are weights of T, the element $w\mu - \lambda - \rho_n$ is a weight of T.
It is convenient to extend the definition of m_μ^λ to an antisymmetric function on $P_{\mathfrak{k}}^r$. As we observed already an element $\mu \in P_{\mathfrak{k}}^r$ can be conjugated to a unique regular element in the corresponding positive chamber $\mathfrak{a}_c \subset \mathfrak{t}^*$ for compact roots, via an element $w \in \mathcal{W}_c$. Thus we define $m_\mu^\lambda = \epsilon(w) m_{w\mu}^\lambda$. Of course with this generalization the multiplicity of the K-type μ is $|m_\mu^\lambda|$ and we can complete our picture for $U(2,1)$, Fig.3, in the following way:

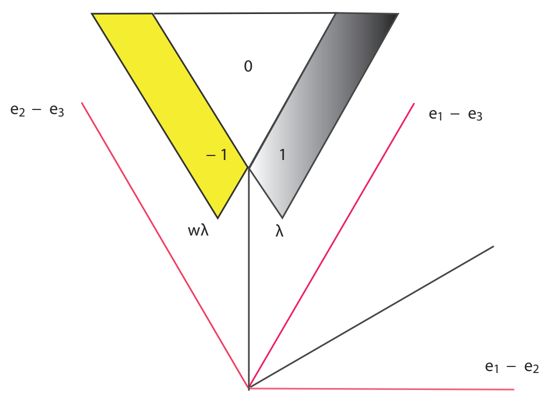

Figure 3: m_μ^λ as antisymmetric function on $U(2,1)$.

The representation τ_{lowest} with Harish-Chandra parameter $\mu_{lowest} = \lambda + \rho_n$ is the lowest K-type of the representation π^λ and occurs with multiplicity 1. It is, in general, difficult to compute m_μ^λ for general μ.

We will use Blattner's formula to compute m_μ^λ. Our algorithm is based on a general scheme for computing partition functions using multidimensional residues. Note that the presence of signs in Blattner formula doesn't even allow to say if a K -type appears without fully computing its multiplicity.

Recall that if $\mathcal{A}^+$ is a positive root system of a semi-simple Lie algebra, the formula for the partition function has been used to compute tensor product decomposition or weight multiplicities ([4],[6]).

2.3 Polynomial behavior of the Duistermaat-Heckman measure

Recall Paradan's results ([16]) on the behavior of the function m_μ^λ and its support. For this it is useful to first recall the semi-classical analog of Blattner formula.

Let us fix a positive system of roots Δ^+ and consider the corresponding (closed) positive chambers $\mathfrak{a}, \mathfrak{a}_c \subset \mathfrak{t}^*$ for Δ^+ and Δ_c^+. Our parameter λ varies in $\mathfrak{a}$ and it is non singular. In this subsection, the integrality condition $\lambda \in P_{\mathfrak{g}}^r$ is not required.

Let $O_\lambda \subset \mathfrak{g}^*$ be the coadjoint orbit of λ. It is a symplectic manifold, and thus is provided with a Liouville measure $d\beta_\lambda$. Let $p : O_\lambda \to \mathfrak{k}^*$ be the projection. This is a proper map. Each coadjoint K-orbit in $\mathfrak{k}^*$ intersect $\mathfrak{a}_c \subset \mathfrak{t}^* \subset \mathfrak{k}^*$. Thus the projection of O_λ on $\mathfrak{k}^*$ is entirely determined by its intersection with $\mathfrak{a}_c$. We recall that the set $p(O_\lambda) \cap \mathfrak{a}_c$ is a closed convex polyhedron.

Definition 10 *The Kirwan polyhedron* $\mathrm{Kirwan}(\lambda)$ *is the polyhedron* $p(O_\lambda) \cap \mathfrak{a}_c$.

As far as we know, there are no algorithm to determine the Kirwan polyhedron.

A weak result on the support of $\mathrm{Kirwan}(\lambda)$ is that $\mathrm{Kirwan}(\lambda)$ is contained in $\lambda + \mathrm{Cone}(\Delta_n^+)$ where $\mathrm{Cone}(\Delta_n^+)$ is the cone generated by positive non compact roots.

The push-forward of the measure $d\beta_\lambda$ along the projection $p : O_\lambda \to \mathfrak{k}^*$ gives us an invariant positive measure on $\mathfrak{k}^*$. By quotienting this measure by the signed Liouville measures $d\beta_{K\mu}$ of the coadjoint orbits $K\mu$ in $\mathfrak{k}^*$, we obtain a $\mathcal{W}_c$-anti-invariant measure dF^λ on $\mathfrak{t}^*$. More precisely, for ϕ a test function on $\mathfrak{k}^*$,

$$\int_{O_\lambda} \phi(p(f)) d\beta_\lambda(f) = \frac{1}{\#\mathcal{W}_c} \int_{\mathfrak{t}^*} dF^\lambda(\mu) \epsilon_\mu \left(\int_{K\mu} d\beta_{K\mu}(f) \phi(f) \right). \tag{2}$$

Here ϵ_μ is the locally constant function on $\mathfrak{t}^*$ anti-invariant by $\mathcal{W}_c$ and equal to 1 on the interior of the positive chamber $\mathfrak{a}_c$. We refer to dF^λ as the Duistermaat-Heckman measure.

If $\mathcal{A}^+ = [\alpha_1, \ldots, \alpha_N]$ is a sequence of elements in $\mathfrak{t}^*$ spanning a pointed cone, the multispline distribution $Y_{\mathcal{A}^+}$ is defined by the following formula. For ϕ a test function on $\mathfrak{t}^*$:

$$\langle Y_{\mathcal{A}^+}, \phi \rangle = \int_0^\infty \cdots \int_0^\infty \phi(\sum_{i=1}^N t_i \alpha_i) dt_1 \cdots dt_N. \tag{3}$$

Then, for $\lambda \in \mathfrak{a}$ and $\mu \in \mathfrak{t}^*$, we have the following result due to Duflo-Heckman-Vergne,[15]:

$$dF^\lambda(\mu) = \sum_{w \in \mathcal{W}_c} \epsilon(w) w(\delta_\lambda * Y_{\Delta_n^+}). \tag{4}$$

where Δ_n^+ is the system of positive noncompact roots defined by $\mathfrak{a}$.

To simplify our next statements, assume that G is semi-simple and has no compact factors. Then $\mathfrak{t}^*$ is generated by non compact roots. Recall that the spline function is given by a locally polynomial function on $\mathfrak{t}^*$ well defined outside a finite number of hyperplanes (see the description later). Choosing the Lebesgue measure dh associated to the root lattice, we may identify the measure $Y_{\Delta_n^+}$ to a function, denoted by Y_n^+. Similarly, we identify the measure dF^λ to a function F^λ on $\mathfrak{t}^*$. Then we have, almost everywhere, the semi-classical analogue of Blattner formula:

$$F^\lambda(\mu) = \sum_{w \in \mathcal{W}_c} \epsilon(w) Y_n^+(w\mu - \lambda)$$

$\lambda \in \mathfrak{a}$ and regular.

The (anti-invariant) function $F^\lambda(\mu)$ restricted to the positive compact chamber $\mathfrak{a}_c$ is a non negative measure with support the Kirwan polyhedron Kirwan(λ).

It follows from the study of spline functions that there exists a finite number of open polyhedral cones R^i in $\mathfrak{a} \times \mathfrak{a}_c$ (so that the union of the cones $\overline{R}^i$ cover $\mathfrak{a} \times \mathfrak{a}_c$) and polynomial functions p^i on $\mathfrak{a} \times \mathfrak{a}_c$ such that $F^\lambda(\mu)$ is given, for $\lambda \in \mathfrak{a}, \mu \in \mathfrak{a}_c$, $(\lambda, \mu) \in R^i$, by the polynomial $p^i(\lambda, \mu)$ on $R^i(\lambda) = \{\mu \in \mathfrak{a}_c,\ (\lambda, \mu) \in R^i\}$. In particular the Kirwan polyhedron Kirwan(λ) is the union of the regions $\overline{R^i(\lambda)}$ for which the polynomial p^i restricted to $\overline{R^i(\lambda)}$ is not equal to 0. In fact the functions $p^i(\lambda, \mu)$ are linear combinations of polynomial functions of $w\lambda - \mu$ where w are some elements of $\mathcal{W}_c$.

If R is an open cone in $\mathfrak{a} \times \mathfrak{a}_c$ such that $F^\lambda(\mu)$ is given by a polynomial formula $p^R(\lambda, \mu)$ when $(\lambda, \mu) \in R$, $\lambda \in \mathfrak{a}$, we say that R is a domain of polynomiality and that p^R is the local polynomial for F^λ on R.

Let us finally recall that the local polynomials p^R belong to some particular space of polynomials satisfying some system of partial differential equations. For α a non compact root, consider the derivative ∂_α. We say that an hyperplane $H \in \mathfrak{t}^*$ is admissible for Δ_n if H is spanned by a subset of $\dim \mathfrak{t} - 1$ non compact roots, that is roots in Δ_n. We denote by $\mathcal{H}_n$ the set of admissible hyperplanes for Δ_n.

Definition 11 *A polynomial p on $\mathfrak{t}^*$ is in the Dahmen-Micchelli space $D(\Delta_n^+)$ if p satisfies the system of equations:*

$$\Big(\prod_{\alpha \in \Delta_n^+ \setminus Q} \partial_\alpha\Big) p = 0$$

for any $Q \in \mathcal{H}_n$.

Remark that the space $D(\Delta_n^+)$ depends only of Δ_n and not of a choice of Δ_n^+. Then, the following result follows from Dahmen-Micchelli theory of the splines.

Proposition 12 *For any domain of polynomiality R, the polynomial $\mu \to p^R(\lambda, \mu)$ belongs to the space $D(\Delta_n^+)$.*

2.4 Quasi-polynomiality results

Let us come back to the discrete setting. Let us fix as before a positive system of roots Δ^+ and consider the corresponding chambers $\mathfrak{a}, \mathfrak{a}_c \subset \mathfrak{t}^*$ for Δ^+ and Δ_c^+. Fix $\lambda \in P_{\mathfrak{g}}^r \cap \mathfrak{a}$ and $\mu \in P_{\mathfrak{k}}^r \cap \mathfrak{a}_c$. We can then define m_μ^λ, the multiplicity of τ_μ in the discrete series π^λ.

By definition, a quasipolynomial function on a lattice L is a function on L which coincides with a polynomial on each coset of some sublattice L' of finite index in L. The subsets $P_{\mathfrak{g}}, P_{\mathfrak{k}}$ are shifted lattices and we may say that a function k on $P_{\mathfrak{k}}^r$ is quasi polynomial on $P_{\mathfrak{g}} \times P_{\mathfrak{k}}$ if the shifted function $k(\lambda - \rho, \mu - \rho_c)$ is quasipolynomial on the lattice $P \times P$.

Theorem 13 *Let R be a domain of polynomiality in $\mathfrak{a} \times \mathfrak{a}_c$ for the Duistermaat-Heckman measure. Then there exists a quasi polynomial function P^R on $P_{\mathfrak{g}} \times P_{\mathfrak{k}}$ such that $m_\mu^\lambda = P^R(\lambda, \mu)$ for any $(\lambda, \mu) \in \overline{R} \cap (P_{\mathfrak{g}}^r \times P_{\mathfrak{k}}^r)$, $\lambda \in \mathfrak{a}, \mu \in \mathfrak{a}_c$.*

(In fact the functions P^R are linear combinations of quasi polynomial functions of $w\lambda - \mu$ where w are some elements of $\mathcal{W}_c$.)

The K-types occurring with non zero multiplicity in π^λ are such that μ is in the interior of the Kirwan polyhedron Kirwan(λ). In particular the lowest K-type μ_{lowest} is in the interior of Kirwan(λ). In particular all the K-types occurring with non zero multiplicity in π^λ are such that μ is in the interior of the cone $\lambda + \mathrm{Cone}(\Delta_n^+)$. We believe they are contained in the cone $\mu_{lowest} + \mathrm{Cone}(\Delta_n^+)$, but we do not know if this assertion is true or not (by Vogan's theorem, they are contained in $\mu_{lowest} + \mathrm{Cone}(\Delta^+)$) .

If $v \in \mathfrak{t}^*$, we say that v is an **asymptotic direction**, if the line $\mu_{\text{lowest}} + tv$ is contained in Kirwan(λ) for all $t \geq 0$. The set of asymptotic directions form a cone, which determines the wave-front set of $\pi^\lambda|_K$.

For the holomorphic discrete series, the descrition of the cone of asymptotic diections is known. In fact if the lowest K-type of π^λ is a one dimensional representation of K, the exact support of the function m_μ^λ has been determined by Schmid.

We will explain in Section 3.7 how to compute regions of polynomiality R and the quasi-polynomial P^R.

The quasi polynomials P^R are in some particular space of quasi polynomials satisfying some system of partial difference equations. For α a non compact root, consider the difference operator ∇_α acting on $\mathbb{Z}$ valued functions on $P_{\mathfrak{k}}$ by

$$(\nabla_\alpha k)(\mu) = k(\mu) - k(\mu - \alpha).$$

Definition 14 *A quasi polynomial L on $P_{\mathfrak{k}}$ is in the Dahmen-Micchelli space $DM(\Delta_n^+)$ if p satisfies the system of equations:*

$$\Big(\prod_{\alpha \in \Delta_n^+ \setminus Q} \nabla_\alpha\Big) L = 0$$

for any $Q \in \mathcal{H}_n$.

Then, the following result follows from Dahmen-Micchelli theory of partition functions.

Proposition 15 *The quasi polynomial $\mu \to P^R(\lambda, \mu)$ belongs to the space $DM(\Delta_n^+)$.*

2.5 Aim of the algorithm: what can we do?

Our algorithm addresses the following questions for $U(p, q)$. All of these questions will be analyzed in more details in Sec.3.7.

