

# EULER-MACLAURIN FORMULA FOR THE MULTIPLICITIES OF THE INDEX OF TRANSVERSALLY ELLIPTIC OPERATORS

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ABSTRACT. Let  $G$  be a connected compact Lie group acting on a manifold  $M$  and let  $D$  be a transversally elliptic operator on  $M$ . The multiplicity of the index of  $D$  is a function on the set  $\hat{G}$  of irreducible representations of  $G$ . Let  $T$  be a maximal torus of  $G$  with Lie algebra  $\mathfrak{t}$ . We construct a finite number of piecewise polynomial functions on  $\mathfrak{t}^*$ , and give a formula for the multiplicity in term of these functions.

## 1. INTRODUCTION

Let  $G$  be a compact connected Lie group acting on a manifold  $M$ . Atiyah-Singer [1] have associated to any  $G$ -transversally elliptic pseudo-differential operator  $D$  on  $M$  a virtual trace class representation of  $G$  :

$$\text{Index}(D) = \sum_{\lambda \in \hat{G}} \text{mult}(D)(\lambda) V_{\lambda}.$$

If  $D$  is elliptic, the preceding sum of irreducible representations  $V_{\lambda}$  of  $G$  is finite. As shown in [1], the computation of the function  $\text{mult}(D) : \hat{G} \rightarrow \mathbb{Z}$  can be reduced to the case where  $G$  is a torus.

Thus we assume that  $G$  is a torus with Lie algebra  $\mathfrak{g}$ , and parameterize the irreducible representations of  $G$  by the lattice  $\Lambda \subset \mathfrak{g}^*$  of characters of  $G$ . Assume, in this introduction, that the stabilizer of any point of  $M$  is connected. We construct a particular ‘‘spline’’ function  $m$  on  $\mathfrak{g}^*$ , extending the function  $\text{mult}(D)$  in a piecewise polynomial function on  $\mathfrak{g}^*$ . The construction of the function  $m$  is based on the notion of infinitesimal index [8] and is canonically associated to the equivariant Chern character  $\text{ch}(\sigma)$  of the principal symbol  $\sigma$  of  $D$ .

A trivial example is when  $G$  acts on  $G = M$  by left translations and  $D$  is the operator 0, with index

$$L^2(G) = \sum_{\lambda \in \Lambda} e^{i\lambda}.$$

Then the function  $\text{mult}(D)$  is identically equal to 1 on  $\Lambda$ , and extended by the constant function  $m = 1$ .

The construction of  $m$  was the object of the article [7], in the case where  $M$  is a vector space with a linear action of  $G$ . In this note, we just state the results. Our proofs are very similar to those of [7]. We use results of [1] (see

also [9]) on generators of the equivariant  $K$ -theory. Then our main tool is (as in [7]) the Dahmen-Micchelli inversion formula for convolution with the box spline, a “Riemann-Roch formula” for approximation theory ([5], see [7]).

Let us briefly explain the origin of our construction. Denote by  $\text{Index}(D)(g)$  the corresponding generalized function.

$$\text{Index}(D)(g) = \sum_{\lambda \in \hat{G}} \text{mult}(D)(\lambda) \text{Tr}_{V_\lambda}(g)$$

on  $G$ . Recall the formula

$$(1) \quad \text{Index}(D)(\exp X) = \frac{1}{(2i\pi)^{\dim M}} \int_{T^*M} \frac{\text{ch}(\sigma)(X)}{J(M)(X)},$$

obtained in ([4], [3]), a “delocalized” version of Atiyah-Bott-Segal-Singer equivariant index formula. Here the inverse of  $J(M)$  is the equivariant Todd class of  $T^*M$  (considered as an almost complex manifold). The formula above is valid when  $X$  varies in a neighborhood of 0 in the Lie algebra  $\mathfrak{g}$  of  $G$  as an equality of generalized functions of  $X$ .

Here we replace the equivariant class  $J(M)$  by its formal series of homogeneous components. By Fourier transform, we obtain a series of generalized functions on  $\mathfrak{g}^*$ . In a limit sense explained in this note, the restriction of this series to  $\Lambda$  coincides with the function  $\lambda \rightarrow \text{mult}(D)(\lambda)$ .

In a subsequent note, we will give a geometric interpretation of the piecewise polynomial function  $m$ , when  $D$  is a Dirac operator twisted by a line bundle  $L$  in term of the moment map associated to an equivariant connection on  $L$ .

I thank Michel Duflo, and Corrado de Concini for comments on this text.

## 2. TRANSVERSALLY ELLIPTIC SYMBOLS

Let  $G$  be a compact Lie group. Let  $M$  be a  $G$ -manifold, and  $T^*M$  its cotangent bundle. We denote by  $p : T^*M \rightarrow M$  the projection.

