MULTIPLE BERNOULLI SERIES AND VOLUMES OF MODULI SPACES OF FLAT BUNDLES OVER SURFACES.

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Abstract. Using Szenes formula for multiple Bernoulli series, we explain how to compute Witten series associated to classical Lie algebras. Particular instances of these series compute volumes of moduli spaces of flat bundles over surfaces, and also multiple zeta values.

Contents

Introduction

Consider a sequence of vectors Φ lying in a lattice Λ of a vector space V. We denote the dual of Λ by Γ, and let $\Gamma_{\mathrm{reg}}(\Phi) = \{\gamma \in \Gamma | \ \langle \phi, \gamma\rangle \neq 0, \text{ for all } \phi \in \Phi\}$.

In this computational paper, we compute

$$\mathcal{B}(\Phi,\Lambda)(v) = \sum_{\gamma\in\Gamma_{\mathrm{reg}}(\Phi)} \frac{e^{\langle 2i\pi v,\gamma\rangle}}{\prod_{\phi\in\Phi}\langle 2i\pi\phi,\gamma\rangle}, \tag{0.0.1}$$

a function on the torus V/Λ. This sum, if not absolutely convergent, has a meaning as a generalized function. If Φ generates V, then $\mathcal{B}(\Phi,\Lambda)$ is *piecewise polynomial.*

For example, for $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ with standard lattice $\Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2$, if we choose $\Phi = [e_1, e_1, e_2, e_1 + e_2, e_1 - e_2]$, then

$$\mathcal{B}(\Phi,\Lambda)(v_1e_1+v_2e_2) = \sum_{n_1,n_2}{}^{'} \frac{e^{2i\pi(v_1n_1+v_2n_2)}}{(2i\pi n_1)^2(2i\pi n_2)(2i\pi(n_1+n_2))(2i\pi(n_1-n_2))},$$

where the summation $\sum'$ means that we sum only over the integers n_1 and n_2 such that $n_1n_2(n_1+n_2)(n_1-n_2) \neq 0$. The formula for $\mathcal{B}(\Phi,\Lambda)(v_1e_1+v_2e_2)$ as a piecewise polynomial function of v_1 and v_2 (of degree 5) is given in Section 2, Equation (2.5.4).

We call $\mathcal{B}(\Phi,\Lambda)$ the *multiple Bernoulli series* associated to Φ and Λ. Multiple Bernoulli series have been extensively studied by A. Szenes ([12],[13]). They are natural generalizations of Bernoulli series: for $V = \mathbb{R}\omega$, $\Lambda = \mathbb{Z}\omega$ and $\Phi_k = [\omega, \omega, \ldots, \omega]$, where ω is repeated k times with $k > 0$, the function

$$\mathcal{B}(\Phi_k,\Lambda)(t\omega) = \sum_{n\neq 0,\ n\in\mathbb{Z}} \frac{e^{2i\pi nt}}{(2i\pi n)^k}$$

is equal to $-\frac{1}{k!}B(k,\{t\})$ where $B(k,t)$ denotes the k^{th} Bernoulli polynomial in variable t, and $\{t\} = t - [t]$ is the fractional part of t. If $k = 2g$ and $t = 0$, due to the symmetry $n \to -n$,

$$\mathcal{B}(\Phi_{2g},\Lambda)(0) = 2\frac{1}{(2i\pi)^{2g}}\zeta(2g).$$

From the residue theorem in one variable, for $k > 0$,

$$\sum_{n\neq 0,\ n\in\mathbb{Z}} \frac{e^{2i\pi nt}}{(2i\pi n)^k} = \mathrm{Res}_{z=0}\left(\frac{1}{z^k}e^{zt}\frac{1}{1-e^z}\right).$$

Szenes multidimensional residue formula (see Theorem 1.33) is the generalization of this formula to higher dimension, and it is the tool that we use in our computations of $\mathcal{B}(\Phi, \Lambda)(v)$ as a piecewise polynomial function.

A particular but crucial instance of multiple Bernoulli series is when Λ is the coroot lattice of a compact connected simple Lie group G, and Φ is comprised of positive coroots of G. The series $\mathcal{B}(\Phi_{2g-2}, \Lambda)$, where the argument Φ_{2g-2} refers to taking elements of Φ with multiplicity $2g-2$, appeared in the work of E. Witten ([16], §3), where Witten shows that its value at $v = 0$ is (up to a scalar depending on G and g) the symplectic volume of the moduli space of flat G-connections on a Riemann surface of genus g. Similarly, for a regular element v of the Cartan Lie algebra of G, Witten shows that the value of $\mathcal{B}(\Phi_{2g-1}, \Lambda)(v)$ is (up to a scalar depending on G and g) the symplectic volume of the moduli space of flat G-connections on a Riemann surface of genus g with one boundary component, around which the holonomy is determined by v.

More generally, for the above choice of Λ and Φ, when $\mathbf{v} = \{v_1, \ldots, v_s\}$ is a collection of s regular elements of the Cartan Lie algebra, certain linear combinations of $\mathcal{B}(\Phi_{2g-2+s}, \Lambda)$ at some particular values (depending on $\mathbf{v}$) is the symplectic volume of the moduli space of flat G-connections on a Riemann surface of genus g with s boundary components, around which the holonomy is determined by $\mathbf{v}$. Then, its dependance on $\mathbf{v}$ is piecewise polynomial.

Y. Komori, K. Matsumoto, H. Tsumura ([6],[7],[8]) studied the restriction of the series (0.0.1) by summing over the cone of dominant regular weights of a semi-simple Lie group G, and defined functions $\zeta(\mathbf{s}, v, G)$ (cf. Section 5.2). They also obtained relations between these functions over $\mathbb{Q}$. When Λ is the coroot lattice of a compact connected simple Lie group G and the sequence Φ is the set of its positive coroots with equal even multiplicity for long roots and (possibly different) equal even multiplicity for short roots, due to the Weyl group symmetry, the summation $\mathcal{B}(\Phi, \Lambda)(0)$ over the full (regular) weight lattice is just (up to multiplication by an appropriate power of (2π)) Komori-Matsumoto-Tsumura zeta function $\zeta(\mathbf{s}, 0, G)$. Thus, the value of $\zeta(\mathbf{s}, 0, G)$ (up to a certain power of (2π)) is a rational number which can be computed explicitly, and we give examples of such computations.

As it is observed in [6], some instances of the series $\zeta(\mathbf{s}, v, G)$ also compute certain multiple zeta values. In the last part of the article we give various such computations of multiple zeta values using $\mathcal{B}(\Phi, \Lambda)$.

Here is the outline of individual sections.

In Section 1, we recall a formula due to A. Szenes, which allows an efficient computation of $\mathcal{B}(\Phi, \Lambda)$.

In Section 2, we give an outline of an algorithm that efficiently computes the needed ingredients of this formula for classical root systems. We also give several simple examples.

In Section 3, we show how this applies to the symplectic volume of the moduli space of flat G-connections on a Riemann surface of genus g with s boundary components. We obtain an expression for the symplectic volume by taking the limit of the Verlinde formula. We then show that our formula thus obtained coincides with that of Witten (including the constants) given in terms of the Riemannian volumes of G and T. We also give examples of these functions.

In Sections 4 and 5, we give several examples and tables of Witten volumes, which include some examples from ([6],[7],[8]). We give an idea of computational limitation of our algorithm (written as a simple Maple program) in terms of the rank of the group G and the number of elements in Φ. Following Y. Komori, K. Matsumoto, H. Tsumura, we also give some examples of rational multiple zeta values.

To compute more examples, our Maple program will soon be available on the webpage of the last author.

Finally, in the appendix, for completeness, we include a proof of Szenes formula.

Notations

U	r-dimensional real vector space.
V	dual of U; $v \in V$.
$\langle\, ,\, \rangle$	the pairing between U and V.
Γ	a lattice in U; $\gamma \in \Gamma$.
$\Lambda := \Gamma^*$	dual lattice in V; $\langle \Gamma, \Lambda \rangle \subset \mathbb{Z}$, $\lambda \in \Lambda$.
Φ	a sequence of vectors in V; $\phi \in \Phi$.
$\mathcal{B}(\Phi, \Lambda)$	multiple Bernoulli series associated to Φ and Λ.
H_ϕ	hyperplane in U comprising of vectors u satisfying $\langle u, \phi \rangle = 0$.
$\mathcal{H}$	arrangement of hyperplanes
Φ^{eq}	a set of equations for $\mathcal{H}$.
$\mathcal{R}_{\mathcal{H}}$	ring of rational functions on U with poles along $\mathcal{H}$.
$\mathcal{S}_{\mathcal{H}}$	a subspace of $\mathcal{R}_{\mathcal{H}}$ given in Definition 1.2.
$\mathcal{G}_{\mathcal{H}}$	a subspace of $\mathcal{R}_{\mathcal{H}}$ given in Definition 1.2.
$\mathbf{R}$	projector from $\mathcal{R}_{\mathcal{H}}$ to $\mathcal{S}_{\mathcal{H}}$.
$\mathcal{T}(\mathcal{H}, \Lambda)$	topes associated to the system $(\mathcal{H}, \Lambda)$; τ a tope.
$\mathfrak{B}(\Phi^{eq})$	the set of subsets of Φ^{eq} forming a basis for V.
$\mathcal{M}_{\mathcal{H}}, \hat{\mathcal{R}}_{\mathcal{H}}$	spaces of functions defined in 1.8.
$V_{reg} = V_{reg}(\mathcal{H}, \Lambda)$	subset of V regular with respect to $(\mathcal{H}, \Lambda)$.

1. Szenes formula for multiple Bernoulli series

1.1. **Functions on complement of hyperplanes.** In this subsection, U is an r-dimensional complex vector space, and we recall briefly some structure theorems for the ring of rational functions that are regular on the complement of a union of hyperplanes [4].

Let V be the dual vector space to U. If $\phi \in V$, we denote by $H_\phi = \{u \in U; \langle \phi, u \rangle = 0\}$.

Let $\mathcal{H} = \{H_1, \ldots, H_N\}$ be a set of hyperplanes in U. Then, we may choose $\phi_k \in V$ such that $H_k = H_{\phi_k}$; the element ϕ_k will be called an equation of H_k. Clearly, an equation ϕ_k is not unique, it is determined up to a nonzero scalar multiple.

We consider

$$U_{\mathcal{H}} := \{u \in U; \langle \phi_k, u \rangle \neq 0 \text{ for all } k\},$$

an open subset of U. An element in $U_{\mathcal{H}}$ will be called *regular*.

Definition 1.1. We denote by $S(V)$ the symmetric algebra of V and identify it with the ring of polynomial functions on U.

We denote by $\mathcal{R}_{\mathcal{H}}$ the ring of rational functions on U regular on $U_{\mathcal{H}}$. That is, the ring generated by $S(V)$ together with inverses of the linear forms ϕ_k.

The ring $\mathcal{D}(U)$ of differential operators on U with polynomial coefficients acts on $\mathcal{R}_{\mathcal{H}}$. In particular, U operates on $\mathcal{R}_{\mathcal{H}}$ by differentiation. We denote by $\partial(U)\mathcal{R}_{\mathcal{H}}$ the space of functions in $\mathcal{R}_{\mathcal{H}}$ obtained by differentiation.

If V is one dimensional, then the ring $\mathcal{R}_{\mathcal{H}}$ is the ring of Laurent polynomials $\mathbb{C}[z, z^{-1}]$, and the function z^i, for $i \neq -1$, is obtained as a derivative $\frac{d}{dz}\frac{1}{i+1}z^{i+1}$. Thus $\mathcal{R}_{\mathcal{H}} = \frac{d}{dz}\mathcal{R}_{\mathcal{H}} \oplus \mathbb{C}z^{-1}$. If $f = \sum_n a_n z^n$ is an element of $\mathbb{C}[z, z^{-1}]$, we denote by $\mathrm{Res}_{z=0} f$ the coefficient a_{-1} of z^{-1} in the expression of f. This linear form is characterized by the fact that it vanishes on $\frac{d}{dz}\mathcal{R}_{\mathcal{H}}$.

By analogy to this one dimensional case, a linear functional on $\mathcal{R}_{\mathcal{H}}$ vanishing on $\partial(U)\mathcal{R}_{\mathcal{H}}$ will be called a 'residue'.

Let us thus analyze the space $\mathcal{R}_{\mathcal{H}}$ modulo $\partial(U)\mathcal{R}_{\mathcal{H}}$.

Let us consider a set $\Phi^{eq} := \{\phi_1, \phi_2, \ldots, \phi_N\}$ of equations for $\mathcal{H}$. A subset σ of Φ^{eq} will be called a *basis* if the elements ϕ_k in σ form a basis of V. We denote by $\mathfrak{B}(\Phi^{eq})$ the set of such subsets σ. A subset ν of Φ^{eq} will be called *generating* if the elements ϕ_k in ν generate the vector space V.

Definition 1.2. • Let $\sigma := \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \in \mathfrak{B}(\Phi^{eq})$. Consider the 'simple fraction'

$$f_\sigma(z) := \frac{1}{\prod_{k=1}^r \alpha_k(z)}.$$

We denote by $\mathcal{S}_{\mathcal{H}}$ the subspace of $\mathcal{R}_{\mathcal{H}}$ generated by the elements f_σ, $\sigma \in \mathfrak{B}(\Phi^{eq})$.

• Let $\nu = [\alpha_1, \ldots, \alpha_k]$ be a sequence of k elements of Φ^{eq} and $\mathbf{n} = [n_1, n_2, \ldots, n_k]$ be a sequence of positive integers. We define

$$\theta(\nu, \mathbf{n}) = \frac{1}{\alpha_1^{n_1} \cdots \alpha_k^{n_k}}.$$

• We denote by $\mathcal{G}_{\mathcal{H}}$ the subspace of $\mathcal{R}_{\mathcal{H}}$ generated by the elements $\theta(\nu, \mathbf{n})$ where ν is generating.

As the notation suggests, the spaces $\mathcal{R}_{\mathcal{H}}$, $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{G}_{\mathcal{H}}$ depend only on $\mathcal{H}$. The term simple fraction comes from the fact that if $\sigma = \{\phi_1, \phi_2, \ldots, \phi_r\}$ is a basis, then we can choose coordinates z_i on U so that $\phi_i(z) = z_i$, so that for this system of coordinates $f_\sigma(z) = \frac{1}{\prod_{i=1}^r z_i}$.

We recall the following 'partial fraction' decomposition theorem.

Lemma 1.3. *Let ν be a subset of Φ^{eq} generating a t dimensional subspace of V. Then $\theta(\nu, \mathbf{n})$ may be written as a linear combination of elements $\theta(\sigma, \mathbf{m}) = \frac{1}{\alpha_{i_1}^{m_1} \cdots \alpha_{i_t}^{m_t}}$ where $\sigma := \{\alpha_{i_1}, \ldots, \alpha_{i_t}\}$ is a subset of ν*

consisting of t *independent elements and* $\mathbf{m} = \{m_1, \ldots, m_t\}$ *a sequence of positive integers.*

Example 1.4. If $\Phi = \{z_1, z_2, z_1 + z_2\}$, then

$$\frac{1}{z_1 z_2 (z_1 + z_2)} = \frac{1}{z_1 (z_1 + z_2)^2} + \frac{1}{z_2 (z_1 + z_2)^2}.$$

Finally, the following theorem is proved in Brion-Vergne [4].

Theorem 1.5.

$$\mathcal{R}_{\mathcal{H}} = \partial(U)\mathcal{R}_{\mathcal{H}} \oplus \mathcal{S}_{\mathcal{H}}.$$

The projector, denoted by $\mathbf{R}$, from $\mathcal{R}_{\mathcal{H}}$ to $\mathcal{S}_{\mathcal{H}}$ will be called the *total residue.*

In view of this theorem, a residue is just a linear form on $\mathcal{S}_{\mathcal{H}}$. When $\mathcal{H}$ is the set of hyperplanes with equations the positive coroots for a simple compact Lie group G, the dimension of $\mathcal{S}_{\mathcal{H}}$ is given by the product of exponents of G ([10]). In Section 2, we will give an explicit basis for $\mathcal{S}_{\mathcal{H}}$ for simple Lie algebras of type A, B and C (which defines the same set of hyperplanes as B).

1.2. **Szenes polynomial.** In this section and for the rest of the article, V will denote a **real** vector space of dimension r.

Let U be the dual vector space of V. Let Λ be a lattice in V with dual lattice Γ in U.

Let $\mathcal{H} := \{H_1, H_2, \ldots, H_N\}$ be a real arrangement of hyperplanes in U. We say that Λ and $\mathcal{H}$ are *compatible* if the hyperplanes in $\mathcal{H}$ are rational with respect to Λ, that is, they can be defined by equations $\phi_k \in \Lambda$.

If Λ' is another lattice commensurable with Λ, then Λ' and $\mathcal{H}$ are also compatible.

Thus we now consider a lattice Λ and a real arrangement of hyperplanes $\mathcal{H} = \{H_1, H_2, \ldots, H_N\}$ in U rational with respect to Λ.

We consider $\Phi^{eq} = \{\phi_1, \phi_2, \ldots, \phi_N\}$ a set of defining equations for $\mathcal{H}$, and we choose these equations in Λ. We sometimes refer to $\mathcal{H}$ only via its set of equations Φ^{eq} and write $\mathcal{H} = \cup\{\phi_k = 0\}$.

We denote the complex arrangement defined by $\cup\{\phi_k = 0\}$ in $U_{\mathbb{C}}$ by the same letter $\mathcal{H}$. We denote by $U_{\mathcal{H}} = \{\prod_k \phi_k \neq 0\}$ the corresponding open subset of $U_{\mathbb{C}}$.

An *admissible hyperplane* W in V (for the system $\mathcal{H}$) is an hyperplane generated by $(r-1)$ linearly independent elements ϕ_k. Such an hyperplane will also be called an (admissible) *wall.* An admissible *affine* wall is a translate of a wall by an element of Λ.

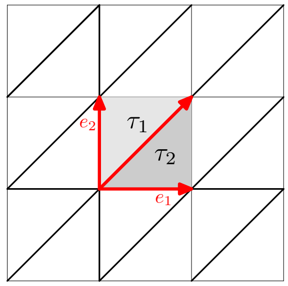

FIGURE 1. $\mathcal{T}(\mathcal{H}, \Lambda)$ for Example 1.6

An element $v \in V$ is called *regular* for $(\mathcal{H}, \Lambda)$ if v is not on any affine wall (we will just say that v is regular). The meaning of the word regular is thus different for elements $v \in V$ (v is not on any affine wall) and $u \in U_{\mathbb{C}}$ (u is such that $\prod_k \langle \phi_k, u \rangle \neq 0$). However, it will be clear what regular means in the context.

A *tope* τ is a connected component of the complement of all affine hyperplanes. Thus a tope τ is an open set of V consisting of regular elements. We denote the set of topes by $\mathcal{T}(\mathcal{H}, \Lambda)$. As the notation indicates, $\mathcal{T}(\mathcal{H}, \Lambda)$ does not depend on the choice of equations for $\mathcal{H}$.

Example 1.6. Let $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Let U be its dual with basis $\{e^1, e^2\}$. We express $z \in U_{\mathbb{C}}$ as $z = z_1 e^1 + z_2 e^2$, and consider the set of hyperplanes

$$\mathcal{H} = \{\{z_1 = 0\}, \{z_2 = 0\}, \{z_1 + z_2 = 0\}\}$$

with the set of equations $\Phi^{eq} = \{e_1, e_2, e_1 + e_2\}$. Figure 1 depicts topes associated to this pair.

Example 1.7. Let $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Let U be its dual with basis $\{e^1, e^2\}$. We express $z \in U_{\mathbb{C}}$ as $z = z_1 e^1 + z_2 e^2$, and consider the set of hyperplanes

$$\mathcal{H} = \{\{z_1 = 0\}, \{z_2 = 0\}, \{z_1 + z_2 = 0\}, \{z_1 - z_2 = 0\}\}$$

with the set of equations $\Phi^{eq} = \{e_1, e_2, e_1 + e_2, e_1 - e_2\}$. Figure 2 depicts topes associated to this pair.