2.5.1 Numeric

We enter as input two parameters $\lambda, \mu \in P_{\mathfrak{g}}^r \times P_{\mathfrak{k}}^r$. The output is the integer m_μ^λ, see Sec.3.7.1.

2.5.2 Regions of polynomiality

The input is two parameters $\lambda_0, \mu_0 \in P_{\mathfrak{g}}^r \times P_{\mathfrak{k}}^r$. Let $\mathfrak{a}, \mathfrak{a}_c$ be the chambers determined by λ_0 and μ_0. We also give two symbolic parameters λ, μ.

Then the output is a closed cone $R(\lambda_0, \mu_0) \subset \mathfrak{a} \oplus \mathfrak{a}_c$ described by linear inequations in λ, μ, containing (λ_0, μ_0) and a quasi-polynomial P in (λ, μ) such that $m_\mu^\lambda = P(\lambda, \mu)$ for any $(\lambda, \mu) \in R(\lambda_0, \mu_0) \cap (P_{\mathfrak{g}}^r \times P_{\mathfrak{k}}^r)$.

We worked out part of this program for $U(p, q)$, but it is still not fully implemented.

In particular, for the moment, we are not able to produce a cover of $\mathfrak{a} \times \mathfrak{a}_c$ by such regions. The number of regions needed grows very fast with the rank. Furthermore, we are not able to decide when we have finished to cover $\mathfrak{a} \times \mathfrak{a}_c$.

2.5.3 Asymptotic directions

We implemented (for $U(p,q)$) a simpler question which gives a test for asymptotic directions.

Let's consider as input parameters λ_0 in $P^r_{\mathfrak{g}}$ and a weight $\vec{v} \in \mathfrak{a}_c$. Let μ_0 be the lowest K-type of π^{λ_0}. The line $t \mapsto (\lambda_0, \mu_0 + t\vec{v})$ cross domains of polynomiality R^i at a certain finite number of points $0 \leq t_1 < t_2 < \cdots < t_s$. Let us define $t_0 = 0, t_{s+1} = \infty$. Then we study the function $P(t) = m^{\lambda_0}_{\mu_0 + tv}$, $t \in \mathbb{N}, t \geq 0$.

We can find polynomials $q_{[t_i,t_{i+1}]}$ of degree bounded by $pq - (p+q+1)$ such that $P(t) = q_{[t_i,t_{i+1}]}(t)$ when $t_i \leq t \leq t_{i+1}$ for $i = 0, \ldots, s$ and $t \in \mathbb{N}$. In particular, the direction $\vec{v}$ belongs to the asymptotic cone of the Kirwan polyhedron, if and only if our last polynomial $q_{[t_s,\infty]}$ is non zero, see Sec.3.7.3.

As we discussed in the first part, if the intervals are two small, these polynomials are not uniquely determined. However the last interval is infinite, and the last polynomial is well determined. If this last data is non zero, then $\vec{v}$ is in the wave front set of π^{λ}. The reciproc is not entirely clear. Indeed for a direction to be in the wave front set, it is sufficient to be approached in the projective space by lines $\mathbb{R}^+\mu_n$ with $\mu_n \in P^r_{\mathfrak{k}} \cap \mathfrak{a}_c$ such that the multiplicity $m^{\lambda}_{\mu_n}$ is non zero, and the sequence μ_n is going to the infinity in $\mathfrak{a}_c$. Thus we do not know if a rational line $\mu_0 + t\vec{v}$ contained in the Kirwan polytope could totally avoid the support of the function m^{λ}_{μ}. We do not think this is possible.

3 Partition functions: the general scheme.

3.1 Definitions

Let U be a r-dimensional real vector space and V be its dual vector space. We fix the choice of a Lebesgue measure dh on V. Consider a list $\mathcal{A}^+$ of non-zero generators for V given by

$$\mathcal{A}^+ = [\alpha_1, \alpha_2, \ldots, \alpha_N].$$

We recall several results concerning partition functions that appear in [3] in the general context. However, let us describe right away the system of vectors $\Delta^+(A,B) \subset A_r$ that will appear in our programs and that describe parabolic subsystem of A_r, (as we said the same method could be applied to other parabolic root systems).

Example 16 • *Let E be an r-dimensional vector space with basis e_i ($i = 1, \ldots, r+1$). Consider the sequence*

$$A_r^+ = [e_i - e_j \,|\, 1 \leq i < j \leq r+1].$$

This is a system of positive roots of type A_r. We let V to be the vector space

$$V = \left\{ h = \sum_{i=1}^{r+1} h_i e_i \in E \,\Big|\, \sum_{i=1}^{r+1} h_i = 0 \right\}.$$

Let $V_{\mathbb{Z}}$ be the lattice spanned by A_r^+. We may identify V with $\mathbb{R}^r$ by $h \mapsto [h_1, h_2, \dots, h_r]$. In this identification the lattice $V_{\mathbb{Z}}$ is identified with $\mathbb{Z}^r$.

- *Let A, B be two complementary subsets of $[1, 2, 3, \dots, p+q]$ with $|A| = p$ and $|B| = q$. Let $r = p+q-1$. We define $\Delta^+(A, B)$ as the sublist of A_r^+ defined by*

  $$\Delta^+(A, B) = [e_i - e_j \,|\, 1 \leq i < j \leq r+1], \textit{with } \ i \in A, j \in B \textit{ or } i \in B, j \in A.$$

  These systems are the system of positive roots for the maximal parabolics of $\mathfrak{gl}(r+1)$, for different choices of orders.

Let us go back to the general scheme.
For any subset S of V, we denote by $\mathcal{C}(S)$ the convex cone generated by non-negative linear combinations of elements of S. We assume that the convex cone $\mathcal{C}(\mathcal{A}^+)$ is acute in V with non-empty interior.
If S is a subset of V, we denote by $< S >$ the vector space spanned by S.

Definition 17 *A hyperplane H in V is $\mathcal{A}^+$-*admissible *if it is spanned by a set of vectors of $\mathcal{A}^+$.*

When $\mathcal{A}^+ = \Delta_n^+(\lambda)$ or $\Delta^+(A, B)$ then an $\mathcal{A}^+$-admissible hyperplane will be also called a noncompact wall.

Chambers
Let $\mathcal{V}_{sing}(\mathcal{A}^+)$ be the union of the boundaries of the cones $\mathcal{C}(S)$, where S ranges over all the subsets of $\mathcal{A}^+$. The complement of $\mathcal{V}_{sing}(\mathcal{A}^+)$ in V is by definition the open set $\mathcal{C}_{reg}(\mathcal{A}^+)$ of *regular* elements. A connected component $\mathfrak{c}$ of $\mathcal{C}_{reg}(\mathcal{A}^+)$ is called a *chamber* of $\mathcal{C}(\mathcal{A}^+)$. Remark that, in our definition, the complement of the cone $\mathcal{C}(\mathcal{A}^+)$ in V is a chamber that we call the *exterior* chamber. The chambers contained in $\mathcal{C}(\mathcal{A}^+)$, that we will call *interior* chambers, are open convex cones. Sometimes chambers are called cells or big cells by other authors.
The faces of the interior chambers span admissible hyperplanes.
The following pictures illustrate the situation for the (interior) chambers in the case of A_3^+, Fig.4, and the (interior) chambers for various subsystems of A_3^+, of type $\Delta^+(A, B)$ relative to $U(2, 2)$, Fig.5.

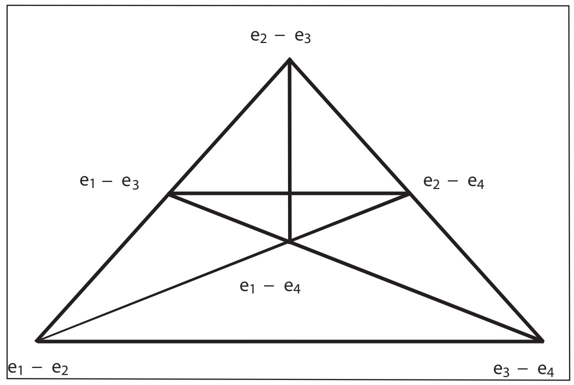

Figure 4: The 7 chambers for A_3^+

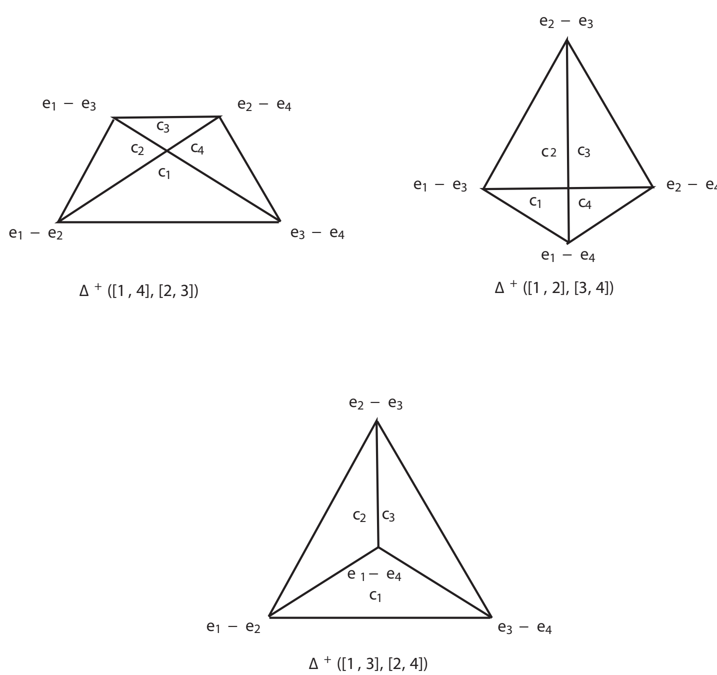

Figure 5: Parabolic subsystems of A_3^+

Polytopes

We consider the space $\mathbb{R}^N$ with its standard basis ω_i and Lebesgue measure dx. If $x = \sum_{i=1}^N x_i\omega_i \in \mathbb{R}^N$, we simply write $x = (x_1, \ldots, x_N)$. Consider the surjective map $A : \mathbb{R}^N \to V$ defined by $A(\omega_i) = \alpha_i$.

If $h \in V$, we define the convex polytope $\Pi_{\mathcal{A}^+}(h)$ consisting of all non-negative solutions of the system of r linear equations $\sum_{i=1}^N x_i\alpha_i = h$ that is

$$\Pi_{\mathcal{A}^+}(h) = \left\{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \,|\, Ax = h,\ x_i \geq 0\right\}.$$

We call $\Pi_{\mathcal{A}^+}(h)$ a partition polytope (associated to $\mathcal{A}^+$ and h).

We identify the spline distribution $Y_{\mathcal{A}^+}$ (by Formula 3) to a function still denoted by $Y_{\mathcal{A}^+}$ using dh.

Recall the following theorem, which follows right away from Fubini theorem, ([10], [11].)

Theorem 18 *The value of the spline function $Y_{\mathcal{A}^+}$ at h is the volume of the partition polytope $\Pi_{\mathcal{A}^+}(h)$ for the quotient measure dx/dh.*

The spline function $Y_{\mathcal{A}^+}$ is given by a polynomial formula on each interior chamber. It is identically equal to 0 on the exterior chamber.

Partition functions

Let $V_{\mathbb{Z}}$ be a lattice in V and suppose now that the elements α_i of our sequence $\mathcal{A}^+$ belong to the lattice $V_{\mathbb{Z}}$. If $h \in V_{\mathbb{Z}}$ we define $N_{\mathcal{A}^+}(h) = |\Pi_{\mathcal{A}^+}(h) \cap \mathbb{Z}^N|$, the number of integral points in the partition polytope $\Pi_{\mathcal{A}^+}(h)$.

Thus $N_{\mathcal{A}^+}(h)$ is the number of solutions $(x_1, x_2, \ldots, x_N)$, in non-negative integers x_j, of the equation $\sum_{j=1}^N x_j\alpha_j = h$.

The function $h \mapsto N_{\mathcal{A}^+}(h)$ is called the partition function of $\mathcal{A}^+$. We refer to it as **Kostant partition function**.

We will see after stating Theorem 24 that $h \mapsto N_{\mathcal{A}^+}(h)$ is quasipolynomial on each chamber.

Let us recall briefly the theory that allows to compute Kostant partition functions.

Jeffrey-Kirwan residue

Let ν be a subset of $\{1, 2, \ldots, N\}$. We will say that ν is *generating* (respectively *basic*) if the set $\{\alpha_i \,|\, i \in \nu\}$ generates (respectively is a basis of) the vector space V. We write $\text{Bases}(\mathcal{A}^+)$ for the set of basic subsets.

Let $\mathcal{R}_{\mathcal{A}^+}$ be the ring of rational functions on U, the dual vector space to V, with poles on hyperplanes determined by kernel of elements $\alpha \in \mathcal{A}^+$.

$\mathcal{R}_{\mathcal{A}^+}$ is $\mathbb{Z}$-graded by degree. Every function in $\mathcal{R}_{\mathcal{A}^+}$ of degree $-r$ decomposes (see [5]) as the sum of basic fractions f_σ, $f_\sigma = \frac{1}{\prod_{i\in\sigma}\alpha_i}$, $\sigma \in \text{Bases}(\mathcal{A}^+)$ and degenerate fractions; here degenerate fractions are those for which the linear forms in the denominator do not span V.

Now having fixed a chamber $\mathfrak{c}$, we define a functional $\text{JK}_\mathfrak{c}(f_\sigma)$ on $\mathcal{R}_{\mathcal{A}^+}$ called the Jeffrey-Kirwan residue (or JK *residue*) as follows:

(5) $$\text{JK}_\mathfrak{c}(f_\sigma) = \begin{cases} \text{vol}(\sigma)^{-1}, & \text{if } \mathfrak{c} \subset \mathcal{C}(\sigma), \\ 0, & \text{if } \mathfrak{c} \cap \mathcal{C}(\sigma) = \emptyset \end{cases}$$

where $\sigma \in \text{Bases}(\mathcal{A}^+)$ and $\text{vol}(\sigma)$ is the volume of the parallelotope $\sum_{i=1}^r [0,1]\alpha_i$ computed for the measure dh.