Let  $\omega$  be the Liouville one form: for  $x \in M$ ,  $\xi \in T_x^*M$  and  $V$  a tangent vector at the point  $(x, \xi) \in T^*M$ ,  $\omega_{x, \xi}(V) = \langle \xi, p_*V \rangle$ . By definition, the symplectic form  $\Omega = -d\omega$  is the symplectic form of  $T^*M$ , and we use the corresponding orientation of  $T^*M$  to compute integrals on  $T^*M$  of differential forms with compact support.

Let  $\mathcal{E}^\pm$  be two  $G$ -equivariant complex vector bundles over  $M$ . A  $G$ -equivariant bundle map  $\sigma : p^*\mathcal{E}^+ \rightarrow p^*\mathcal{E}^-$  will be called a symbol. The support  $\text{supp}(\sigma) \subset T^*M$  of  $\sigma$  is the set of elements  $n \in T^*M$  such that the linear map  $\sigma(n) : \mathcal{E}_{p(n)}^+ \rightarrow \mathcal{E}_{p(n)}^-$  is not invertible. An elliptic symbol is a symbol such that  $\text{supp}(\sigma)$  is compact.

Denote by  $T_G^*M \subset T^*M$  the union of the space of covectors conormal to the  $G$  orbit through  $x$ . If  $X \in \mathfrak{g}$ , we denote by  $v_X$  the vector field on  $N$  generated by  $-X \in \mathfrak{g}$ . Let  $\mu : T^*M \rightarrow \mathfrak{g}^*$  be the moment map

$\langle \mu(x, \xi), X \rangle = -\langle \xi, v_X \rangle$ . Then the zero fiber  $Z = \mu^{-1}(0)$  of the moment map  $\mu$  is the space  $T_G^*M$ . If the support of the symbol  $\sigma$  intersects  $Z$  in a compact set, we say that  $\sigma$  is a transversally elliptic symbol (it is elliptic in the directions transverse to the  $G$ -orbits). Then  $\sigma$  determines an element in the topological equivariant  $K$  group  $K_G^0(Z)$ , still denoted by  $\sigma$ .

If  $D$  is a pseudo-differential operator with principal symbol  $\sigma$ , we say that  $D$  is transversally elliptic if its principal symbol is transversally elliptic. The index of  $D$  depends only of the class of  $\sigma \in K_G^0(Z)$ .

Atiyah-Singer [1] have associated to any transversally elliptic symbol  $\sigma$  a virtual trace class representation of  $G$  :

$$\text{Index}(\sigma) = \sum_{\lambda \in \hat{G}} \text{mult}(\sigma)(\lambda) V_\lambda.$$

If  $\sigma$  is elliptic, the preceding sum of irreducible representations  $V_\lambda$  of  $G$  is finite. By construction, if  $D$  is a pseudo-differential operator with principal symbol  $\sigma$ , then  $\text{Index}(D) = \text{Index}(\sigma)$  (and  $\text{mult}(D) = \text{mult}(\sigma)$ ).

Let us give examples of such operators.

**Example 2.1.** Elliptic operators are transversally elliptic. If  $M$  is an even dimensional compact manifold and is oriented, then, up to stable homotopy, any elliptic symbol is the principal symbol of a twisted Dirac operator.

**Example 2.2.** Let  $M = G$  and let  $H \subset G$  be a compact connected subgroup of  $G$ . Then the 0 operator is transversally elliptic with respect to the action of  $G \times H$ , acting by left and right multiplication, and  $\text{Index}(0) = \sum_{\lambda \in \hat{G}, \mu \in \hat{H}} m(\lambda, \mu) V_\lambda^G \otimes V_\mu^H$ , where  $m(\lambda, \mu)$  is the multiplicity of the irreducible representation  $V_\mu^H$  of  $H$  in the irreducible representation  $(V_\lambda^G)^*$  of  $G$ .

**Example 2.3.** Let  $M$  be a compact manifold and let  $G$  be a group acting on  $M$ , and such that  $G$  acts infinitesimally freely on  $M$ . Then  $M/G$  is an orbifold. Let  $\sigma$  be a  $G$ -transversally elliptic symbol on  $M$ . Denote by  $\lambda_0$  the trivial representation of  $G$ . If  $D$  is a  $G$ -invariant operator with principal symbol  $\sigma$ ,  $\text{mult}(\sigma)(\lambda_0)$  is the index of the elliptic operator associated to  $D$  on the orbifold  $M/G$ .

**Example 2.4.** Let  $V$  be a Hermitian vector space and  $G$  a subgroup of  $U(V)$ . Let  $c$  be the Clifford representation of  $V^*$  in the superspace  $E = \Lambda V^*$ .