We denote by $V_{reg}(\mathcal{H}, \Lambda)$ (or simply V_{reg}) the set of $(\mathcal{H}, \Lambda)$ regular elements of V. It is an open subset of V which is the disjoint union of all topes.

A locally constant function on V_{reg} is a function on V_{reg} which is constant on each tope. A piecewise polynomial function on V_{reg} is a function on V_{reg} which is given by a polynomial formula on each tope.

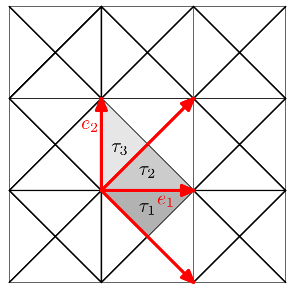

FIGURE 2. $\mathcal{T}(\mathcal{H},\Lambda)$ for Example 1.7

If $t \in \mathbb{R}$, we denote by $[t]$ the integral part of t, and by $\{t\} = t - [t]$ the fractional part of t. If $\gamma \in \Gamma$ vanishes on an admissible hyperplane W, and c is a constant, the function $v \to \{\langle\gamma, v\rangle + c\}$ is a piecewise polynomial (piecewise linear) and is periodic with respect to Λ. Szenes residue formula provides an algorithm to describe Bernoulli series in terms of these basic functions.

Definition 1.8. Let $\mathcal{M}_{\mathcal{H}}$ be the space of functions h/Q where Q is a product of linear forms belonging to Φ^{eq}, and h a holomorphic function defined in a neighborhood of 0 in $U_{\mathbb{C}}$.

We define the space $\hat{\mathcal{R}}_{\mathcal{H}}$ as the space of functions $\hat{h}/Q$ where $\hat{h} = \sum_{k=0}^{\infty} P_k$ is a formal power series and Q is a product of linear forms belonging to Φ^{eq} as before.

Taking the Taylor series $\hat{h}$ of h at 0 defines an injective map from $\mathcal{M}_{\mathcal{H}}$ to $\hat{\mathcal{R}}_{\mathcal{H}}$. The projector $\mathbf{R}$ from $\mathcal{R}_{\mathcal{H}}$ to $\mathcal{S}_{\mathcal{H}}$ extends to $\hat{\mathcal{R}}_{\mathcal{H}}$. Indeed $\mathbf{R}$ vanishes outside the homogeneous components of degree $-r$ of the graded space $\mathcal{R}_{\mathcal{H}}$. Thus if h/Q is an element in $\mathcal{M}_{\mathcal{H}}$, with Q a product of N elements of Φ^{eq}, we take the Taylor series $[h]_{N-r}$ of h up to order $N-r$, and define $\mathbf{R}(\frac{h}{Q}) = \mathbf{R}(\frac{[h]_{N-r}}{Q})$. For example, the equality

$$\frac{e^{zt}}{e^z - 1} = \frac{1}{z}(z\frac{e^{zt}}{e^z - 1})$$

identifies the function $\frac{e^{zt}}{e^z-1}$ to an element of $\mathcal{M}_{\mathcal{H}}$ with $\mathcal{H} = \{0\}$. Note that each homogeneous term of the Taylor series expansion

$$z\frac{e^{zt}}{e^z - 1} = \sum_{k=0}^{\infty} B(k,t)\frac{z^k}{k!},$$

where $B(k,t)$ is the k^{th} Bernoulli polynomial in t as before, is a polynomial in t.

Let $f \in \mathcal{R}_{\mathcal{H}}$, $z \in U_{\mathcal{H}}$ and $\gamma \in \Gamma$. Then if z is small, $2i\pi\gamma - z$ is still a regular element of U. Consider the series

$$S(f, z, v) = \sum_{\gamma \in \Gamma} f(2i\pi\gamma - z) e^{\langle v, 2i\pi\gamma \rangle}.$$

When f decreases sufficiently quickly at infinity, the series $S(f, z, v)$ is absolutely convergent and defines a continuous function of v. In general, as f is of at most polynomial growth, the series

$$\sum_{\gamma \in \Gamma} f(2i\pi\gamma - z) e^{\langle v, 2i\pi\gamma \rangle}$$

is the Fourier series of a generalized function on V/Λ.

Multiplying $S(f, z, v)$ by the exponential $e^{-\langle z, v \rangle}$ we introduce the following definition.

Definition 1.9. Let $f \in \mathcal{R}_{\mathcal{H}}$, $z \in U_{\mathcal{H}}$ and small. We define the generalized function of v by

$$A^{\Lambda}(f)(z, v) = \sum_{\gamma \in \Gamma} f(2i\pi\gamma - z) e^{\langle v, 2i\pi\gamma - z \rangle}.$$

The meaning of $A^{\Lambda}(f)$ is clear : average the function $z \mapsto f(-z)e^{-\langle v, z \rangle}$ over $2i\pi\Gamma$ in order to obtain a function on the complex torus $U_{\mathbb{C}}/2i\pi\Gamma$.

We consider $A^{\Lambda}(f)(z, v)$ as a generalized function of $v \in V$ with coefficients meromorphic functions of z on $U_{\mathbb{C}}/2i\pi\Gamma$. In fact, as we will see, when f is in $\mathcal{S}_{\mathcal{H}}$, the convergence of the series

$$\sum_{\gamma \in \Gamma} f(2i\pi\gamma - z) e^{\langle v, 2i\pi\gamma \rangle}$$

holds in the sense of the Fourier series of an L^2- periodic function of $v \in V/\Lambda$, and

$$v \to e^{-\langle v, z \rangle} \left(\sum_{\gamma \in \Gamma} f(2i\pi\gamma - z) e^{\langle v, 2i\pi\gamma \rangle} \right) = A^{\Lambda}(f)(z, v)$$

is a locally constant function of $v \in V_{reg}$ with values in $\mathcal{M}_{\mathcal{H}}$.

Note the covariance relation. For $\lambda \in \Lambda$,

$$A^{\Lambda}(f)(z, v + \lambda) = e^{-\langle \lambda, z \rangle} A^{\Lambda}(f)(z, v). \tag{1.9.1}$$

It is easy to compare $A^{\Lambda}(f)(z, v)$ when we change the lattice Λ.

Lemma 1.10. *Let $f \in \mathcal{R}_{\mathcal{H}}$. If $\Lambda^1 \subset \Lambda^2$, then*

$$A^{\Lambda^2}(f)(z,v) = |\Lambda^2/\Lambda^1|^{-1} \sum_{\lambda \in \Lambda^2/\Lambda^1} A^{\Lambda^1}(f)(z, v+\lambda). \tag{1.10.1}$$

Proof. Denote the dual of Λ^i by Γ^i. Then,

$$A^{\Lambda^1}(f)(z, v+\lambda) = \sum_{\gamma \in \Gamma^1} f(2i\pi\gamma - z) e^{\langle v+\lambda, 2i\pi\gamma - z\rangle}$$

and the sum over $\lambda \in \Lambda^2/\Lambda^1$ of $e^{\langle \lambda, 2i\pi\gamma\rangle}$ is equal to 0 except when $\gamma \in \Gamma^2$. ☐

Example 1.11. Let $V = \mathbb{R}$, $\Lambda = \mathbb{Z}$, $z \in \mathbb{C}$ small, $z \neq 0$, and $f(z) = \frac{1}{z}$. For $v \in \mathbb{R}$,

$$A^{\Lambda}(f)(z,v) = \sum_{n \in \mathbb{Z}} \frac{e^{v(2i\pi n - z)}}{(2i\pi n - z)} = e^{-zv} \sum_{n \in \mathbb{Z}} \frac{e^{2i\pi n v}}{(2i\pi n - z)}.$$

This series is not absolutely convergent, but the oscillatory factor $e^{2i\pi n v}$ insures the convergence in the distributional sense as a function of v. We have

$$A^{\Lambda}(f)(z,v) = \frac{e^{-[v]z}}{1-e^z} \tag{1.11.1}$$

(recall that $[v]$ denotes the integral part of v).

Indeed, let us compute the L^2-expansion of the periodic function $v \mapsto \frac{e^{(v-[v])z}}{1-e^z}$. By definition, this is

$$\sum_{n \in \mathbb{Z}} \left(\int_0^1 \frac{e^{(v-[v])z}}{1-e^z} e^{-2i\pi n v} dv \right) e^{2i\pi n v} = \sum_{n \in \mathbb{Z}} \left(\int_0^1 \frac{e^{v(z-2i\pi n)}}{1-e^z} dv \right) e^{2i\pi n v}$$

$$= \sum_{n \in \mathbb{Z}} \frac{e^{(z-2i\pi n)} - 1}{(1-e^z)(z-2i\pi n)} e^{2i\pi n v} = \sum_{n \in \mathbb{Z}} \frac{1}{(2i\pi n - z)} e^{2i\pi n v}.$$

We see in this one dimensional example that $A^{\Lambda}(f)(z,v)$ is a locally constant function of v.

In general, we have the following proposition.

Proposition 1.12. *If $f \in \mathcal{S}_{\mathcal{H}}$, the function $v \to A^{\Lambda}(f)(z,v)$ is a locally constant function on V_{reg}, with values in $\mathcal{M}_{\mathcal{H}}$.*

We prove this by computing $A^{\Lambda}(f)(z,v)$ explicitly for a simple fraction $f = f_\sigma$. Recall that the set of equations Φ^{eq} is a subset of Λ. Let $\sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be an element of $\mathfrak{B}(\Phi^{eq})$. The elements α_k belong to Λ. Let $Q_\sigma := \oplus_{k=1}^r [0,1)\alpha_k$ be the semi-open parallelepiped spanned by σ.

Definition 1.13. Let v be regular in V, and let $\sigma \in \mathfrak{B}(\Phi^{eq})$ be a basis. Define $T(v,\sigma)$ to be the set of elements $\lambda \in \Lambda$ such that $v+\lambda \in Q_\sigma$.

This set depends only on the tope τ where v belongs, hence we denote it by $T(\tau,\sigma)$.

Let Λ_σ be the sublattice of Λ generated by the elements in the basis σ. Then the set $T(\tau,\sigma)$ contains exactly Λ/Λ_σ elements.

Proposition 1.14. *If $v \in \tau$ and $\sigma = \{\alpha_1, \alpha_2, \dots, \alpha_r\} \in \mathfrak{B}(\Phi^{eq})$,*

$$A^\Lambda(f_\sigma)(z,v) = \frac{1}{|\Lambda/\Lambda_\sigma|} \sum_{\lambda \in T(\tau,\sigma)} \frac{e^{\langle \lambda, z\rangle}}{\prod_{i=1}^r (1-e^{\langle \alpha_i, z\rangle})}.$$

Proof. If $\Lambda = \Lambda_\sigma$, the formula reduces to the one dimensional case. Otherwise, we use Lemma 1.10 and the covariance formula 1.9.1. □

The dependance of $A^\Lambda(f_\sigma)(z,v)$ on v is only via the tope τ where v belongs. Thus we see that, for any $f \in \mathcal{S}_\mathcal{H}$, the function $v \mapsto A^\Lambda(f)(z,v)$ is a locally constant function on V_{reg} with value in $\mathcal{M}_\mathcal{H}$.

Example 1.15. We return to Example 1.6, where $\Phi^{eq} = \{e_1, e_2, e_1 + e_2\}$. Using the covariance relation (1.9.1), to describe completely the function $v \mapsto A^\Lambda(f)(z,v)$ on V_{reg}, it is sufficient to describe its values on the topes τ_1 and τ_2 depicted in Figure 1, as any element of V_{reg} can be brought into τ_1 or τ_2 by an element of Λ and then use the covariance relation (1.9.1).

Choose $\sigma = \{e_1, e_1 + e_2\}$ a basis of Φ^{eq}. Write $z = z_1 e^1 + z_2 e^2$ in the dual space, then $f_\sigma(z) = \frac{1}{z_1(z_1+z_2)}$ is in $S_\mathcal{H}$.

For $v \in \tau_1$

$$A^\Lambda(f_\sigma)(z,v) = \frac{e^{z_1}}{(1-e^{z_1})(1-e^{z_1+z_2})},$$

while if $v \in \tau_2$

$$A^\Lambda(f_\sigma)(z,v) = \frac{1}{(1-e^{z_1})(1-e^{z_1+z_2})}.$$

For $v \in V_{reg}$, denote by $Z^\Lambda(v) : \mathcal{S}_\mathcal{H} \to \mathcal{M}_\mathcal{H}$ the map given by

$$(Z^\Lambda(v)f)(z) = A^\Lambda(f)(z,v).$$

This operator is locally constant. We denote by $Z^\Lambda(\tau)$ its value on τ:

$$(Z^\Lambda(\tau)f)(z) = A^\Lambda(f)(z,v)$$

for any choice of $v \in \tau$.

We now define a piecewise polynomial function of v associated to a function $g(z)$ in $\mathcal{R}_{\mathcal{H}}$.

First, the operator on $\mathcal{M}_{\mathcal{H}}$ given by multiplication by a function $h(z)$, that is $f(z) \mapsto h(z)f(z)$, is simply denoted by $f \mapsto hf$.

If $v \in V$ and $g \in \mathcal{R}_{\mathcal{H}}$, then $g_v(z) = g(z)e^{\langle z,v\rangle}$ is a function in $\mathcal{M}_{\mathcal{H}}$ depending on v.

Let $v \in V_{reg}$. Consider the map $\mathcal{S}_{\mathcal{H}} \to \mathcal{M}_{\mathcal{H}}$ which associates to $f \in \mathcal{S}_{\mathcal{H}}$ the function $g(z)e^{\langle z,v\rangle}(Z^{\Lambda}(v)f)(z)$. We project back this function on $\mathcal{S}_{\mathcal{H}}$ using the projector $\mathbf{R}$. Thus the map

$$f(z) \to \mathbf{R}\left(g(z)e^{\langle z,v\rangle}(Z^{\Lambda}(v)f)(z)\right) \tag{1.15.1}$$

is a map from $\mathcal{S}_{\mathcal{H}}$ to $\mathcal{S}_{\mathcal{H}}$ depending on v. As $\mathcal{S}_{\mathcal{H}}$ is finite dimensional, we can take the trace of this operator. We obtain a function of $v \in V_{reg}$. Let us record this definition.

Definition 1.16. Let $g \in \mathcal{R}_{\mathcal{H}}$. Define the function $P(\mathcal{H}, \Lambda, g)$ on $V_{reg}(\mathcal{H}, \Lambda)$ by

$$P(\mathcal{H}, \Lambda, g)(v) := Tr_{\mathcal{S}_{\mathcal{H}}}\left(\mathbf{R}\, g_v\, Z^{\Lambda}(v)\right).$$

Let us see that $P(\mathcal{H}, \Lambda, g)(v)$ is a polynomial function of v on each tope τ. Indeed, to compute $P(\mathcal{H}, \Lambda, g)(v)$ using (1.15.1), we have to compute the total residue of functions $g(z)e^{\langle z,v\rangle}A^{\Lambda}(f_i)(z,v)$ with f_i varying over a basis of $\mathcal{S}_{\mathcal{H}}$. If $v \in \tau$, then $A^{\Lambda}(f_i)(z,v) = Z^{\Lambda}(\tau)f_i(z)$ is constant in v. So when v stays in a tope τ, the dependance of $g(z)e^{\langle z,v\rangle}A^{\Lambda}(f_i)(z,v) = g(z)e^{\langle z,v\rangle}(Z^{\Lambda}(\tau)f_i)(z)$ on v is via $e^{\langle z,v\rangle}$, and the map $\mathbf{R}$ involves only the Taylor series of this function up to some order.

Thus we have associated to $g \in \mathcal{R}_{\mathcal{H}}$ (and Λ) a piecewise polynomial function $P(\mathcal{H}, \Lambda, g)$ on V_{reg}.

It is easy to compare piecewise polynomial functions $P(\mathcal{H}, \Lambda, g)$ associated to different lattices.

Let $\Lambda^1 \subset \Lambda^2$, then $V_{reg}(\mathcal{H}, \Lambda^2) \subset V_{reg}(\mathcal{H}, \Lambda^1)$.

Lemma 1.17. *If* $\Lambda^1 \subset \Lambda^2$, *then*

$$P(\mathcal{H}, \Lambda^2, g)(v) = |\Lambda^2/\Lambda^1|^{-1} \sum_{\lambda \in \Lambda^2/\Lambda^1} P(\mathcal{H}, \Lambda^1, g)(v+\lambda). \tag{1.17.1}$$

This follows right away from Lemma 1.10.

Our next aim is to compute the piecewise polynomial function $P(\mathcal{H}, \Lambda, g)$ by residues. We need more definitions.

An ordered basis of Φ^{eq} is a sequence $[\alpha_1, \alpha_2, \ldots, \alpha_r]$ of elements of Φ^{eq} such that the underlying set is in $\mathfrak{B}(\Phi^{eq})$. We denote the set of ordered bases by $\overrightarrow{\mathfrak{B}}(\Phi^{eq})$.

Let $\overrightarrow{\sigma} = [\alpha_1, \alpha_2, \ldots, \alpha_r] \in \overrightarrow{\mathfrak{B}}(\Phi^{eq})$. Then, to this data, one associates an iterated residue functional $\mathrm{Res}^{\overrightarrow{\sigma}}$ on $\mathcal{R}_{\mathcal{H}}$ as follows. For $z \in U$, let $z_j = \langle z, \alpha_j \rangle$. Then a function f in $\mathcal{R}_{\mathcal{H}}$ can be expressed as a function $f(z_1, z_2, \ldots, z_r)$. We define

$$\mathrm{Res}^{\overrightarrow{\sigma}}(f) := \mathrm{Res}_{z_1=0}(\mathrm{Res}_{z_2=0} \cdots (\mathrm{Res}_{z_r=0}\, f(z_1, z_2, \ldots, z_r)) \cdots).$$

Clearly $\mathrm{Res}^{\overrightarrow{\sigma}}(f_\sigma) = 1$.

The functional $\mathrm{Res}^{\overrightarrow{\sigma}}$ factors through the canonical projection $\mathbf{R} : \mathcal{R}_{\mathcal{H}} \to \mathcal{S}_{\mathcal{H}}$: $\mathrm{Res}^{\overrightarrow{\sigma}} = \mathrm{Res}^{\overrightarrow{\sigma}}\, \mathbf{R}$.

Definition 1.18. A *diagonal subset of* $\overrightarrow{\mathfrak{B}}(\Phi^{eq})$ is a subset $\overrightarrow{\mathcal{D}}$ of $\overrightarrow{\mathfrak{B}}(\Phi^{eq})$ such that the set of simple fractions f_σ, $\overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}$, forms a basis of $\mathcal{S}_{\mathcal{H}}$:

$$\mathcal{S}_{\mathcal{H}} = \oplus_{\overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}} \mathbb{C} f_\sigma$$

and the dual basis to the basis $\{f_\sigma, \overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}\}$ of $\mathcal{S}_{\mathcal{H}}$ is the set of linear forms $\mathrm{Res}^{\overrightarrow{\sigma}}$, that is, $\mathrm{Res}^{\overrightarrow{\tau}}(f_\sigma) = \delta_\sigma^\tau$, for $\overrightarrow{\sigma}, \overrightarrow{\tau} \in \overrightarrow{\mathcal{D}}$.

A total order on Φ^{eq} allows us to construct the set of Orlik-Solomon bases (see [4]), which provides diagonal basis of $\mathcal{S}_{\mathcal{H}}$. However we will use also some other diagonal subsets.