There exists a linear form $\text{JK}_\mathfrak{c}$, that we call the Jeffrey-Kirwan residue, on $\mathcal{R}_{\mathcal{A}^+}$ such that $\text{JK}_\mathfrak{c}$ takes the above values on the elements f_σ, and is equal to 0 on a degenerate fraction or on a rational function of pure degree different from $-r$.

If $\mathfrak{c}$ is the exterior chamber, then clearly $\text{JK}_\mathfrak{c}$ is equal to 0, as $\mathfrak{c}$ is not contained in $\mathcal{C}(\mathcal{A}^+)$.

We may go further and extend the definition of the Jeffrey-Kirwan residue to the space $\widehat{\mathcal{R}}_\mathcal{A}$ which is the space consisting of functions P/Q where Q is a product of powers of the linear forms α_i and $P = \sum_{k=0}^\infty P_k$ is a formal power series. Then we just define, if Q is of degree q,

$$\text{JK}_\mathfrak{c}(P/Q) = \text{JK}_\mathfrak{c}(P_{q-r}/Q)$$

as the JK residue of the component of degree $-r$ of P/Q.

3.2 Spline functions and Kostant partition function

Let us recall the formulae for the spline function $Y_{\mathcal{A}^+}$ and for $N_{\mathcal{A}^+}(h)$.

Definition 19 *Let $\mathfrak{c}$ be an chamber contained in the cone $\mathcal{C}(\mathcal{A}^+)$. Define the function $\mathbf{Y}^\mathfrak{c}$ on V by*

$$\mathbf{Y}^\mathfrak{c}_{\mathcal{A}^+}(h) = \text{JK}_\mathfrak{c}\left(\frac{e^h}{\prod_{i=1}^N \alpha_i}\right).$$

More explicitly, as $\text{JK}_\mathfrak{c}$ vanishes outside the degree $-r$, we have

$$\mathbf{Y}^\mathfrak{c}_{\mathcal{A}^+}(h) = \frac{1}{(N-r)!}\text{JK}_\mathfrak{c}\left(\frac{h^{N-r}}{\prod_{i=1}^N \alpha_i}\right).$$

We thus see $\mathbf{Y}^\mathfrak{c}_{\mathcal{A}^+}(h)$ is an homogeneous polynomial on V.

The proof of the following theorem is immediate ([10]).

Theorem 20 *Let $Y_{\mathcal{A}^+}(h)$ be the multispline function associated to $\mathcal{A}^+$. Let $\mathfrak{c}$ be a chamber contained in the cone $\mathcal{C}(\mathcal{A}^+)$.*

We have for $h \in \mathfrak{c}$:

$$Y_{\mathcal{A}^+}(h) = \mathbf{Y}^{\mathfrak{c}}_{\mathcal{A}^+}(h).$$

Remark 21 *According to Theorem 18, this theorem gives the formula for the volume $V_{\mathcal{A}^+}(h)$ of the partition polytope $\Pi_{\mathcal{A}^+}(h)$.*

Let us now give the residue formula for the number of integral points $N_{\mathcal{A}^+}$ of the partition polytope $\Pi_{\mathcal{A}^+}(h)$.

Consider the torus $T = U/U_{\mathbb{Z}}$ where U is the dual vector space to V and $U_{\mathbb{Z}} \subset U$ is the dual lattice to $V_{\mathbb{Z}}$. If $G \in U$, we denote by g its image in T.

For $\sigma \in \text{Bases}(\mathcal{A}^+)$ we denote by $T(\sigma)$ the subset of T defined by

$$T(\sigma) = \left\{ g \in T \,\middle|\, e^{\langle \alpha, 2\pi\sqrt{-1}G\rangle} = 1 \text{ for all } \alpha \in \sigma, \quad G \text{ a representative of } g \in U/U_{\mathbb{Z}} \right\}.$$

The set $T(\sigma)$ is a finite subset of T.

For $G \in U$ and $h \in V$, consider the **Kostant function** $K(G,h)$ on U defined by

$$K(G,h)(u) = \frac{e^{\langle h, 2\pi\sqrt{-1}G + u\rangle}}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, 2\pi\sqrt{-1}G + u\rangle})}. \tag{6}$$

Remark 22 *If $h \in V_{\mathbb{Z}}$, the function $K(G,h)$ depends only of the class g of G in $U/U_{\mathbb{Z}}$.*

The function $K(G,h)(u)$ is an element of $\widehat{\mathcal{R}}_{\mathcal{A}}$.
Indeed if we write $I(g) = \left\{ i \,\middle|\, 1 \le i \le N, e^{-\langle \alpha_i, 2\pi\sqrt{-1}G\rangle} = 1 \right\}$, then

$$K(G,h)(u) = e^{\langle h, 2\pi\sqrt{-1}G\rangle} \frac{e^{\langle h,u\rangle}\psi^g(u)}{\prod_{i \in I(g)} \langle \alpha_i, u\rangle} \tag{7}$$

where $\psi^g(u)$ is the holomorphic function of u (in a neighborhood of zero) defined by

$$\psi^g(u) = \prod_{i \in I(g)} \frac{\langle \alpha_i, u\rangle}{(1 - e^{-\langle \alpha_i, u\rangle})} \times \prod_{i \notin I(g)} \frac{1}{(1 - e^{-\langle \alpha_i, 2\pi\sqrt{-1}G + u\rangle})}.$$

By taking the Taylor series of $e^{\langle h,u\rangle}\psi^g(u)$ at $u = 0$, we see that the function $u \to K(G,h)(u)$ on U defines an element of $\widehat{\mathcal{R}}_{\mathcal{A}}$. If $\mathfrak{c}$ is a chamber of $\mathcal{C}(\mathcal{A}^+)$, the Jeffrey-Kirwan residue $\text{JK}_{\mathfrak{c}}(K(g,h))$ is thus well defined.

Definition 23 *Let $\mathfrak{c}$ be a chamber. Let F be a finite subset of U. We define the function $\mathbf{N}_{\mathcal{A}^+}^{\mathfrak{c},F}$ on V by*

$$\mathbf{N}_{\mathcal{A}^+}^{\mathfrak{c},F}(h) = \mathrm{vol}(V/V_{\mathbb{Z}}, dh) \sum_{G \in F} \mathrm{JK}_{\mathfrak{c}}(K(G,h))$$

where $\mathrm{vol}(V/V_{\mathbb{Z}}, dh)$ is the volume of the fundamental domain of $V_{\mathbb{Z}}$ for dh.

Finally introduce the zonotope $Z(\mathcal{A}^+)$ to be the convex polyhedra defined by

$$Z(\mathcal{A}^+) := \{\sum_{i=1}^{N} t_i \alpha_i ; 0 \leq t_i \leq 1\}.$$

When $\mathcal{A}^+$ is fixed, we just write $Z = Z(\mathcal{A}^+)$, and if C is a set, we denote by $C - Z$ the set of elements $\{\xi - z\}$ where $\xi \in C$ and $z \in Z$.

The following theorem is due to Szenes-Vergne [9]. It generalizes [7], [13] and [5].

Theorem 24 *Let $\mathfrak{c}$ be a chamber. Let F be a finite subset of U. Assume that for any $\sigma \in \mathrm{Bases}(\mathcal{A}^+)$ such that $\mathfrak{c} \subset \mathcal{C}(\sigma)$, we have $T(\sigma) \subset F/U_{\mathbb{Z}}$.*

Then for $h \in V_{\mathbb{Z}} \cap (\mathfrak{c} - Z)$, we have

$$N_{\mathcal{A}^+}(h) = \mathbf{N}_{\mathcal{A}^+}^{\mathfrak{c},F}(h).$$

We choose for any chamber $\mathfrak{c}$ such a finite set F such that all elements $g \in F/U_{\mathbb{Z}}$ have finite order and such that F satisfies the condition:

(**C**) for any $\sigma \in \mathrm{Bases}(\mathcal{A}^+)$ such that $\mathfrak{c} \subset \mathcal{C}(\sigma)$, we have $T(\sigma) \subset F/U_{\mathbb{Z}}$.

It is possible to achieve this, for example choosing a set F of representatives of $\frac{1}{p}U_{\mathbb{Z}}$ modulo $U_{\mathbb{Z}}$, where p is that $pU_{\mathbb{Z}}$ is contained in $\sum_{i \in \sigma} \mathbb{Z}\alpha_i$ for any basis σ.

We now simply denote $\mathbf{N}^{\mathfrak{c},F}$ by $\mathbf{N}^{\mathfrak{c}}$, leaving implicit the choice of the finite set F.

Remark 25 • *Observe that $\mathbf{N}_{\mathcal{A}^+}^{\mathfrak{c}}(h)$ does not depend on the measure dh, as it should be.*

• *If $\mathfrak{c}$ is the exterior chamber, then $\mathbf{N}_{\mathcal{A}^+}^{\mathfrak{c}}(h) = 0$. In our algorithm, we are not knowing in advance if the point h belongs to the cone $\mathcal{C}(\mathcal{A}^+)$ or not, so that this remark is not as stupid as it looks.*

• *Observe also that if $\mathfrak{c}$ is an interior chamber, then $\mathfrak{c} - Z$ contains the closure $\overline{\mathfrak{c}}$ of $\mathfrak{c}$, while if $\mathfrak{c}$ is the exterior chamber $\mathfrak{c} - Z = \mathfrak{c}$. For an interior chamber, usually the set $\mathfrak{c} - Z$ intersected with the lattice $V_{\mathbb{Z}}$ is strictly larger than $\overline{\mathfrak{c}}$ intersected with $V_{\mathbb{Z}}$. This fact will be important for computing shifted partition functions, as we will explain later.*

Let us explain the behavior of the partition function $N_{\mathcal{A}^+}$ on the domain $\mathfrak{c} - Z$. We first explain the case of an unimodular system.

Definition 26 *The system $\mathcal{A}^+$ is unimodular if each $\sigma \in \mathrm{Bases}(\mathcal{A}^+)$ is a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}$.*

Example 27 *It is easy to see that A_r^+ is unimodular, so is any subsystem.*

Thus if $\mathcal{A}^+$ is unimodular, the set $F = \{0\}$ satisfies the condition (**C**) and we choose this set F.

Proposition 28 *If $\mathcal{A}^+$ is unimodular, the function $\mathbf{N}^{\mathfrak{c}}_{\mathcal{A}^+}(h)$ is a polynomial function on V.*

Proof. We have just to consider $K(G, h) = K(0, h)$ and we can write

$$K(0,h)(u) = \frac{e^{\langle h,u\rangle}}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, u\rangle})} = \frac{e^{\langle h,u\rangle}}{\prod_{i=1}^N \langle \alpha_i, u\rangle} \times \frac{\prod_{i=1}^N \langle \alpha_i, u\rangle}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, u\rangle})}$$

where $\frac{\prod_{i=1}^N \langle \alpha_i, u\rangle}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, u\rangle})} = \sum_{k=0}^{+\infty} \psi_k(u)$ is a holomorphic function of u in a neighborhood of 0 with $\psi_0(u) = 1$.

It follows that $\mathbf{N}^{\mathfrak{c}}_{\mathcal{A}^+}(h)$ is given by the following polynomial function of h

$$\begin{aligned} \mathbf{N}^{\mathfrak{c}}_{\mathcal{A}^+}(h) &= \mathrm{vol}\,(V/V_{\mathbb{Z}}, dh)\, \mathrm{JK}_{\mathfrak{c}} \left(\frac{e^{\langle h,u\rangle}}{\prod_{i=1}^N \langle \alpha_i, u\rangle} \times \sum_{k=0}^{+\infty} \psi_k(u) \right) \\ &= \mathrm{vol}\,(V/V_{\mathbb{Z}}, dh) \sum_{k=0}^{N-r} \frac{1}{(N-r-k)!}\, \mathrm{JK}_{\mathfrak{c}} \left(\frac{\langle h, u\rangle^{N-r-k} \psi_k(u)}{\prod_{i=1}^N \langle \alpha_i, u\rangle} \right). \end{aligned} \tag{8}$$

Note that the function $\mathbf{N}^{\mathfrak{c}}_{\mathcal{A}^+}$ is a polynomial function of degree $N - r$ whose homogeneous component of degree $N - r$ is the function $\mathbf{Y}^{\mathfrak{c}}_{\mathcal{A}^+}(h)$, that is the volume of the polytope.

Let us now consider the general case where F is no longer reduced to $\{0\}$. For example for parabolic root systems of B_r, C_r, D_r, the set F satisfying the condition (**C**) cannot longer be taken as equal to $\{0\}$.

We recall that an exponential polynomial function is a linear combination of exponential functions multiplied by polynomials.