Consider the moment map  $\Phi : V \rightarrow \mathfrak{g}^*$ . Assume  $\Phi^{-1}(0) = \{0\}$ . Identify the Kirwan vector field associated to the moment map  $\Phi$  to a  $G$ -invariant one form  $\lambda$  on  $V$ . Then the pushed symbol  $\sigma(v, \xi) = c(\xi + \lambda_v)$  is a transversally elliptic symbol on  $V$ . The function  $\text{mult}(\sigma)(\lambda)$  is the multiplicity of the representation  $V_\lambda$  of  $G$  in the space  $S(V^*)$  of polynomial functions on  $V$  (see [6]). In particular, if  $G$  is a torus, then  $\lambda \rightarrow \text{mult}(\sigma)(\lambda)$  is the vector partition function associated to the list of weights of  $G$  in  $V$ .

## 3. DEFINITIONS

**3.1. Equivariant cohomology.** Let  $G$  be a compact Lie group, with Lie algebra  $\mathfrak{g}$ , acting on a manifold  $N$ . Let  $\mathcal{A}(N)$  be the space of differential forms on  $N$ . If  $X \in \mathfrak{g}$ , we denote by  $\iota_X$  the contraction of a differential form on  $N$  by  $v_X$ . Let  $D = d - \iota(v_X)$  be the equivariant differential. In the Cartan model, a representative of an element of the equivariant cohomology  $H_G^*(N)$  is an equivariant form  $\alpha : \mathfrak{g} \rightarrow \mathcal{A}(N)$  such that  $D(\alpha) = 0$ . The dependance of  $\alpha$  in  $X \in \mathfrak{g}$  is polynomial. Then  $H_G^*(N)$  is a  $\mathbb{Z}$ -graded algebra.

Let  $q$  be a formal variable. If  $F$  is a vector space and  $f([q]) = \sum_{k=0}^{\infty} q^k f_k$  is a formal series of elements of  $F$ , we write  $f \in F[[q]]$ . If  $f(q, x)$  is a smooth function of  $q$ , defined near  $q = 0$ , and depending of some parameters  $x$ , we denote by  $f([q], x) = \sum_{k=0}^{\infty} q^k f_k(x)$  its Taylor series at  $q = 0$ , a formal series of functions of  $x$ . If the series  $f([q])$  is finite, we write  $f([1])$  or  $f([q])_{q=1}$  for the sum  $\sum_k f_k$ .

We introduce formal series

$$\alpha([q], X) = \sum_{k=0}^{\infty} q^k \alpha_k(X)$$

of equivariant forms on  $N$ . If the constant term  $\beta_0$  of  $\beta([q], X) = \sum_{k=0}^{\infty} q^k \beta_k(X)$  is a non zero constant, we can define the formal series  $\alpha([q], X)/\beta([q], X)$ . If  $N$  is oriented, we can integrate such series against any equivariant form  $c(X)$  with compact support, and obtain a formal power series  $\int_N c(X) \alpha([q], X) = \sum_{k=0}^{\infty} q^k \int_N c(X) \alpha_k(X)$  of functions of  $X \in \mathfrak{g}$ .

If  $Z$  is a  $G$ -invariant closed subset of  $N$ , we have defined in [8] a Cartan model for the space  $H_{G,c}^{\infty}(Z)$  of equivariant cohomology with compact supports. A representative is an equivariant form  $\alpha : \mathfrak{g} \rightarrow \mathcal{A}_c(N)$  such that  $D(\alpha) = 0$  in a neighborhood of  $Z$ . Here  $\mathcal{A}_c(N)$  is the space of differential forms with compact supports and the dependance of  $\alpha$  in  $X$  is  $C^{\infty}$ .

Let  $M$  be a  $G$ -manifold,  $T^*M$  its cotangent bundle and  $Z = T_G^*M$ . If  $\alpha$  is an equivariant cohomology class on  $M$ , its pull back  $p^*\alpha$  is an equivariant cohomology class on  $T^*M$ , that we denote simply by  $\alpha$ .

Let  $\sigma : p^*\mathcal{E}^+ \rightarrow p^*\mathcal{E}^-$  be a transversally elliptic symbol. Choosing  $G$ -invariant connections on  $p^*\mathcal{E}^{\pm}$ , coinciding via  $\sigma$  outside a small neighborhood of  $\text{supp}(\sigma)$ , we can construct the equivariant Chern character  $\text{ch}(\sigma) \in H_{G,c}^{\infty}(Z)$  of  $\sigma$ , an equivariant cohomology class, represented by a differential form with compact support on  $T^*M$ , still denoted by  $\text{ch}(\sigma)$ .

Let  $\mathcal{V} \rightarrow M$  be a real or complex vector bundle on  $M$  with typical fiber a vector space  $V$ . The Chern-Weil map  $W$  associates to an invariant polynomial  $f$  on  $\text{End}(V)$  an equivariant characteristic class  $W(f)$  in  $H_G^*(M)$ . If  $f$  is homogeneous of degree  $k$ , then  $W(f)$  is homogeneous of degree  $2k$ . Our conventions for the Chern-Weil homomorphism  $W$  and the Chern character are as in [2].