If $B : \mathcal{S}_{\mathcal{H}} \to \mathcal{M}_{\mathcal{H}}$ is an operator, the trace of the operator $A := \mathbf{R}B$ is thus

$$Tr(A) := \sum_{\overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}} \mathrm{Res}^{\overrightarrow{\sigma}}\, B f_\sigma.$$

Definition 1.19. Let $g \in \mathcal{R}_{\mathcal{H}}$ and τ a connected component of V_{reg}. We denote by $P(\mathcal{H}, \Lambda, g, \tau)(v)$ the polynomial function on V such that

$$P(\mathcal{H}, \Lambda, g)(v) = P(\mathcal{H}, \Lambda, g, \tau)(v)$$

for $v \in \tau$.

Hence, we may give a more explicit formula for the polynomial $P(\mathcal{H}, \Lambda, g, \tau)(v)$ using a set $\overrightarrow{\mathcal{D}}$.

Proposition 1.20. *Let $g \in \mathcal{M}_{\mathcal{H}}$. Let $\tau \in \mathcal{T}(\mathcal{H}, \Lambda)$ be a tope. Let $\overrightarrow{\mathcal{D}}$ be a diagonal subset of $\overrightarrow{\mathcal{B}}(\Phi^{eq})$. Then*

$$P(\mathcal{H}, \Lambda, g, \tau)(v) = \sum_{\overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}} \mathrm{Res}^{\overrightarrow{\sigma}} \left(e^{\langle z, v \rangle} g(z) Z^{\Lambda}(\tau)(f_\sigma)(z) \right).$$

Furthermore $Z^{\Lambda}(\tau)(f_\sigma)(z)$ is given explicitly by Proposition 1.14. Thus, in principle, the formula above allows us to compute $P(\mathcal{H}, \Lambda, g)$.

It is important to remark that the determination of a diagonal subset $\overrightarrow{\mathcal{D}}$ depends essentially only on the system of hyperplanes $\mathcal{H}$ and not on the choice of Φ^{eq}. The difficulties in writing an algorithm for $P(\mathcal{H},\Lambda,g)$ lies in the description of a diagonal subset $\overrightarrow{\mathcal{D}}$, and also for each $\sigma \in \overrightarrow{\mathcal{D}}$, in the computation of $A^\Lambda(f_\sigma)$. The difficulty of this last computation depends on the lattice Λ.

Definition 1.21. Let $\Phi^{eq} \subset \Lambda$. A basis $\sigma \in \mathfrak{B}(\Phi^{eq})$ is called unimodular (with respect to Λ) if σ is a basis of the lattice Λ. A set Φ^{eq} is called unimodular, if any basis $\sigma \in \mathfrak{B}(\Phi^{eq})$ is unimodular.

Example 1.22. The set

$$\Phi^{eq} = \{e^1, e^2, (e^1+e^2), (e^1-e^2)\}$$

is contained in $\Lambda = \mathbb{Z}e^1 + \mathbb{Z}e^2$. Then $\sigma = \{e^1+e^2, e^1-e^2\}$ belongs to $\mathfrak{B}(\Phi^{eq})$, and the index of Λ_σ in Λ is 2. So Φ^{eq} is not unimodular.

Definition 1.23. Let $\sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a basis. For $1 \leq i \leq r$, the linear form $v \to c_i^\sigma(v)$ is the coefficient of v with respect to α_i.

Consider the function $\{t\} = t - [t]$. On each open interval $\tau =]n, n+1[$, the function $\{t\}$ coincide with the linear function $t \mapsto (t-n)$.

If σ is a unimodular basis, we express $v = \sum_{i=1}^r c_i^\sigma(v)\alpha_i$. Then

$$v - \sum_{i=1}^r [c_i^\sigma(v)]\alpha_i = \sum_{i=1}^r \{c_i^\sigma(v)\}\alpha_i$$

is in Q_σ. Thus the set $T(\tau,\sigma)$ contains exactly the element $\lambda = -\sum_{i=1}^r [c_i^\sigma(v)]\alpha_i$ (which depends only on the tope τ where v lies).

Corollary 1.24. Let σ be a unimodular basis in $\mathfrak{B}(\Phi^{eq})$. Let $v \in \tau$, and $\lambda = -\sum_{i=1}^r [c_i^\sigma(v)]\alpha_i$. Then

$$A^\Lambda(f_\sigma)(z,v) = \frac{e^{\langle \lambda, z\rangle}}{\prod_{i=1}^r (1 - e^{\langle \alpha_i, z\rangle})}.$$

It may happen that even when the system Φ^{eq} is not unimodular for the lattice Λ, we can choose $\overrightarrow{\mathcal{D}}$ to consist of unimodular bases. In particular, using Proposition 1.20, we can give an explicit algorithm for computing the piecewise polynomial function $P(\mathcal{H},\Lambda,g)$ for classical root systems in the form of a step polynomial. Let us define what this means.

Definition 1.25. Let $\overrightarrow{\mathcal{D}}$ be a subset of $\overrightarrow{\mathcal{B}}(\Phi^{eq})$. We denote by $Step(\overrightarrow{\mathcal{D}})$ the algebra of functions on V generated by the piecewise linear functions $v \to \{c_i^\sigma(v)\}$ with σ running over $\overrightarrow{\mathcal{D}}$ and $1 \leq i \leq r$. An element of the algebra $Step(\overrightarrow{\mathcal{D}})$ will be called a step polynomial (associated to $\overrightarrow{\mathcal{D}}$).

It is clear that a step polynomial is a periodic function on V, which is expressed by a polynomial formula on each tope.

Proposition 1.26. *Let $g \in \mathcal{G}_{\mathcal{H}}$. Assume that $\overrightarrow{\mathcal{D}}$ is a diagonal subset of $\overrightarrow{\mathfrak{B}}(\Phi^{eq})$ consisting of unimodular basis (with respect to Λ). Then the piecewise polynomial function $P(\mathcal{H},\Lambda,g)$ belongs to the algebra $Step(\overrightarrow{\mathcal{D}})$.*

Proof. This is clear, as we have the formula
(1.26.1)
$$P(\mathcal{H},\Lambda,g)(v) = \sum_{\overrightarrow{\sigma}\in\overrightarrow{\mathcal{D}}} \mathrm{Res}^{\overrightarrow{\sigma}}\, g(z) e^{\sum_{i=1}^{r}\{c_i^{\sigma}(v)\}\langle\alpha_i,z\rangle} \frac{1}{\prod_{i=1}^{r}(1-e^{\langle\alpha_i,z\rangle})},$$
and the dependance on v is through the Taylor expansion (in z) of $e^{\sum_{i=1}^{r}\{c_i^{\sigma}(v)\}\langle\alpha_i,z\rangle}$ up to some order. □

1.3. **Multiple Bernoulli series.** We return to our main object of study: the multiple Bernoulli series.

Let V, Λ and $\mathcal{H}$ be as before. We denote by $\Gamma\subset U$ the dual lattice to Λ, and by $\Gamma_{reg}(\mathcal{H})$ the set $\Gamma\cap U_{\mathcal{H}}$. If $\gamma\in\Gamma_{reg}(\mathcal{H})$, a function g in $\mathcal{R}_{\mathcal{H}}$ is defined on $2i\pi\gamma$.

Definition 1.27. If $g\in\mathcal{R}_{\mathcal{H}}$, the generalized function $\mathcal{B}(\mathcal{H},\Lambda,g)(v)$ on V is defined by
$$\mathcal{B}(\mathcal{H},\Lambda,g)(v) = \sum_{\gamma\in\Gamma_{\mathrm{reg}}(\mathcal{H})} g(2i\pi\gamma)e^{2i\pi\langle v,\gamma\rangle}.$$

The above series converges in the space of generalized functions on V.

We state some obvious properties of $\mathcal{B}(\mathcal{H},\Lambda,g)$.

Lemma 1.28. *If $\Lambda^1\subset\Lambda^2$, then $\Gamma^2\subset\Gamma^1$, and*
$$\mathcal{B}(\mathcal{H},\Lambda^2,g)(v) = |\Lambda^2/\Lambda^1|^{-1}\sum_{\lambda\in\Lambda^2/\Lambda^1}\sum_{\gamma\in\Gamma^1_{\mathrm{reg}}(\mathcal{H})} g(2i\pi\gamma)e^{2i\pi\langle v+\lambda,\gamma\rangle}$$
so that
$$\mathcal{B}(\mathcal{H},\Lambda^2,g)(v) = |\Lambda^2/\Lambda^1|^{-1}\sum_{\lambda\in\Lambda^2/\Lambda^1}\mathcal{B}(\mathcal{H},\Lambda^1,g)(v+\lambda). \tag{1.28.1}$$

If we dilate a lattice Λ by ℓ, and if g is homogeneous of degree h, we clearly have

$$\mathcal{B}(\mathcal{H},\ell\Lambda,g)(v) = \ell^{-h}\mathcal{B}(\mathcal{H},\Lambda,g)(\frac{v}{\ell}). \tag{1.28.2}$$

With these two properties, we can compare $\mathcal{B}(\mathcal{H}, \Lambda, g)$ over commensurable lattices.

Definition 1.29. A generalized function b on V will be called piecewise polynomial relative to $\mathcal{H}$ and Λ if
- the function b is locally L^1,
- for each tope τ in $\mathcal{T}(\mathcal{H}, \Lambda)$, there exists a polynomial function b^τ on V such that the restriction of b to τ coincides with the restriction of the polynomial b^τ to τ.

As an L^1-function is entirely determined by its restriction to V_{reg}, we will not distinguish between piecewise polynomial generalized functions on V and piecewise polynomial functions on V_{reg} as defined in the preceding section.

Be careful: the restriction to any tope τ of a piecewise polynomial generalized function b is polynomial. However, the converse is not true. For example the δ function of the lattice Λ restricts to 0 on any tope τ, but is not a piecewise polynomial generalized function, as it is not an L^1-function.

Recall the definition of $\mathcal{G}_{\mathcal{H}}$, given in Definition 1.2. If we multiply g by a polynomial p, the function $v \mapsto \mathcal{B}(\mathcal{H}, \Lambda, pg)(v)$ is obtained from the function $\mathcal{B}(\mathcal{H}, \Lambda, g)(v)$ by differentiation (in the distribution sense). Any function f in $\mathcal{R}_{\mathcal{H}}$ is of the form p/g, with $g \in \mathcal{G}_{\mathcal{H}}$. Thus we can reduce the computation of $\mathcal{B}(\mathcal{H}, \Lambda, f)$ to the computation of $\mathcal{B}(\mathcal{H}, \Lambda, g)$ for $g \in \mathcal{G}_{\mathcal{H}}$. Thus the following proposition follows from calculations in dimension one, Lemma 1.3 and comparison formulae on different lattices as given in Lemma 1.28.

Proposition 1.30. *If $f \in \mathcal{R}_{\mathcal{H}}$, the restriction to any tope τ of $\mathcal{B}(\mathcal{H}, \Lambda, f)$ is given by a polynomial function.*

Furthermore, if $f \in \mathcal{G}_{\mathcal{H}}$, the generalized function $\mathcal{B}(\mathcal{H}, \Lambda, f)$ is a piecewise polynomial generalized function.

Let us emphasize on the subtle difference between the conditions $f \in \mathcal{R}_{\mathcal{H}}$, or $f \in \mathcal{G}_{\mathcal{H}}$. Consider $f = 1$ in the one dimensional space U and $\mathcal{H} = \{0\}$. The function f is not in $\mathcal{G}_{\mathcal{H}}$. Let $v \in V$. Then

$$\mathcal{B}(\mathcal{H}, \Lambda, f)(v) = \sum_{n \neq 0} e^{2i\pi nv} = -1 + \sum_{n \in \mathbb{Z}} e^{2i\pi nv}.$$

Thus $\mathcal{B}(\mathcal{H}, \Lambda, f)(v)$ is the constant function equal to -1 on any tope. However, it has some singular part $\delta_{\mathbb{Z}}$.

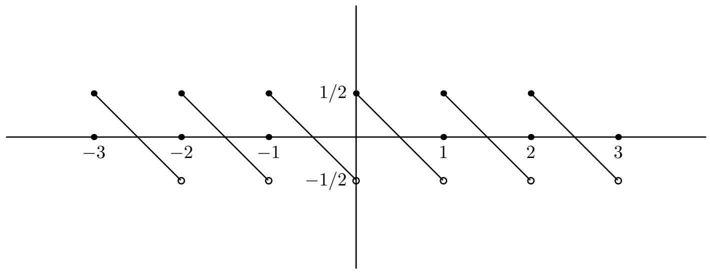

FIGURE 3. Graph of $\mathcal{B}(\{0\}, \mathbb{Z}, 1/z)(v) = \frac{1}{2} - \{v\}$

In contrast, if $f = \dfrac{1}{z}$, the generalized function

$$\mathcal{B}(\mathcal{H}, \Lambda, f)(v) = \sum_{n \neq 0} \frac{e^{2i\pi n v}}{2i\pi n}$$

is locally L^1 and equal to the piecewise polynomial function $-B(1, \{v\}) = 1/2 - \{v\}$ (see Figure 3).

Definition 1.31. Let $f \in \mathcal{R}_{\mathcal{H}}$. Given a tope τ in $\mathcal{T}(\mathcal{H}, \Lambda)$, we denote by $\mathcal{B}(\mathcal{H}, \Lambda, f, \tau)$ the polynomial function on V which coincides with $\mathcal{B}(\mathcal{H}, \Lambda, f)$ on the tope τ.

Remark 1.32. It is interesting to understand the space of polynomials generated by the polynomial functions $b^\tau = \mathcal{B}(\mathcal{H}, \Lambda, f, \tau)$, when τ runs over the topes, and the wall crossing formula between b^{τ_1} and b^{τ_2} when τ_1 and τ_2 are adjacent. We addressed some aspects of these theoretical questions in [3].

Consider the piecewise polynomial function $P(\mathcal{H}, \Lambda, f)$ on $V_{reg}(\mathcal{H}, \Lambda)$ as given in Definition 1.16.

Theorem 1.33. *(Szenes) Let $f \in \mathcal{R}_{\mathcal{H}}$. On $V_{\mathrm{reg}}(\mathcal{H}, \Lambda)$, we have the equality*

$$\mathcal{B}(\mathcal{H}, \Lambda, f) = P(\mathcal{H}, \Lambda, f).$$

For completeness, we give a proof of this theorem in the Appendix.

Our Maple program computes, given data $\mathcal{H}$, Λ, f, where $\mathcal{H}$ is the hyperplane arrangement associated to a classical root system, a piecewise polynomial function on V in terms of step polynomials. Naturally, we can also evaluate this function at any point $v \in V_{reg}$.

We return to the definition of multiple Bernoulli series in the way we introduced them in the introduction.

Let V be a vector space with a lattice Λ with dual lattice Γ. We considered Φ in the introduction as a **list** of elements of Λ. We then introduced the following definition. Let $\Gamma_{\mathrm{reg}}(\Phi) = \{\gamma \in \Gamma; \langle \phi, \gamma \rangle \neq 0 \text{ for all } \phi \in \Phi\}$ and defined

$$\mathcal{B}(\Phi, \Lambda)(v) = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(\Phi)} \frac{e^{\langle 2i\pi v, \gamma \rangle}}{\prod_{\phi \in \Phi} \langle 2i\pi\phi, \gamma \rangle}. \tag{1.33.1}$$

Consider $\mathcal{H} = \cup_k \{\phi_k = 0\}$ (some elements of the list Φ might define the same hyperplane) and $g(z) = 1/\prod_{\phi \in \Phi} \langle \phi, z \rangle$, then

$$\mathcal{B}(\Phi, \Lambda)(v) = \mathcal{B}(\mathcal{H}, \Lambda, g)(v).$$

We will also call the functions $\mathcal{B}(\mathcal{H}, \Lambda, g)(v)$ multiple Bernoulli series.

2. Classical root systems

Let G be a simple, simply connected, compact Lie group of rank r with maximal torus T. We denote the Lie algebra of T and G by $\mathfrak{t}$ and $\mathfrak{g}$ respectively. Then the complexification $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

For $\alpha \in \mathfrak{h}^*$, define $(\mathfrak{g}_{\mathbb{C}})_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}}; [H, X] = \langle \alpha, H \rangle X \text{ for all } H \in \mathfrak{h}\}$. If $\alpha \neq 0$ and $(\mathfrak{g}_{\mathbb{C}})_\alpha \neq 0$, then α is called a *root* of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$. Let $R = R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}) \subset \mathfrak{h}^*$ be the set of roots; roots $\alpha \in R$ take imaginary values on $\mathfrak{t}$. We denote the root lattice by Q and its dual, the coweight lattice, by $\check{P}$.

For $\alpha \in R$, there exists a unique element H_α in $[(\mathfrak{g}_{\mathbb{C}})_\alpha, (\mathfrak{g}_{\mathbb{C}})_{-\alpha}]$ satisfying $\langle \alpha, H_\alpha \rangle = 2$; it is called the *coroot* associated to the root α. For any $\alpha \in R$, iH_α is in $\mathfrak{t}$, and for any $\alpha, \beta \in R$, $\beta(H_\alpha)$ is integral. The lattice spanned by H_α is called the coroot lattice and denoted by $\check{Q}$.

Define the weight lattice $P = \{\lambda \in \mathfrak{h}^*; \lambda(H_\alpha) \in \mathbb{Z}, \ \forall \alpha \in R\}$; it is the dual of the coroot lattice $\check{Q}$. A *regular* weight $\lambda \in P^{reg}$ is such that $\lambda(H_\alpha) \neq 0$ for all H_α.

We denote by $\mathfrak{h}_{\mathbb{R}} := \sum_\alpha \mathbb{R} H_\alpha$, the real span of coroots. In this section we have $V = \mathfrak{h}_{\mathbb{R}}$, and its dual $\mathfrak{h}_{\mathbb{R}}^*$ is denoted by U as before. We follow the notation of Bourbaki for root data.

2.1. **Diagonal subsets.** To compute multiple Bernoulli series associated to classical root systems we need to construct explicit diagonal bases for the corresponding $\mathcal{S}_{\mathcal{H}}$. Such bases can be constructed by an algorithmic procedure, based on Orlik-Solomon construction. However in some cases we can describe a diagonal subset $\overrightarrow{\mathcal{D}}$ of $\overrightarrow{\mathcal{B}}(\Phi^{eq})$ whose associated simple fractions form a basis for $\mathcal{S}_{\mathcal{H}}$ in a direct way, and that we present now.

2.1.1. *The system of type A_r.* Let $n = r+1$. We consider $\mathbb{R}^n$ with standard basis $\{e_1, e_2, \ldots, e_n\}$. Let

$$A_r := [(e_i - e_j);\ 1 \leq i < j \leq n]$$

be the root system of type A and rank r.

Let e^i be the dual basis to e_i and

$$V = \{v = \sum_{i=1}^{n} v_i e^i;\ \sum_{i=1}^{n} v_i = 0\}.$$

Let $z = \sum_{i=1}^{n} z^i e_i$ be in U (hence $\sum_{i=1}^{n} z^i = 0$) and let $\mathcal{H}_r^A$ be the system of hyperplanes in U given by

$$\mathcal{H}_r^A = \cup_{1 \leq i < j \leq n} \{z^i - z^j = 0\}.$$

We take the set

$$\Phi^{eq}(A_r) = \{e^i - e^j; 1 \leq i < j \leq n\}$$

of positive coroots as equations of $\mathcal{H}_r^A$.

One way to find a diagonal basis of $\mathcal{S}_{\mathcal{H}_r^A}$ is as follows.

Let $\Sigma = [e^1 - e^2, e^2 - e^3, \ldots, e^r - e^{r+1}]$ be the set of simple coroots. For a permutation w, we denote by $\overrightarrow{\sigma}_w = [e^{w(i)} - e^{w(i+1)},\ i = 1, \ldots, r]$. Then $\overrightarrow{\sigma}_w$ is an ordered basis associated to w, and the corresponding simple fraction is $f_w(z) := \frac{1}{\prod_{i=1}^{r}(z^{w(i)} - z^{w(i+1)})}$.