Proposition 29 *The function $\mathbf{N}^{\mathfrak{c}}_{\mathcal{A}^+}(h)$ is an exponential polynomial function on V and the restriction of $\mathbf{N}^{\mathfrak{c}}_{\mathcal{A}^+}(h)$ to $V_{\mathbb{Z}}$ is a quasipolynomial function on $V_{\mathbb{Z}}$.*

Proof. Let us denote by $\psi^g(u) = \sum_{k=0}^{+\infty} \psi_k^g(u)$ the series development of the holomorphic function ψ^g appearing in formula (7). Then we see that $\mathrm{JK}_{\mathfrak{c}}(K(G,h))$ equals

$$
\begin{aligned}
(9) \qquad & \left(e^{\langle h, 2\pi\sqrt{-1}G\rangle} JK_{\mathfrak{c}} \frac{e^{\langle h,u\rangle}}{\prod_{i\in I(g)} \langle \alpha_i, u\rangle} \psi^g(u) \right) \\
= \quad & e^{\langle h, 2\pi\sqrt{-1}G\rangle} \sum_{k=0}^{|I(g)|-r} \frac{1}{(|I(g)|-r-k)!} \mathrm{JK}_{\mathfrak{c}} \left(\frac{\langle h,u\rangle^{|I(g)|-r-k}}{\prod_{i\in I(g)} \langle \alpha_i, u\rangle} \psi_k^g(u) \right).
\end{aligned}
$$

The function

$$
h \mapsto \mathrm{JK}_{\mathfrak{c}} \left(\frac{\langle h,u\rangle^{|I(g)|-r-k}}{\prod_{i\in I(g)} \langle \alpha_i, u\rangle} \psi_k^g(u) \right)
$$

is a polynomial function of h of degree $|I(g)|-r-k$. Thus we see that $\mathrm{JK}_{\mathfrak{c}}(K(G,h))$ is the product of the exponential function $e^{\langle h, 2\pi\sqrt{-1}G\rangle}$ by a polynomial function of h.

Furthermore, if g is of order p and h varies in $V_{\mathbb{Z}}$, the function $h \mapsto e^{\langle h, 2\pi\sqrt{-1}G\rangle}$ is constant on each coset $h + pV_{\mathbb{Z}}$ of the lattice $pV_{\mathbb{Z}}$.

Return to the computation of the partition function $N_{\mathcal{A}^+}(h)$. Thus we see that when h varies in $(\mathfrak{c} - Z) \cap V_{\mathbb{Z}}$, we have that $N_{\mathcal{A}^+}(h)$ coincide with the quasi polynomial function $\mathbf{N}^{\mathfrak{c}}_{\mathcal{A}^+}(h)$ above. Note that its highest degree component is polynomial and is again the function $\mathbf{Y}^{\mathfrak{c}}_{\mathcal{A}^+}(h)$, the volume of the polytope $\Pi_{\mathcal{A}^+}(h)$.

The quasipolynomial nature of the integral-point counting functions $N_{\mathcal{A}}^+$ stems precisely from the root of unity in formula (9).

Furthermore for parabolic root systems of type B, C, and D, these roots of unity are of order 2, as in the following example. Thus we summarize the properties of our partition functions in the following remark:

Remark 30

- *$\mathcal{A}_r^+$ is unimodular, that is we can choose $F = 0$ in Theorem 24, and thus the partition function N_Φ for any subset Φ of $\mathcal{A}_r^+$ coincide with a polynomial function on each domain $\mathfrak{c} - Z(\Phi)$.*
- *The integral-point counting functions N_Φ for any subsystem of B_r, C_r, D_r coincide with quasipolynomials with period 2 on each domain $\mathfrak{c} - Z(\Phi)$.*

We now compute the number of integral points in two different situations: a non unimodular case and a unimodular one. We treat the non unimodular case first.

Example 31 *Here V is a vector space with real coordinates and basis e_1, e_2 and $U = V^*$ has dual basis e^1, e^2. We write $v = \sum_{i=1}^2 v_i e_i \in V$ and $u = \sum_{i=1}^2 h_i e^i \in U$*

for elements in V *and* U *respectively. Let us compute the number of integral points for the positive non compact root system occuring for the holomorphic discrete series of* $SO(5,\mathbb{C})$ *: that is we fix* $\Delta^+ := \{e_1, e_2, e_1+e_2, e_1-e_2\}$ *and* $\mathcal{A}^+ = \Delta_n^+ := \{e_1, e_1+e_2, e_1-e_2\}$*. We also write a vector* $h = h_1e_1 + h_2e_2$ *in the cone* $\mathcal{C}(\mathcal{A}^+)$ *as* (h_1, h_2)*. Of course, the calculation can be done by hand, but we illustrate the method in this very simple example.*

Observe that the root lattice is $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ *and* $\mathrm{vol}\,(V/V_{\mathbb{Z}}, dh) = 1$ *for the measure* $dh = dh_1dh_2$*.*

There are two chambers, namely $\mathfrak{c}_1 = \mathcal{C}(\{e_1+e_2, e_1\})$ *and* $\mathfrak{c}_2 = \mathcal{C}(\{e_1, e_1-e_2\})$*.*

Now let us compute the Jeffrey-Kirwan residues on the chambers.

We have for example:

$$\begin{array}{rclrcl} \mathrm{JK}_{\mathfrak{c}_1}\left(\frac{1}{u_1(u_1+u_2)}\right) &=& 1, & \mathrm{JK}_{\mathfrak{c}_2}\left(\frac{1}{u_1(u_1+u_2)}\right) &=& 0,\\ \mathrm{JK}_{\mathfrak{c}_1}\left(\frac{1}{(u_1+u_2)(u_1-u_2)}\right) &=& \frac{1}{2}, & \mathrm{JK}_{\mathfrak{c}_2}\left(\frac{1}{(u_1+u_2)(u_1-u_2)}\right) &=& \frac{1}{2},\\ \mathrm{JK}_{\mathfrak{c}_1}\left(\frac{1}{u_1(u_1-u_2)}\right) &=& 0, & \mathrm{JK}_{\mathfrak{c}_2}\left(\frac{1}{u_1(u_1-u_2)}\right) &=& 1 \end{array}$$

For the number of integral points, we first note that $F = \{(0,0), (1/2, 1/2)\}$*. Consequently* $N_{\Delta_n^+}(h)$ *is equal to the Jeffrey-Kirwan residue of* $f_1 = K((1,1), h)$ *plus* $f_2 = K((1/2, 1/2), h)$*. We rewrite the series* f_j *(*$j = 1,\ 2$*) as* $f_j = f_j' \times e^{u_1h_1+u_2h_2}/u_1(u_1+u_2)(u_1-u_2)$ *where*

$$\begin{array}{rcl} f_1' &=& \frac{u_1}{1-e^{-u_1}} \times \frac{u_1+u_2}{1-e^{-(u_1+u_2)}} \times \frac{u_1-u_2}{1-e^{-(u_1-u_2)}},\\ f_2' &=& \frac{u_1}{1+e^{-u_1}} \frac{u_1+u_2}{1-e^{-(u_1+u_2)}} \times \frac{u_1-u_2}{1-e^{-(u_1-u_2)}} \times (-1)^{h_1+h_2}. \end{array}$$

Using the series expansions $\frac{x}{1-e^{-x}} = 1+\frac{1}{2}x+\frac{1}{12}x^2+O(x^3)$ *and* $\frac{x}{1+e^{-x}} = \frac{1}{2}x+O(x^2)$*, we obtain that the number of integral points is the JK residue of*

$$\begin{array}{l} \frac{h_1+\frac{1}{2}}{(u_1-u_2)(u_1+u_2)} + \frac{\frac{1}{2}}{u_1(u_1+u_2)} + \frac{\frac{1}{2}}{u_1(u_1-u_2)} + \frac{h_2u_2}{u_1(u_1-u_2)(u_1+u_2)} + \frac{\frac{1}{2}(-1)^{h_1+h_2}}{(u_1+u_2)(u_1-u_2)}\\ = \frac{h_1+\frac{1}{2}+\frac{1}{2}(-1)^{h_1+h_2}}{(u_1-u_2)(u_1+u_2)} + \frac{\frac{1}{2}}{u_1(u_1+u_2)} + \frac{\frac{1}{2}}{u_1(u_1-u_2)} - \frac{h_2}{u_1(u_1+u_2)} + \frac{h_2}{(u_1+u_2)(u_1-u_2)} \end{array}$$

We then obtain:

$$\begin{array}{rcll} N_{\Delta_n^+}(h) &=& \frac{1}{2}h_1 + \frac{1}{4}(-1)^{h_1+h_2} + \frac{3}{4} - \frac{1}{2}h_2, & \textit{if } h \in \mathfrak{c}_1,\\ N_{\Delta_n^+}(h) &=& \frac{1}{2}h_1 + \frac{1}{4}(-1)^{h_1+h_2} + \frac{3}{4} + \frac{1}{2}h_2, & \textit{if } h \in \mathfrak{c}_2, \end{array}$$

Note that the functions $N_{\Delta_n^+}$ *agree on walls, that is* $h_2 = 0$*, and the formulae above are valid on the closures of the chambers.*

The second example treats the unimodular case of A_r^+, see Example 16. Since we have identified V with $\mathbb{R}^r$, then we have a canonical identification of $U = V^*$ with $\mathbb{R}^r$ defined by duality: $u \in \mathbb{R}^r$ to $u = \sum_{i=1}^r u_i e^i \in E^*$, where e^i is the dual basis to e_i. Thus the root $e_i - e_j$ $(1 \leq i < j \leq r)$ produces the linear function $u_i - u_j$ on U, while the root $e_i - e_{r+1}$ produces the linear function u_i. Recall also the identification $h = \sum_{i=1}^{r+1} h_i e_i = [h_1, \ldots, h_r]$,

We compute the number of integral points for the parabolic subsystems of $U(2,2)$ illustrated in Fig.5.

Example 32 *We consider the 3 different systems of non compact roots as described in Fig.5 and give the formulae for the partition function.*

1. *If* $\Delta_n^+ = \Delta^+([1,4],[2,3])$ *then*

$$N_{\Delta_n^+}(h) = \begin{cases} h_1 + h_2 + 1 & \textit{if} \quad h \in \mathfrak{c}_1, \\ h_1 + h_2 + h_3 + 1 & \textit{if} \quad h \in \mathfrak{c}_2, \\ h_1 + h_3 + 1 & \textit{if} \quad h \in \mathfrak{c}_3, \\ h_1 + 1 & \textit{if} \quad h \in \mathfrak{c}_4 \end{cases}$$

2. *If* $\Delta_n^+ = \Delta^+([1,2],[3,4])$ *then*

$$N_{\Delta_n^+}(h) = \begin{cases} 1 + h_2 & \textit{if} \quad h \in \mathfrak{c}_1, \\ 1 + h_1 + h_2 + h_3 & \textit{if} \quad h \in \mathfrak{c}_2, \\ 1 + h_1 & \textit{if} \quad h \in \mathfrak{c}_3, \\ 1 - h_3 & \textit{if} \quad h \in \mathfrak{c}_4 \end{cases}$$

3. *If* $\Delta_n^+ = \Delta^+([1,3],[2,4])$ *then*

$$N_{\Delta_n^+}(h) = \begin{cases} 1 + h_1 + h_2 & \textit{if} \quad h \in \mathfrak{c}_1 \\ 1 + h_1 + h_2 + h_3 & \textit{if} \quad h \in \mathfrak{c}_2, \\ 1 + h_1 & \textit{if} \quad h \in \mathfrak{c}_3, \end{cases}$$

We have to compute the Jeffrey-Kirwan residue of the function

$f = f_1 \times \frac{1}{\prod_{\alpha \in \Delta_n^+} \alpha}$ *where* $f_1(h)(u) = \prod_{\alpha \in \Delta_n^+} \frac{\langle \alpha, u \rangle}{1 - e^{-\langle \alpha, u \rangle}} \times e^{u_1 h_1 + u_2 h_2 + u_3 h_3}$. *The computation is immediate since we need only term of degree one for the expansion of* f_1. *We omit the details. Remark though that once again the formulae agree on walls as it should be.*

3.3 Shifted partition functions

Let us consider as before our lattice $V_{\mathbb{Z}}$ and our sequence $\mathcal{A}^+$ of elements of $V_{\mathbb{Z}}$. Let

$$\rho_n = \frac{1}{2} \sum_{\alpha \in \mathcal{A}^+} \alpha.$$

We introduce

$$P_n = \rho_n + V_{\mathbb{Z}}.$$

Thus for any $\mu \in P_n$, the function $N_{\mathcal{A}^+}(\mu - \rho_n)$ is well defined.

Let $\mathcal{H}$ be the complement of all admissible hyperplanes, that is hyperplanes generated by elements of $\mathcal{A}^+$, Def.17.

Definition 33 *A tope is a connected component of the open subset $V - \mathcal{H}$ of V.*

We choose once for all a finite set F of elements G of U, so that the image of elements g cover all groups $T(\sigma)$.

If τ is a tope, then τ is contained in a unique chamber $\mathfrak{c}$, and we denote by $\mathbf{N}^{\tau}_{\mathcal{A}^+}$ the exponential polynomial function $\mathbf{N}^{\mathfrak{c},F}_{\mathcal{A}^+}$ given in Definition 23. If τ is not contained in $\mathcal{C}(\mathcal{A}^+)$, then $\mathbf{N}^{\tau}_{\mathcal{A}^+} = 0$.

The closures of the topes τ form a cover of V. A consequence of Theorem 24, is the following.

Theorem 34 *For any tope τ such that $\mu \in \overline{\tau} \cap P_n$, we have*

$$N_{\mathcal{A}^+}(\mu - \rho_n) = \mathbf{N}^{\tau}_{\mathcal{A}^+}(\mu - \rho_n).$$

3.4 A formula for the Jeffrey-Kirwan residue

Having stated a formula for partition functions (or shifted partition functions) in terms of $\mathrm{JK}_{\mathfrak{c}}$, we will explicit it using the notion of maximal proper nested sets, as developed in [12], and the notion of iterated residues. The algorithmic implementation of this formula is working in a quite impressive way, at least for low dimension.

This general scheme will be then be applied to Blattners' formula.

3.4.1 Iterated residue

If f is a meromorphic function of one variable z with a pole of order less than or equal to k at $z = 0$, then we can write $f(z) = Q(z)/z^k$, where $Q(z)$ is a holomorphic function near $z = 0$. If the Taylor series of Q is given by $Q(z) = \sum_{s=0}^{\infty} q_s z^s$, then as usual the residue at $z = 0$ of the function $f(z) = \sum_{s=0}^{\infty} q_s z^{s-k}$ is the coefficient of $1/z$, that is, q_{k-1}. We will denote it by $\mathrm{res}_{z=0} f(z)$. To compute this residue we can either expand Q into a power series and search for the coefficient of z^{-1}, or employ the formula

$$\mathrm{res}_{z=0} f(z) = \frac{1}{(k-1)!} (\partial_z)^{k-1} \left(z^k f(z) \right) \Big|_{z=0}. \tag{10}$$

We now introduce the notion of iterated residue on the space $\mathcal{R}_{\mathcal{A}^+}$.

