HIGHEST COEFFICIENTS OF WEIGHTED EHRHART QUASI-POLYNOMIALS FOR A RATIONAL POLYTOPE

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Abstract. We describe a method for computing the highest degree coefficients of a weighted Ehrhart quasi-polynomial for a rational simple polytope.

1. Introduction

Let $\mathfrak{p}$ be a rational polytope in $V = \mathbb{R}^d$ and $h(x)$ a polynomial function on V. A classical problem is to compute the sum of values of $h(x)$ over the set of integral points of $\mathfrak{p}$,

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

The function $h(x)$ is called the weight. When $\mathfrak{p}$ is dilated by an integer $n \in \mathbb{N}$, we obtain a function of n which is quasi-polynomial, the so-called weighted Ehrhart quasi-polynomial of the pair $(\mathfrak{p}, h)$

$$S(n\mathfrak{p}, h) = \sum_{m=0}^{d+M} E_m n^m.$$

It has degree $d+M$, where $N = \deg h$. The coefficients E_m are periodic functions of $n \in \mathbb{N}$, with period the smallest integer q such that $q\mathfrak{p}$ is a lattice polytope.

In [4], Barvinok obtained a formula relating the kth coefficient of the (unweighted) Ehrhart quasi-polynomial of a rational polytope to volumes of sections of the polytope by affine lattice subspaces parallel to k-dimensional faces of the polytope. As a consequence, he proved that the k_0 highest degree coefficients of the unweighted Ehrhart quasi-polynomial of a *rational simplex* can be computed by a polynomial algorithm, when the dimension d is part of the input, but k_0 is fixed.

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The sum $S(\mathfrak{p}, h)$ has natural generalizations, the *intermediate* sums $S^L(\mathfrak{p}, h)$, where $L \subseteq V$ is a rational vector subspace. For a polytope $\mathfrak{p} \subset V$ and a polynomial $h(x)$

$$S^L(\mathfrak{p}, h) = \sum_x \int_{\mathfrak{p}\cap(x+L)} h(y)dy,$$

where the summation index x runs over the projected lattice in V/L. In other words, the polytope $\mathfrak{p}$ is sliced along lattice affine subspaces parallel to L and the integrals of h over the slices are added up. For $L = V$, there is only one term and $S^V(\mathfrak{p}, h)$ is just the integral of $h(x)$ over $\mathfrak{p}$, while, for $L = \{0\}$, we recover $S(\mathfrak{p}, h)$. Barvinok's method was to introduce particular linear combinations of the intermediate sums,

$$\sum_{L\in\mathcal{L}} \rho(L)S^L(\mathfrak{p}, h).$$

It is natural to replace the polynomial weight $h(x)$ with an exponential function $x \mapsto e^{\langle\xi,x\rangle}$, and consider the corresponding holomorphic functions of ξ in the dual V^*. Moreover, one can allow $\mathfrak{p}$ to be unbounded, then the sums

$$S^L(\mathfrak{p})(\xi) = \sum_x \int_{\mathfrak{p}\cap(x+L)} e^{\langle\xi,y\rangle}dy$$

still make sense as meromorphic functions on V^*. The map $\mathfrak{p} \mapsto S^L(\mathfrak{p})(\xi)$ is a valuation. In [2], we proved that Barvinok's valuation $\sum_{L\in\mathcal{L}} \rho(L)S^L(\mathfrak{p})(\xi)$ approximates $S(\mathfrak{p})(\xi)$ in a sense which is made precise below. As a consequence, we recovered Barvinok's main theorem of [4] and we sketched a method for computing the highest degree coefficients of the Ehrhart quasipolynomial of a rational simplex which is hopefully easier to implement. The proof in [2] relied on our Euler-Maclaurin expansion of these functions.

The main interest of the present article is to give a simpler formulation and an elementary proof of the approximation result of [2], in the case of a **simple polytope**.

In a forthcoming article, in collaboration with J. De Loera and M. Koeppe, we plan to apply the results of this article to derive a polynomial algorithm for computing the k_0 highest degree coefficients of a weighted Ehrhart quasi-polynomial relative to a simplex, when k_0 and the degree of the weight $h(x)$ are fixed, but the dimension of the simplex is part of the input. The article [1] dealt with the integral $\int_\mathfrak{p} h(x)dx$, which is of course the highest coefficient, if $h(x)$ is homogeneous.

To explain the main idea, let us assume in this introduction that the vertices s of $\mathfrak{p}$ are integral. Using a theorem of Brion, one writes

$S(\mathfrak{p})(\xi)$ as a sum of the generating functions of the supporting cones at the vertices s of $\mathfrak{p}$,

$$S(\mathfrak{p})(\xi) = \sum_{s\in\mathcal{V}(\mathfrak{p})} S(s+\mathfrak{c}_s)(\xi).$$

Then the dilated polytope $n\mathfrak{p}$ has vertices ns with the same cone of feasible directions $\mathfrak{c}_s$, thus

$$S(n\mathfrak{p})(\xi) = \sum_{s\in\mathcal{V}(\mathfrak{p})} e^{n\langle\xi,s\rangle} S(\mathfrak{c}_s)(\xi).$$

The generating function of a cone $\mathfrak{c}$ is a meromorphic function of a particular type, namely, near $\xi = 0$, it is a quotient of a holomorphic function by a product of linear forms. Hence, it admits a decomposition into homogeneous components. One shows that the lowest degree is $\geq -d$.