Let W_r be the subset of the Weyl group Σ_{r+1} of permutations of $\{e^1, e^2, \ldots, e^{r+1}\}$ leaving the last element $e^{r+1} = e^n$ fixed. Recall the following result (see for example Baldoni-Vergne [2] for a proof).

Proposition 2.1. *The set $\overrightarrow{\mathcal{D}}_W$ consisting of the ordered bases $\overrightarrow{\sigma}_w$ for $w \in W_r$ is a diagonal subset of $\overrightarrow{\mathfrak{B}}(\Phi^{eq}(A_r))$.*

We use the above basis in our Maple program.

We now give another interesting diagonal subset.

Consider a sequence $\overrightarrow{\sigma} = [\alpha_2, \alpha_3, \ldots, \alpha_n]$ where $\alpha_i = e^i - e^j$ with $j < i$. That is, $\alpha_2 = e^2 - e^1$, $\alpha_3 = e^3 - e^2$ or $e^3 - e^1$, $\alpha_4 = e^4 - e^3$, or $e^4 - e^2$, or $e^4 - e^1$, etc. Clearly, $\overrightarrow{\sigma}$ is in $\overrightarrow{\mathfrak{B}}(\Phi^{eq}(A_r))$. We call such $\overrightarrow{\sigma}$ a $f$lag basis; there are $r!$ such sequences $\overrightarrow{\sigma}$.

Lemma 2.2. *The set $\overrightarrow{\mathcal{D}}(A_r)$ consisting of flag bases is a diagonal subset of $\overrightarrow{\mathfrak{B}}(\Phi^{eq}(A_r))$.*

We only need to prove that if $\overrightarrow{\sigma}$ and $\overrightarrow{\tau}$ are two flag bases, then $\mathrm{Res}^{\overrightarrow{\sigma}} f_\tau = 0$ unless $\overrightarrow{\sigma} = \overrightarrow{\tau}$. But this is evident.

2.1.2. *The systems of type B_r or C_r.* We consider $V = \mathbb{R}^r$ with standard basis $\{e^1, e^2, \ldots, e^r\}$.

Let
$$B_r = [\pm e_i, \pm(e_i \pm e_j),\, 1 \leq i \leq r,\, 1 \leq i < j \leq r]$$
be the root system of type B and rank r.

Let
$$C_r = [\pm 2e_i, \pm(e_i \pm e_j),\, 1 \leq i \leq r,\, 1 \leq i < j \leq r]$$
be the root system of type C and rank r.

As roots of systems of type B and C are proportional, the system of hyperplanes in $U = \mathfrak{h}^*_{\mathbb{R}}$ defined by coroots of B and C are the same, and we denote it by $\mathcal{H}^{BC}_r$. More precisely, let $z = \sum_{i=1}^r z^i e_i$ in U, then the system of hyperplanes $\mathcal{H}^{BC}_r$ in U is given by
$$\mathcal{H}^{BC}_r = \cup_{1\leq i<j\leq r}\{z^i \pm z^j = 0\} \cup \cup_{1\leq i\leq r}\{z^i = 0\}.$$

We take the set
$$\Phi^{eq}(BC_r) = \cup_{1\leq i<j\leq r}\{e^i \pm e^j = 0\} \cup \cup_{1\leq i\leq r}\{e^i = 0\}$$
as equations of $\mathcal{H}^{BC}_r$.

We define similarly a flag basis $\overrightarrow{\sigma}$ of $\Phi^{eq}(BC_r)$. This is a basis of the form $\overrightarrow{\sigma} = [\alpha_1, \alpha_2, \ldots, \alpha_r]$ of r elements of $\Phi^{eq}(BC_r)$ so that $\alpha_i = e^i$, or $e^i - e^j$ or $e^i + e^j$ with $j < i$. That is, $\alpha_1 = e^1$, $\alpha_2 = e^2$ or $e^2 - e^1$, or $e^2 + e^1$, $\alpha_3 = e^3$ or $e^3 - e^2$, or $e^3 + e^2$, or $e^3 - e^1$, or $e^3 + e^1$, etc. Clearly, there are $(1)(3)(5)\cdots(2r-1)$ such sequences $\overrightarrow{\sigma}$.

Lemma 2.3. *The set $\overrightarrow{\mathcal{D}}(BC_r)$ consisting of flag bases is a diagonal subset of $\overrightarrow{\mathfrak{B}}(\Phi^{eq}(BC_r))$.*

Proof. We first prove, by induction on r, that simple fractions f_b associated to a flag basis b generate $\mathcal{S}_{\mathcal{H}^{BC}_r}$. We use the identities
$$\frac{1}{(x_r - x_i)}\frac{1}{(x_r + x_i)} = \frac{1}{(x_r + x_i)}\frac{1}{2x_r} + \frac{1}{(x_r - x_i)}\frac{1}{2x_r},$$
$$\frac{1}{x_r}\frac{1}{(x_r + x_i)} = -\frac{1}{(x_r + x_i)}\frac{1}{x_i} + \frac{1}{x_r}\frac{1}{x_i},$$
$$\frac{1}{x_r}\frac{1}{(x_r - x_i)} = \frac{1}{(x_r - x_i)}\frac{1}{x_i} - \frac{1}{x_r}\frac{1}{x_i},$$
to reduce to the case where a simple fraction f_b contains a linear form of type e^r, or $e^r + e^i$ or $e^r - e^i$ in the denominator, but not any two at the same time. Then, by induction on r, we see that a simple fraction f_b

associated to flag basis b generates the space $\mathcal{S}_{\mathcal{H}_r^{BC}}$. The dual property on the elements of $\overrightarrow{\mathcal{D}}(BC_r)$ is evident. □

Remark 2.4. Although the system $\Phi^{eq}(BC_r)$ is not unimodular for the lattice $\Lambda = \oplus \mathbb{Z}e^i$, we see that any $\overrightarrow{\sigma}$ in the set $\overrightarrow{\mathcal{D}}(BC_r)$ above is unimodular, so that the computation of $Z^\Lambda(\tau)(f_\sigma)$ is easy.

2.1.3. *The system of type D_r.* We consider $V = \mathbb{R}^r$ with standard basis $\{e^1, e^2, \ldots, e^r\}$. Let

$$D_r = [\pm(e_i \pm e_j);\ 1 \leq i < j \leq r]$$

be the root system of type D and rank r.

Let $z = \sum_{i=1}^r z^i e_i$ in U. We consider the system of hyperplanes

$$\mathcal{H}_r^D = \cup_{1\leq i<j\leq r}\{z^i \pm z^j = 0\}.$$

The dimension of $\mathcal{S}_{\mathcal{H}_r^D}$ is known to be $(1)(3)(5)\cdots(2r-3)(r-1)$. However, we did not find a nice diagonal basis for $\mathcal{S}_{\mathcal{H}_r^D}$.

The set $U_{\mathcal{H}_r^D}$ of regular elements for $\mathcal{H}_r^D$ contains $U_{\mathcal{H}_r^{BC}}$. Indeed, for any z in $U_{\mathcal{H}_r^D}$, we have $z^i \pm z^j \neq 0$, but z^i may equal zero.

We define the set

$$U_{k,r} := \{z^k = 0,\ z^i \pm z^j \neq 0 \text{ for } 1 \leq i < j \leq r \text{ and } z^i \neq 0 \text{ for } 1 \leq i \leq r,\ i \neq k\}.$$

Then, we have the following disjoint decomposition

$$U_{\mathcal{H}_r^D} = U_{\mathcal{H}_r^{BC}} \bigcup \cup_{k=1}^r U_{k,r}.$$

The set $U_{k,r}$ is clearly isomorphic to the open set $U_{\mathcal{H}_{r-1}^{BC}}$ in rank $r-1$ via the map i_k which inserts a zero coordinate in position k, and hence,

$$U_{\mathcal{H}_r^D} = U_{\mathcal{H}_r^{BC}} \bigcup \cup_{k=1}^r i_k(U_{\mathcal{H}_{r-1}^{BC}}). \tag{2.4.1}$$

The above decomposition allows us to reduce calculations in system of type D to that of systems of type B or C.

2.2. **Calculations of multiple Bernoulli series for type A.** We use the same notation as in Section 2.1.1.

Let $Q_A \subset U$ be the root lattice generated by A_r, and $P_A \subset U$ be the weight lattice. Then P_A is generated by Q_A and $e_1 - \frac{1}{r+1}(e_1 + e_2 + \cdots + e_{r+1})$ and P_A/Q_A is of cardinality $r+1$.

Let Γ be a lattice such that $Q_A \subset \Gamma \subset P_A$. We denote by $\Gamma_{reg} = \Gamma \cap U_{\mathcal{H}_r^A}$ the set of regular elements in Γ.

Let $\Lambda \subset V$ be the dual lattice to Γ, and let $\mathbf{s} = [s_\alpha]$ be a series of exponents. Define

$$g_{\mathbf{s}}^A(z) = \frac{1}{\prod_{\alpha>0} \langle H_\alpha, z\rangle^{s_\alpha}},$$

where the set $\{H_\alpha, \alpha > 0\}$ is the set of positive coroots $\Phi^{eq}(A_r)$.

If $v \in V$,

$$\mathcal{B}(\mathcal{H}_r^A, \Lambda, g_{\mathbf{s}}^A)(v) = \sum_{\gamma\in\Gamma_{reg}} \frac{e^{2i\pi\langle v,\gamma\rangle}}{\prod_{\alpha>0}(2i\pi\langle H_\alpha, \gamma\rangle)^{s_\alpha}}.$$

If we use the diagonal basis $\overrightarrow{\mathcal{D}}_W$ for $\Phi^{eq}(A_r)$ as defined in Proposition 2.1, then the diagonal basis consists of elements $[e^{w(1)}-e^{w(2)}, \ldots, e^{w(r)}-e^{r+1}]$ where w is a permutation leaving e^{r+1} stable. Thus if we express $v = \sum_{i=1}^r v_i(e^i - e^{r+1})$, the algebra $\mathit{Step}(\overrightarrow{\mathcal{D}}_W)$ consists of functions $\{\sum_I v_i\}$ where I runs over subsets of $\{1, 2, \ldots, r\}$.

We now discuss two simple cases, where Γ is either the weight lattice P_A, or the root lattice Q_A.

2.2.1. *Bernoulli series for the weight lattice.* If P_A is the weight lattice, then the dual of P_A is the coroot lattice $\check{Q}_A$ generated by simple coroots H_α, and the system $\Phi^{eq}(A_r)$ of equations (the positive coroots) is unimodular with respect to $\check{Q}_A$.

Thus $\mathcal{B}(\mathcal{H}_r^A, \check{Q}_A, g_{\mathbf{s}}^A)(v)$ is a piecewise polynomial function of degree $\sum_\alpha s_\alpha$ and lies in the algebra $\mathit{Step}(\overrightarrow{\mathcal{D}}_W)$. Our program then gives $\mathcal{B}(\mathcal{H}_r^A, \check{Q}_A, g_{\mathbf{s}}^A)(v)$ as a polynomial expression of the functions $\{\sum_I v_i\}$. It also computes numerically the value of this function at any point v.

Example 2.5. Consider the root system of type A and of rank $r = 2$. We assume all multiplicities $s_\alpha = 1$, and compute $\mathcal{B}(\mathcal{H}_2^A, \check{Q}_A, g_{\mathbf{s}}^A)(v)$ for $v = v_1e^1 + v_2e^2 + v_3e^3$ with $v_1 + v_2 + v_3 = 0$. The simple coroots are $H_{\alpha_1} = e^1 - e^2$ and $H_{\alpha_2} = e^2 - e^3$, and the remaining positive coroot is their sum $H_{\alpha_1} + H_{\alpha_2}$. The dual lattice has basis dual to H_{α_1}, H_{α_2}. Thus, for $\mathbf{s} = [1, 1, 1]$,

$$\mathcal{B}(\mathcal{H}_2^A, \check{Q}_A, g_{\mathbf{s}}^A)(v) = {\sum_{m,n\in\mathbb{Z}}}' \frac{e^{2i\pi m v_1 - 2i\pi n v_3}}{(2i\pi m)(2i\pi n)(2i\pi(m+n))}. \tag{2.5.1}$$

The symbol $\sum'$ above means that we sum over the integers m, n with $mn(m + n) \neq 0$.

Denoting the fractional part of t with $\{t\} \in [0, 1[$ as before, we obtain that $P(v_1, v_2, v_3) = \mathcal{B}(\mathcal{H}_2^A, \check{Q}_A, g_{\mathbf{s}}^A)(v_1e^1 + v_2e^2 + v_3e^3)$ is equal to

(2.5.2)

$$\frac{1}{6}(\{v_2\}-\{v_1\})(\{v_1\}^2-3\{v_1+v_2\}\{v_1\}+\{v_2\}\{v_1\}+3\{v_1+v_2\}-1-3\{v_1+v_2\}\{v_2\}+\{v_2\}^2).$$

We remark that the series (2.5.1) is not absolutely convergent, but the sum has a meaning and is a piecewise polynomial function.

Let us give some numerical examples. Consider again A_2. Suppose $\mathbf{s} = [10, 10, 10]$ and $v_1 = v_2 = 0$. Then

$$\mathcal{B}(\mathcal{H}_2^A, \check{Q}_A, g_{\mathbf{s}}^A)(0) = {\sum_{m,n}}' \frac{1}{(2i\pi m)^{10}(2i\pi n)^{10}(2i\pi(m+n))^{10}}$$

is equal to

$$-\frac{27739097}{41746719321210992766914396160000000}.$$

Consider now the system of type A and rank $r = 4$. Suppose $v = [0, 0, 0, 0, 0]$. We list the exponents with respect to the following order on the roots

$[e_1 - e_2, e_1 - e_3, e_1 - e_4, e_1 - e_5, e_2 - e_3, e_2 - e_4, e_2 - e_5, e_3 - e_4, e_3 - e_5, e_4 - e_5]$.

Then for $\mathbf{s} = [6, 6, 6, 6, 4, 2, 2, 2, 2, 2]$ we have

$$\mathcal{B}(\mathcal{H}_4^A, \check{Q}_A, g_{\mathbf{s}}^A)(0) = \frac{1}{(2i\pi)^{38}} \times$$

$${\sum}' \frac{1}{m_1^6 m_2^4 m_3^2 m_4^2 (m_1+m_2)^6 (m_1+m_2+m_3)^6 (m_1+m_2+m_3+m_4)^6 (m_2+m_3)^2 (m_2+m_3+m_4)^2 (m_3+m_4)^2}$$

$$= \frac{66581757}{20814165388976983019020695652962140160000000000},$$

while for $\mathbf{s} = [4, 4, 4, 4, 4, 4, 4, 4, 4, 4]$ we obtain

$$\mathcal{B}(\mathcal{H}_4^A, \check{Q}_A, g_{\mathbf{s}}^A)(0) = \frac{3998447009863}{193188341191020986049682108358620340866252800000000000}.$$

2.2.2. *Bernoulli series for the root lattice.* Let $\xi = \sum_{j=1}^{r}(e^j - e^{r+1})$. Then a system of representatives for $\check{P}_A/\check{Q}_A$ consists of the elements $\lambda_j = \frac{j}{r+1}\xi$, with j varying between 0 and r. Thus, using Formula (1.28.1),

$$\mathcal{B}(\mathcal{H}_r^A, \check{P}_A, g_{\mathbf{s}}^A)(v) = \frac{1}{r+1} \sum_{j=0}^{r} \mathcal{B}(\mathcal{H}_r^A, \check{Q}_A, g_{\mathbf{s}}^A)(v + \lambda_j).$$

We obtain an expression for $\mathcal{B}(\mathcal{H}_r^A, \check{P}_A, g_{\mathbf{s}}^A)$ in terms of the functions $\{(\sum_I v_i) + c/(r+1)\}$ where c are integers between 0 and r.

Recall Example 2.5. For the same data, we now compute $\mathcal{B}(\mathcal{H}_2^A, \check{P}_A, g_{\mathbf{s}}^A)$ for $v = v_1e^1 + v_2e^2 + v_3e^3$ with $v_1 + v_2 + v_3 = 0$. Hence, we express $v = v_1(e^1 - e^2) - v_3(e^2 - e^3)$, and compute for $\mathbf{s} = [1, 1, 1]$:

$$\mathcal{B}(\mathcal{H}_2^A, \check{P}_A, g_{\mathbf{s}}^A)(v) = \sum_{m,n}{}' \frac{e^{2i\pi m(v_1-v_2)+2i\pi n(v_2-v_3)}}{(2i\pi(2m-n))(2i\pi(2n-m))(2i\pi(m+n))}.$$

This is equal to

$$\frac{1}{3}\left(P(v_1, v_2, v_3) + P(v_1 + \frac{1}{3}, v_2 + \frac{1}{3}, v_3 - \frac{2}{3}) + P(v_1 + \frac{2}{3}, v_2 + \frac{2}{3}, v_3 - \frac{4}{3})\right)$$

where P is the piecewise polynomial function given in Equation (2.5.2).

2.3. **Calculation of multiple Bernoulli series for type C and B.**

We use the same notation as in Section 2.1.2.

We now consider the system $\mathcal{H}_r^{BC}$ of hyperplanes in U

$$\mathcal{H}_r^{BC} = \cup_{1\leq i\leq r}\{z^i = 0\} \cup \cup_{1\leq i<j\leq r}\{z^i \pm z^j = 0\}.$$

Let Λ be a lattice commensurable with $\oplus\mathbb{Z}e^i$, with dual lattice Γ. Denote simply by $\Gamma_{reg} = \Gamma_{\mathrm{reg}}(\mathcal{H}_r^{BC})$. If $g \in \mathcal{R}_{\mathcal{H}_r^{BC}}$,

$$\mathcal{B}(\mathcal{H}_r^{BC}, \Lambda, g)(v) = \sum_{\gamma\in\Gamma_{\mathrm{reg}}} g(2i\pi\gamma)e^{2i\pi\langle v,\gamma\rangle}. \tag{2.5.3}$$

2.3.1. *Root system C_r.* Let P_C be the weight lattice of the root system C_r. We thus have the coroot lattice $\check{Q}_C = \oplus_{i=1}^r \mathbb{Z}e^i$.

Let $\mathbf{s} = [s_\alpha]$ be a series of exponents and let

$$g_{\mathbf{s}}^C(z) = \frac{1}{\prod_{\alpha>0}\langle H_\alpha, z\rangle^{s_\alpha}}.$$

Here $\{H_\alpha,\ \alpha > 0\}$ are positive coroots of the system C_r, which are explicitly

$$\{e^i, 1 \leq i \leq r, (e^i \pm e^j), 1 \leq i < j \leq r\}.$$

Clearly, the function $g_{\mathbf{s}}^C$ belongs to $\mathcal{R}_{\mathcal{H}_r^{BC}}$.