Let  $A \in \text{End}(V)$ . Introduce a variable  $q$ , and consider the Taylor expansion

$$\det_V\left(\frac{e^{qA} - 1}{qA}\right) = \sum_{k=0}^{\infty} q^k T_k(A) = 1 + \frac{q}{2} \text{Tr}(A) + \dots$$

Thus  $T_k$  is an invariant homogeneous polynomial of degree  $k$  on  $\text{End}(V)$ .

Our main new concept is the introduction of the following formal equivariant characteristic class of  $M$ .

**Definition 3.2.** The formal  $J$ -class of  $M$  is the series of elements of  $H_G^*(M)$  defined by

$$J([q], M) = \sum_{k=0}^{\infty} q^k W(T_k)$$

obtained by applying the Chern-Weil map for the real vector bundle  $\mathcal{V} = TM \rightarrow M$  to the series  $\det_V\left(\frac{e^{qA}-1}{qA}\right) = \sum_{k=0}^{\infty} q^k T_k(A)$ .

Here  $W(T_k)$  is homogeneous of degree  $2k$ .

When  $G = \{1\}$ , then  $p^*J([1], M) = J(M)$  is the inverse of the usual Todd class of the tangent bundle to  $T^*M$  (considered as an almost complex manifold). Furthermore, if  $\sigma$  is elliptic, Atiyah-Singer formula for  $\text{Index}(\sigma) \in \mathbb{Z}$  is

$$\text{Index}(\sigma) = \frac{1}{(2i\pi)^{\dim M}} \int_{T^*M} \frac{\text{ch}(\sigma)}{J(M)}.$$

**3.3. Piecewise polynomial generalized functions.** In this subsection,  $G$  is a torus.

Let  $V = \mathfrak{g}^*$  equipped with the lattice  $\Lambda \subset \mathfrak{g}^*$  of weights of  $G$ . If  $g = \exp X$ , we denote by  $g^\lambda = e^{i\langle \lambda, X \rangle}$ . The function  $g \mapsto g^\lambda$  is a character of  $G$ .

We denote by  $\mathcal{C}(\Lambda)$  the space of (complex valued) functions on  $\Lambda$ . If  $g \in G$ , we denote by  $\hat{g}$  the function  $g \rightarrow g^\lambda$  on  $\Lambda$ . If  $m \in \mathcal{C}(\Lambda)$ , then  $\hat{g}m \in \mathcal{C}(\Lambda)$  is defined by  $(\hat{g}m)(\lambda) = g^\lambda m(\lambda)$ .

Using the Lebesgue measure  $d\xi$  associated to  $\Lambda$ , we identify generalized functions on  $\mathfrak{g}^*$  and distributions on  $\mathfrak{g}^*$ . Let  $f$  be a test function on  $\mathfrak{g}^*$ , we define  $\hat{f}(X) = \int_{\mathfrak{g}^*} e^{i\langle \xi, X \rangle} f(\xi) d\xi$ . The Lebesgue measure  $dX$  on  $\mathfrak{g}$  is such that the Fourier inversion  $\int_{\mathfrak{g}} e^{-i\langle \xi, X \rangle} \hat{f}(X) dX = f(\xi)$  holds. If  $h$  is a generalized function on  $\mathfrak{g}^*$ , we denote by  $\int_{\mathfrak{g}^*} h(\xi) f(\xi) d\xi$  its value on the test function  $f$ . We denote by  $\delta_v$  the Dirac function at the point  $v \in V$ .

Let  $\mathcal{H}$  be a finite collection of rational hyperplanes in  $\mathfrak{g}^*$ . An element of  $\mathcal{H}$  will be called an admissible hyperplane. An element  $v \in V$  is called  $\mathcal{H}$ -generic if  $v$  is not on any hyperplane of the collection  $\mathcal{H}$ . We just say that  $v$  is generic.

An admissible wall is a translate of an hyperplane in  $\mathcal{H}$  by an element of  $\Lambda$ . A *tope*  $\tau$  is a connected component of the complement of all admissible walls and we denote by  $V_{reg}$  the union of topes.

A piecewise polynomial function is a function on  $V_{reg}$  which is given by a polynomial formula on each tope. We denote by  $PW$  the space of piecewise polynomial functions.

Consider  $f \in PW$  (defined on  $V_{reg}$ ) as a locally  $L^1$ -function on  $V$ , thus  $f$  defines a generalized function on  $V$ . An element of  $PW$ , considered as a generalized function on  $V$ , will be called a piecewise polynomial generalized function.

**Definition 3.4.** The space  $\mathcal{S}$  is the space of generalized functions on  $V$  generated by the action of constant coefficients differential operators on piecewise polynomial generalized functions.

For example, the Heaviside function on  $\mathbb{R}$  is a piecewise polynomial function. Its derivative in the sense of generalized functions is the Dirac function at 0.