Let us fix an integer k_0 , $0 \leq k_0 \leq d$. Our goal is to compute the k_0+1 highest degree coefficients of the Ehrhart quasi-polynomial of $\mathfrak{p}$ for the weight $h(x) = \frac{\langle\xi,x\rangle^M}{M!}$. As we explain in Section 3, this computation amounts to computing the lowest homogeneous components

$$S(\mathfrak{c}_s)_{[-d+k]}(\xi), \quad k = 0, \dots, k_0,$$

of the generating functions of the cones $\mathfrak{c}_s$.

Our main result, Theorem 16, is an expression, depending on k_0, for the components $S(s+\mathfrak{c})_{[-d+k]}(\xi)$, $k = 0, \dots, k_0$ in the particular case where the cone $s + \mathfrak{c}$ is simplicial. For a unimodular cone, the generating function is given by a "short" formula, thus its lowest degree components are readily computed. In general, let $v_i \in \mathbb{Z}^d$, $i = 1, \dots, d$, be integral generators of the edges of $\mathfrak{c}$. The finite sum

$$f(\xi) = \sum_{x\in\left(\sum_{i=1}^d [0,1[\, v_i\right)\cap\mathbb{Z}^d} e^{\langle\xi,x\rangle}$$

is an analytic function of ξ. (If $\mathfrak{c}$ is unimodular and the generators v_i are primitive, then $f(\xi) = 1$). The term of degree k of $f(\xi)$ is given by

$$\sum_{x\in\left(\sum_{i=1}^d [0,1[\, v_i\right)\cap\mathbb{Z}^d} \frac{\langle\xi,x\rangle^k}{k!}.$$

A monomial of degree k in $(\xi_1, \dots, \xi_d)$ can involve at most k variables among the ξ_i. From this elementary remark, we deduce that the terms of degree $\leq k_0$ of $f(\xi)$ can be computed using a Moebius type combination of sums similar to $f(\xi)$, in dimension $\leq k_0$, and determinants. As a consequence, we obtain an expression for the terms

$S(s+\mathfrak{c})_{[-d+k]}(\xi), \quad k=0,\dots,k_0$, which involves the generating functions of cones in dimension $\leq k_0$ only. This feature is useful because, when the dimension is fixed, there is an efficient algorithm for decomposing a simplicial cone into a signed combination of unimodular cones, due to Barvinok.

2. Notations and basic facts

2.1. We consider a rational vector space V of dimension d, that is to say a finite dimensional real vector space with a lattice denoted by Λ. We will need to consider subspaces and quotient spaces of V, this is why we cannot just let $V=\mathbb{R}^d$ and $\Lambda=\mathbb{Z}^d$. The set $\Lambda\otimes\mathbb{Q}$ of rational points in V is denoted by $V_{\mathbb{Q}}$. A subspace L of V is called rational if $L\cap\Lambda$ is a lattice in L. If L is a rational subspace, the image of Λ in V/L is a lattice in V/L, so that V/L is a rational vector space. The image of Λ in V/L is called the projected lattice.

A rational space V, with lattice Λ, has a canonical Lebesgue measure dx, for which V/Λ has measure 1.

A convex rational polyhedron $\mathfrak{p}$ in V (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces bounded by rational affine hyperplanes. We say that $\mathfrak{p}$ is solid (in V) if the affine span of $\mathfrak{p}$ is V.

In this article, a cone is a polyhedral cone (with vertex 0) and an affine cone is a translated set $s+\mathfrak{c}$ of a cone $\mathfrak{c}$.

A polytope $\mathfrak{p}$ is a compact polyhedron. The set of vertices of $\mathfrak{p}$ is denoted by $\mathcal{V}(\mathfrak{p})$. For each vertex s, the cone of feasible directions at s is denoted by $\mathfrak{c}_s$.

A cone $\mathfrak{c}$ is called simplicial if it is generated by independent elements of V. A simplicial cone $\mathfrak{c}$ is called unimodular if it is generated by independent integral vectors $v_1,\dots,v_k$ such that $\{v_1,\dots,v_k\}$ can be completed to an integral basis of V. An affine cone $\mathfrak{a}$ is called simplicial (resp. simplicial unimodular) if it is the translate of a simplicial (resp. simplicial unimodular) cone.

2.2. **Generating functions.**

Definition 1. *We denote by $\mathcal{H}(V^*)$ the ring of holomorphic functions defined around $0\in V^*$. We denote by $\mathcal{M}(V^*)$ the ring of meromorphic functions defined around $0\in V^*$ and by $\mathcal{M}_\ell(V^*)\subset\mathcal{M}(V^*)$ the subring consisting of those meromorphic functions $\phi(\xi)$ such that there exists a product of linear forms $D(\xi)$ such that $D(\xi)\phi(\xi)$ is holomorphic.*

A function $\phi(\xi) \in \mathcal{M}_\ell(V^*)$ has a unique expansion into homogeneous rational functions

$$\phi(\xi) = \sum_{m \geq m_0} \phi_{[m]}(\xi).$$

If P is a homogeneous polynomial on V^* of degree p, and D a product of r linear forms, then $\frac{P}{D}$ is an element in $\mathcal{M}_\ell(V^*)$ homogeneous of degree $m = p - r$. For instance, $\frac{\xi_1}{\xi_2}$ is homogeneous of degree 0. On this example, we observe that a function in $\mathcal{M}_\ell(V^*)$ which has no negative degree terms need not be analytic.