If $v \in V$,

$$\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v) = \sum_{\gamma\in(P_C)_{reg}} \frac{e^{2i\pi\langle v,\gamma\rangle}}{\prod_{\alpha>0}(2i\pi\langle H_\alpha, \gamma\rangle)^{s_\alpha}}.$$

The function $\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)$ is a piecewise polynomial function on V of degree $\sum_\alpha s_\alpha$. We use the diagonal basis constructed in Section 2.1.2 to compute it. Let us now compute the example given in the introduction, which corresponds to C_2, and the exponent $\mathbf{s} = [2, 1, 1, 1]$

for the order of roots $[2e_1, 2e_2, e_1+e_2, e_1-e_2]$ (so that coroots H_α are in order $[e^1, e^2, e^1+e^2, e^1-e^2]$) . We express $v = v_1 e^1 + v_2 e^2$ and compute

$$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v) = \sum_{m,n}{}' \frac{e^{2i\pi m v_1 + 2i\pi n v_2}}{(2i\pi m)^2 (2i\pi n)(2i\pi(m+n))(2i\pi(m-n))}.$$

This piecewise polynomial function is given by

(2.5.4)

$$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v) = Q(v_1, v_2) =$$

$$-\frac{1}{160}\{-v_2+v_1\}^5 - 1/48\{v_1\}^2 + 1/24\{v_1\}^3 + 1/24\{v_2+v_1\}^3\{v_2\} - 1/48\{v_2+v_1\}^4\{v_2\} - 1/48\{v_2+v_1\}^2\{v_2\} - \frac{1}{960}\{v_2+v_1\} + \frac{1}{96}\{v_2+v_1\}^2 - \frac{1}{96}\{v_2+v_1\}^3 - \frac{1}{192}\{v_2+v_1\}^4 + \frac{1}{960}\{-v_2+v_1\} + \frac{1}{96}\{-v_2+v_1\}^2 - 1/32\{-v_2+v_1\}^3 + \frac{5}{192}\{-v_2+v_1\}^4 - 1/48\{v_1\}^4 + 1/24\{v_1\}^2\{v_2\} - 1/12\{v_1\}^3\{v_2\} + 1/24\{v_1\}^4\{v_2\} + \frac{1}{160}\{v_2+v_1\}^5 + 1/24\{-v_2+v_1\}^3\{v_2\} - 1/48\{-v_2+v_1\}^4\{v_2\} - 1/48\{-v_2+v_1\}^2\{v_2\}.$$

Let's see what happens on a tope. Figure 2 depicts topes associated to the pair $\Phi^{eq}(BC_2) = \{e^1, e^2, e^1+e^2, e^1-e^2\}$ and $\Lambda = \mathbb{Z}e^1 \oplus \mathbb{Z}e^2$.

Consider for example the tope

$$\tau_2 = \{v_1 > 0,\ v_2 > 0,\ v_1 > v_2,\ v_1 + v_2 < 1\}.$$

Then on τ_2, the piecewise polynomial function $\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v)$ coincides with the polynomial

(2.5.5) $$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v) = Q_{\tau_2}(v_1, v_2) =$$

$$\frac{1}{8}\Big(-\frac{1}{60}v_2 + 1/2\, {v_1}^2 v_2 - {v_1}^3 v_2 + 1/6\, {v_2}^2 - v_1 {v_2}^2 + v_1 {v_2}^3 + {v_1}^2 {v_2}^2 - 1/6\, {v_2}^3 + 1/6\, {v_2}^4 + 1/2\, {v_1}^4 v_2 - {v_1}^2 {v_2}^3 - \frac{7}{30} {v_2}^5\Big)$$

If we compute $Q_{\tau_2}(v_1, v_2)$ for $v_1 = \frac{1}{15},\ v_2 = \frac{1}{30}$ we obtain $-\frac{276037}{5832000000}$.

We give some more numerical examples with different exponents.

For example, we may compute with exponents $\mathbf{s} = [s_1, s_2, s_3, s_4]$ associated to the order of roots $[2e_1, 2e_2, e_1+e_2, e_1-e_2]$ and $v = [v_1, v_2]$

$$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v_1 e^1 + v_2 e^2) = \sum_{m,n}{}' \frac{e^{2i\pi m v_1 + 2i\pi n v_2}}{(2i\pi m)^{s_1}(2i\pi n)^{s_2}(2i\pi(m+n))^{s_3}(2i\pi(m-n))^{s_4}}$$

$$= \begin{cases} \frac{810650239}{132316540312500} & if \quad \mathbf{s}=[2,2,1,1] \quad and \quad \mathrm{v}=[1/5,\ 1/19] \\ \frac{4703611043885476130163645994 1}{15291744295791972509433253459771267822387 20} & if \quad \mathbf{s}=[2,3,4,5] \quad and \quad \mathrm{v}=[1/7,\ 1/17] \end{cases}$$

2.3.2. *Root system B_r.* We consider the root system of type B and rank r. Let $\Gamma = P_B$ be the lattice of weights of B, and as before the dual lattice generated by the coroots is denoted by $\check{Q}_B$.

Let $\mathbf{s} = [s_\alpha]$ be a series of exponents and

$$g_{\mathbf{s}}^B(z) = \frac{1}{\prod_{\alpha>0} \langle H_\alpha, z\rangle^{s_\alpha}}.$$

Here, again, $\{H_\alpha,\ \alpha > 0\}$ are positive coroots of the system B_r.

If $v \in V$, $\mathbf{s} = [s_\alpha]$ then

$$\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) = \sum_{\gamma\in(P_B)_{reg}} \frac{e^{2i\pi\langle v,\gamma\rangle}}{\prod_{\alpha>0}(2i\pi\langle H_\alpha,\gamma\rangle)^{s_\alpha}}.$$

Clearly, as long coroots of B are twice the short coroots of C, and short coroots of B are long coroots of C, we have

$$g_{\mathbf{s}}^B = c_1 g_{\mathbf{s}}^C, \text{ where } c_1 = \frac{1}{2^{s_{2e_1}+\cdots+s_{2e_r}}}.$$

We then use the comparison formula with two lattices. Indeed, we have $2\check{Q}_C \subset \check{Q}_B$. The lattice $2\check{Q}_C$ is of index 2^{r-1} in $\check{Q}_B$ and a set of representatives is given, for example, by

$$F := \{0, e^{i_1}+e^{i_2}+\cdots+e^{i_k}, 1 \le i_1 < i_2 < \cdots < i_k,\ k = 2j,\ 1 \le j \le [r/2]\}.$$

We then use Formulae (1.28.1) and (1.28.2). Since $g_{\mathbf{s}}^C$ is homogeneous of degree $-\sum_\alpha s_\alpha$, we obtain

$$\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) = \frac{1}{2^{r-1}} c_2 \left(\sum_{\lambda\in F} \mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(\frac{v+\lambda}{2})\right)$$

where $c_2 = 2^{\sum_{1\le i<j\le r} s_{e_i-e_j}+s_{e_i+e_j}}$. In particular, if $\mathbf{s} = [m,\ldots,m]$, then $c_2 = 2^{r(r-1)m}$.

For example, for B_2, we compute for $v = v_1e^1+v_2e^2$ and multiplicities $\mathbf{s} = [2,1,1,1]$ with respect to the order $[e_1-e_2, e_2, e_1+e_2, e_1]$ of roots,

$$(2i\pi)^5 \mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) = \sum_{m_1,m_2}{}' \frac{e^{2i\pi((m_1+1/2\,m_2)v_1+1/2\,m_2v_2)}}{m_1^2 m_2\,(2m_1+\ m_2)\,(m_2+m_1)}$$

where the symbol $\sum'$ means that we sum over the m_1, m_2 with

$$(2m_1+\ m_2)\,(m_2+m_1)\,m_2m_1 \neq 0.$$

We obtain

$$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) = 2\left(Q(\frac{v_1}{2}, \frac{v_2}{2}) + Q(\frac{v_1+1}{2}, \frac{v_2+1}{2})\right)$$

where $Q = \mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)$ is given in Equation (2.5.4).

In particular, for $u = [1/15, 1/30]$ we obtain

$$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(u) = \frac{-276037}{5832000000}.$$

For B_3, we compute for $v = v_1 e^1 + v_2 e^2 + v_3 e^3$ and $\mathbf{s} = [1, 1, 1, 1, 1, 1, 1, 1, 1]$:

$$(2i\pi)^9 \mathcal{B}(\mathcal{H}_3^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) =$$

$$\sum_{m_1, m_2, m_3}' \frac{e^{2i\pi((m_1+m_2+1/2\, m_3)v_1 + (m_2+1/2\, m_3)v_2 + 1/2\, m_3 v_3)}}{(2m_1 + 2m_2 + m_3)(2m_2 + m_3)\, m_3 m_1\, (m_1 + 2\, m_2 + m_3)(m_1 + m_2)(m_1 + m_2 + m_3)\, m_2\, (m_2 + m_3)}.$$

We obtain

$$\mathcal{B}(\mathcal{H}_3^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) =$$

$$2^4 \left(Q(\frac{v_1}{2}, \frac{v_2}{2}, \frac{v_3}{2}) + Q(\frac{v_1+1}{2}, \frac{v_2+1}{2}, \frac{v_3}{2}) + Q(\frac{v_1+1}{2}, \frac{v_2}{2}, \frac{v_3+1}{2}) + Q(\frac{v_1}{2}, \frac{v_2+1}{2}, \frac{v_3+1}{2}) \right)$$

where $Q = \mathcal{B}(\mathcal{H}_3^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)$ is a piecewise polynomial that is too long to be included here.

2.4. **Calculation of multiple Bernoulli series for type D.** We follow the same notation as in Section 2.1.3.

Let $\mathcal{H}_r^D = \cup_{i,j}\{z^i \pm z^j = 0, 1 \le i < j \le r\}$.

Let $\mathbf{s} = [s_\alpha]$ be a list of exponents for positive roots of D_r. The ordering of s_α in the list $\mathbf{s}$ is taken to match the following ordering of the roots $[e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_r, e_2 - e_3, \ldots, e_1 + e_2, \ldots, e_{r-1} + e_r]$.

We denote by

$$g_{\mathbf{s}}^D(z) = \frac{1}{\prod_{\alpha>0} \langle H_\alpha, z\rangle^{s_\alpha}}.$$

Here $\{H_\alpha,\ \alpha > 0\}$ are positive coroots of the system D_r.

We embed the list of roots of D_r in the list of roots of B_r, writing the short roots e_i at the end. We denote by $\mathbf{S} = [s_\alpha, 0, \ldots, 0]$ the list obtained from $\mathbf{s}$ by adding to it a list of r zeros. Thus

$$g_{\mathbf{s}}^D(z) = g_{\mathbf{S}}^B(z).$$

We now associate to the list $\mathbf{s}$ a list of exponents $\mathbf{s}_k$ for the system B of rank $r-1$. We eliminate the position corresponding to the roots $e_i \pm e_k$, then add the element $s_{e_i+e_k} + s_{e_i-e_k}$ as exponent of the root e_i. We also let $i_k(v)$ to be the vector with $r-1$ coordinates obtained from $v = \sum_{i=1}^r v_i e^i$ by putting $v_k = 0$.

Let $\Gamma = P_D$ be the weight lattice of D and $\check{Q}_D$ the dual lattice generated by the coroots. Since P_D is the weight lattice of the simply connected group $Spin(2r)$, $\gamma = \sum_{i=1}^r \gamma^i e_i$ is in P_D if $\gamma^i \pm \gamma^j \in \mathbb{Z}$ and $P_D = P_B$. Consider the intersection of P_D with the hyperplane $z^k = 0$. Then, we see that this intersection is isomorphic to the weight lattice of a system $C_{r-1,k}$ of type C, rank $r-1$, embedded in C of rank r with simple roots $\{e_1 - e_2, e_2 - e_3, \ldots, e_{k-1} - e_{k+1}, e_{k+1} - e_{k+2}, \ldots, e_r\}$.

Using the decomposition (2.4.1), we decompose the set of regular elements of the lattice P_D as a disjoint set of regular elements of the lattice P_B, union a disjoint set of regular elements of the lattice P_C of rank $r-1$.

In particular, if $\gamma \in (P_D)_{reg}$ and $\gamma_k = 0$, then

$$\frac{1}{\prod\limits_{\alpha>0}(2i\pi\langle H_\alpha, \gamma\rangle)^{s_\alpha}} =$$

$$\frac{1}{\prod\limits_{\gamma_i \pm \gamma_j \neq 0, \gamma_k = 0}(2i\pi(\gamma_i - \gamma_j))^{s_{e_i - e_j}}(2i\pi(\gamma_i + \gamma_j))^{s_{e_i + e_j}}} = c_k g^C_{\mathbf{s}_k}(i_k(\gamma))$$

with $c_k = (-1)^{\sum_{j=k+1}^r s_{e_k - e_j}}$. In particular, if $\mathbf{s} = [m, \ldots, m]$, then $c_k = (-1)^{(r-k)m}$.

Thus, we can compute multiple Bernoulli series for a system of type D_r by using computations for types B_r and C_{r-1} with appropriate exponents. More explicitly,

$$\mathcal{B}(\mathcal{H}^D_r, \check{Q}_D, g^D_{\mathbf{s}})(v) = \sum_{\gamma \in (P_D)_{reg}} \frac{e^{2i\pi\langle v, \gamma\rangle}}{\prod_{\alpha>0}(2i\pi\langle H_\alpha, \gamma\rangle)^{s_\alpha}} =$$

$$\mathcal{B}(\mathcal{H}^{BC}_r, \check{Q}_B, g^B_{\mathbf{S}})(v) + \sum_{k=1}^r c_k \mathcal{B}(\mathcal{H}^{BC}_{r-1}, \check{Q}_{C_{r-1,k}}, g^C_{\mathbf{s}_k})(i_k(v)).$$

3. Witten formula for volumes of moduli spaces of flat connections on surfaces

Let G be a simple, simply connected, compact Lie group of rank r with maximal torus T. For $g_1, g_2 \in G$, we denote by $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ the commutator of g_1, g_2. Let Σ be a compact connected oriented surface of genus g and let $p := \cup_j \{p_j\}$ be a set of s points on Σ. Let $\mathcal{C} := (\mathcal{C}_j)$ be a set of s conjugacy classes in G. We consider the representation variety

$$\mathcal{M}(G, g, s, \mathcal{C}) := \{(a, c) \in G^{2g} \times \mathcal{C};\ \prod_{i=1}^g [a_{2i-1}, a_{2i}] = \prod_{j=1}^s c_j\}/G.$$

If the adjoint orbits $\mathcal{C}_j$ are generic, this is an orbifold of dimension $(2g-2)\dim G + s \dim G/T$. It parameterizes the set of flat G-valued connections on $\Sigma - p$, with holonomy around p_j belonging to the conjugacy class $\mathcal{C}_j$ modulo gauge equivalence. As shown by Atiyah-Bott [1], once a G-invariant inner product on $\mathfrak{g}$ is chosen, the manifold

$\mathcal{M}(G, g, s, \mathcal{C})$ carries a natural symplectic form, and Witten gave a formula for the volume of $\mathcal{M}(G, g, s, \mathcal{C})$ that we recall.

We use the notation of Section 2. We need some more definitions.

Let R^+ be a choice of positive roots; denote the highest root of R by θ. Let $\mathfrak{h}_+ := \{h \in \mathfrak{h}_{\mathbb{R}};\ \alpha(h) \geq 0 \text{ for all } \alpha \in R^+\}$ be the positive chamber (closed) in $\mathfrak{h}_{\mathbb{R}}$. Let $\mathfrak{A} := \{h \in \mathfrak{h}_+;\ \theta(h) \leq 1\}$ be the fundamental alcove. An element of $\mathfrak{A}$ is said to be *regular* if it lies strictly inside the alcove.

Let $W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ be the Weyl group (identified with $N_G(T)/T$).

We now give the Witten formula.

Let $\mathbf{a} = \{a_1, a_2, \ldots, a_s\}$ be a set of regular elements in $\mathfrak{A} \subset \mathfrak{h}_+$. Let $\mathcal{C}_j$ be the adjoint orbit of $\exp(a_j)$; we denote the collection of orbits $\mathcal{C}_j$ by $\mathcal{C}$.

Consider the function on $\mathfrak{h}^*$ given by

$$N_{\mathbf{a}}(\lambda) = \prod_{j=1}^{s} \sum_{w \in W} \varepsilon(w) e^{\langle w a_j, \lambda \rangle}.$$

Let $\Phi = \Phi(G)$ be the list of positive coroots H_α. Define

$$W(\Phi(G), P, g, s)(\mathbf{a}) := \sum_{\gamma \in P^{reg}} \frac{N_{\mathbf{a}}(2i\pi\gamma)}{\prod_{H_\alpha \in \Phi} (2i\pi \langle H_\alpha, \gamma \rangle)^{2g-2+s}}.$$

The above expression is always meaningful as a generalized function of the parameters a_j. If $s = 0$, this formula has to be understood as

$$W(\Phi(G), P, g) = \sum_{\gamma \in P^{reg}} \frac{1}{\prod_{H_\alpha \in \Phi} (2i\pi \langle H_\alpha, \gamma \rangle)^{2g-2}}$$

which is meaningful if $g \geq 2$.

Interchanging the sum and the product, $N_{\mathbf{a}}(\lambda)$ may be expressed as

$$N_{\mathbf{a}}(\lambda) = \sum_{(w_1, w_2, \ldots, w_s) \in W^s} \prod_{j=1}^{s} \varepsilon(w_j) e^{\sum_{j=1}^{s} \langle w_j a_j, \lambda \rangle}.$$

Hence the function $W(\Phi(G), P, g, s)(\mathbf{a})$ can be expressed as a sum over W^s with signs of Bernoulli series $\mathcal{B}(\Phi_{2g-2+s}, \check{Q})(\sum_j w_j a_j)$. Here, as before, Φ_{2g-2+s} means that each coroot in Φ is taken with multiplicity $2g - 2 + s$.

As is well known, the series $W(\Phi(G), P, g, s)(\mathbf{a})$ computes the symplectic volume of $\mathcal{M}(G, g, s, \mathcal{C})$ up to a scalar factor, which we will give in the next section.

Let us now recall the normalization of the volume as the limit of the Verlinde formula.

We need some more notation.

Let $(\ |\ )$ denote the G-invariant symmetric form on $\mathfrak{g}_{\mathbb{C}}$ normalized such that $(H_\theta|H_\theta) = 2$. We will use the same notation for the restricted form on $\mathfrak{h}$, and the induced form on $\mathfrak{h}^*$. We call $(\ |\ )$ the *basic invariant form.* It is positive definite on the real span $\mathfrak{h}_{\mathbb{R}}$, and negative definite on $\mathfrak{t}$.

Let $\check{h} := \rho(H_\theta) + 1$ be the dual Coxeter number, where ρ is the half sum of positive roots. Let $Q_{long} \subset Q$ be the lattice spanned by long roots. The basic invariant form identifies $\mathfrak{h}_{\mathbb{R}}$ and ${\mathfrak{h}_{\mathbb{R}}}^*$; under this isomorphism the coroot lattice $\check{Q}$ is identified to Q_{long}. Let q be the index of Q_{long} in Q, and let f be the index of Q in P. Let $Z = Z(G)$ denote the center of G.

For a positive integer ℓ, define the set

$$P_\ell := \{\mu \in P \cap \mathfrak{h}_+^*; \mu(H_\theta) \leq \ell\}.$$

An element of P_ℓ is said to be a weight of *level* ℓ. We denote by P'_ℓ the subset of P_ℓ consisting of elements μ satisfying $\mu(H_\theta) < \ell$ and $\mu(H_\alpha) > 0$ for any simple root α. By definition of $\check{h}$, there is a bijection between sets P_ℓ and $P'_{\ell+\check{h}}$ via $\mu \mapsto \mu + \rho$.

Consider the maximal torus T of G with Lie algebra $\mathfrak{t}$. If $t = \exp X \in T$, with $X \in \mathfrak{t}$, and α is a root (which takes imaginary value on $\mathfrak{t}$), we denote by $e^\alpha(t) = e^{\langle \alpha, X\rangle}$. Let $\Delta(t) = \prod_{\alpha\in R}(e^\alpha(t) - 1)$. An element of T is said to be regular if $\Delta(t) \neq 0$. Denote by T_ℓ the subgroup of elements t of T such that $e^\alpha(t)$ is $\ell + \check{h}$ root of unity for each long root α. We denote the set of regular elements in T_ℓ by T_ℓ^{reg}.

We now give the Verlinde formula.

Consider the set $\underline{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ with $\lambda_i \in P_\ell$. Then to this collection of weights $\underline{\lambda}$ of level ℓ, the group G, and a nonnegative integer g, is associated a vector space $\mathcal{V}(G, g, s, \underline{\lambda}, \ell)$ (see [15]), called the space of conformal blocks, whose dimension is given by the Verlinde formula $V(G, \underline{\lambda}, g, \ell)$:

$$V(G, \underline{\lambda}, g, \ell) = (fq)^{g-1}(\ell + \check{h})^{r(g-1)} \sum_{t\in T_\ell^{\mathrm{reg}}/W} \frac{\chi_{V(\underline{\lambda})}(t)}{\Delta(t)^{g-1}}.$$

Above r is the rank of G, $V(\underline{\lambda}) = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_s}$ where V_{λ_i} denotes the simple $\mathfrak{g}$ module with highest weight λ_i, and χ_{V_λ} denotes the character of V_λ. By Weyl character formula $\chi_{V_\lambda} = J(e^{\lambda+\rho})/J(e^\rho)$ where $J(e^\nu) = \sum_{w\in W} \varepsilon(w)e^{w\nu}$.