Let $\vec{\nu} = [\alpha_1, \alpha_2, \ldots, \alpha_r]$ be an ordered basis of V consisting of elements of $\mathcal{A}^+$ (here we have implicitly renumbered the elements of $\mathcal{A}^+$ in order that the elements of our basis are listed first). We choose a system of coordinates on U such that $\alpha_i(u) = u_i$. A function $\phi \in \mathcal{R}_{\mathcal{A}^+}$ is thus written as a rational fraction $\phi(u_1, u_2, \ldots, u_r) = \frac{P(u_1,u_2,\ldots,u_r)}{Q(u_1,u_2,\ldots,u_r)}$ where the denominator Q is a product of linear forms.

Definition 35 *If $\phi \in \mathcal{R}_{\mathcal{A}}$, the iterated residue* $\mathrm{Ires}_{\vec{\nu}}(\phi)$ *of ϕ for $\vec{\nu}$ is the scalar*

$$\mathrm{Ires}_{\vec{\nu}}(\phi) = \mathrm{res}_{u_r=0}\mathrm{res}_{u_{r-1}=0} \cdots \mathrm{res}_{u_1=0}\phi(u_1, u_2, \ldots, u_r)$$

where each residue is taken assuming that the variables with higher indices are considered constants.

Keep in mind that at each step the residue operation augments the homogeneous degree of a rational function by $+1$ (as for example $\mathrm{res}_{x=0}(1/xy) = 1/y$) so that the iterated residue vanishes on homogeneous elements $\phi \in \mathcal{R}_{\mathcal{A}}$, if the homogeneous degree of ϕ is different from $-r$.

Observe that the value of $\mathrm{Ires}_{\vec{\nu}}(\phi)$ depends on the order of $\vec{\nu}$. For example, for $f = 1/(x(y-x))$ we have $\mathrm{res}_{x=0}\mathrm{res}_{y=0}(f) = 0$ and $\mathrm{res}_{y=0}\mathrm{res}_{x=0}(f) = 1$.

Remark 36 *Choose any basis $\gamma_1, \gamma_2, \ldots, \gamma_r$ of V such that $\oplus_{k=1}^{j}\alpha_j = \oplus_{k=1}^{j}\gamma_j$ for every $1 \leq j \leq r$ and such that $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_r = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r$. Then, by induction, it is easy to see that for $\phi \in \mathcal{R}_{\mathcal{A}^+}$*

$$\mathrm{res}_{\alpha_r=0} \cdots \mathrm{res}_{\alpha_1=0}\phi = \mathrm{res}_{\gamma_r=0} \cdots \mathrm{res}_{\gamma_1=0}\phi.$$

Thus given an ordered basis, we may modify α_2 by $\alpha_2 + c\alpha_1, \ldots,$ with the purpose of getting easier computations.

As for the usual residue, the iterated residue can be expressed as an integral as explained in [3]. This fact allows change of variables.

3.4.2 Maximal proper nested sets adapted to a vector

We recall briefly the notion of maximal proper nested set, $MNPS$ in short, and some of their properties (see [12]).

A subset S of $\mathcal{A}^+$ is *complete* if $S = \langle S \rangle \cap \mathcal{A}^+$: here recall that $\langle S \rangle$ is the vector space spanned by S. A complete subset S is called *reducible* if we can find a decomposition $V = V_1 \oplus V_2$ such that $S = S_1 \cup S_2$ with $S_1 \subset V_1$ and $S_2 \subset V_2$. Otherwise S is said to be *irreducible*.

A set $M = \{I_1, I_2, \dots, I_k\}$ of irreducible subsets of $\mathcal{A}^+$ is called *nested* if, given any subfamily $\{I_1, \dots, I_m\}$ of M such that there exists no i, j with $I_i \subset I_j$, then the set $I_1 \cup \cdots \cup I_m$ is *complete* and the elements I_j are the irreducible components of $I_1 \cup I_2 \cup \cdots \cup I_m$. Then every maximal nested set M, MNS in short, contains $\mathcal{A}^+$ and has exactly r elements.

We now recall how to construct all maximal nested sets. We may assume that $\mathcal{A}^+$ is irreducible, otherwise just take one of the irreducible components. If M is a maximal nested set, the vector space $\langle M \setminus \mathcal{A}^+ \rangle$ is an hyperplane H, thus an admissible hyperplane.

Definition 37 *Let H be a $\mathcal{A}^+$-admissible hyperplane. A maximal nested set M such that $\langle M \setminus \mathcal{A}^+ \rangle = H$ is said attached to H.*

Given M a MNS for $\mathcal{A}^+$ attached to H, then $\langle M \setminus \mathcal{A}^+ \rangle$ is a MNS for $H \cap \mathcal{A}^+$. Therefore maximal nested sets for an irreducible set $\mathcal{A}^+$ can be determined by induction over the set of $\mathcal{A}^+$-admissible hyperplanes.

For computing the Jeffrey-Kirwan residue, we only need some particular MNS's. Let us briefly review the main ingredients.

Fix a total order ht on $\mathcal{A}^+$. Let $M = \{S_1, S_2, \dots, S_k\}$ be a set of subsets of $\mathcal{A}^+$ and choose in each S_j the element α_j maximal for the order given by ht. This defines a map Θ from M to $\mathcal{A}^+$ and we say that M is *proper* if $\Theta(M) = \overrightarrow{M}$ is a basis of V. We denote by $\mathcal{P}(\mathcal{A}^+)$ the set of $MPNS$.

So we have associated to every maximal proper nested set M an ordered basis, by sorting the set $\overrightarrow{M} = [\alpha_1, \alpha_2, \dots, \alpha_r]$ of elements of $\mathcal{A}^+$.

Let v be an element in V not belonging to any admissible hyperplane.

Definition 38 *Define $\mathcal{P}(v, \mathcal{A}^+)$ to be the set of $M \in \mathcal{P}(\mathcal{A}^+)$ such that $v \in \mathcal{C}(M) = \mathcal{C}(\alpha_1, \dots, \alpha_r)$.*

When there is no possibility of confusion we will drop simply write $\mathcal{P}(v)$ for $\mathcal{P}(v, \mathcal{A}^+)$.

We are now ready to state the basic formula for our calculations.

Theorem 39 ([12]) *Let $\mathfrak{c}$ be a chamber and let $v \in \mathfrak{c}$. Then, for $\phi \in \widehat{\mathcal{R}}_{\mathcal{A}^+}$, we have*

$$\mathrm{JK}_{\mathfrak{c}}(\phi) = \sum_{M \in \mathcal{P}(v, \mathcal{A}^+)} \frac{1}{\mathrm{vol}(M)} \mathrm{Ires}_{\overrightarrow{M}} \phi.$$

Let us finally sketch the algorithm to determine $\mathcal{P}(v, \mathcal{A}^+)$ without going to construct all the MNS's. If $\mathcal{A}^+ = \mathcal{A}_1^+ \times \mathcal{A}_2^+$ is reducible, then $\mathcal{P}(v, \mathcal{A}^+)$ is the product of the corresponding sets $\mathcal{P}(v_i, \mathcal{A}_i^+)$.

Assume $\mathcal{A}^+$ is irreducible. Let θ be the highest root of the system $\mathcal{A}^+$ (for our order ht). We start by constructing all possible $\mathcal{A}^+$-admissible hyperplanes H for which v and θ are strictly on the same side of H. In particular, the hyperplane H does not contain the highest root.

Then we compute the projected vector $proj_H v$ on H parallel to θ: $v = \mathrm{proj}_H v + t\theta$, with $\mathrm{proj}_H v \in H$ and $t > 0$ and compute $\mathcal{A}^+ \cap H$. If M_H is in $\mathcal{P}(proj_H v, \mathcal{A}^+ \cap H)$, then $M = \{op(M_H), \mathcal{A}^+\}$ is in $\mathcal{P}(v, \mathcal{A}^+)$. Running through all hyperplanes H, for which v and θ are strictly on the same side of H, we obtain the set $\mathcal{P}(v, \mathcal{A}^+)$. Let us summarize the scheme of the algorithm in Figure 6. Recall that we have as input a regular vector v, and as output the list of all $MPNS$'s belonging to $\mathcal{P}(v, \mathcal{A}^+)$.

```
for each hyperplane H do
  check if v and θ are on the same side of H
    if not, then skip this hyperplane
    define the projection proj_H(v) of v on H along θ
    write A ∩ H as the union of its irreducible components I_1 ∪ ··· ∪ I_k
    write v as v_1 ⊕ ··· ⊕ v_k according to the previous decomposition
      for ēach I_j do
      compute all MPNS's for v_j and I_j
      collect all these MPNS's for v_j and I_j
      end of loop running across I_j's
  collect all MPNS's for the hyperplane H by taking the product of P(I_j, v_j)
end of loop running across H's
return the set of all MPNS's for all hyperplanes
```

Figure 6: Algorithm for MPNS's computation (general case)

In our program, we run this algorithm for an element v not in any admissible hyperplane, without knowing in advance if v belongs to the cone $\mathcal{C}(\mathcal{A}^+)$. The algorithm returns a non empty set if and only if v belongs to $\mathcal{C}(\mathcal{A}^+)$.

3.5 The Kostant function: another formula for subsystems of A_r^+

In this article, we will be using partition functions for lists $\Delta^+(A, B)$ described in the Example 16. These lists are sublists of a system of type A_r^+ (with $r = p+q-1$). In residue calculation, we can use change of variables and thus use a formula for which iterated residues will be easier to compute.

Let us describe this formula. We will describe it for sublists, eventually, with multiplicities of a system A_r^+. We take the notations of Example 16.

Let Φ be a sequence of vectors generating V and of the form $(e_i - e_j), 1 \leq i < j \leq (r+1)$, eventually with multiplicities. Let $m_{i,j}$ $(i < j)$ be the multiplicity of the vector $e_i - e_j$ in Φ and define $t_j = m_{j,j+1} + \cdots + m_{j,r+1} - 1$. We recall our identification of V with $\mathbb{R}^r$ and of $U = V^*$ with $\mathbb{R}^r$ defined by duality. In this way, as we already observed the root $e_i - e_j$ $(1 \leq i < j \leq r)$ produces the linear function $u_i - u_j$ on U, while the root $e_i - e_{r+1}$ produces the linear function u_i.

We are now ready to give another formula for the Kostant function in this situation.

Theorem 40 *Let $\mathfrak{c}$ be a chamber of $\mathcal{C}(\Phi)$. Let $h = \sum_{i=1}^{r+1} h_i e_i = [h_1, \ldots, h_r]$, then*

$$\mathbf{N}_\Phi^{\mathfrak{c}}(h) = \mathrm{vol}\,(V/V_{\mathbb{Z}}, dh)\, \mathrm{JK}_{\mathfrak{c}}(f_\Phi(h)(u)), \text{where}$$

$$f_\Phi(h)(u) = \frac{\prod_{i=1}^{r}(1+u_i)^{h_i+t_i}}{\prod_{i=1}^{r} u_i^{m_{i,r+1}} \prod_{1\leq i<j\leq r}(u_i-u_j)^{m_{i,j}}}$$

Thus, when $h \in V_{\mathbb{Z}} \cap (\mathfrak{c} - Z(\Phi))$, we have

$$N_\Phi(h) = \mathbf{N}_\Phi^{\mathfrak{c}}(h).$$

Example 41

- *If $\Phi = A_r^+$, then*

$$f_{A_r^+}(h)(u) = \frac{\prod_{i=1}^{r}(1+u_i)^{h_i+r-i}}{\prod_{1\leq i<j\leq r}(u_i-u_j) \times \prod_{i=1}^{r} u_i}.$$

- *Let p, q integers such that $p + q = r + 1$ and Φ be the system of positive noncompact root for A_r^+ defined by $\Phi = \{e_i - e_j, 1 \leq i \leq p,\ p+1 \leq j \leq r+1\}$. That is $\Phi = \Delta^+(A, B)$ with $A = [1, \ldots, p]$ and $B = [p+1, \ldots, p+q]$.*

  Then

$$f_\Phi(h_1, h_2, \ldots, h_r)(u) =$$
$$\frac{(1+u_1)^{h_1+q-1} \cdots (1+u_p)^{h_p+q-1}(1+u_{p+1})^{h_{p+1}-1} \cdots (1+u_{p+q-1})^{h_{p+q-1}-1}}{(u_1-u_{p+1})\cdots(u_1-u_{p+q-1})\cdots(u_p-u_{p+1})\cdots(u_p-u_{p+q-1})u_1u_2\cdots u_p}$$

Proof. The function $K(0,h)(u) = e^{\langle h,u\rangle} / \prod_{\alpha\in\Phi}(1 - e^{-\langle\alpha,u\rangle})$ computed for the system Φ is

$$K(0,h)(u) = \frac{e^{h_1u_1}e^{h_2u_2}\cdots e^{h_ru_r}}{\prod_{i=1}^{r}(1-e^{-u_i})^{m_{i,r+1}} \prod_{1\leq i<j\leq r}(1-e^{-(u_i-u_j)})^{m_{i,j}}}$$

Note that the change of variable $1 + z_i = e^{u_i}$ preserves the hyperplanes $u_i = 0$ and $u_i = u_j$ and that $z_i = e^{u_i} - 1$ leads to $dz_i = e^{u_i} du_i = (1 + z_i)du_i$. Thus after the change of variable we get the required formulae.