Let us recall the definition of the functions $I(\mathfrak{p})$ and $S(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$ associated to a polyhedron $\mathfrak{p}$, (see for instance the survey [5]).

Proposition 2. *There exists a unique map I which to every polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function $I(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold:*

(a) If $\mathfrak{p}$ is not solid or if $\mathfrak{p}$ contains a straight line, then $I(\mathfrak{p})$=0.

(b) If $\xi \in V^*$ is such that $e^{\langle \xi, x\rangle}$ is integrable over $\mathfrak{p}$, then

$$I(\mathfrak{p})(\xi) = \int_{\mathfrak{p}} e^{\langle \xi, x\rangle} dx.$$

(c) For every point $s \in V_{\mathbb{Q}}$, one has

$$I(s + \mathfrak{p})(\xi) = e^{\langle \xi, s\rangle} I(\mathfrak{p})(\xi).$$

Proposition 3. *There exists a unique map S which to every polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function $S(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold:*

(a) If $\mathfrak{p}$ contains a straight line, then $S(\mathfrak{p})$=0.

(b) If $\xi \in V^*$ is such that $e^{\langle \xi, x\rangle}$ is summable over the set of lattice points of $\mathfrak{p}$, then

$$S(\mathfrak{p})(\xi) = \sum_{x \in \mathfrak{p} \cap \Lambda} e^{\langle \xi, x\rangle}.$$

(c) For every point $s \in \Lambda$, one has

$$S(s + \mathfrak{p})(\xi) = e^{\langle \xi, s\rangle} S(\mathfrak{p})(\xi).$$

Moreover the maps $\mathfrak{p} \mapsto I(\mathfrak{p})$ and $\mathfrak{p} \mapsto S(\mathfrak{p})$ have additivity properties, with consequence the fundamental Brion's theorem.

Theorem 4 (Brion, [7])**.** *Let $\mathfrak{p}$ be a polyhedron with set of vertices $\mathcal{V}(\mathfrak{p})$. For each vertex s, let $\mathfrak{c}_s$ be the cone of feasible directions at s. Then*

$$S(\mathfrak{p}) = \sum_{s \in \mathcal{V}(\mathfrak{p})} S(s + \mathfrak{c}_s) \text{ and } I(\mathfrak{p}) = \sum_{s \in \mathcal{V}(\mathfrak{p})} I(s + \mathfrak{c}_s).$$

2.3. **Notations and basic facts in the case of a simplicial cone.** Let $v_i, i = 1, \dots, d$ be linearly independent integral vectors and let $\mathfrak{c} = \sum_{i=1}^d \mathbb{R}^+ v_i$ be the cone they span.

Definition 5. *The* semi-closed unit cell $\mathbf{B}$ *of the cone (with respect to the generators* $v_i, i = 1, \dots, d$*) is the set*

$$\mathbf{B} = \sum_{i=1}^{d} [0, 1[v_i.$$

We recall the following elementary but crucial lemma.

Lemma 6. *(i) The affine cone* $(s + \mathfrak{c}) \cap \Lambda$ *is the disjoint union of the translated cells* $s + \mathbf{B} + v$*, for* $v \in \sum_{j=1}^d \mathbb{N} v_j$*.*
(ii) The set of lattice points in the affine cone $s+\mathfrak{c}$ *is the disjoint union of the sets* $x + \sum_{i=1}^d \mathbb{N} v_i$ *when* x *runs over the set* $(s + \mathbf{B}) \cap \Lambda$*.*
(iii) The number of lattice points in the cell $s+\mathbf{B}$ *is equal to the volume of the cell with respect to the Lebesgue measure defined by the lattice, that is*

$$\operatorname{Card}((s + \mathbf{B}) \cap \Lambda) = |\det(v_i)|.$$

Let $s \in V_{\mathbb{Q}}$. We have immediately

$$I(s + \mathfrak{c})(\xi) = e^{\langle \xi, s \rangle} \frac{(-1)^d |\det(v_i)|}{\prod_{i=1}^d \langle \xi, v_i \rangle}. \tag{1}$$

The study of the function $S(s + \mathfrak{c})(\xi)$ will be the main point of this article. It reduces to the study of the holomorphic function $S(s+\mathbf{B})(\xi)$ defined by the following finite sum, over the lattice points of the unit cell.