A function in  $\mathcal{S}$  can be evaluated at any point of  $V_{reg}$ . If  $v \in V$ , and  $\epsilon$  is a generic vector, then  $v + t\epsilon$  is in  $V_{reg}$  if  $t > 0$  and sufficiently small.

**Definition 3.5.** Let  $v \in V$ , and  $f \in \mathcal{S}$ . Let  $\epsilon$  a generic vector. Define  $(\lim_{\epsilon} f)(v) = \lim_{t>0, t \rightarrow 0} f(v + t\epsilon)$ .

Remark that this definition depends only of the restriction of  $f$  to  $V_{reg}$ .

Introduce formal series  $m([q]) = \sum_{k=0}^{\infty} q^k m_k$  of generalized functions on  $V$ . Then if  $f$  is a test function

$$\int_{\mathfrak{g}^*} m([q])(\xi) f(\xi) d\xi = \sum_{k=0}^{\infty} q^k \int_{\mathfrak{g}^*} m_k(\xi) f(\xi) d\xi$$

is a formal power series in  $q$ . It may be evaluated at  $q = 1$  if the preceding series is finite (or convergent).

If  $\epsilon$  is generic, we define a map  $\lim_{\epsilon}^{\Lambda} : \mathcal{S}[[q]] \rightarrow \mathcal{C}(\Lambda)[[q]]$  by

$$(\lim_{\epsilon}^{\Lambda} m([q]))(\lambda) = \sum_{k=0}^{\infty} q^k (\lim_{\epsilon} m_k)(\lambda).$$

If all, but a finite number, the functions  $m_k$  are equal to 0 on  $V_{reg}$ , then  $\lim_{\epsilon}^{\Lambda} m([q])|_{q=1}$  is an element of  $\mathcal{C}(\Lambda)$ .

Formal series of distributions occur naturally in the context of Euler-MacLaurin formula.

**3.6. Fourier transforms of equivariant integrals.** In this section,  $G$  is a torus acting on a manifold  $M$ ,  $V = \mathfrak{g}^*$  and  $Z = T_G^*M$ . We assume that  $M$  admits a  $G$ -equivariant embedding in a vector space provided with a linear representation of  $G$ . For  $x \in M$ , denote by  $\mathfrak{g}_x \subset \mathfrak{g}$  the infinitesimal stabilizer of  $x \in M$ . We assume that the generic infinitesimal stabilizer for the action of  $G$  on  $M$  is equal to 0. We denote by  $\mathcal{I}^1$  the set of infinitesimal stabilizers of dimension 1. Let  $\mathcal{H}$  be the finite collection of hyperplanes  $\ell^{\perp}$  where the line  $\ell$  varies in  $\mathcal{I}^1$ . Let  $\mathcal{S}$  be the corresponding space of generalized functions on  $V = \mathfrak{g}^*$ .

Consider a transversally elliptic symbol  $\sigma$ . We then use the notion of infinitesimal index to perform the integration on  $T^*M$ .

If  $\alpha$  is a closed equivariant form with **polynomial** coefficients, and  $f$  a test function on  $\mathfrak{g}^*$ , the double integral

$$\int_{T^*M} \int_{\mathfrak{g}} e^{isD\omega(X)} \text{ch}(\sigma)(X) \alpha(X) \hat{f}(X) dX$$

is independent of  $s \in \mathbb{R}$ , for  $s$  positive and sufficiently large. We denote it by

$$\int_{T^*M}^{\omega} \int_{\mathfrak{g}} \text{ch}(\sigma)(X) \alpha(X) \hat{f}(X) dX.$$

**Definition 3.7.** If  $\sigma$  is a transversally elliptic symbol and  $\alpha$  is a closed equivariant form with **polynomial** coefficients, we define the generalized function  $m(\sigma, \alpha)$  on  $\mathfrak{g}^*$  so that

$$\int_{T^*M}^{\omega} \int_{\mathfrak{g}} \text{ch}(\sigma)(X) \alpha(X) \hat{f}(X) dX = \int_{\mathfrak{g}^*} m(\sigma, \alpha)(\xi) f(\xi) d\xi$$

for any test function  $f(\xi)$  on  $\mathfrak{g}^*$ .

Then  $m(\sigma, \alpha)$  depends only of the cohomology class of  $\alpha$  (still denoted by  $\alpha$ ).

**Proposition 3.8.** • *The generalized function  $m(\sigma, \alpha)$  belongs to  $\mathcal{S}$ .*

• *If  $\alpha$  is an homogeneous equivariant class of degree  $2k$ , then  $m(\sigma, \alpha)$  restricts to a polynomial of degree less or equal to  $\dim M - \dim G - k$  on each connected component of  $V_{reg}$ . In particular, when  $k$  is greater than  $\dim M - \dim G$ ,  $m(\sigma, \alpha)$  restricts to 0 on  $V_{reg}$ .*

#### 4. THE ELLIPTIC CASE

Let  $G$  be a torus acting on a connected manifold  $M$ . To explain the flavor of our formula, we assume that  $\sigma$  is an elliptic symbol and we make a further simplification. We assume that the stabilizer of any point of  $M$  is connected and that the generic stabilizer is trivial.