We remark that if $\sum_{i=1}^s \lambda_i$ is not in the root lattice, then $V(G,\underline{\lambda},g,\ell)$ is zero.

Under the isomorphism given by the basic invariant form, an element a lying in $\mathfrak{A} \subset \mathfrak{h}_+$ defines an element $\hat{a}$ of $\mathfrak{h}_+^*$. We now consider a collection $\{a_1, a_2, \ldots, a_s\}$ of rational elements in $\mathfrak{A}$, that is, each a_j lies in the dense subset $\mathfrak{A}\cap(\check{Q}\otimes\mathbb{Q}) \subset \mathfrak{A}$. We may choose ℓ large enough so that each $\lambda_j := \ell\hat{a}_j$ is a weight; which then lies in P_ℓ. We furthermore choose ℓ so that $\sum_{j=1}^s \lambda_j$ is in the root lattice and consider the space of conformal blocks $\mathcal{V}(G,g,s,\underline{\lambda},\ell)$ associated to this collection $\underline{\lambda} = \{\lambda_1,\lambda_2,\ldots,\lambda_s\}$. We can dilate simultaneously the weights λ_j and the level ℓ by a factor k. Then, the function

$$k \to \dim(\mathcal{V}(G,g,s,[k\lambda_1,k\lambda_2,\cdots,k\lambda_s],k\ell))$$

is a quasi-polynomial in k of degree $m = \dim(G)(g-1) + s|R^+|$, the complex dimension of the moduli space $\mathcal{M}(G,g,s,\mathcal{C})$. The volume computes the highest term of this quasi-polynomial. More precisely,

$$\mathrm{vol}(\mathcal{M}(G,g,s,\mathcal{C})) = \lim_{k\to\infty} (\ell k)^{-m} dim(\mathcal{V}(G,g,s,[k\ell\hat{a}_1,\ldots,k\ell\hat{a}_s],k\ell).$$

Proposition 3.1. *Let $\mathbf{a} = \{a_1,\cdots,a_s\}$ be a collection of regular rational elements in $\mathfrak{A}$. Let $\mathrm{vol}(G,g)(\mathbf{a})$ denote the symplectic volume of the moduli space $\mathcal{M}(G,g,s,\mathcal{C})$. Then,*

$$\mathrm{vol}(G,g)(\mathbf{a}) = (fq)^{g-1}\frac{|Z|}{|W|}2^{p(2g-2+s)}(-1)^{(g-1)|\Phi(G)|}W(\Phi(G),P,g,s)(\mathbf{a}),$$

where p is the number of short positive roots of G.

We recall that for simply laced groups $p = 0$ since all roots are considered as long.

Proof. Choose ℓ so that each $\lambda_j := \ell\hat{a}_j$ lies in P_ℓ and $\sum_{j=1}^s \lambda_j$ is a root. Then,

$$\mathrm{vol}(G,g)(\mathbf{a}) = \lim_{k\to\infty} \frac{1}{(k\ell)^m} V(G,k\underline{\lambda},g,k\ell),$$

where $m = \dim(G)(g-1)+s|R^+|$ is the dimension of the moduli space $\mathcal{M}(G,g,s,\mathcal{C})$ and

$$V(G,k\underline{\lambda},g,k\ell) = (fq)^{g-1}(k\ell+\check{h})^{r(g-1)} \sum_{t\in T_{k\ell}^{\mathrm{reg}}/W} \frac{\chi_{V(k\underline{\lambda})}(t)}{\Delta(t)^{g-1}}.$$

An element $\mu \in P_\ell$ determines a unique regular element $h_\mu \in \mathfrak{A}$, the image of $\frac{\mu+\rho}{\ell+\check{h}}$ under the identification given by the basic invariant form. Denote the image of h_μ under the exponential map by $t_\mu \in T_{\mathbb{C}}$. The set $\{t_\mu : \mu \in P_\ell\}$ form a set of representatives for T_ℓ^{reg}/W. Using also the bijection between sets P_ℓ and $P'_{\ell+\check{h}}$ via $\mu \mapsto \mu+\rho$,

$$\begin{aligned} V(G, k\underline{\lambda}, g, k\ell) &= (fq)^{g-1}(k\ell+\check{h})^{r(g-1)} \sum_{\mu+\rho \in P'_{k\ell+\check{h}}} \frac{\chi_{V(k\underline{\lambda})}(t_\mu)}{\Delta(t_\mu)^{g-1}} \\ &= (fq)^{g-1}(k\ell+\check{h})^{r(g-1)}|W|^{-1} \sum_{\mu+\rho \in W\cdot P'_{k\ell+\check{h}}} \frac{\prod_{j=1}^s J(e^{k\lambda_j+\rho})(t_\mu)}{(J(e^\rho)(t_\mu))^s \Delta(t_\mu)^{g-1}} \\ &= \tfrac{(fq)^{g-1}}{|W|}(k\ell+\check{h})^{r(g-1)}(-1)^{|\Delta^+|(g-1)} \sum_{\mu+\rho \in W\cdot P'_{k\ell+\check{h}}} \frac{\prod_{j=1}^s J(e^{k\lambda_j+\rho})(t_\mu)}{(J(e^\rho)(t_\mu))^{2g-2+s}}. \end{aligned}$$

The second line above follows from the fact that both $\chi_{V(\lambda)}(t)$ and $\Delta(t)$ are W-invariant. The third line follows from the second by the identity $\Delta(t) = J(e^\rho)(t)\overline{J(e^\rho)(t)} = (-1)^{|R^+|}(J(e^\rho)(t))^2$.

We now analyze the above formula as k gets large.

The expression

$$\prod_{j=1}^s J(e^{k\lambda_j+\rho})(t_\mu) = \prod_{j=1}^s \sum_{w\in W} \varepsilon(w) e^{w(k\lambda_j+\rho)}(t_\mu)$$

$$= \prod_{j=1}^s \sum_{w\in W} \varepsilon(w) \exp\left(2i\pi\left(\mu+\rho | \frac{w(k\ell\hat{a}_j+\rho)}{k\ell+\check{h}}\right)\right).$$

Now as k gets large, the expression $\exp\left(2i\pi\left(\mu+\rho|\frac{w(k\ell\hat{a}_j+\rho)}{k\ell+\check{h}}\right)\right)$ approaches to $\exp(2i\pi(\mu+\rho|w\hat{a}_j))$. Observe also that the set $W\cdot P'_{k\ell+\check{h}}$ approaches P^{reg}. Denote an element $\mu+\rho$ of this limiting set by γ. Hence, $\prod_{j=1}^s J(e^{k\lambda_j+\rho})(t_\mu)$ approaches $\prod_{j=1}^s \sum_{w\in W} \varepsilon(w) e^{\langle 2i\pi\gamma, wa_j\rangle} = N_{\mathbf{a}}(2i\pi\gamma)$.

Now we analyze the denominator of the summand,

$$\frac{1}{J(e^\rho)(t_\mu)} = \frac{1}{\prod_{\alpha>0}(e^{\alpha/2}(t_\mu) - e^{-\alpha/2}(t_\mu))} = \frac{1}{\prod_{\alpha>0} 2i\sin(\pi\frac{(\alpha|\mu+\rho)}{k\ell+\check{h}})}.$$

This expression explodes at each central vertex and the contribution from each, as k gets large, is $\frac{(k\ell+\check{h})^{|R^+|}}{\prod_{\alpha>0} 2i\pi(\alpha|\mu+\rho)}$ Also observe that, for $z \in Z(G)$ both T_ℓ^{reg} and $\Delta(t)$ is invariant under $t \mapsto tz$. Moreover, since $\sum_{i=1}^s \lambda_i$ is in the root lattice by construction, $\chi_{V(k\underline{\lambda})}(t)$ is also invariant. Therefore, we may add all these equal contributions from central vertices. (see also Remark 5.8. [14]). Hence, we get that the expression

$$\sum_{\mu+\rho\in W\cdot P'_{k\ell+\check{h}}} \frac{\prod_{j=1}^{s} J(e^{k\lambda_j+\rho})(t_\mu)}{(J(e^{\rho})(t_\mu))^{2g-2+s}}$$

approaches

$$|Z(G)|(k\ell+\check{h})^{|R^+|(2g-2+s)} \sum_{\gamma\in P^{\mathrm{reg}}} \frac{N_{\mathbf{a}}(2i\pi\gamma)}{\prod_{\alpha>0}(2i\pi(\alpha|\gamma))^{2g-2+s}}.$$

By virtue of the normalization in the basic invariant form, if α is a long root we have $\langle H_\alpha,\gamma\rangle = (\alpha|\gamma)$; otherwise $\langle H_\alpha,\gamma\rangle = 2(\alpha|\gamma)$. Using also that the dimension of G is $r+2|R^+|$, and that $|R^+| = |\Phi(G)|$, we obtain

$$\lim_{k\to\infty} \frac{1}{(k\ell)^m} V(G, k\underline{\lambda}, g, k\ell) = (fq)^{g-1}\frac{|Z(G)|}{|W|}(-1)^{|\Phi(G)|(g-1)} 2^{p(2g-2+s)} W(\Phi(G), P, g, s)(\mathbf{a})$$

as claimed. $\square$

Remark 3.2. In the case of one marking, the Verlinde formula reduces to

$$V(G,\lambda,g,\ell) = (fq)^{g-1}(\ell+\check{h})^{r(g-1)} \sum_{t\in T_\ell^{\mathrm{reg}}} \frac{e^{\lambda}(t)}{D(t)\Delta(t)^{g-1}},$$

where $D(t) = \prod_{\alpha>0}(1-e^{-\alpha}(t))$.

Following the same line of arguments as in the proof of Proposition 3.1 we get that for $s=1$ with $\lambda = \ell\hat{a}$ lying in the root lattice,

(3.2.1)

$$\mathrm{vol}(G,g)(a) = (fq)^{g-1}|Z|2^{p(2g-1)}(-1)^{(g-1)|\Phi|} \sum_{\gamma\in P^{\mathrm{reg}}} \frac{e^{\langle a,2i\pi\gamma\rangle}}{\prod_{H_\alpha\in\Phi}(2i\pi\langle H_\alpha,\gamma\rangle)^{2g-1}}.$$

That is,

$$\mathrm{vol}(G,g)(a) = (fq)^{g-1}|Z|2^{p(2g-1)}(-1)^{(g-1)|\Phi|}\mathcal{B}(\Phi_{2g-1},\check{Q})(a).$$

Let us demonstrate this with an example.

Example 3.3. We consider the moduli space of SU(2) bundles on a Riemann surface of genus g with one marking.

Let $a = tH_\alpha$ be a regular element in $\mathfrak{A}$; in other words, $0 < t < 1/2$. Let $\hat{a} \in \mathfrak{h}^*_{\mathbb{R}}$ denote the dual of $a \in \mathfrak{A}$ under the isomorphism given by the basic invariant form. The Verlinde formula for SU(2) with marking $\lambda = \ell\hat{a} = \ell t\alpha \in P_\ell$ such that $t\ell$ is a positive integer (that is, λ lies in the root lattice) is

$$V(\mathrm{SU}(2), t\ell\alpha, g, \ell) = 2^{g-1}(\ell+2)^{g-1}2(-1)^g \sum_{n=0}^{\infty} c_{g,t}(n)(\ell+2)^{2g-1-n}\frac{B(2g-1-n,t)}{(2g-1-n)!},$$

where $c_{g,t}(n)$ is the n^{th} coefficient of the Taylor series expansion of $e^{x(g-2t)}\left(\frac{x}{e^x-1}\right)^{2g-1}$ in x around zero, and $B(p,t)$ denotes the p^{th} Bernoulli polynomial in t. Clearly $c_{g,t}(0) = 1$.

In the expression for $V(\mathrm{SU}(2), tk\ell\alpha, g, k\ell)$ the highest term in k occurs when $n = 0$. Hence,

$$\begin{aligned}\mathrm{vol}(\mathrm{SU}(2), g)(a) &= \lim_{k\to\infty}\frac{V(\mathrm{SU}(2), tk\ell\alpha, g, k\ell)}{(k\ell+2)^{3g-2}}\\ &= 2^{g-1}2(-1)^g c_{g,t}(0)\tfrac{B(2g-1,t)}{(2g-1)!}\\ &= 2^g(-1)^g\tfrac{B(2g-1,t)}{(2g-1)!}\end{aligned}$$

We now calculate the volume using Equation (3.2.1).

We have $P = \mathbb{Z}\rho$ with $\langle\rho, H_\alpha\rangle = 1$. In this case, with the notation of Proposition 3.1, $s = 1$, $p = 0$, $q = 1$, $f = 2$ and $|Z(SU(2))| = 2$, hence $2^{p(2g-1)}(fq)^{g-1}|Z(G)| = 2^g$. Using the expression (3.2.1) of the volume in the one marking case,

$$\begin{aligned}2^g(-1)^{(g-1)|\Phi|}\sum_{\gamma\in P^{\mathrm{reg}}}\frac{e^{\langle a, 2i\pi\gamma\rangle}}{\prod_{H_\alpha\in\Phi}(2i\pi\langle H_\alpha,\gamma\rangle)^{2g-1}} &= 2^g(-1)^{g-1}\sum_{n\neq 0}\frac{e^{2i\pi tn}}{(2i\pi n)^{2g-1}}\\ &= 2^g(-1)^g\frac{B(2g-1,t)}{(2g-1)!}.\end{aligned}$$

Clearly we get the same formula.

3.1. **Volume of the moduli space as a function of the volume of T and G.** Let us recall the formula for the symplectic volume of the moduli space $\mathcal{M}(G, g, s, \mathcal{C})$ as given by Witten ([16] equation 4.1.14),

(3.3.1)
$$\mathrm{vol}(\mathcal{M}(G,g,s,\mathcal{C})) = \frac{|Z(G)|}{(2\pi)^{2m}}\mathrm{vol}(G)^{2g-2}\mathrm{vol}(G/T)^s\sum_{\lambda\in\mathrm{IrrG}}\frac{\prod_{j=1}^s[\chi_{V_\lambda}(\mathcal{C}_j)\sqrt{\Delta(\mathcal{C}_j)}]}{\dim V_\lambda{}^{2g-2+s}}$$

where $2m$ is the real dimension of $\mathcal{M}(G, g, s, \mathcal{C})$, and IrrG denotes the set of irreducible representations of G. Above $\mathrm{vol}(G)$, $\mathrm{vol}(G/T)$ are

Riemannian volumes of G and G/T which we now express following Bourbaki (Ch. IX, pages 396-411):

Choose a $\mathfrak{g}$-invariant scalar product. This determines a Lebesgue measure μ on $\mathfrak{g}$, via identification of $\mathfrak{g}$ with $\mathbb{R}^n$ by an orthonormal basis. Similarly let τ be the Lesbegue measure on $\mathfrak{t}$ corresponding to the restriction of the scalar product on $\mathfrak{t}$. We can construct from μ and τ Haar measures μ_G and μ_T on G and T respectively.

Since we aim to compare the volume formula in Proposition 3.1 with that of Witten in Equation (3.3.1), we choose the normalized Killing form as the $\mathfrak{g}$-invariant scalar product in the above construction, as this was our choice in the previous section. Then, for this choice, with respect to μ_G and μ_T constructed as above, we get that

$$\mathrm{vol}(G) = (fq)^{1/2}(2\pi)^{|R^+|+r}\frac{1}{\prod_{\alpha>0}(\alpha|\rho)}, \quad \mathrm{vol}(G/T) = \frac{(2\pi)^{|R^+|}}{\prod_{\alpha>0}(\alpha|\rho)}.$$

Recall from the previous section that

$$\Delta(t) = J(e^\rho)(t)\overline{J(e^\rho)(t)} = (-1)^{|R^+|}(J(e^\rho)(t))^2,$$

hence it takes positive values on a regular element t. Then, parametrizing irreducible representation of G with the cone of dominant weights P^+, for $\mathcal{C}_j$ the adjoint orbit of $\exp(a_j)$, we may write

$$\chi_{V_\lambda}(\mathcal{C}_j) = \frac{\sum_{w\in W} \varepsilon(w)e^{2i\pi\langle w(\lambda+\rho),a_j\rangle}}{i^{|R^+|}\sqrt{\Delta(\mathcal{C}_j)}}.$$

Let $d(\gamma) = \prod_{\alpha>0}\frac{\langle\gamma, H_\alpha\rangle}{\langle\rho, H_\alpha\rangle}$; it computes the dimension of $V_{\gamma-\rho}$.

Thus

$$\sum_{\lambda\in\mathrm{IrrG}}\frac{\prod_{j=1}^s\chi_{V_\lambda}(\mathcal{C}_j)\sqrt{\Delta(\mathcal{C}_j)}}{\dim {V_\lambda}^{2g-2+s}} = \sum_{\lambda\in P^+}\frac{\prod_{j=1}^s N_{a_j}(2i\pi(\lambda+\rho))}{i^{s|R^+|}d(\lambda+\rho)^{2g-2+s}}.$$

Observe that the summand above is invariant under the Weyl group (both the numerator and the denominator are anti-invariant by factor $(\mathrm{sign}(w))^s$ for a Weyl group element w). We get,

$$\sum_{\lambda\in\mathrm{IrrG}}\frac{\prod_{j=1}^s\chi_{V_\lambda}(\mathcal{C}_j)\sqrt{\Delta(\mathcal{C}_j)}}{\dim {V_\lambda}^{2g-2+s}} = \frac{1}{|W|}\sum_{\gamma\in P^{reg}}\frac{\prod_{j=1}^s N_{a_j}(2i\pi\gamma)}{i^{s|R^+|}d(\gamma)^{2g-2+s}}.$$

Inserting the explicit expressions for the volume of G and G/T above into Equation (3.3.1), all (2π) factors cancel, and combining the terms we get

$$\begin{aligned}\mathrm{vol}(\mathcal{M}(G,g,s,\mathcal{C})) &= \tfrac{|Z|}{|W|}(fq)^{g-1}\tfrac{(\prod_{\alpha>0}\langle\rho,H_\alpha\rangle)^{2g-2+s}}{(\prod_{\alpha>0}(\alpha|\rho))^{2g-2+s}}(-1)^{(g-1)|R^+|}W(\Phi(G),P,g)(a)\\ &= \tfrac{|Z|}{|W|}(fq)^{g-1}2^{p(2g-2+s)}(-1)^{(g-1)|R^+|}W(\Phi(G),P,g)(a),\end{aligned}$$

which is precisely the formula that we obtained in Proposition 3.1.

4. Various examples of volume calculations

Example 4.1. We now compute the volume of the moduli space of SU(3) bundles on a Riemann surface of genus one using the Witten series.

Simple roots are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, and fundamental weights are $\varpi_1 = \frac{2e_1-e_2-e_3}{3}$, $\varpi_2 = \frac{e_1+e_2-2e_3}{3}$. The positive coroots are

$$\Phi(\mathrm{SU}(3)) = \{H_{\alpha_1} = e^1 - e^2, H_{\alpha_2} = e^2 - e^3, H_{\alpha_1+\alpha_2} = e^1 - e^3\}$$

and $P = \mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2$. Let $\gamma = n_1\varpi_1 + n_2\varpi_2$. Then $\gamma \in P^{\mathrm{reg}}$ if and only if $n_1 \neq 0$, $n_2 \neq 0$ and $n_1 + n_2 \neq 0$.