Definition 7.

$$S(s + \mathbf{B})(\xi) = \sum_{x \in (s+\mathbf{B}) \cap \Lambda} e^{\langle \xi, x \rangle}.$$

Lemma 8.

$$S(s + \mathfrak{c})(\xi) = S(s + \mathbf{B})(\xi) \frac{1}{\prod_{j=1}^d (1 - e^{\langle \xi, v_j \rangle})}. \tag{2}$$

In particular, $S(s + \mathfrak{c}) \in \mathcal{M}_\ell(V^*)$*, thus it admits a decomposition into homogeneous components,*

$$S(s + \mathfrak{c})(\xi) = S_{[-d]}(s + \mathfrak{c})(\xi) + S_{[-d+1]}(s + \mathfrak{c})(\xi) + \dots, \tag{3}$$

and the lowest degree term $S_{[-d]}(s + \mathfrak{c})(\xi)$ *is equal to* $I(\mathfrak{c})(\xi)$

Proof. (2) follows from Lemma 6 (ii). Next, we write

$$\prod_{j=1}^{d} \frac{1}{1-e^{\langle \xi, v_j\rangle}} = \prod_{j=1}^{d} \frac{\langle \xi, v_j\rangle}{1-e^{\langle \xi, v_j\rangle}} \frac{1}{\prod_{j=1}^{d} \langle \xi, v_j\rangle}. \tag{4}$$

The function $\frac{x}{1-e^x}$ is holomorphic with value -1 for $x=0$. Thus $S(s+\mathfrak{c}) \in \mathcal{M}_\ell(V^*)$. The value at $\xi = 0$ of the sum over the cell is the number of lattice points of the cell, that is the volume $|\det(v_i)|$. This proves the last assertion. □

3. Weighted Ehrhart quasipolynomials

Let $\mathfrak{p} \subset V$ be a rational polytope and let $h(x)$ be a polynomial function of degree M on V. We consider the following weighted sum over the set of lattice points of $\mathfrak{p}$,

$$\sum_{x \in \mathfrak{p}\cap\Lambda} h(x).$$

When $\mathfrak{p}$ is dilated by a non negative integer n, we obtain the quasi-polynomial of the pair $(\mathfrak{p}, h)$,

$$\sum_{x \in n\mathfrak{p}\cap\Lambda} h(x) = \sum_{m=0}^{d+M} E_m\, n^m.$$

The coefficients E_m actually depend on n, but they depend only on n mod q, where q is the smallest integer such that $q\mathfrak{p}$ is a lattice polytope. If $h(x)$ is homogeneous of degree M, the highest degree coefficient E_{d+M} is equal to the integral $\int_{\mathfrak{p}} h(x)dx$.

Let us fix a number k_0. Our goal is to compute the k_0+1 highest degree coefficients E_m, for $m = M+d, \dots, M+d-k_0$.

We concentrate on the special case where the polynomial $h(x)$ is a power of a linear form

$$h(x) = \frac{\langle \xi, x\rangle^M}{M!}.$$

Of course, any polynomial can be written as a linear combination of powers of linear forms.

We will explain our results with the simplifying assumption that the vertices of the polytope are lattice points.

Definition 9. *We define the coefficients* $E_m(\mathfrak{p}, \xi, M), m = 0, \dots, M+d$ *by*

$$\sum_{x \in n\mathfrak{p}\cap\Lambda} \frac{\langle \xi, x\rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(\mathfrak{p}, \xi, M) n^m.$$

Proposition 10. *Let $\mathfrak{p}$ be a lattice polytope. Then, for $k \geq 0$, we have*

$$E_{M+d-k}(\mathfrak{p}, \xi, M) = \sum_{s \in \mathcal{V}(\mathfrak{p})} \frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!} S_{[-d+k]}(\mathfrak{c}_s)(\xi). \tag{5}$$

The highest degree coefficient is just the integral

$$E_{M+d}(\mathfrak{p}, \xi, M) = \int_{\mathfrak{p}} \frac{\langle \xi, x\rangle^M}{M!} dx.$$

Remark 11. *As functions of ξ, the coefficients $E_m(\mathfrak{p}, \xi, M)$ are polynomial, homogeneous of degree M. However, in (5), they are expressed as linear combinations of rational functions of ξ, whose poles cancel out.*

Proof. The starting point is Brion's formula. As the vertices are lattice points, we have

$$\sum_{x \in \mathfrak{p} \cap \Lambda} e^{\langle \xi, x\rangle} = \sum_{s \in \mathcal{V}(\mathfrak{p})} S(s + \mathfrak{c}_s)(\xi) = \sum_{s \in \mathcal{V}(\mathfrak{p})} e^{\langle \xi, s\rangle} S(\mathfrak{c}_s)(\xi). \tag{6}$$

When $\mathfrak{p}$ is replaced with $n\mathfrak{p}$, the vertex s is replaced with ns but the cone $\mathfrak{c}_s$ does not change. We obtain

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} e^{\langle \xi, x\rangle} = \sum_{s \in \mathcal{V}(\mathfrak{p})} e^{n\langle \xi, s\rangle} S(\mathfrak{c}_s)(\xi).$$

We replace ξ with $t\xi$,

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} e^{t\langle \xi, x\rangle} = \sum_{s \in \mathcal{V}(\mathfrak{p})} e^{nt\langle \xi, s\rangle} S(\mathfrak{c}_s)(t\xi).$$

The decomposition into homogeneous components gives

$$S(\mathfrak{c}_s)(t\xi) = t^{-d} I(\mathfrak{c}_s)(\xi) + t^{-d+1} S_{[-d+1]}(\mathfrak{c}_s)(\xi) + \cdots + t^k S_{[k]}(\mathfrak{c}_s)(\xi) + \cdots .$$

Hence, the t^M-term in the right-hand side is equal to

$$\sum_{k=0}^{M+d} (nt)^{M+d-k}\, t^{-d+k} \frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!} S_{[-d+k]}(\mathfrak{c}_s)(\xi).$$