We write

$$\text{Index}(\sigma) = \sum_{\lambda \in \Lambda} \text{mult}(\sigma)(\lambda) e^{i\lambda}$$

with  $\text{mult}(\sigma) \in \mathcal{C}(\Lambda)$ .

To the elliptic symbol  $\sigma$ , we associate a series  $m([q], \sigma)$  of generalized functions on  $\mathfrak{g}^*$ .

**Definition 4.1.** Define  $m([q], \sigma) = \sum_{k=0}^{\infty} q^k m_k$  to be the series of generalized functions on  $\mathfrak{g}^*$  such that, for any test function  $f$  on  $\mathfrak{g}^*$ ,

$$(2) \quad (2i\pi)^{-\dim M} \int_{T^*M} \int_{\mathfrak{g}} \frac{\text{ch}(\sigma)(X)}{J([q], M)(X)} \hat{f}(X) dX = \int_{\mathfrak{g}^*} m([q], \sigma)(\xi) f(\xi) d\xi.$$

All distributions  $m_k$  are compactly supported, as we are in the elliptic case. The restriction of  $m_k$  to each connected component of  $V_{reg}$  is a polynomial of degree less or equal to  $\dim M - \dim G - k$ .

Compare with Formula (1) (for  $D$  with principal symbol  $\sigma$ ). The equivariant form  $\frac{\text{ch}(\sigma)(X)}{J([q], M)(X)}$  is for  $q = 1$  equal to  $\frac{\text{ch}(\sigma)(X)}{J(M)(X)}$ . The left hand side of the equality (2) cannot be evaluated for  $q = 1$ , as  $J(M)(X)$  is not invertible for  $X$  large. Here comes the miracle. The right hand side can be evaluated at  $q = 1$ , when restricted to  $\Lambda$  and we have the following theorem.

**Theorem 4.2.** *For any generic vector  $\epsilon$ ,*

$$\text{mult}(\sigma) = \lim_{\epsilon}^{\Lambda} m([q], \sigma)|_{q=1}.$$

When  $f$  is a polynomial,  $\hat{f}(X)$  is supported at 0, and the two formulae (1), (2) coincide at  $q = 1$ , thus we have the following Euler-MacLaurin formula.

**Theorem 4.3.** *For any polynomial function  $f$  on  $\mathfrak{g}^*$ ,  $\int_{\mathfrak{g}^*} m_k(\xi) f(\xi) d\xi$  is equal to 0 when  $k$  is sufficiently large, and*

$$\sum_{\lambda} \text{mult}(\sigma)(\lambda) f(\lambda) = \sum_{k=0}^{\infty} \int_{\mathfrak{g}^*} m_k(\xi) f(\xi) d\xi.$$

Let us give a simple example. Let  $M = P_1(\mathbb{C})$ , let  $A$  be a non negative integer and  $\bar{\partial}_A$  the  $\bar{\partial}$  operator on sections of  $\mathcal{L}^A$ , where  $\mathcal{L}$  is the dual of the tautological bundle. Let  $S^1$  be the circle group with generator  $J$ , acting on  $[x, y] \in P^1(\mathbb{C})$  by  $\exp(\theta J)[x, y] = [e^{i\theta}x, y]$ . Here  $[x, y]$  are the homogeneous coordinates on  $P_1(\mathbb{C})$ . Then

$$\text{Index}(\bar{\partial}_A)(\exp(\theta J)) = \sum_{d=0}^A e^{id\theta}.$$

We identify  $\mathfrak{g}^*$  with  $\mathbb{R}$  and  $\Lambda$  with  $\mathbb{Z}$ . The multiplicity function  $\text{mult}(\bar{\partial}_A)$  is such that  $\text{mult}(\bar{\partial}_A)(n) = 1$  if  $0 \leq n \leq A$ , otherwise is equal to 0.

Consider the function

$$F(q, \theta) = \frac{(1 - e^{-i\theta})}{(1 - e^{iq\theta})(1 - e^{-iq\theta})} + \frac{e^{iA\theta}(1 - e^{i\theta})}{(1 - e^{iq\theta})(1 - e^{-iq\theta})}.$$

The Taylor series  $F([q], \theta)$  is a series of analytic functions of  $\theta$  and it follows from the localisation formula on  $T^*P_1(\mathbb{C})$  that, for  $X = \theta J$ ,

$$(2i\pi)^{-\dim M} \int_{T^*M} \frac{\text{ch}(\sigma)(X)}{J([q], M)(X)} = F([q], \theta).$$

Let us write the first terms of the expansion of  $F([q], \theta)$ . This is

$$(e^{iA\theta} + 1 - e^{i(A+1)\theta} - e^{-i\theta}) \left( -\frac{1}{i^2\theta^2} + \frac{1}{12}q^2 - \frac{1}{240}i^2\theta^2q^4 + \frac{1}{6048}i^4\theta^4q^6 + \dots \right)$$