3.6 Computation of Kostant partition function: general scheme

3.6.1 Numeric

We have as input $\mathcal{A}^+$ a sequence of vectors in our lattice $V_{\mathbb{Z}}$, a vector $h \in V_{\mathbb{Z}}$, and we want to compute $N_{\mathcal{A}^+}(h)$. We will compute it by !!

$$N_{\mathcal{A}^+}(h) = N_{\mathcal{A}^+}(h + \rho_n - \rho_n).$$

We mean: Let τ be any tope such that $h' = h + \rho_n$ belongs to the closure of τ. Using Theorem 34 then

$$N_{\mathcal{A}^+}(h) = N_{\mathcal{A}^+}(h' - \rho_n) = \mathbf{N}^{\tau}_{\mathcal{A}^+}(h' - \rho_n).$$

To compute a tope τ containing h', we can move $h + \rho_n$ in any generic direction ϵ.

Here is an outline of the steps needed to compute the number $N_{\mathcal{A}^+}(h)$ by the formula $N_{\mathcal{A}^+}(h) = \mathbf{N}^{\tau}_{\mathcal{A}^+}(h)$.

Input: a vector $h \in V_{\mathbb{Z}}$, and $\mathcal{A}^+$ a sequence of vectors in $V_{\mathbb{Z}}$.

Output: the number $N_{\mathcal{A}^+}(h)$.

- **Step 1** Compute the Kostant function

  $$K(h) = K(0, h) = \frac{e^h}{\prod_{\alpha \in \mathcal{A}^+}(1 - e^{-\alpha})}$$

  or more generally compute a set F and the functions $K(G, h)$ for $G \in F$.

- **Step 2** Find a small vector ϵ so that if h is in $V_{\mathbb{Z}}$, the vector $h + \rho_n + \epsilon$ does not belong to any admissible hyperplane. Thus the vector $h_{reg} = h + \rho_n + \epsilon$ is in a unique tope τ. The procedure to obtain h_{reg} is called $DefVector_{nc}(h)$.

- **Step 3** Compute the set $All := \mathcal{P}(h_{reg}, \mathcal{A}^+)$ as explained in Fig.6.

- **Step 4** Compute $\mathbf{N}^{\tau}_{\mathcal{A}^+}(h)$ by computing the iterated residues of $K(G, h)$ associated to the various ordered basis $\overrightarrow{M}$ for M varying in the set All. That is compute the number

  $$\boxed{out := \sum_{G \in F} e^{\langle h, 2\pi\sqrt{-1}G\rangle} \sum_{M \in All} \mathrm{Ires}_{\overrightarrow{M}} K(G, h)}$$

  where $\overrightarrow{M}$ is the ordered basis attached to M.

  Then $\boxed{N_{\mathcal{A}^+}(h) = out}$

3.6.2 Symbolic

The previous calculation runs with symbolic parameters. If $hfix$ is an element in $V_{\mathbb{Z}}$, we might want to find a tope τ such that $hfix$ belongs to the closure of τ. Then

$$N_{\mathcal{A}^+}(h) = \mathbf{N}^{\tau}_{\mathcal{A}^+}(h)$$

will be valid whenever h is in the closure of τ. Here is the outline of the algorithm.

Input: $hfix$ is an element in $V_{\mathbb{Z}}$ and $\mathcal{A}^+$ a sequence of vectors.

Output: A domain $D \subset V$ and an exponential polynomial function $P(h)$ on V.

The domain D is a closed convex cone in V (described by linear inequations) such that $hfix$ is in D. The formula $P(h) = N_{\mathcal{A}^+}(h)$ is valid whenever $h \in D \cap V_{\mathbb{Z}}$.

- **Step 1** Consider h as a parameter and compute the Kostant function $K(h)(u)$ as a function of (h, u) given by

  $$K(h)(u) = K(0, h)(u) = \frac{e^{<h,u>}}{\prod_{\alpha \in \mathcal{A}^+}(1 - e^{-<\alpha,u>})}$$

  or more generally compute a set F and the functions $K(G, h)(u)$ for $G \in F$, as function of (h, u).

- **Step 2** Find a small vector ϵ so that if $hfix$ is in $V_{\mathbb{Z}}$, then the vector $hfix_{reg} := hfix + \rho_n + \epsilon$ does not belong to any admissible hyperplane.

  Compute the domain $D := \overline{\tau}$ where τ is the unique tope τ containing $hfix_{reg}$.

- **Step 3** Compute the set $All := \mathcal{P}(hfix_{reg}, \mathcal{A}^+)$ as explained in Fig.6.

- **Step 4** Compute $\mathbf{N}^{\tau}_{\mathcal{A}^+}(h)$ by computing the iterated residues of $K(G, h)$ associated to the various ordered basis $\overrightarrow{M}$ for M varying in the set All, here h is treated now as a parameter.

  That is we compute

  $$out := \sum_{G \in F} e^{\langle h, 2\pi\sqrt{-1}G\rangle} \sum_{M \in All} \mathrm{Ires}_{\overrightarrow{M}} K(G, h)$$

  where $\overrightarrow{M}$ is the ordered basis attached to M. The output out is an exponential polynomial function $P(h)$ of h and once again we compute

  $$\boxed{N_{\mathcal{A}^+}(h) = P(h) = out, \ \forall h \in D \cap V_{\mathbb{Z}}}$$

  The domain D is a rational polyhedral cone which includes $hfix$.

In practice, this works only for small dimensions and when $\mathcal{A}^+$ is not too big.
We will program variations of these algorithms, with less ambitious goals.

3.7 Computation of Blattner formula: general scheme.

In this subsection, we summarize the steps to compute Blattner's formula and the general scheme to obtain the region of polynomiality. The relative algorithms will be outlined in Section 6.

Let G, K, T be given as in Section 2.2. Let $\Delta_n \subset \mathfrak{t}^*$ be the list of noncompact roots.

Our inputs are $\lambda \in P_{\mathfrak{g}}^r$ and $\mu \in P_{\mathfrak{k}}^r$. The goal is the study of the function $\mu \to m_\mu^\lambda$. Let $\mathcal{A}^+ = \Delta_n^+(\lambda)$ and recall that in this case a $\mathcal{A}^+$-admissible hyperplane is called a noncompact wall.

We use **Blattner's formula**. In our notations:

$$m_\mu^\lambda = \sum_{w \in \mathcal{W}_c} \epsilon(w) N_{\Delta_n^+(\lambda)}(w\mu - \lambda - \rho_n) \tag{11}$$

3.7.1 Numeric

Input $\lambda \in P_{\mathfrak{g}}^r$ and $\mu \in P_{\mathfrak{k}}^r$.
Output a number.

The algorithm is clear:

- Compute $\mathcal{A}^+ = \Delta_n^+(\lambda)$ and ρ_n.
- Compute the Kostant function $K(G, h)$ for this system $\mathcal{A}^+$.
- Compute a finite set F satisfying condition (**C**).
- Compute a small element ϵ such that $\mu_{reg} = \mu + \epsilon$ does not belong to any affine hyperplane of the form $w\lambda + H$ where H is a noncompact wall, $w \in \mathcal{W}_c$.
- Compute for all $w \in \mathcal{W}_c$ the number

  $$contribution_w := N_{\mathcal{A}^+}(w\mu - \lambda - \rho_n)$$

  using the algorithm described in 3.6.1.

  That is compute

  $All_w := \mathcal{P}(w * (\mu_{reg}) - \lambda, \mathcal{A}^+)$ as explained in Fig.6.

  Then compute

  $$contribution_w := \sum_{G \in F} e^{\langle h, 2\pi\sqrt{-1}G\rangle} \sum_{M \in Allw} \mathrm{Ires}_{\overrightarrow{M}} K(G, w\mu - \lambda - \rho_n).$$

  ENDs

- Finally compute

$$out := m^\lambda_\mu = \sum_{w\in\mathcal{W}_c} \epsilon(w) * contribution_w$$

Let us comment briefly: If $w(\mu_{reg}) - \lambda$ is not in the cone generated by non compact positive roots, the set All_w is an empty set. In particular we may restrict the computation by diverse consideration to **valid permutations** which have some chance to give a non empty set, see Section 4.3.)

3.7.2 Symbolic

The preceding calculation runs with symbolic parameter and we take advantage of this to find regions of polynomiality.

Let's explain how.

Let $\mathfrak{a}$ be a chamber in $\mathfrak{t}^*$ for the system Δ of roots of $\mathfrak{g}$. Let U be the open set of $(\lambda, \mu) \in \mathfrak{a} \times \mathfrak{t}^*$ such that $w\lambda - \mu$ does not belong to any non compact wall. Let $(\lambda_0, \mu_0) \in U$. Then we define $R(\lambda_0, \mu_0)$ to be the closure of connected component of U containing (λ_0, μ_0). This region is a cone in $\mathfrak{t}^* \times \mathfrak{t}^*$ with non empty interior and can be described by linear inequalities in λ, μ.

For this domain we can compute a polynomial formula and state the following result.

If λ varies in $\mathfrak{a}$, the systems $\Delta^+_n(\lambda), \Delta^+_c(\lambda)$ determined by λ remains the same. We denote it by Δ^+_n, Δ^+_c.

Furthermore, if $(\lambda, \mu) \in R(\lambda_0, \mu_0)$, for any $w \in \mathcal{W}_c$, the element $w\lambda - \mu$ lies in a tope τ_w for the system Δ^+_n which depends only of w.

Theorem 42 *The domain $R(\lambda_0, \tau_0)$ is a domain of polynomiality for the Duistermaat-Heckman measure and thus is a domain of quasi-polynomiality for the multiplicity function m^λ_μ.*

More precisely, for $(\lambda, \mu) \in R(\lambda_0, \mu_0) \cap (P^r_{\mathfrak{g}} \times P^r_{\mathfrak{k}})$, we have

$$m^\lambda_\mu = \sum_{w\in\mathcal{W}_c} \epsilon(w) \mathbf{N}^{\tau_w}_{\Delta^+_n}(w\lambda - \mu - \rho_n). \tag{12}$$

The right hand side of Equation 12 is a quasi polynomial function of λ, μ, antisymmetric in μ. It takes positive values if μ is dominant for Δ^+_c. Recall that, in this case, the multiplicity of μ on λ is the absolute value of the function m^λ_μ above, (Sec.2.2).

Of course, the symbolic calculation above, with the present approach, is limited to very small examples.

Also, we are not able to determine the largest domains where the function m_μ^λ is given by a quasi polynomial formula.

3.7.3 Asymptotic directions

We address now a simpler problem. We have the same setting that in the previous section, but we are now testing only the noncompact walls crossing in one fixed direction $\vec{v}$.

Let μ_0, λ_0 be given, with $\lambda_0 \in \mathfrak{a} \cap P_{\mathfrak{g}}^r$ an Harish-Chandra parameter, and $\mu_0 \in \mathfrak{a}_c \cap P_{\mathfrak{k}}^r$. Let $\vec{v} \in \mathfrak{a}_c$ be integral. We will do the calculation of $m_{\mu_t}^{\lambda_0}$ when $\mu_t = \mu_0 + t\vec{v}$, with $t \geq 0$, is in the half-line in the direction $\vec{v}$. In the application μ_0 will be the lowest K-type $\lambda_0 - \rho_n$ of our discrete series π^{λ_0}. We compute the values t_i where $\mu_0 + t\vec{v} - w\lambda_0$ cross a non compact wall (other than the ones which may contain the line $\mu_0 + t\vec{v} - w\lambda_0$). These are the values where the line μ_t may cross the domains of quasipolynomiality described above. We order this finite set of values $0 \leq t_1 < t_2 < t_i < \cdots < t_s$. Consider the interval $I_i = [t_i, t_{i+1}], \quad 0 \leq i \leq s$, where $t_0 = 0$ and $t_{s+1} = \infty$.

Consider $I_i \cap \mathbb{Z}$, an "interval" in $\mathbb{Z}$, described by two integers $[a_i, b_i]$, with $a_i = ceil(t_i)$ and $b_i = floor(t_{i+1})$.

The "interval" $I_i \cap \mathbb{Z}$ can also be reduced to a point.

Then we find exponential polynomial function $P^i(t)$ on $\mathbb{R}$ such that $m_{\mu_t}^\lambda$ is equal to $P^i(t)$ for $t \in I_i \cap \mathbb{Z}$.

If particular, $\vec{v}$ is an asymptotic direction, if and only if the last quasipolynomial $P^s(t)$ does not vanish.

The algorithm is as follows.

For each consecutive value t_i, t_{i+1}, choose $\mu_r = \mu + t_r v$ with $t_i < t_r < t_{i+1}$. Then, move very slightly μ_r in μ_r^ϵ. Then for each $w \in \mathcal{W}_c$, $w\mu_r^\epsilon - \lambda_0$ lies in a tope τ_i^w for $\Delta_n^+(\lambda_0)$. Then

$$P_i(t) = \sum_w \epsilon(w) \mathbf{N}_{\Delta_n^+(\lambda)}^{\tau_i^w}(w\mu_t - \lambda - \rho_n).$$

The right hand side of this formula is an exponential polynomial function of $t \in \mathbb{Z}$.

The algorithm implementing this procedure is described in Fig.9. Let us remark that our algorithm implementation is for type A_r and thus the P_i are polynomials.

4 Blattner's formula for $U(p,q)$

4.1 Non compact positive roots

With the notation of Section 1 we let $G = U(p,q)$ and $K = U_p \times U_q$ be a maximal compact subgroup.