Thus we have

$$\begin{aligned} \sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x\rangle^M}{M!} = \sum_{s \in \mathcal{V}(\mathfrak{p})} n^{M+d} \frac{\langle \xi, s\rangle^{M+d}}{(M+d)!} I(\mathfrak{c}_s)(\xi) \\ + n^{M+d-1} \frac{\langle \xi, s\rangle^{M+d-1}}{(M+d-1)!} S_{[-d+1]}(\mathfrak{c}_s)(\xi) + \cdots + S_{[M]}(\mathfrak{c}_s)(\xi). \end{aligned} \tag{7}$$

On this relation, we read immediately that $\sum_{x\in n\mathfrak{p}\cap\Lambda}\frac{\langle\xi,x\rangle^M}{M!}$ is a polynomial function of n of degree $M+d$, and that the coefficient of n^{M+d-k} is given by (5). The highest degree coefficient is given by

$$E_{M+d}(\mathfrak{p},\xi,M)=\sum_{s\in\mathcal{V}(\mathfrak{p})}\frac{\langle\xi,s\rangle^{M+d}}{(M+d)!}I(\mathfrak{c}_s)(\xi).$$

Applying Brion's formula for the integral, this is equal to the term of ξ-degree M in $I(\mathfrak{p})(\xi)$, which is indeed the integral $\int_{\mathfrak{p}}\frac{\langle\xi,x\rangle^M}{M!}dx$. $\square$

From Proposition 10, we draw an important consequence: in order to compute the k_0+1 highest degree terms of the weighted Ehrhart polynomial for the weight $h(x)=\frac{\langle\xi,x\rangle^M}{M!}$, we **need only** the k_0+1 lowest degree homogeneous terms of the meromorphic function $S(\mathfrak{c}_s)(\xi)$, for every vertex s of $\mathfrak{p}$. We compute such an approximation in the next section.

4. Approximation of the generating function of a simplicial affine cone

Let $\mathfrak{c}\subset V$ be a simplicial cone with integral generators $(v_j, j=1,\ldots,d)$, and let $s\in V_{\mathbb{Q}}$. Let $k_0\leq d$. In this section we will obtain an expression for the k_0+1 lowest degree homogeneous terms of the meromorphic function $S(s+\mathfrak{c})(\xi)$. Recall that if $\mathfrak{c}$ is unimodular, the function $S(s+\mathfrak{c})(\xi)$ has a "short" expression, given by (2),

$$S(s+\mathfrak{c})(\xi)=e^{\langle\xi,\bar{s}\rangle}\prod_{j=1}^{d}\frac{1}{1-e^{\langle\xi,v_j\rangle}},$$

where $(v_i, i=1,\ldots,d)$ are the primitive integral generators of the edges and $\bar{s}$ is the unique lattice point in the corresponding cell $s+\mathbf{B}$. Thus in the unimodular case, computing the lowest degree components is immediate.

When $\mathfrak{c}$ is not unimodular, it is not possible to compute efficiently the Taylor expansion of the function $S(s+\mathbf{B})(\xi)$ at order k_0, if the order is part of the input as well as the dimension d. In contrast, **if the order k_0 is fixed**, we are going to obtain an expression for the terms of degree $\leq k_0$ which involves discrete summation over cones in dimension $\leq k_0$ only, and determinants. For example, for $k_0=0$, the constant term $S(s+\mathbf{B})(0)$ is the number of lattice points in the cell, which is equal to a determinant, by Lemma 6 (iii).

We need some notations.

For $I\subseteq\{1,\ldots,d\}$, we denote by L_I the linear span of the vectors $(v_i, i\in I)$. We denote by $\mathbf{B}_I=\sum_{i\in I}[0,1[v_i$ the unit cell in L_I.

We denote by I^c the complement of I in $\{1,\dots,d\}$. We have $V = L_I \oplus L_{I^c}$. For $x \in V$ we denote the components by

$$x = x_I + x_{I^c}.$$

Thus we identify the quotient V/L_{I^c} with L_I and we denote the projected lattice by $\Lambda_I \subset L_I$. Note that $L_I \cap \Lambda \subseteq \Lambda_I$, but the inclusion is strict in general.

Example 12. *$v_1 = (1,0), v_2 = (1,2)$. Take $I = \{1\}$. Then $\Lambda_I = \mathbb{Z}\frac{v_1}{2}$.*

We denote by $\mathfrak{c}_I$ the projection of the cone $\mathfrak{c}$ on the space L_I. Its edges are generated by v_j, $j \in I$, and the corresponding unit cell $\mathbf{B}_I$ is the projection of $\mathbf{B}$. Remark that v_j may be non primitive for the projected lattice Λ_I, even if it is primitive for Λ, as we see in the previous example. This is the reason why in Lemma 6, we did not make the (unnecessary) assumption that the generators v_j are primitive.

For $u = (u_1,\dots,u_d)$, we denote $u_I = \sum_{i\in I} u_i$.

We denote the binomial coefficient $\frac{p!}{k!(p-k)!}$ by $\binom{p}{k}$.