Consider

$$a = a_1 H_{\alpha_1} + a_2 H_{\alpha_2} = a_1(e^1 - e^3) + (a_2 - a_1)(e^2 - e^3) \in \mathfrak{h}_{\mathbb{R}}.$$

Suppose that a is a regular element in $\mathfrak{A}$, in other words, $2a_1 - a_2 > 0$ $2a_2 - a_1 > 0$ (in particular $a_1 > 0$ and $a_2 > 0$) and $\theta(a_1 H_{\alpha_1} + a_2 H_{\alpha_2}) = a_1 + a_2 < 1$.

We compute the volume using the Formula (3.2.1). In this case, $s = 1$, $p = 0$, $q = 1$, $f = 3$ and $|Z(SU(3))| = 3$; hence, for $g = 1$, $2^{p(2g-1)}(fq)^{g-1}|Z(G)| = 3$.

$$\begin{aligned}\mathrm{vol}(\mathrm{SU}(3), g=1)(a) &= 3\sum_{\gamma\in P_{reg}}\frac{e^{2i\pi\langle a,\gamma\rangle}}{\prod_{H_\alpha\in\Phi}2i\pi\langle H_\alpha,\gamma\rangle}\\ &= 3\sum_{n_1\neq0,n_2\neq0,n_1+n_2\neq0}\frac{e^{2i\pi(n_1a_1+n_2a_2)}}{(2i\pi n_1)(2i\pi n_2)(2i\pi(n_1+n_2))}\end{aligned}$$

and we obtain

$$\mathrm{vol}(\mathrm{SU}(3), g=1)(a) = \begin{cases} -1/2(1+a_1-2a_2)(a_1-1+a_2)(2a_1-a_2), & a_1 \leq a_2\\ -1/2(a_1-2a_2)(a_1-1+a_2)(2a_1-1-a_2) & a_1 \geq a_2\end{cases}$$

Example 4.2. With the notation of Example 4.1, we make similar computations for SU(3) when genus $g = 2$.

We compute the volume employing the Formula (3.2.1). In this case, $s = 1$, $p = 0$, $q = 1$, $f = 3$, $|Z(\mathrm{SU}(3))| = 3$, hence, for $g = 2$, $2^{s(2g-1)}(fq)^{g-1}|Z(G)|(-1)^{(g-1)|\Phi|} = -3^2$.

$$\begin{aligned}\mathrm{vol}(\mathrm{SU}(3), g=2)(a) &= -9 \sum_{\gamma \in P_{reg}} \frac{e^{2i\pi\langle a,\gamma\rangle}}{\prod_{H_\alpha\in\Phi}(2i\pi\langle H_\alpha,\gamma\rangle)^3} \\ &= -9 \sum_{n_1\neq 0, n_2\neq 0, n_1+n_2\neq 0} \frac{e^{2i\pi(n_1a_1+n_2a_2)}}{(2i\pi n_1)^3(2i\pi n_2)^3(2i\pi(n_1+n_2))^3}\end{aligned}$$

and we obtain

$$\mathrm{vol}(\mathrm{SU}(3), g=2)(a) = \begin{cases} 1/40320(a_2-2a_1)(a_2-1+a_1)(-1+2a_2-a_1)P_1, & a_1 \leq a_2 \\ 1/40320(a_2+1-2a_1)(2a_2-a_1)(a_2-1+a_1)P_2, & a_1 \geq a_2 \end{cases}$$

where the polynomials P_1 and P_2 above are too long to be included in here.

Example 4.3. We now give an example of the volume of the moduli space of Spin(5) bundles on a Riemann surface of genus $g = 1$ with one marking.

Positive roots are $\{\alpha_1 + \alpha_2 = e_1, \alpha_2 = e_2, \theta = e_1 + e_2, \alpha_1 = e_1 - e_2\}$, with associated coroots $H_{e_1} = 2e^1$, $H_{e_2} = 2e^2$, $H_{e_1-e_2} = e^1 - e^2$, $H_{e_1+e_2} = e^1 + e^2$.

Let $a = a_1H_{\alpha_1} + a_2H_{\alpha_2}$ be a regular element in $\mathfrak{A}$; in other words, $a_1 > a_2$, $2a_2 > a_1$, $2a_2 < 1$. We can express a as $a = t_1e^1 + t_2e^2$ (with $t_1 = a_1$ and $t_2 = 2a_2 - a_1$), t_1 and t_2 satisfy $t_1 > t_2, t_2 > 0, t_1 + t_2 < 1$.

We calculate the volume for B_2 and genus $g = 1$ employing the Formula (3.2.1). In this case, $s = 1$, $p = 2$, $q = 2$, $f = 2$, $|Z(Spin5)| = 2$. Hence, for $g = 1$,

$$2^{s(2g-1)}(fq)^{g-1}|Z(Spin)|(-1)^{(g-1)|\Phi|} = 8.$$

We get,

$$\begin{aligned}\mathrm{vol}(B_2, g=1)(a) &= \frac{1}{2}t_2(t_1-1)(t_1-1+t_2)(t_1-t_2) \\ &= (2a_2-a_1)(a_1-1)(-1+2a_2)(a_1-a_2).\end{aligned}$$

Example 4.4. With the notation of Example 4.3, we compute the volume of the moduli space of Spin(5) bundles on a Riemann surface of genus one and two markings.

Let $\mathbf{a} = \{a_1, a_2\}$, where a_1 and a_2 are regular elements in $\mathfrak{A}$. Write $a_1 = t_1e^1 + t_2e^2$ and $a_2 = u_1e^1 + u_2e^2$. Then the function $\mathrm{vol}(B_2, 1)(\mathbf{a})$ is a piecewise polynomial function of t_1, t_2, u_1, u_2. For example, choose $v_1 = \frac{1}{2}e^1 + \frac{1}{5}e^2$, $v_2 = \frac{1}{7}e^1 + \frac{1}{9}e^2$ and consider $\tau(\mathbf{v}) \subset \mathfrak{A} \times \mathfrak{A}$, the open set determined by the condition that $a_1 + w(a_2)$ is in the same tope as $v_1 + w(v_2)$ for each element w in the Weyl group of B_2. Then for $\mathbf{a} \in \tau(\mathbf{v})$, we have

$$\mathrm{vol}(B_2, 1)(\mathbf{a}) = 4W(\Phi(B_2), P, 1, 2)(\mathbf{a}) =$$

TABLE 1. Value tables

G-type	q	f	p	$\lvert Z(G)\rvert$	$\lvert W\rvert$
A_r	1	$r+1$	0	$r+1$	$(r+1)!$
B_r	2	2	r	2	$2^r r!$
C_r	2^{r-1}	2	$r(r-1)$	2	$2^r r!$
D_r	1	4	0	4	$2^{r-1} r!$

$$-\frac{1}{60}\, u_2 u_1{}^5 + 1/6\, t_2 u_2 t_1 u_1{}^3 - 1/6\, t_2 u_2{}^3 t_1 u_1 + 1/6\, u_2{}^3 u_1 t_2{}^2 - 1/6\, u_2 t_2{}^2 u_1{}^3 - 1/12\, u_2{}^3 u_1 t_2{}^3 +$$
$$1/12\, u_2 t_2{}^3 u_1{}^3 - \frac{1}{60}\, u_2{}^5 t_1 u_1 + 1/12\, t_2 u_2 t_1{}^3 u_1{}^3 - 1/4\, t_2 u_2 t_1{}^2 u_1{}^3 + 1/4\, t_2 u_2{}^3 t_1{}^2 u_1 + 1/6\, t_2{}^2 u_2 t_1 u_1{}^3 -$$
$$1/6\, t_2{}^2 u_2{}^3 t_1 u_1 + \frac{1}{60}\, u_2 t_1 u_1{}^5 - 1/12\, t_2{}^3 u_2 t_1 u_1{}^3 + 1/12\, t_2{}^3 u_2{}^3 t_1 u_1 - 1/12\, t_2 u_2{}^3 t_1{}^3 u_1 + \frac{1}{60}\, u_2{}^5 u_1.$$

We compute some values.

$$\mathrm{vol}(B_2, 1)(\mathbf{a}) =$$
$$\begin{cases} \frac{141791}{372163703625} & if \quad a_1 = \frac{1}{2}e_1 + \frac{1}{5}e_2 \quad and \quad a_2 = \frac{1}{7}e_1 + \frac{1}{9}e_2 \\ \frac{1418037104720960397931}{3721637036250000000000000000} & if \quad a_1 = (\frac{1}{2} + \frac{1}{10000})e_1 + (\frac{1}{5} + \frac{1}{100000})e_2 \quad and \quad a_2 = \frac{1}{7}e_1 + \frac{1}{9}e_2 \end{cases}$$

5. MORE EXAMPLES

In this section we will compute some instances of the Witten volume using the formula

$$\mathrm{vol}(G, g)(\mathbf{a}) = W(\Phi(G), P, g, s)(\mathbf{a}) 2^{p(2g-2+s)} (fq)^{g-1} |Z(G)| (-1)^{(g-1)|\Phi(G)|} |W|^{-1}$$

as given in Proposition 3.1. We will denote by c_{vol} the factor

$$c_{\mathrm{vol}} := 2^{p(2g-2+s)} (fq)^{g-1} |Z(G)| (-1)^{(g-1)|\Phi(G)|} |W|^{-1}.$$

For convenience, we list values of the parameters in c_{vol} for each type of classical Lie group in Table 1.

5.1. **Tables of volumes of moduli spaces.** We simply denote by $\mathrm{vol}(G, g)$ the Witten volume in the case of no marking, that is, when $s = 0$. We will list some values of $\mathrm{vol}(G, g)$ for classical Lie groups in Tables 2 and 3. We will also list some values of the factor c_{vol} that we will need in Section 5.2 to compare our computations with the other numerical results in literature.

In the tables the column with heading G refers to the simple simply connected Lie group type: for instance A_r means of type A and rank r. Computations are very quick for rank less or equal to 4 (and relatively small genus). Beyond rank 5, computations cannot be made within a time limit of half-hour with our method.

5.2. **Comparison results.** In this section we compare some of our computations of $\mathrm{vol}(G, g)$ with that of Komori-Matsumoto-Tsumura ([6],[7],[8]). The setting is as follows.

As before, G is a simple, compact Lie group of rank r. We do not assume that G is simply-connected. Let L be the weight lattice of G. Let P be the weight lattice of the simply connected group covering G and let Q be its root lattice. Then, $Q \subset L \subset P$. Let P^+ be the 'cone' of dominant weights, and let $L^+ = L \cap P^+$.

Let $\mathbf{s} = [s_\alpha]$ be a sequence of real variables indexed by the positive roots R^+. For $v \in \mathfrak{h}_\mathbb{R}$, Komori-Matsumoto-Tsumura introduced

$$\zeta(\mathbf{s}, v, G) = \sum_{\gamma \in \rho + L^+} e^{2i\pi\langle v,\gamma\rangle} \prod_{\alpha \in R^+} \frac{1}{\langle \gamma, H_\alpha\rangle^{s_\alpha}}.$$

If G is simply connected, then $L = P$, and we may denote $\zeta(\mathbf{s}, v, G)$ by $\zeta(\mathbf{s}, v, \mathfrak{g})$, or for the Lie algebra $\mathfrak{g}$ of type X_r by $\zeta(\mathbf{s}, v, X_r)$ as in [8].

Example 5.1. Consider the simply connected group $G = \mathrm{SU}(4)$; its positive roots are

$$[e_1 - e_2, e_2 - e_3, e_3 - e_4, e_1 - e_3, e_2 - e_4, e_1 - e_4].$$

The cone of dominant weights is the simplicial cone generated by fundamental weights ω_1, ω_2 and ω_3 that are dual to simple coroots $e^1 - e^2$, $e^2 - e^3$ and $e^3 - e^4$ respectively. Then, if we order the exponents $\mathbf{s} = [s_i]$ with respect to the order of the roots as given above,

$$\zeta(\mathbf{s}, v, \mathrm{SU}(4)) = \zeta(\mathbf{s}, v, A_3)$$
$$= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \frac{e^{2i\pi\langle v, m_1\omega_1 + m_2\omega_2 + m_3\omega_3\rangle}}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1 + m_2)^{s_4} (m_2 + m_3)^{s_5} (m_1 + m_2 + m_3)^{s_6}}.$$

The series $\zeta(\mathbf{s}, v, G)$ converges when the elements s_α are sufficiently large. It can be shown that $\zeta(\mathbf{s}, v, G)$ can be continued as a meromorphic function of $\mathbf{s}$. Let $S = \sum s_\alpha$. Suppose s_α are the same for all short roots, respectively for all long roots, and both are equal to positive even integers (that are not necessarily the same positive even integers). Then $(2\pi)^{-S}\zeta(\mathbf{s}, 0, G)$ is rational. Indeed, using the invariance of the sum under the Weyl group W, $(2\pi)^{-S}\zeta(\mathbf{s}, 0, G)$ is proportional to a Bernoulli series (with repetition of coroots in Φ matching the exponent data) which is obtained by summing over all the regular elements of the full lattice L. More precisely,

$$\frac{\zeta(\mathbf{s}, 0, G)}{(2\pi)^S} = |W|^{-1} i^S \sum_{\gamma \in L_{reg}} \frac{1}{\prod_{\alpha \in R^+} (2i\pi\langle \gamma, H_\alpha\rangle)^{s_\alpha}}, \tag{5.1.1}$$

G	g	vol(G,g)	c_{vol}
A_2	2	$\frac{1}{20160}$	$-\frac{3}{2}$
	3	$\frac{19}{41513472000}$	$\frac{9}{2}$
	4	$\frac{1031}{189225711747072000}$	$-\frac{27}{2}$
	5	$\frac{32293}{48744543346045747200000}$	$\frac{81}{2}$
	6	$\frac{27739097}{3435943977054402696865382400000}$	$-\frac{243}{2}$
	7	$\frac{29835840687589}{30319572290041089305610122050928640000000000}$	$\frac{729}{2}$
	8	$\frac{71810708985991}{59867814661023533299285522596881655398400000000}$	$-\frac{2187}{2}$
	9	$\frac{221137132669842886663}{151246026314426013297816671756013269041872371712000000000000}$	$\frac{6561}{2}$
	10	$\frac{725206220511587536480143}{4069132057827757380938528031496082686769123090733843087360000000000000}$	$-\frac{19683}{2}$
A_3	2	$\frac{23}{653837184000}$	$\frac{2}{3}$
	3	$\frac{14081}{6481469910963304857600000}$	$\frac{8}{3}$
	4	$\frac{17634884778757}{111550917631548518367564040765093969920000000000}$	$\frac{32}{3}$
	5	$\frac{4257463829989959473}{367215613941141638310663592914134282922173229170688000000000000000}$	$\frac{128}{3}$
A_4	2	$\frac{1}{27303661403504640000}$	$\frac{5}{24}$
	3	$\frac{3998447009863}{185460807543380146607694824024275527231602688000000000}$	$\frac{25}{24}$
	4	$\frac{638972940367595659107077016 77}{45362078256453588666977436687320229387024422032399906211071677239139256762368000000000000}$	$\frac{125}{24}$
A_5	2	$\frac{46511}{1266317152743400113929736683520000000}$	$-\frac{1}{20}$
D_2	2	$\frac{1}{36}$	4
	3	$\frac{1}{32400}$	16
	4	$\frac{1}{14288400}$	64
	5	$\frac{1}{5715360000}$	256
	6	$\frac{1}{2240649734400}$	1024
	7	$\frac{477481}{417483460137744000000}$	4096
	8	$\frac{1}{340802824602240000}$	16384
	9	$\frac{13082689}{173739916770923543040000000 0}$	65536
	10	$\frac{1924313689}{9957444656345207631183974400000 0}$	262144
D_3	2	$\frac{23}{653837184000}$	$\frac{2}{3}$
	3	$\frac{14081}{6481469910963304857600000}$	$\frac{8}{3}$
	4	$\frac{17634884778757}{111550917631548518367564040765093969920000000000}$	$\frac{32}{3}$
	5	$\frac{4257463829989959473}{367215613941141638310663592914134282922173229170688000000000000000}$	$\frac{128}{3}$
D_4	2	$\frac{68227}{1084047447508315948449792000000}$	$\frac{1}{12}$
	3	$\frac{3727283292300079}{416352722747556555298704434232929841878747087946890870784000000000000}$	$\frac{1}{3}$
	4	$\frac{1}{10^{18}}\,\frac{5372550944533148798111597103943896132463}{401595287497255375910389668105337327730608192879514113863554560467009086512201613320691542243934208}$	$\frac{4}{3}$
D_5	2	$\frac{24136975469}{949559334622106350631397499004290699425805762560000000000000}$	$\frac{1}{120}$

TABLE 2. Witten volumes with $s = 0$ type A, D

G	g	$vol(G,g)$	c_{vol}
B_2	2	$\frac{1}{604800}$	16
	3	$\frac{4799}{444609285120000}$	1024
	4	$\frac{830287}{105864008545733979340800 0}$	65536
	5	$\frac{15426208283}{268823571386585938271797248000000000}$	4194304
	6	$\frac{116091757826572267}{2764256255594966032075350680497658265600000000000}$	268435456
	7	$\frac{890103918575782649595995 77}{28958731319990765227072551720618310178972852079820800000000000000}$	17179869184
	8	$\frac{1189513900446338789485454 1173}{52877369206125057710964488249573327181156553092261268684800000000000000000}$	1099511627776
	9	$\frac{6338083843161543552557928205114 21021}{3849636942831172456365873314390564274746568066276449044888380984262656000000000000000000}$	70368744177664
	10	$\frac{372724427371281986645902844259956399824270911 28187}{30932222350398276966127751813424212594377684046869506826237858166456278052464135608729600000000000000000000}$	4503599627370496
B_3	2	$\frac{19}{4302395130249216000}$	$\frac{-32}{3}$
	3	$\frac{82197390067823}{91382511723764546246708462194764980158464000000000}$	$\frac{8192}{3}$
	4	$\frac{494277616807840302658 8037}{25937228483121418723635585219476686166387446269585025372918068019200000000000000}$	$\frac{-2097152}{3}$
	5	$\frac{35490828246735714849071318814579272233062 6811}{10^{17}8784896914002461348039773989804254044107054219239371240765256010215823767454410291390127486586060 8}$	$\frac{536870912}{3}$
B_4	2	$\frac{4999}{10993395862420168735562234773635072000 00}$	$\frac{16}{3}$
	3	$\frac{2905251157055121225414764 28433}{29110138980741515852319493982928220072033854458512007601770585360017103402900354236416000000000000000}$	$\frac{16384}{3}$
B_5	2	$\frac{673653727654631}{114503199208433739395536475504707052302608351142027798548106444800000000000000000}$	$-\frac{32}{15}$
C_2	2	$\frac{1}{604800}$	16
	3	$\frac{479}{444609285120000}$	1024
	4	$\frac{830287}{105864008545733979340800 0}$	65536
	5	$\frac{15426208283}{268823571386585938271797248000000000}$	65536
	6	$\frac{116091757826572267}{2764256255594966032075350680497658265600000000000}$	268435456
	7	$\frac{890103918575782649595995 77}{28958731319990765227072551720618310178972852079820800000000000000}$	17179869184
	8	$\frac{1189513900446338789485454 1173}{52877369206125057710964488249573327181156553092261268684800000000000000000}$	1099511627776
	9	$\frac{6338083843161543552557928205114 21021}{3849636942831172456365873314390564274746568066276449044888380984262656000000000000000000}$	70368744177664
	10	$\frac{372724427371281986645902844259956399824270911 28187}{30932222350398276966127751813424212594377684046869506826237858166456278052464135608729600000000000000000000}$	4503599627370496
C_3	2	$\frac{19}{33612461955072000}$	$\frac{-4096}{3}$
	3	$\frac{328842292593389}{22310183526309703673512808153018793984000000000}$	$\frac{134217728}{3}$
	4	$\frac{790851606011875419185575 93}{1978853491449082849398466889913687604247089101378252057870336000000000000000}$	$\frac{-4398046511104}{3}$
	5	$\frac{22714138337998023353982263960846269191446635633}{2094482639790168129930442330790580283190501742181628046218217852167087499488451550338298675200000000000000000}$	$\frac{144115188075855872}{3}$
C_4	2	$\frac{2668859}{22366121139363950064213531033600000 00}$	$\frac{4194304}{3}$
	3	$\frac{29752108821608265723542364 4965697}{43377487332733744162797173832249969351342103568482410314337291359926914279014400000000000000000}$	$\frac{1125899906842624}{3}$
C_5	2	$\frac{5395475362754527}{10663941428850753176541670738226459576059910575116779520000000000000000}$	$\frac{-274877906944/15}{15}$

Table 3. Witten volumes with $s=0$ type B, C

where the series on the right hand side is a multiple Bernoulli series which has (in the case that it converges absolutely) rational value.