Let E be a $p+q$-dimensional vector space with basis e_i ($i = 1, \dots, p+q$) and V as in Ex. 16). Let $r = p+q-1$. Consider the set of roots

$$\Delta = \pm\{e_i - e_j \,|\, 1 \le i < j \le p+q\}.$$

We then choose T to be the diagonal subgroup of $U(p,q)$, and identify $\mathfrak{t}^*$ with E. In this identification the lattice of weights is identified with $\mathbb{Z}^{p+q}$: the element $(n_1, \dots, n_{p+q})$ giving rise to the character

$$t = (\exp(i\theta_1), \dots, \exp(i\theta_{p+q})) \to e^{in_1\theta_1} \cdots e^{in_{p+q}\theta_{p+q}}.$$

The system of compact roots Δ_c is

$$\Delta_c = \pm\{e_i - e_j \,|\, 1 \le i < j \le p\} \cup \pm\{e_i - e_j \,|\, p+1 \le i < j \le p+q\}.$$

The system of non compact roots is

$$\Delta_n = \pm\{e_i - e_j \,|\, 1 \le i \le p, p+1 \le j \le p+q\}.$$

Let λ be the Harish Chandra parameter of a discrete series for G and μ the Harish-Chandra parameter of a finite dimensional irreducible representation of K.

Because discrete series are equivalent under the action of the Weyl group of K, then we may assume that $\lambda = \;[\alpha, \beta]\;$ where $\alpha = \sum_{i=1}^{p} \alpha_i e_i = [\alpha_1, \dots, \alpha_p], \alpha_1 > \alpha_2 > \cdots > \alpha_p$ and $\beta = \sum_{i=p+1}^{p+q} \beta_i e_i = [\beta_1, \dots, \beta_q], \beta_1 > \beta_2 \cdots > \beta_q$.

Here α_i, β_j are integers if $p+q$ is odd, or half-integers if $p+q$ is even, that is we fix as system of positive compact roots the system $\Delta_c^+ = \{e_i - e_j \,|\, 1 \le i < j \le p\} \cup \{e_i - e_j \,|\, p+1 \le i < j \le p+q\}$.

We parametrize $\mu \in P_{\mathfrak{k}}^r$ by another couple

$$\mu := [a, b] = [[a_1, a_2, \dots, a_p], [b_1, \dots, b_q]]$$

with $a_1 > \cdots > a_p$ and $b_1 > \cdots > b_q$.

Here a_i are integers if p is odd, half-integers if p is even. Similarly b_j are integers if q is odd, half-integers if q is even.

As the center of G acts by a scalar in an irreducible representation, we need that the sum of the coefficients of λ has to be equal to the sum of the coefficients of μ for the multiplicity of μ in π^λ to be non zero. Thus $\lambda - \mu$ is in V, (see Ex. 16).

We now parametrize the different dominant chambers of $\mathfrak{t}^*$ modulo the Weyl group of K by a subset A of $[1, 2, \ldots, r+1]$ of cardinal p. Let B its complementary subset in $[1, 2, \ldots, r+1]$.

To visualize A, B we write a sequence of lenght $p+q$ of elements a, b with a in the places of A, b in the places of B: for example if $A = [3,5]$ and $B = [1,2,4]$, then we write $[b, b, a, b, a]$ or simply *bbaba*. Now we use this visual aid and describe a permutation w_A of the index $[1, 2, \ldots, p+q]$, by putting the index $[1, \ldots, p]$ in order and in the places marked by a, and the remaining indices $[p+1, \ldots, p+q]$ in order and in the places marked by b, precisely $w_A : [1,2,3,4,5] \to [3,4,1,5,2]$. The elements w_A where A varies describe a system of representatives of $\Sigma_{p+q}/(\Sigma_p \times \Sigma_q)$, ($\Sigma_n$ being the permutations on n letters), that is also the chambers of $\mathfrak{t}^*$ for $\Delta(\mathfrak{g}, \mathfrak{t})$ modulo $\mathcal{W}_c$. In the above the chamber is described by $\{h = [h_1, h_2, h_3, h_4, h_5] \quad h_3 > h_4 > h_1 > h_5 > h_2\}$

Let $\mathfrak{a}_{standard}$ be the chamber $\alpha_1 > \alpha_2 > \cdots > \alpha_p > \beta_1 > \beta_2 \cdots > \beta_q$.

Then if $\lambda \in w_A \mathfrak{a}_{standard}$, we have $\Delta_c^+(\lambda) = \Delta_c^+$ and $\Delta_n^+(\lambda)$ is isomorphic to $\Delta^+(A, B)$, by relabeling the roots via w_A^{-1}. The next example will clarify the situation. The subset A can be read from λ: we reorder completely the sequence λ and define A as the indices where the first p elements of λ are relocated.

Example 43 *Let $G = U(2,3)$ with compact roots $\Delta_c = \pm\{e_1 - e_2, e_3 - e_4, e_3 - e_5, e_4 - e_5\}$ and noncompact roots $\Delta_n = \pm\{e_1 - e_3, e_1 - e_4, e_1 - e_5, e_2 - e_3, e_2 - e_4, e_2 - e_5\}$. Let $\lambda = [\alpha, \beta]$ with $\alpha = [4,2] = 4e_1 + 2e_2$ and $\beta = [6,5,3] = 6e_3 + 5e_4 + 3e_5$.*

Then $A = [3,5]$, $B = [1,2,4]$ that is the configuration bbaba. The system of non compact positive roots for λ is $e_3 - e_1, e_4 - e - 1, e - 1 - e_5, e_3 - e_2, e_4 - e_2, e_5 - e_2$, isomorphic to $\Delta^+(A, B)$ by the relabeling of the roots suggested by w_A^{-1}, that is $e_3 = f_1, e_4 = f_2, e_1 = f_3, e_5 = f_4, e_2 = f_5$.

Thus relabeling the roots, our calculations will be done for $\Delta^+(A, B)$ inside A_r^+, where $\Delta^+(A, B)$ is given in Example 16.

Remark here that $\Delta^+(A, B)$ is irreducible if p and q are strictly greater than 1. In contrast, when p or $q = 1$, the system is fully reducible. Consider for example the case $p = 1$.

Example 44

In this case A has only 1 element and the system $\Delta^+(A, B)$ has r elements and is a base of V. Thus $\Delta^+(A, B)$ is fully reducible in the direct sum of $p+q-1$ one dimensional systems.

For example take $U(1, r)$ with $A = [1]$ and $B = [2, \ldots, r+1]$. Then $\Delta^+(A, B) = \{e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_{r+1}\}$ is isomorphic to $A_1^+ \times A_1^+ \times \cdots \times A_1^+$.

Remark that when A_1, A_2 have the same number of elements, although the system of noncompact roots $\pm\Delta_n^+(A_1, B_1)$ and $\pm\Delta_n^+(A_2, B_2)$ are clearly isomorphic, the combinatorial properties of $\Delta_n^+(A, B)$ may vary.
For example, (see Figure 5), if $A = [1, 2]$, $B = [3, 4]$, the cone generated by the non compact roots has basis a square and is not a simplicial cone. If $A = [1, 3]$ and $B = [2, 4]$, then the cone generated by the non compact roots is the simplicial cone generated by $e_1 - e_2, e_2 - e_3, e_3 - e_4$.

4.2 Algorithm to compute $MPNS$: the case of $\Delta^+(A, B)$

With the notations of Ex.16, we denote by A a proper subset of $[1, 2, \ldots, r + 1]$ (with $r = p + q - 1$) and by B the complementary subset to A in $[1, 2, \ldots r + 1]$.

Given $v \in V$, and not on any admissible hyperplane, we describe the algorithm to compute $\mathcal{P}(v, \Delta^+(A, B))$.

If p or $q = 1$, roots α in $\Delta^+(A, B)$ form a basis on V, thus there is only one maximal nested set $M = \{\{\alpha\}, \alpha \in \Delta^+(A, B)\}$. Thus $\mathcal{P}(v, \Delta^+(A, B))$ is empty or equal to $\{\vec{M}\}$ depending if v belongs to the cone generated by $\Delta^+(A, B)$, or not. This is very easy to check.

If $p > 1$ and $q > 1$, we determine the set $\mathcal{P}(v, \Delta^+(A, B))$ by induction, going to admissible hyperplanes.

If $L \subsetneq [1, 2, \ldots, r + 1]$ is a proper subset of $[1, 2, \ldots, r + 1]$, we will also use the notation $L' = [i \notin L \,|\, 1 \leq i \leq r + 1]$ for the complement of L. We denote by $H_L := \{v \in V \,|\, \sum_{i \in L} v_i = 0\}$ the hyperplane determined by L; the hyperplane H_L is equal to the hyperplane $H_{L'}$ determined by L'.
It is very simple to describe $\Delta^+(A, B)$-admissible hyperplanes, that is noncompact walls. The description is an adaptation of the A_r^+-admissible hyperplanes that appear in [3].

Keeping A fixed, with $|A| \neq 1, r$, we consider hyperplanes H_L indexed by subsets $L \subset 1, 2, \ldots, r + 1]$ with the following properties:

- if $|L| \neq 1$ or r, then H_L is a noncompact wall if and only if both A and B intersect L and L'. In this case $\Delta_n^+(A, B) \cap H_L$ is the product of two systems $\Delta_n^+(A \cap L, B \cap L) \times \Delta_n^+(A \cap L', B \cap L')$ and thus reducible.

- if L is of cardinal 1, then H_L is a noncompact wall. In this case $\Delta_n^+(A, B) \cap H_L$ is $\Delta_n^+(A \cap L', B \cap L')$ and thus irreducible.

At this point to compute the $MNPS$ or better, as we explained the $\overrightarrow{M}'s$, we can proceed as in Fig.6. The algorithm is outlined in Fig.7.

We conclude with the following observation. A necessary and sufficient condition for the set $MPNS(v, \Delta_n^+(A, B))$ to be non empty is that v belongs to the cone generated by $\Delta_n^+(A,B)$. As far as we know, the equations of this cone are not known, except in a few cases. It is clearly necessary that v belongs to the simplicial cone generated by all positive roots. To speed up the calculations, we check this condition at each step of the algorithm.

We conclude with a simple example with $p = 2, q = 2$ and $A = [1,2]$, $B = [3,4]$. We follow the outline described in Fig.6. The highest non compact root is $\theta := e_1 - e_4$. There are 3 noncompact walls not containing the highest root. $L = [1], [4], [1,3]$ We choose a vector $v = [4,3,-2,-5]$ not on any noncompact walls. Then $[1], [4], [1,3]$ are all such that v and θ are on the same side.

For $L = [1]$, the v projection do not belong to the cone generated by $\Delta_n^+(A,B) \cap H_L = [e_2 - e_3, e_3 - e_4]$.

For $L = [4]$, we obtain the element $M := \{[1,2,3,4],[1,3],[2,3]\}$ in $\mathcal{P}(v, \Delta_n^+(A,B))$.

For $L := [1,3]$, we obtain the element $M = \{[1,2,3,4],[1,3],[2,3]\}$ in $\mathcal{P}(v, \Delta_n^+(A,B))$.

4.3 Valid permutations

Let $w \in \mathcal{W}_c$. Remark that if $w\mu - \lambda$ does not belong to the cone of non compact positive roots, then the term corresponding to w in Blattner formula is equal to 0. It is important to minimize the number of terms in Blattner formula. To this purpose, we use a weaker condition: we say that $w \in \mathcal{W}_c$ is a valid element if $w\mu - \lambda$ is in the cone spanned by (all) positive roots. Thus if w is not valid, the corresponding term to w in Blattner formula is equal to 0. As there is a simple description of the faces of cone spanned by all positive roots (it is the simplicial cone dual to the simplicial cone generated by fundamental weights), there is a simple algorithm that constructs valid permutations one at the time depending on the conditions they have to satisfy, instead of listing all the elements of $\mathcal{W}_c$. The corresponding algorithm is used in [4],[6], and we just reproduced it.

5 Examples

Example 45

We consider the discrete series representation indexed by λ and we test for the multiplicity m_μ^λ where μ is a K type. We write μ_{lowest} for the lowest K-type. We use the algorithm whose command is :

>discretemult(λ,μ,p,q)

m_μ^λ: **numeric case**			
Group	**Input**	**Output**	**Time**
U(3,3)	λ=[[31/2,15/2,9/2],[5/2,3/2,-5/2]]		
	μ_{lowest}=[[17, 9, 6], [1, 0, -4]]	1	0.026 sec.
	μ=[[1017, 1009, 1006], [-999, -1000, -1004]]	9	0.97 sec.
	μ=[[100017, 10009, 10006], [-9999, -10000, -100004]]	9	0.91 sec.
U(3,3)	λ=[[31/2, 19/2, 11/2], [15/2, 7/2, -37/2]]		
	μ_{lowest}=[[17, 11, 6], [7, 2, -20]]	1	0.073 sec.
	μ=[[1017, 1011, 1006], [-993, -998, -1020]]	275	0.529 sec.
	μ=[[100017, 10011, 10006], [-9993, -9998, -100020]]	11700255	0.538 sec.
U(4,4)	λ=[[11/2,7/2,3/2,-1/2],[9/2,5/2,1/2,-3/2]]		
	μ_{lowest}=[[15/2, 9/2, 3/2, -3/2], [11/2, 5/2, -1/2, -7/2]]	1	0.565 sec.
	μ=[[2015/2, 9/2, 3/2, -3/2], [11/2, 5/2, -1/2, -2007/2]]	120495492015	3.493 sec.
U(4,4)	λ=[[11/2, 9/2, 7/2, 5/2], [3/2, 1/2, -1/2, -3/2]]		
	μ_{lowest}=[[15/2, 13/2, 11/2, 9/2], [-1/2, -3/2, -5/2, -7/2]]	1	0.334 sec.
	μ=[[20015/2, 2013/2, 211/2, 29/2], [-21/2, -203/2, -2005/2, -20007/2]]	1	273.719 sec.
U(5,4)	λ=[[5, 3, 1, -1, -3], [4, 2, 0, -2]]		
	μ_{lowest}=[[7, 4, 1, -2, -5], [11/2, 5/2, -1/2, -7/2]]	1	3.952 sec.
	μ=[[1007, 4, 1, -2, -5], [11/2, 5/2, -1/2, -2007/2]]	120495492015	13.752 sec.
U(5,5)	λ=[[11/2,7/2,3/2,-1/2,-5/2],[9/2,5/2,1/2,-3/2,-7/2]]		
	μ_{lowest}=[[8, 5, 2, -1, -4], [6, 3, 0, -3, -6]]	1	51.910 sec.
	μ=[[106,4,2,0,-102],[104,2,0,-2,-104]]	1458704380546472381	163.104 sec.