Definition 13. *Given a function $I \mapsto \lambda(I)$ on the set of subsets $I \subseteq \{1,\dots,d\}$ with cardinal $|I| \le k_0$, we denote*

$$T(s,\mathfrak{c},k_0,\lambda)(\xi) = \sum_{|I|\le k_0} \lambda(I)\operatorname{vol}(\mathbf{B}_{I^c}) S(s_I + \mathfrak{c}_I)(\xi)(-1)^{d-|I|} \prod_{j\in I^c} \frac{1}{\langle \xi, v_j\rangle}.$$

Remark 14. *The function $S(s_I + \mathfrak{c}_I)(\xi)$ is a meromorphic function on the space L_I^*. We extend it to V^* by the decomposition $V = L_I \oplus L_{I^c}$.*

It is easy to see that the function $T(s,\mathfrak{c},k_0,\lambda)(\xi)$ lies in $\mathcal{M}_\ell(V^*)$, and its expansion into homogeneous components has lowest degree $-d$. Thus

$$T(s,\mathfrak{c},k_0,\lambda)(\xi) = T_{[-d]}(s,\mathfrak{c},k_0,\lambda)(\xi) + T_{[-d+1]}(s,\mathfrak{c},k_0,\lambda)(\xi) + \cdots.$$

We will use functions $I \mapsto \lambda(I)$ which have the following property.

Definition 15. *A (d,k_0)-patchfunction is a function $I \mapsto \lambda(I)$ on the set of subsets $I \subseteq \{1,\dots,d\}$ of cardinal $|I| \le k_0$ which satisfies the following condition.*

$$e^{u_1+\cdots+u_d} \equiv \sum_{|I|\le k_0} \lambda(I) e^{u_I} \text{ mod terms of } u\text{-degree } \ge k_0+1. \tag{8}$$

Theorem 16. *Let $I \mapsto \lambda(I)$ be a k_0-patchfunction. Then we have*

$$S(s+\mathbf{B})(\xi) \equiv \sum_{|I|\leq k_0} \lambda(I)\,\mathrm{vol}(\mathbf{B}_{I^c}) S(s_I+\mathbf{B}_I)(\xi) \quad \text{mod terms of } \xi\text{-degree } \geq k_0+1. \tag{9}$$

$$S(s+\mathfrak{c})(\xi) \equiv T(s,\mathfrak{c},k_0,\lambda)(\xi) \quad \text{mod terms of } \xi\text{-degree } \geq -d+k_0+1. \tag{10}$$

Proof. Using (2), we write

$$S(s+\mathfrak{c})(\xi) = \left(S(s+\mathbf{B})(\xi) \prod_{j=1}^{d} \frac{\langle \xi, v_j\rangle}{1-e^{\langle \xi, v_j\rangle}} \right) \frac{1}{\prod_{j=1}^{d} \langle \xi, v_j\rangle}.$$

Thus we need only the terms of ξ-degree at most k_0 in the Taylor expansion of the holomorphic function $S(s+\mathbf{B})(\xi) \prod_{j=1}^{d} \frac{\langle \xi, v_j\rangle}{1-e^{\langle \xi, v_j\rangle}}$, and finally we need only the the terms of ξ-degree at most k_0 in the Taylor expansion of $S(s+\mathbf{B})(\xi)$. Applying (8) to each term $e^{\langle \xi, x\rangle} = e^{\xi_1 x_1 + \cdots + \xi_d x_d}$ of the finite sum $S(s+\mathbf{B})(\xi)$, we have

$$S(s+\mathbf{B})(\xi) \equiv \sum_{|I|\leq k_0} \lambda(I) \sum_{x\in(s+\mathbf{B})\cap\Lambda} e^{\langle \xi, x_I\rangle} \text{ mod terms of } \xi\text{-degree } \geq k_0+1.$$

For each I, the term $\sum_{x\in(s+\mathbf{B})\cap\Lambda} e^{\langle \xi, x_I\rangle}$ is the sum, over $x \in (s+\mathbf{B})\cap\Lambda$, of a function of x which depends only on the projection x_I of x in the decomposition $x = x_I + x_{I^c} \in L^I \oplus L^{I^c}$. When x runs over $(s+\mathbf{B})\cap\Lambda$, its projection x_I runs over $(s_I+\mathbf{B}_I)\cap\Lambda_I$. Let us show that the fibers have the same number of points, equal to $\mathrm{vol}(\mathbf{B}_{I^c})$. For a given $x_I \in (s_I+\mathbf{B}_I)\cap\Lambda_I$, let us compute the fiber

$$\{y \in (s+\mathbf{B})\cap\Lambda;\ y_I = x_I\}.$$

Fix a point $x_I + x_{I^c}$ in this fiber. Then $y = x_I + y_{I^c}$ lies in the fiber if and only if $y_{I^c} - x_{I^c} \in (s_{I^c} - x_{I^c} + \mathbf{B}_{I^c})\cap\Lambda$. By Lemma 6(ii), the cardinal of the fiber is equal to $\mathrm{vol}(\mathbf{B}_{I^c})$. Thus, we have obtained (9).