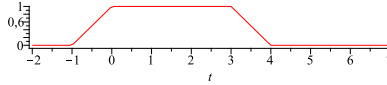


FIGURE 1.  $m_0(\xi)$  for  $A = 3$

The Fourier transform of  $F([q], \theta)$  is the series of generalized functions

$$m([q], \xi) = \sum_{k=0}^{\infty} q^k m_k(\xi).$$

We see that only  $m_0(\xi)$  is non zero on  $\mathbb{R} \setminus \mathbb{Z}$ , and given by the piecewise polynomial function

$$m_0(\xi) = \begin{cases} 0 & \xi \leq -1 \\ \xi + 1 & -1 \leq \xi \leq 0 \\ 1 & 0 \leq \xi \leq A \\ (A + 1) - \xi & A \leq \xi \leq A + 1 \\ 0 & (A + 1) \leq \xi \end{cases}$$

Remark that  $m_0(\xi)$  continuous and coincide with the function  $\text{mult}(\bar{\partial}_A)$  on  $\mathbb{Z}$ . This is the content of Theorem 4.2.

We can compute further terms of the expansion,

$$m([q], \xi) = m_0(\xi) + \frac{q^2}{12}(\delta_0 + \delta_A - \delta_{-1} - \delta_{A+1}) - \frac{q^4}{240}\left(\frac{d}{d\xi}\right)^2(\delta_0 + \delta_A - \delta_{-1} - \delta_{A+1}) + \dots$$

Then for any polynomial function  $f$ , we obtain a version of Euler-MacLaurin formula:

$$\sum_{d=0}^A f(d) = \int_{\mathbb{R}} f(\xi) m_0(\xi) d\xi$$

$$+\frac{1}{12}(f(0) + f(A) - f(-1) - f(A + 1)) - \frac{1}{240}(f''(0) + f''(A) - f''(-1) - f''(A + 1)) + \dots$$

This is the content of Theorem 4.3.

## 5. THE GENERAL FORMULA FOR A TORUS

Let  $G$  be a torus acting on  $M$ . The collection of hyperplanes  $\mathcal{H}$  in  $\mathfrak{g}^*$  is as in Subsection 3.6 and  $Z = T_G^*M$ .

Let  $\sigma \in p^*\mathcal{E}^+ \rightarrow p^*\mathcal{E}^-$  be a transversally elliptic symbol on  $M$ . We assume that the generic infinitesimal stabilizer for the action of  $G$  on  $M$  is equal to 0. We can always reduce the problem to this case.

For  $g \in G$ , denote by  $M^g = \{x \in M; gx = x\}$ . This is a manifold, which might not be connected. We say that  $g$  is a vertex if there exists  $x \in M^g$  with a finite stabilizer under the action of  $G$  (if stabilizers are all connected, then the only vertex is  $g = 1$ ). We denote by  $\mathcal{V}(M)$  the set of vertices of the action of  $G$  on  $M$ . This is a finite subset of  $G$ .

Let  $g \in G$ . Then  $g$  acts by a fiberwise transformation on  $\mathcal{E}^\pm \rightarrow M^g$  still denoted  $g$ . The morphism  $\sigma$  commutes with  $g$  over  $T^*M^g$ . The equivariant twisted Chern character  $\text{ch}^g(\sigma)$  is defined in [7] as an element of  $H_{G,c}^\infty(Z^g)$ .

If  $\alpha \in H_G^*(M^g)$ , we can define a generalized function  $m(g, \sigma, \alpha)$  on  $\mathfrak{g}^*$  by the formula

$$\int_{T^*M^g}^{\omega} \int_{\mathfrak{g}} \text{ch}^g(\sigma)(X) \alpha(X) \hat{f}(X) dX = \int_{\mathfrak{g}^*} m(g, \sigma, \alpha)(\xi) f(\xi) d\xi$$

for any test function  $f$  on  $\mathfrak{g}^*$ .

**Lemma 5.1.** *The generalized function  $m(g, \sigma, \alpha)$  belong to  $\mathcal{S}$ .*

*If  $g$  is not a vertex, the generalized function  $m(g, \sigma, \alpha)$  vanishes on  $V_{reg}$ .*

If  $E$  is a vector space and  $s \in \text{End}(E)$  is an invertible and semi-simple transformation of  $E$ , we denote by  $GL(s)$  the group of invertible linear transformation of  $E$  commuting with  $s$ . We consider for  $A \in \text{End}(E)$

$$D(q, s, A) = \det_E(1 - se^{qA}),$$

an analytic function of  $A \in \text{End}(E)$ . Write the Taylor series

$$D([q], s, A) = \sum_{k=0}^{\infty} q^k D_k^s(A).$$

Then  $A \rightarrow D_k^s(A)$  are homogeneous polynomials of degree  $k$ , invariant under  $GL(s)$ .