If all s_α are equal to an even integer $2k$, we denote the sequence $\mathbf{s} = [s_\alpha]$ by $\mathbf{s}_{2k}$. Then, for exponents $\mathbf{s}_{2k}$, and G simply connected, we may compute $\zeta(\mathbf{s}_{2k}, 0, G)$ using the Witten volume formula for $g = k+1$,

$$\begin{aligned}\zeta(\mathbf{s}_{2k}, 0, G) &= |W|^{-1}(2\pi)^{2k|R^+|}(-1)^{k|R^+|}W(\Phi(G), L, k+1)(0) \\ &= |W|^{-1}(2\pi)^{2k|R^+|}(-1)^{k|R^+|}\frac{1}{c_{\mathrm{vol}}}\mathrm{vol}(G, k+1)(0).\end{aligned} \tag{5.1.2}$$

Above $\Phi(G)$ denotes the set of positive coroots as before. Thus we can use the values of the volume listed in the tables of the previous section to compute some instances of the series $\zeta(\mathbf{s}_{2k}, 0, G)$.

We now demonstrate some computations of $\zeta(\mathbf{s}_{2k}, 0, G)$.

5.2.1. *Examples of type A_r.* Let $n = r+1$. We consider the simply connected group $G = \mathrm{SU}(n)$. If we write $N = |R^+| = \frac{n(n-1)}{2}$, then Equation (5.1.2) is

$$\zeta(\mathbf{s}_{2k}, 0, A_r) = (-1)^{kN}(2\pi)^{2kN}\frac{1}{n!}\frac{1}{c_{\mathrm{vol}}}\mathrm{vol}(\mathrm{SU}(n), k+1)(0),$$

where $c_{\mathrm{vol}} = n^{k+1}(-1)^{k\frac{n(n-1)}{2}}\frac{1}{n!}$.

Thus we can recover, the values of $\zeta(\mathbf{s}_{2k}, 0, A_r)$ for $n = 3, 4, 5, 6$ using Table 2.

For instance, if $n = 3$ (that is $r = 2$), and $k = 1$, then we have $N = 3$, $\mathrm{vol}(\mathrm{SU}(3), 2)(0) = \frac{1}{20160}$ and $c_{\mathrm{vol}} = -3/2$, and we obtain

$$\zeta(\mathbf{s}_2, 0, A_2) = (2\pi)^6(-1)^3\frac{1}{3!}\frac{1}{9(-1)^3\frac{1}{3!}}\frac{1}{20160} = \pi^6\frac{1}{2835}$$

as in [9] equation 7.11.

We give one other example whose parameters are not contained in the tables. Consider $n = 4$, $k = 5$. Then, $N = 6$ and

$$\zeta(\mathbf{s}_{2k}, 0, A_3) = (2\pi)^{60}\frac{13936140662907425134123100958 46}{58203152419058513584890890509712229288124323632762771449711578369140625}$$

5.2.2. *Examples of type B_r, C_r and D_r.* For root systems of type B_r and C_r, the number of positive roots is $N = r^2$ and the order of the Weyl group is $|W| = r!2^r$. For example, for B_r when all exponents $s_\alpha = 2k$,

$$\zeta(\mathbf{s}_{2k}, 0, B_r) = \frac{1}{r!2^r}(2\pi)^{2kN}(-1)^{kN}\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_B, g^B_{\mathbf{s}_{2k}})(0)$$

Explicitly for C_2, fundamental weights are e_1 and e_1+e_2, and positive roots are $[e_1-e_2, 2e_2, 2e_1, e_1+e_2]$. We consider the multiple zeta series

$$\zeta([s_1,s_2,s_3,s_4],0,C_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+2n)^{s_4}},$$

where we order the exponents with respect to the order given in the list of roots above. In the particular case that all $s_i=2$, using $|W|=8$ and values in Table 3, we find that for $\mathbf{s_2}=[2,2,2,2]$

$$\zeta(\mathbf{s_2},0,C_2)=\frac{1}{8}(-1)^4(2\pi)^8\frac{1}{16}\frac{1}{604800}=\frac{1}{302400}\,\pi^8,$$

which is the equation (7.23) of [9].

We also give an example of D_4 with all exponents equal to 6 (that is $k=3$ and $\mathbf{s_6}=[6,6,6,6,6,6,6,6,6,6,6,6]$).

$$\zeta(\mathbf{s_6},0,D_4)=$$

$$\frac{5372550944533148798111597103943896132463}{21770524158223250767856810653451043131130341521323218291199402843808716814637088000000000000000000}\pi^{72}$$

It is also possible to compute $\zeta(\mathbf{s},0,G)$ when the exponents in the list $\mathbf{s}=[s_1,s_2,s_3,s_4]$ are different positive even integers for short and long roots. We conclude with one example of this kind.

Consider the list of exponents $[2,4,4,2]$ corresponding to the list of positive roots $[e_1-e_2, 2e_2, 2e_1, e_1+e_2]$. Then,

$$\zeta([2,4,4,2],0,C_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2n^4(m+n)^4(m+2n)^2}=\pi^{12}\frac{53}{6810804000},$$

which coincides with equation (4.30) of [9].

5.3. **Some multiple zeta values.** Let k be a positive integer. Consider the multiple zeta series

$$\zeta_r(2k,2k,\ldots,2k):=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{m_1^{2k}}\frac{1}{(m_1+m_2)^{2k}}\cdots\frac{1}{(m_1+m_2+\cdots+m_r)^{2k}}.$$

Following [6], we want to demonstrate how the above series can be computed using the Bernoulli series $\mathcal{B}(\mathcal{H}_r^{BC},\check{Q}_C,g_{\mathbf{s}}^C)(0)$ for the root system of type C_r, where the exponents $\mathbf{s}=[s_\alpha]$ are taken to be 0 for long positive roots, and $2k$ for short positive roots. Using the invariance of the sum under the Weyl group, which is of order $2^r r!$ for C_r, we may write

$$\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(0) = 2^r r! \sum_{\gamma \in (P_C^+)_{reg}} \frac{1}{\prod_{\alpha>0}(2i\pi\langle H_\alpha, \gamma\rangle)^{s_\alpha}}.$$

A dominant integral regular weight $\gamma \in (P_C^+)_{reg}$ is of the form $\gamma = \sum_{i=1}^r m_i\omega_i$ with $m_i \geq 1$ (as before ω_i denotes the fundamental weights). Also recall that the root system of type C_r admits r long roots $\{2e_i\}_{1\leq i\leq r}$, with corresponding (short) coroots $\{H_{2e_i} = e^i\}_{1\leq i\leq r}$. If we express $H_{2e_i} = e^i = (e^i - e^{i+1}) + (e^{i+1} - e^{i+2}) + \cdots + e^r$, then $\langle H_{2e_i}, \gamma\rangle = m_i + m_{i+1} + \cdots + m_r$. Thus,

$$\zeta_r(2k, 2k, \ldots, 2k) = (-1)^{kr}(2\pi)^{2kr}\frac{1}{2^r r!}\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(0).$$

For example, $\zeta_2(4,4) = \frac{\pi^8}{113400}$, $\zeta_5(4,4,4,4,4) = \frac{\pi^{20}}{548828480360160000}$, $\zeta_5(6,6,6,6,6) = \frac{\pi^{30}}{134782828682597206525476 5625}$.

6. Appendix: Szenes formula

Let $\mathcal{H}$ be an arrangement of hyperplanes compatible with a lattice Λ. Let $g \in \mathcal{R}_{\mathcal{H}}$. Consider

$$\mathcal{B}(\mathcal{H}, \Lambda, g)(v) = \sum_{\gamma\in\Gamma_{reg}(\mathcal{H})} g(2i\pi\gamma)e^{2i\pi\langle v,\gamma\rangle}.$$

This function (a generalized function on V) coincide with a polynomial function $\mathcal{B}(\mathcal{H}, \Lambda, g, \tau)$ on a tope τ (see Proposition 1.30). The piecewise polynomial function $P(\mathcal{H}, \Lambda, g)$ has been defined in Definition 1.16. Following Szenes [12], we prove the following formula.

Theorem 6.1. *(Szenes) Let $g \in \mathcal{R}_{\mathcal{H}}$. On $V_{reg}(\mathcal{H}, \Lambda)$ we have the equality*

$$\mathcal{B}(\mathcal{H}, \Lambda, g) = P(\mathcal{H}, \Lambda, g).$$

We recall that, for $f \in S_{\mathcal{H}}$,

$$Z^\Lambda(v)(f)(z) = \sum_{\gamma\in\Gamma} f(2i\pi\gamma - z)e^{\langle v, 2i\pi\gamma - z\rangle},$$

and $P(\mathcal{H}, \Lambda, g)(v)$ is the trace on $S_{\mathcal{H}}$ of the operator $A(v, g) : \mathcal{S}_{\mathcal{H}} \to \mathcal{S}_{\mathcal{H}}$ defined by

$$f(z) \mapsto \mathbf{R}(e^{\langle z,v\rangle}g(z)(Z^\Lambda(v)f)(z)). \tag{6.1.1}$$

Here $\mathbf{R} : R_{\mathcal{H}} \to S_{\mathcal{H}}$ is the total residue.

We first consider the one dimensional case where $V=\mathbb{R}$, and $\Lambda=\mathbb{Z}$. Here $\mathcal{H}=\{0\}$, with equation $z=0$. The topes are the intervals $]-n,n+1[$, and the space $\mathcal{S}_{\mathcal{H}}$ is one dimensional with basis $f_\sigma=\frac{1}{z}$.

Let $\tau=]0,1[$. Assume $v\in\tau$ so that $[v]=0$. If we consider $g(z)=\frac{1}{z^k}$, the formula to be proven is

$$\sum_{n\neq 0}\frac{e^{2i\pi nv}}{(2i\pi n)^k}=\mathrm{Res}_{z=0}(\frac{1}{z^k}e^{zv}(Z^\Lambda(\tau)f_\sigma)(z)). \tag{6.1.2}$$

As $Z^\Lambda(\tau)(f_\sigma)(z)=\frac{1}{1-e^z}$ (see Example 1.11), we have thus to verify that

$$\sum_{n\neq 0}\frac{e^{2i\pi nv}}{(2i\pi n)^k}=\mathrm{Res}_{z=0}(\frac{1}{z^k}e^{zv}\frac{1}{1-e^z}).$$

The poles of the function $\frac{1}{1-e^z}$ consist of the elements $2i\pi n$, with $n\in\mathbb{Z}$, and , when $k\geq 0$, the equality above follows from the residue theorem in one variable. If $k<0$, both sides vanish (the left hand side gives a generalized function supported on $\mathbb{Z}$, the right hand side has no poles).

Szenes formula generalizes this result in higher dimensions, which we aim to demonstrate below.

Proof. We first remark that using both the comparison formulae (1.17) and (1.28) over commensurable lattices, it suffices to prove the equality for any lattice Λ (compatible with $\mathcal{H}$) of our choice.

We will prove Theorem 6.1 by the standard 'deletion-contraction' argument on arrangement of hyperplanes.

Choose a set Φ^{eq} of equations for $\mathcal{H}$. For $\phi\in\Phi^{eq}$, we consider the following two arrangements:

- $\mathcal{H}'=\mathcal{H}\setminus H_\phi$.
- $\mathcal{H}_0=\{H\cap H_\phi, H\in\mathcal{H}'\}$, the trace of the arrangement $\mathcal{H}'$ on H_ϕ.

Consider the vector space $V_0:=V/\mathbb{R}\phi$, let $p:V\to V_0$ be the projection. The dual space U_0 of the vector space V_0 is the hyperplane H_ϕ.

We now compare the spaces $\mathcal{S}_{\mathcal{H}}$, $\mathcal{S}_{\mathcal{H}_0}$ and $\mathcal{S}_{\mathcal{H}'}$.

Definition 6.2. We say that a function $f\in\mathcal{M}_{\mathcal{H}}$ has at most a simple pole along the hyperplane $\phi=0$ if $\phi f\in\mathcal{M}_{\mathcal{H}'}$. In this case, we define $res_\phi f\in\mathcal{M}_{\mathcal{H}_0}$ by $res_\phi f=(\phi f)|_{H_\phi}$

In other words, the meromorphic function f has at most a simple pole on H_ϕ if the denominator of f contains the factor ϕ at most once. Then we multiply f by ϕ, eliminating ϕ from the denominator of f, and we can restrict ϕf to $\phi = 0$. This operation kills the functions f having no poles of $\phi = 0$.

If $f = \frac{1}{\phi} f'$ with $f' \in \mathcal{M}_{\mathcal{H}'}$, then

$$res_\phi \mathbf{R} f = \mathbf{R} res_\phi f. \tag{6.2.1}$$

This is easy to verify using for example a decomposition of f' with denominator on a set of independent hyperplanes (see Lemma 1.3).

The map res_ϕ is well defined on $\mathcal{S}_{\mathcal{H}}$, as elements in $\mathcal{S}_{\mathcal{H}}$ have at most a simple pole on $\phi = 0$. It is easy to prove that we have the exact sequence

$$0 \longrightarrow \mathcal{S}_{\mathcal{H}'} \xrightarrow{\ i\ } \mathcal{S}_{\mathcal{H}} \xrightarrow{res_\phi} \mathcal{S}_{\mathcal{H}_0} \longrightarrow 0. \tag{6.2.2}$$

Let $v \in V_{reg}(\Lambda, \mathcal{H})$. Its projection $v_0 = p(v)$ belongs to $V_{reg}(\Lambda_0, \mathcal{H}_0)$.

Lemma 6.3. *Let $v \in V_{reg}(\Lambda, \mathcal{H})$ and $f \in \mathcal{S}_{\mathcal{H}}$. Then*

$$res_\phi Z^\Lambda(v)(f) = -Z^{\Lambda_0}(v_0)(res_\phi f),$$

with $v_0 = p(v)$.

Proof. We have

$$Z^\Lambda(v)(f)(z) = \sum_{\gamma \in \Gamma} f(2i\pi\gamma - z) e^{\langle v, 2i\pi\gamma - z\rangle}.$$

If γ is such that $\langle \phi, \gamma\rangle \neq 0$, then the term $f(2i\pi\gamma - z)$ has no pole on $\phi = 0$. Thus we obtain, for $z \in H_\phi$,

$$\begin{aligned} res_\phi Z^\Lambda(v)(f)(z) &= \sum_{\gamma\in\Gamma, \langle\gamma,\phi\rangle = 0} (\phi(z) f(2i\pi\gamma - z))|_{H_\phi} e^{\langle v_0, 2i\pi\gamma - z\rangle} \\ &= -\sum_{\gamma\in\Gamma, \langle\gamma,\phi\rangle = 0} \phi(2\pi\gamma - z) f(2i\pi\gamma - z)|_{H_\phi} e^{\langle v_0, 2i\pi\gamma - z\rangle}. \end{aligned}$$

□

Let $g \in \mathcal{R}_{\mathcal{H}'}$, and let g_0 be its restriction to H_ϕ. Then the operator $A(v, g)$ leaves $\mathcal{S}_{\mathcal{H}'}$ stable.

If F has at most a simple pole on $\phi = 0$, then gF also has at most a simple pole on $\phi = 0$, as g has no pole on $\phi = 0$. Thus the maps in the diagram below are well defined. Its commutativity follows from Lemma 6.3.

Lemma 6.4. *Let $g \in \mathcal{R}_{\mathcal{H}'}$. Then the following diagram is commutative.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}_{\mathcal{H}'} & \longrightarrow & \mathcal{S}_{\mathcal{H}} & \longrightarrow & \mathcal{S}_{\mathcal{H}_0} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle A(v,g)} & & \downarrow{\scriptstyle A(v,g)} & & \downarrow{\scriptstyle -A(v_0,g_0)} & & \\ 0 & \longrightarrow & \mathcal{S}_{\mathcal{H}'} & \longrightarrow & \mathcal{S}_{\mathcal{H}} & \longrightarrow & \mathcal{S}_{\mathcal{H}_0} & \longrightarrow & 0 \end{array} \tag{6.4.1}$$

We are now ready to prove Theorem 6.1 by induction on the number of hyperplanes in $\mathcal{H}$. If there are less than r hyperplanes, then $\mathcal{S}_{\mathcal{H}} = \{0\}$, the generalized function $\mathcal{B}(\mathcal{H}, \Lambda, g)$ is supported on affine walls, so both sides of the equation of Theorem 6.1 vanish.

Assume that $\mathcal{H}$ consists of r independent hyperplanes intersecting on $\{0\}$. Changing the lattice Λ, we can eventually assume that Λ is the lattice generated by the equations ϕ_k of the hyperplanes. Then, the theorem follows from Formula (6.1.2) in the one dimensional case.

Assume that $\mathcal{H}$ have more than r hyperplanes. Then by the Lemma 1.3, we can write a function in $\mathcal{R}_{\mathcal{H}}$ as a sum of functions g whose poles lie on an independent subset of hyperplanes of $\mathcal{H}$, thus in number less or equal to r. Thus $\mathcal{R}_{\mathcal{H}}$ is linearly generated by functions g such that some equation $\phi \in \Phi^{eq}$ is not a pole of g. We consider such a couple (g, ϕ) and the arrangements $\mathcal{H}'$ and $\mathcal{H}_0$ associated to ϕ by deletion and contraction. The function g is in $\mathcal{R}_{\mathcal{H}'}$.

Let $g_0 \in \mathcal{R}_{\mathcal{H}_0}$ be the restriction of g to H_ϕ. Thus $\mathcal{B}(\mathcal{H}_0, \Lambda_0, g_0)$ is a generalized function on $H_\phi^* = V/\mathbb{R}\phi$ and $p^*\mathcal{B}(\mathcal{H}_0, \Lambda_0, g_0)$ is a function on V (constant in the direction ϕ).

We have the following recurrence relation for the function (eventually generalized) $\mathcal{B}(\mathcal{H}, \Lambda, g)$ associated to an element $g \in R_{\mathcal{H}'}$.

Proposition 6.5. *If $g \in \mathcal{R}_{\mathcal{H}'}$, then*

$$\mathcal{B}(\mathcal{H}, \Lambda, g) = \mathcal{B}(\mathcal{H}', \Lambda, g) - p^*\mathcal{B}(\mathcal{H}_0, \Lambda_0, g_0).$$

This is clear. Indeed the set $\Gamma_{reg}(\mathcal{H}')$ is larger than $\Gamma_{reg}(\mathcal{H})$ as it may contain also elements γ with $\langle \gamma, \phi \rangle = 0$. This additional summation gives rise to the term $\mathcal{B}(\mathcal{H}_0, \Lambda_0, g_0)$.

Let $v \in V_{reg}(\mathcal{H}, \Lambda)$. As $P(\mathcal{H}, \Lambda, g)(v)$ is the trace of the operator $A(v, g)$ defined in (6.1.1), the commutativity of the diagram (6.4.1) above implies that

$$P(\mathcal{H}, \Lambda, g)(v) = P(\mathcal{H}', \Lambda, g)(v) - P(\mathcal{H}_0, \Lambda_0, g_0)(v_0).$$

Comparing with Proposition 6.5, we see by induction that Szenes formula holds. $\square$

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