Next we write the sum $S(s_I+\mathfrak{c}_I)(\xi)$ over the projected cone $s_I+\mathfrak{c}_I$ in terms of the sum over the projected cell $s_I+\mathbf{B}_I$. We obtain

$$\begin{aligned} S(s+\mathfrak{c})(\xi) &\equiv \sum_{|I|\leq k_0} \lambda(I)\,\mathrm{vol}(\mathbf{B}_{I^c}) S(s_I+\mathfrak{c}_I)(\xi) \prod_{j\in I^c} \frac{1}{(1-e^{\langle \xi, v_j\rangle})} \\ &\equiv \sum_{|I|\leq k_0} \lambda(I)\,\mathrm{vol}(\mathbf{B}_{I^c}) S(s_I+\mathfrak{c}_I)(\xi)(-1)^{d-|I|} \prod_{j\in I^c} \frac{1}{\langle \xi, v_j\rangle} \\ &\qquad \text{mod terms of } \xi\text{-degree } \geq -d+k_0+1. \end{aligned}$$

□

Next we compute an explicit (d, k_0)-patchfunction.

Lemma 17. *If $I \subseteq \{1, \dots, d\}$ has cardinal $|I| \leq k_0$, let*
$$\lambda_{d,k_0}(I) = (-1)^{k_0-|I|}\binom{d-|I|-1}{d-k_0-1}.$$
Then λ_{d,k_0} satisfies Condition (8).

Proof. The trick is to write $e^u = 1 + t(e^u - 1)|_{t=1}$. Thus
$$e^{u_1+\cdots+u_d} = \prod_1^d e^{u_i} = \prod_1^d (1 + t(e^{u_i} - 1))|_{t=1}$$
Let us consider $P(t) := \prod_1^d (1+t(e^{u_i}-1)) = \sum_{q=0}^d C_q(u)t^q$ as a polynomial in the indeterminate t. As $e^{u_i} - 1$ is a sum of terms of u_i-degree > 0, we have

$$e^{u_1+\cdots+u_d} \equiv \sum_{q=0}^{k_0} C_q(u) \text{ mod terms of } u\text{-degree } \geq k_0 + 1. \tag{11}$$

In order to compute the coefficient $C_q(u)$, we write
$$P(t) = \prod_1^d (1 + t(e^{u_i} - 1)) = \prod_1^d ((1-t) + te^{u_i}).$$
By expanding the product, we obtain
$$C_q(u) = \sum_{|I|\leq q} (-1)^{q-|I|}\binom{d-|I|}{q-|I|} e^{u_I}.$$
Summing up these coefficients for $0 \leq q \leq k_0$, we obtain
$$\sum_{q=0}^{k_0} C_q(u) = \sum_{|I|\leq k_0}\left(\sum_{q=|I|}^{k_0}(-1)^{q-|I|}\binom{d-|I|}{q-|I|}\right) e^{u_I}.$$
For $m_0 \leq d_0$, let us denote
$$F(m_0, d_0) = \sum_{j=0}^{m_0}(-1)^j\binom{d_0}{j}.$$
Thus
$$\sum_{q=0}^{k_0} C_q(u) = \sum_{|I|\leq k_0} F(k_0 - |I|, d - |I|)e^{u_I}.$$

The sum $F(m_0, d_0)$ is easy to compute by induction on m_0, using the recursion relation

$$\binom{d_0}{j} = \binom{d_0 - 1}{j} + \binom{d_0 - 1}{j-1}.$$

We obtain

$$F(m_0, d_0) = (-1)^{m_0}\binom{d_0 - 1}{m_0}.$$

Hence,

$$F(k_0 - |I|, d - |I|) = (-1)^{k_0 - |I|}\binom{d - |I| - 1}{k_0 - |I|} = (-1)^{k_0 - |I|}\binom{d - |I| - 1}{d - k_0 - 1}.$$

The claim follows now from Equation (11). ☐

Remark 18. *As promised, the main feature of Formula (10) is that the right-hand-side* $T(s, \mathfrak{c}, k_0, \lambda)$ *involves discrete summations in dimension* $|I| \leq k_0$ *only.*

Theorem 16 can be reformulated in terms of the intermediate valuations introduced by Barvinok in [4]. The reformulation relies on the next lemma, which shows that the (d, k_0)-patchfunction condition is equivalent to a Moebius-type condition for the function $I \mapsto \lambda(I)$.

Lemma 19. *Let* $0 \leq k_0 \leq d$ *be two integers. Let* λ *be a function on the set of subsets* $I \subseteq \{1, \dots, d\}$ *of cardinal* $|I| \leq k_0$. *The following conditions are equivalent.*
(i) For every I_0 *of cardinal* $|I_0| \leq k_0$,

$$\sum_{\{I;\, |I| \leq k_0,\, I_0 \subseteq I\}} \lambda(I) = 1.$$

(ii) For every integer k *such that* $0 \leq k \leq k_0$, *we have the equality of polynomials*

$$(u_1 + \cdots + u_d)^k = \sum_{|I| \leq k_0} \lambda(I) u_I^k \tag{12}$$

(iii) The function λ *is a* (d, k_0)*-patchfunction.*

Proof. Conditions (ii) and (iii) are clearly equivalent. Let us show that (i) and (ii) are equivalent. We expand $(u_1 + \cdots + u_d)^k$ into a sum of monomials. A monomial of degree k can involve at most k variables u_i, with $k \leq k_0$. Therefore we obtain

$$\frac{1}{k!}(u_1 + \cdots + u_d)^k = \sum_{|I| \leq k_0} \sum_{\substack{(k_i)_{i \in I} \\ \sum k_i = k}} \prod_{i \in I} \frac{u_i^{k_i}}{k_i!}. \tag{13}$$

We expand similarly each term in the right-hand side of (12). A given monomial $\prod_{i\in I_0}\frac{u_i^{k_i}}{k_i!}$, with $k_i\neq 0$ for all $i\in I_0$, occurs in the right-hand side of (12) exactly for the subsets I such that $I_0\subseteq I$. Thus (i) implies (ii). Conversely, Equation (12) for $k=k_0$ implies (i). □

5. Computation of Ehrhart quasi-polynomials

We now apply the approximation of the generating functions of the cones at vertices to the computation of the highest coefficients for a weighted Ehrhart polynomial, when the weight is a power of a linear form, as we explained in section 3.