Consider the normal bundle  $\mathcal{N} \rightarrow M^g$ . Thus  $g$  produces an invertible linear transformation of  $\mathcal{N}_x$  at any  $x \in M^g$ . The Chern Weil homomorphism for the bundle  $\mathcal{N}$  (with structure group  $GL(g)$ ) produces a series  $D([q], g, M/M^g) := \sum_{k=0}^{\infty} q^k W(D_k^g)$  of closed equivariant forms on  $M^g$ . The coefficient in  $q^0$  of this series is just the function  $x \rightarrow \det_{\mathcal{N}_x}(1-g)$ , a function which is a non zero constant on each connected component of  $M^g$ . Similarly the function  $\dim M^g$  is constant on each connected component of  $M^g$ .

Recall that we have defined the class of  $J([q], M^g)$  on  $M^g$ .

**Definition 5.2.** Define the series of generalized functions  $m([q], g, \sigma)$  of generalized functions on  $\mathfrak{g}^*$  such that

$$\begin{aligned} \int_{T^*M^g}^\omega \int_{\mathfrak{g}} (2i\pi)^{-\dim M^g} \frac{\text{ch}^g(\sigma)(X)}{J([q], M^g)(X)D([q], g, M/M^g)(X)} \hat{f}(X) dX \\ = \int_{\mathfrak{g}^*} m([q], g, \sigma)(\xi) f(\xi) d\xi \end{aligned}$$

for any test function  $f(\xi)$  on  $\mathfrak{g}^*$ .

Here is the multiplicity formula for the index of a transversally elliptic symbol  $\sigma$ .

**Theorem 5.3.** *For any  $\epsilon$  generic*

$$\text{mult}(\sigma) = \sum_{g \in \mathcal{V}(M)} \hat{g} \lim_{\epsilon}^{\Lambda} m([q], g^{-1}, \sigma)|_{q=1}.$$

## 6. COMPACT GROUPS

Let  $G$  be a compact simply connected Lie group acting on a manifold  $M$  and let  $Z = T_G^*M$ . Let  $T$  be a maximal torus of  $G$ . The action of  $T$  on  $M$  defines a set of hyperplanes  $\mathcal{H}$  in  $\mathfrak{t}^*$  and a space  $\mathcal{S}$  of distributions on  $\mathfrak{t}^*$ . We denote by  $\mathcal{V}(M) \subset T$  the set of vertices for the action of  $T$  on  $M$ .

We parameterize the set of irreducible representations of  $G$  as follows. We consider  $\Lambda \subset \mathfrak{t}^*$  to be the set of weights of  $T$ . We choose a system of positive roots  $\Delta^+ \subset \mathfrak{t}^*$ . Let  $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ , and  $\mathfrak{t}_{>0}^*$  be the positive Weyl chamber. For  $\lambda \in \Lambda$ , regular and dominant, we denote by  $V_\lambda$  the irreducible representation of  $G$  of highest weight  $\lambda - \rho$ .

Let  $\sigma \in K_G^0(Z)$ . We write

$$\text{Index}(\sigma) = \sum_{\lambda \in \Lambda \cap \mathfrak{t}_{>0}^*} \text{mult}(\sigma)(\lambda) V_\lambda$$

and we extend the function  $\text{mult}(\sigma)$  in an antiinvariant (under the action of the Weyl group  $W$ ) function  $\tilde{\text{mult}}(\sigma)$  on  $\Lambda$ . The function  $\tilde{\text{mult}}(\sigma)$  is the multiplicity index of a  $T$ -transversally elliptic symbol  $\tilde{\sigma}$  on  $M$ , constructed in [1]. (If  $\sigma$  is itself  $T$  transversally elliptic, denote by  $\sigma^T$  the symbol  $\sigma$  considered as a  $T$ -transversally elliptic symbol. Then  $\tilde{\sigma}$  is the symbol  $\sigma^T$  twisted by the representation of  $T$  in the spinor superspace of  $\mathfrak{g}/\mathfrak{t}$ , and then  $\tilde{\text{mult}}(\sigma) = \sum_{w \in W} \epsilon(w) \text{mult}(\sigma^T)(\lambda + w\rho)$ .)

The multiplicity of  $\tilde{\sigma}$  is an anti-invariant function on  $\Lambda$ . Thus we construct from  $\tilde{\sigma}$  the series  $m([q], g, \tilde{\sigma})$  of anti-invariant generalized functions on  $\mathfrak{t}^*$ , belonging to the space  $\mathcal{S}$ .

**Theorem 6.1.** • For any  $\epsilon$  generic,

$$\tilde{\text{mult}}(\sigma) = \sum_{g \in \mathcal{V}(M)} \hat{g} \lim_{\epsilon}^{\Lambda} (m([q], g, \tilde{\sigma}))|_{q=1}.$$

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