Example 46

We consider the discrete series representation indexed by λ, a direction $\vec{v}$ and we test for the multiplicity $m^\lambda_{\mu+t\vec{v}}$ where $\mu = \mu_{lowest}$ is the lowest K-type. We use the algorithm whose command is
> **function_discrete_mu_direction_lowest_**$(\lambda,\vec{v},$**p,q**$)$
For completeness we list μ_{lowest} relative to each example.

$m^\lambda_{\mu+t\vec{v}}$, $t\in\mathbb{N}$: **asymptotic case**				
Group	**Input**	$m^\lambda_{\mu+t\vec{v}}$	**Output**	**Time**
U(2,3)	λ=[[9, 7], [-1, -2, -13]]			
	μ_{lowest}=[[21/2, 17/2], [-2, -3, -14]]			
	$\vec{v}$=[[1, 0], [-1, 0, 0]]	$m^\lambda_{\mu+t\vec{v}} = \begin{cases} 1 & \text{if } t=0 \\ 0 & \text{if } t\geq 1 \end{cases}$	A1	0.054 sec.
	$\vec{v}$=[[6, 1], [-1, -1, -5]]	$m^\lambda_{\mu+t\vec{v}} = \begin{cases} 1 & \text{if } t\leq 10 \\ 0 & \text{if } t\geq 11 \end{cases}$	A2	0.736 sec.
	$\vec{v}$=[[1, 0], [0, 0, -1]]	$m^\lambda_{\mu+t\vec{v}} = 1 \text{ if } t\geq 0$	A3	0.26 sec.
U(2,3)	λ=[[59, 39], [51, 7, -156]]			
	μ_{lowest}=[[121/2, 79/2], [51, 6, -157]]			
	$\vec{v}$=[[1,0], [0,0,-1]]	$m^\lambda_{\mu+t\vec{v}} = t+1$	A4	0.71 sec.
U(2,4)	λ=[[341/2, 49/2], [-3/2, -5/2, -11/2, -371/2]]			
	μ_{lowest}=[[345/2, 53/2], [-5/2, -7/2, -13/2, -373/2]]			
	$\vec{v}$=[[6, 1], [-1, -1, -1, -4]]	$m^\lambda_{\mu+t\vec{v}} = \begin{cases} 1 & \text{if } t\leq 2 \\ 0 & \text{if } t\geq 3 \end{cases}$	A5	46.754 sec.
U(3,3)	λ=[[343/2, 31/2, 21/2], [-13/2,-19/2,-363/2]]			
	μ_{lowest}=[[173, 17, 12], [-8, -11, -183]]			
	$\vec{v}$=[[6, 1, 0], [-1, -1, -5]]	$m^\lambda_{\mu+t\vec{v}} = \begin{cases} 1 & \text{if } t=0 \\ 3 & \text{if } t=1 \\ 6 & \text{if } 2\leq t\leq 171 \\ 3 & \text{if } t=172 \\ 1 & \text{if } t=173 \\ 0 & \text{if } t\geq 174 \end{cases}$	A6	52.020 sec.
	$\vec{v}$=[[1, 1, 0], [0,-1, -1]]	$m^\lambda_{\mu+t\vec{v}} = \begin{cases} t+1 & \text{if } 0\leq t\leq 155 \\ 156 & \text{if } t\geq 156 \end{cases}$	A7	7.1730 sec.

where

$A1 := [[-(1/2)*t^2+(1/2)*t+1,[0,0]],[1+(1/2)*t^2-(3/2)*t,[1,1]],[0,[2,inf]]]$
$A2 := [[-(15/2)*t^2+(7/2)*t+1,[0,0]],[1,[1,10]],[66+(1/2)*t^2-(23/2)*t,[11,11]],[0,[12,inf]]]$
$A3 := [[-(1/2)*t^2+(1/2)*t+1,[0,0]],[1,[1,inf]]]$
$A4 := [[t+1,[0,inf]]$ $A5 := [1+(10/3)*t-(7/2)*t^2+(25/6)*t^3,[0,0]],[1+(1/3)*t-(1/2)*t^2+(1/6)*t^3,[1,1]],[1,[1,2]],$ $[6-(7/2)*t+(1/2)*t^2,[2,3]],[0,[3,inf]]$
$A6 := [1+(13/4)*t-(51/8)*t^2+(61/4)*t^3-(89/8)*t^4,[0,0]],[1+(3/2)*t+(1/2)*t^2,[1,1]],$ $[-3+(27/4)*t-(1/8)*t^2-(3/4)*t^3+(1/8)*t^4,[2,3]],[6,[4,170]],$ $[32664996-(9380059/12)*t+(168227/24)*t^2-(335/12)*t^3+(1/24)*t^4,[171,172]],$ $[15225-(349/2)*t+(1/2)*t^2,[172,173]]$ $[78155000-(10718575/6)*t+(183749/12)*t^2-(175/3)*t^3+(1/12)*t^4,[174,175]],[0,[176,inf]]$
$A7 := [1+(5/4)*t+(1/24)*t^2-(1/4)*t^3-(1/24)*t^4,[0,0]],[t+1,[1,154]],$ $[23726781-(2457885/4)*t+(143207/24)*t^2-(103/4)*t^3+(1/24)*t^4,[155,156]],[156,[156,inf]]$

Remark that in some of the examples above, it can happen that although the polynomials P_i and P_{i+1} (displayed in the last table giving the A_i) are different, the polynomial P_{i+1} may coincide with P_i on I_i (recall that several polynomials can have the same values on $I_i \cap \mathbb{N}$). Thus in this case, we join the two intervals I_i and I_{i+1} and give only the polynomial P_{i+1}. This streamlining of the function $m^\lambda_{\mu+t\vec{v}}$ is given in the third column of the table describing the asymptotic behavior of $m^\lambda_{\mu+t\vec{v}}$.

We now give an example on $U(3,4)$.

Example 47

```
discrete:=[[473, 39, 1], [3, 51, 5, -572]];
direction:=,[[1, 0, 0], [0, 0, 0, -1]];

>function_discrete_mul_direction_lowest(discrete,direction,3,4);

[1+51/20 t-1/120 t^5-1/360 t^6+851/360 t^2+23/24 t^3+5/36 t^4,[0,0]],
[1+31/12 t+19/8 t^2+11/12 t^3+1/8 t^4,[1,37]],
[-3262622+2687514/5 t+73/240 t^5-1/720 t^6-13275857/360 t^2+64795/48 t^3-3977/144 t^4,[38,39]],
[-265030+27790 t-1090 t^2+20 t^3,[39,44]],
[-9631849+79305707/60 t+29/80 t^5-1/720 t^6-3399664/45 t^2+110609/48 t^3-5675/144 t^4,[45,45]],
[27182687-212385511/60 t-31/40 t^5+1/360 t^6+69073219/360 t^2-132929/24 t^3+1619/18 t^4,[46,46]],
[-784945+886169/12 t-20959/8 t^2+511/12 t^3-1/8 t^4,[47,83]],
[469370132-337238937/10 t-167/240 t^5+1/720 t^6+363465857/360 t^2-774005/48 t^3+20897/144 t^4,[84,85]],
[5790400-235000 t+2820 t^2,[86,inf]]
```

Thus the multiplicity $m^{\lambda}_{\mu+t\vec{v}}$ can be completely described by the following piecewise polynomial function:

$$m^{\lambda}_{\mu+t\vec{v}} = \begin{cases} 1+(31/12)*t+(1/8)*t^4+(11/12)*t^3+(19/8)*t^2 & \text{if} \quad 0 \leq t \leq 39 \\ -265030+27790*t+20*t^3-1090*t^2 & \text{if} \quad 40 \leq t \leq 46 \\ -784945+(886169/12)*t-(1/8)*t^4+(511/12)*t^3-(20959/8)*t^2 & \text{if} \quad 47 \leq t \leq 85 \\ 5790400-235000*t+2820*t^2 & \text{if} \quad 86 \leq t. \end{cases}$$

The time to compute the example is $TT := 19.487$ and the formula says for instance that, for $\lambda =$discrete and $\mu = [[475, 40, 0], [103/2, 9/2, 5/2, -1147/2]]$ the lowest K-type, then

$$m^{\lambda}_{\mu+20000000\vec{v}} = 1127995300005790400.$$

6 The program: "Discrete series and K multiplicities for type A_r"

We give a brief sketch of the main steps for the algorithms involved in Blattner's formula.

6.1 $MNPS$ non compact

We outline the algorithm that computes directly $\overrightarrow{M}$ for $M \in \mathcal{P}(v, \Delta^+(A,B))$. We are taking advantage of the fact that we know the $\overrightarrow{M}'s$ in the case of $|A| = 1,$ r, as we saw in Ex. 44. In the following scheme p, q are integers, $A \subset [1, 2, \ldots, p+q]$ is a set of cardinality p, B is the complement subset defining $U(A,B)$, θ_I is the highest noncompact root for I. If $L \subset [1, 2, \ldots, p+q]$ we denote by L' the complement set.

Input $[v, A, I]$, v a vector and $A \subset I = [1, 2, \ldots, p+q]$, $|A| = p$
proceed by induction on the cardinality of A.
if $|A| = 1$ or $|A| = r$ write the unique $\overrightarrow{M}$, $M \in MNPS$ determined by the situation
if $v \in C(\overrightarrow{M})$ then the output is $\overrightarrow{M}$,
construct the hyperplane H_L.
check if v and θ_I are on the same side then H_L
if not skip the hyperplane
define the projection $v' = \mathrm{proj}_{H_L}(v)$ of v on H_L along θ_I
compute $[v_1, A_1, I_1], [v_2, A_2, I_2]$
where $A_1 = L \cap A$, $A_2 = L' \cap A$, $I_1 = L$, $I_2 = L'$ and
v_1, v_2 are the components of v' on L and L' respectively.
if v_1(resp. v_2) is not in the positive cone for $\Delta(I_1)$ then skip the hyperplane
if $|A_1| = 1$, apply the induction and compute $\overrightarrow{M_1}$, $M_1 \in MNPS(v_1, A_1, I_1)$,
add to M_1 the root θ_{I_1} (do the same if $|A_2| = 1$)
else apply the induction and compute $\overrightarrow{M_i}$, $M_i \in \mathcal{P}(v_i, A_i, I_i), i = 1, 2$
do the cartesian product $\overrightarrow{M_1} \times \overrightarrow{M_2}$ and add to each set the root θ_I
collect all $\overrightarrow{M}'s$, for the wall L
end of loop running across L's
end induction
return the set of all $\overrightarrow{M}'s$, $M \in MPNS$ for all hyperplanes

Figure 7: $\mathcal{P}(v, \Delta^+(A, B))$

6.2 Numeric

The scheme is described in Fig.8.

6.3 Asymptotic directions

We fix the parameter λ_0 and μ_0, regular in the chambers $\mathfrak{a}, \mathfrak{a}_c$ and a weight $\vec{v}$. We want to compute $m^{\lambda_0}_{\mu+t\vec{v}}$. In the application μ_0 will be the lowest K-type. The scheme is described in Fig.9.

Subroutines:

- Procedure to find $\mathcal{A}^+$-admissible hyperplanes.
- Procedure to deform a vector: $DefVec_nc(v, \mathcal{A}^+)$
- Procedure to compute $\overrightarrow{M}$, $M \in \mathcal{P}(v, \mathcal{A}^+)$ as in Fig.7.
- Procedure to compute Kostant function $K(h) = K(0,h) = \frac{e^h}{\prod_{\alpha\in\Delta_n^+}(1-e^{-\alpha})}$ or more generally $K(g,h)$.
- Compute the valid permutation $Valid(u, v) \subset \mathcal{W}_c$

```
Input: λ, μ
Compute A⁺ = Δ_n⁺(λ):
Compute Valid(λ, μ)
   for each w ∈ Valid(λ, μ),
   compute μ_reg = DefVec_nc(v, A⁺)
   compute All_w := P(wμ_reg − λ, Δ_n⁺)
   compute cont_w = Σ_{M⃗ ⊂ All_w} Ires_{M⃗} K(wμ − λ − ρ_n)
   end of loop running across w's
collect all the terms and return
m^λ_μ = Σ_{w ∈ Valid(λ,μ)} ε(w) cont_w
```

Figure 8: Blattner's algorithm (numeric case)

Subroutines:

- Procedure to find $\mathcal{A}^+$-admissible hyperplanes.
- Procedure to deform a vector: $DefVec_{nc}(v, \mathcal{A}^+)$
- Procedure to compute $\overrightarrow{M}$, $M \in |CP(v, \mathcal{A}^+)$ as in Fig.7.
- Procedure to compute Kostant function $K(h) = K(0,h) = \frac{e^h}{\prod_{\alpha\in\Delta_n^+}(1-e^{-\alpha})}$ or more generally $K(g,h)$.

```
Input λ_0 and v⃗    for each H noncompact wall
      if (H, v⃗) = 0 then skip H
      else if (H, λ_0 − wμ_0)(H, v⃗) < 0 then skip H else
      collect t_H = (H, λ_0 − wμ_0)/(H, v⃗)
   end of loop running across H's
   order list t'_H s as [t_0, t_1, ..., t_s] where t_0 = 0, t_s = ∞
   choose an interior point t̄_i in each interval [t_i, t_{i+1}]
   Compute polynomial on each [t_i, t_{i+1}] following the scheme Fig.8 and the ordered basis determined by t̄_i
output: the sequence of values m^{λ_0}_{μ_0+t v⃗} valid on [t_i, t_{i+1}], t ∈ ℕ
```

Figure 9: Blattner's algorithm: asymptotic case

.

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