Corollary 20. *Let $\mathfrak{p}$ be a simple lattice polytope. Fix $\xi\in\mathbb{R}^d$ and $M\in\mathbb{N}$. Let $E_m(\mathfrak{p},\xi,M), m=0,\dots,d+M$, be the coefficients of the weighted Ehrhart polynomial*

$$\sum_{x\in n\mathfrak{p}\cap\Lambda}\frac{\langle\xi,x\rangle^M}{M!}=\sum_{m=0}^{M+d}E_m(\mathfrak{p},\xi,M)n^m.$$

Fix $0\le k_0\le d$. Let λ be a (d,k_0)-patchfunction. Then, for $k=0,\dots,k_0$, the Ehrhart coefficient $E_{M+d-k}(\mathfrak{p},\xi,M)$ is given by the following formula.

$$E_{M+d-k}(\mathfrak{p},\xi,M) = \sum_{s\in\mathcal{V}(\mathfrak{p})}\frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!}T(0,\mathfrak{c}_s,k_0,\lambda)_{[-d+k]}(\xi). \tag{14}$$

In the general case, when the vertices are not assumed to be lattice points, we state the result without going through the details of the computation.

Theorem 21. *Let $\mathfrak{p}$ be a simple polytope. For each vertex s of $\mathfrak{p}$, let $q_s\in\mathbb{N}$ be the smallest integer such that $q_s s\in\Lambda$. For $n\in\mathbb{N}$, let n_s be the residue of n mod q_s, so that $0\le n_s\le q_s-1$. Fix $\xi\in V^*$ and M a nonnegative integer. Fix $0\le k_0\le d$. Let λ be a (d,k_0)-patchfunction.*

Then the Ehrhart quasi-polynomial

$$\sum_{x\in n\mathfrak{p}\cap\Lambda}\frac{\langle\xi,x\rangle^M}{M!}$$

coincides in degree $\ge M+d-k_0$ with the following quasi-polynomial

$$\sum_{k=0}^{k_0}\sum_{s\in\mathcal{V}(\mathfrak{p})}(n-n_s)^{M+d-k}\frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!}T_{[-d+k]}(n_s s,\mathfrak{c}_s,k_0,\lambda)(\xi). \tag{15}$$

Observe that (15) is clearly a quasi-polynomial in n with coefficients which depend only on the residues $(n \mod q_s) = n_s, s \in \mathcal{V}(\mathfrak{p})$.

Remark 22. *In practice, we first reduce the vertices s mod Λ by using*

$$S(s + \mathfrak{c}_s)(\xi) = e^{\langle \xi, v \rangle} S(s - v + \mathfrak{c}_s)(\xi), \text{ for } v \in \Lambda.$$

Then we write an approximation similar to (15).

6. Conclusion

Let $\mathfrak{p} \subset \mathbb{R}^d$ be a rational simplex. Let $\langle \xi, x \rangle$ be a rational linear form on $\mathbb{R}^d$, and consider a power $\langle \xi, x \rangle^M$. Let $E_m(\mathfrak{p}, \xi, M), m = 0, \ldots, d + M$, be the coefficients of the weighted Ehrhart quasi-polynomial

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(\mathfrak{p}, \xi, M) n^m.$$

Fix an integer k_0, $0 \leq k_0 \leq d$. The main consequence of this study is a method for efficiently computing the $k_0 + 1$ highest degree coefficients $E_m(\mathfrak{p}, \xi, M)$, for $m = M + d, \ldots, M + d - k_0$. The method relies on expanding (15) in Theorem 21 as a power series in ξ.

Furthermore, one can write any homogeneous polynomial weight $h(x)$ as a linear combination of powers of linear forms,

$$h(x) = \sum_k c_k \langle \xi_k, x \rangle^M.$$

In a forthcoming article by the authors of [1], we will show how to derive

- first, an algorithm for computing $E_m(\mathfrak{p}, \xi, M)$, for $m = M + d, \ldots, M + d - k_0$. Hopefully this algorithm is polynomial, when the input consists of the dimension d and the degree M , the rational simplex $\mathfrak{p} \subset \mathbb{R}^d$, the rational linear form ξ on $\mathbb{R}^d$, provided k_0 is fixed;

- second, an algorithm for computing the $k_0 + 1$ highest degree coefficients of a weighted Ehrhart quasi-polynomial relative to a simplex. Hopefully this algorithm is polynomial when k_0 and the degree of the weight $h(x)$ are fixed, but the dimension of the simplex is part of the input.

[1] dealt with the case of the highest Ehrhart coefficient which is just the integral of the weight over the simplex.

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