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ARRANGEMENT OF HYPERPLANES, II: THE SZENES FORMULA AND EISENSTEIN SERIES

MICHEL BRION AND MICHÈLE VERGNE

To Victor Guillemin, for his 60th birthday

1. Introduction. Consider a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of linear forms in r complex variables, with integral coefficients. The linear forms α_j need not be distinct. For example, $r = 2$ and $\alpha_1 = \alpha_2 = z_1$, $\alpha_3 = \alpha_4 = z_2$, $\alpha_5 = \alpha_6 = z_1 + z_2$. For any such sequence, D. Zagier [5] introduced the series

$$\sum_{n \in \mathbb{Z}^r, \langle \alpha_j, n\rangle \neq 0} \frac{1}{\prod_{j=1}^k \langle \alpha_j, n\rangle}.$$

Assuming convergence, its sum is a rational multiple of π^k. For example (see [5]), we have

$$\sum_{n_1 \neq 0, n_2 \neq 0, n_1+n_2 \neq 0} \frac{1}{n_1^2 n_2^2 (n_1+n_2)^2} = \frac{(2\pi)^6}{30240}.$$

These numbers are natural multidimensional generalizations of the value of the Riemann zeta function at even integers. A. Szenes gave in [3, Theorem 4.4] a residue formula for these numbers, relating them to Bernoulli numbers. The formula of Szenes [3] is the multidimensional analogue of the residue formula

$$\sum_{n \neq 0} \frac{1}{n^{2l}} = (2\pi)^{2l} \frac{B_{2l}}{(2l)!} = (-1)^l (2\pi)^{2l} \operatorname{Res}_{z=0} \left(\frac{1}{z^{2l}(1-e^z)} \right).$$

A motivation for computing such sums comes from the work of E. Witten [4]. In the special case where α_j are the positive roots of a compact connected Lie group G, each of these roots being repeated with multiplicity $2g-2$, Witten expressed the symplectic volume of the space of homomorphisms of the fundamental group of a Riemann surface of genus g into G, in terms of these sums. In [2], L. Jeffrey and F. Kirwan proved a special case of the Szenes formula leading to the explicit computation of this symplectic volume, when G is $\mathrm{SU}(n)$.

Our interest in such series comes from a different motivation. Let us consider first the 1-dimensional case. By the Poisson formula, for $\mathrm{Re}(z) > 0$, the convergent series $\sum_{m=1}^{\infty} m e^{-mz}$ is also equal to $\sum_{n \in \mathbb{Z}} 1/(z + 2i\pi n)^2$. Similarly, sums of products

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of polynomial functions with exponential functions over all integral points of an r-dimensional rational convex cone are related to functions of r complex variables of the form

$$\psi(z) = \sum_{n\in\mathbb{Z}^r} \frac{1}{\prod_{j=1}^k \langle \alpha_j, z + 2i\pi n\rangle}.$$

When this series is not convergent, introduce the oscillating factor $e^{\langle t, 2i\pi n\rangle}$ and define the Eisenstein series

$$\psi(t,z) = \sum_{n\in\mathbb{Z}^r} \frac{e^{\langle t, z+2i\pi n\rangle}}{\prod_{j=1}^k \langle \alpha_j, z + 2i\pi n\rangle},$$

a generalized function of $t \in \mathbb{R}^r$.

In Section 3, we construct a decomposition of an open dense subset of $\mathbb{R}^r$ into alcoves such that $t \mapsto \psi(t,z)$ is given on each alcove by a polynomial in t, with rational functions of e^z as coefficients. Our first theorem (see Theorem 19) gives an explicit residue formula for $\psi(t,z)$. It follows easily from the obvious behaviour of $\psi(t,z)$ under differentiation in z.

This formula allows us to give a residual meaning "$\psi(t,0)$" for the value of $\psi(t,z)$ at $z = 0$, although $\psi(t,z)$ clearly has poles along all hyperplanes $\langle \alpha_j, z\rangle = 0$. An alternate way to define $\psi(t,0)$ is to remove all infinities $1/\alpha_j$ in the series

$$\psi(t,0) = \sum_{n\in\mathbb{Z}^r} \frac{e^{\langle t, 2i\pi n\rangle}}{\prod_{j=1}^k \langle \alpha_j, 2i\pi n\rangle}.$$

Indeed, we prove that the residue formula for "$\psi(t,0)$" coincides with the renormalized sum:

$$\text{“}\psi(t,0)\text{”} = \sum_{n\in\mathbb{Z}^r,\, \langle\alpha_j, n\rangle \neq 0} \frac{e^{\langle t, 2i\pi n\rangle}}{\prod_{j=1}^k \langle \alpha_j, 2i\pi n\rangle}.$$

This equality gives another proof of the Szenes residue formula, as a "limit" of a natural formula for $\psi(t,z)$ when $z \to 0$ along a generic line.

To illustrate our method, let us consider the 1-dimensional case. For $k \geq 2$, we can define the Eisenstein series

$$E_k(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z+2i\pi n)^k}.$$

Clearly, $E_k(z)$ is periodic in z with respect to translation by the lattice $2i\pi\mathbb{Z}$. From the residue theorem, when y is not in $2i\pi\mathbb{Z}$, we have the kernel formula

$$E_k(y) = \mathrm{Res}_{z=0}\left(\frac{1}{z^k(1-e^{z-y})}\right). \tag{1}$$

Observe that the right-hand side has a meaning when $y = 0$, and equals, by definition, the Bernoulli number $B_k/k!$. The function

$$E_k(y) = \frac{1}{y^k} + \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{(y+2i\pi n)^k}$$

has a Laurent expansion at $y=0$, with $1/y^k$ as Laurent negative part. We see from the residue formula that the constant term $\mathrm{CT}(E_k) = \sum_{n\in\mathbb{Z},n\neq 0} 1/(2i\pi n)^k$ equals $\mathrm{Res}_{z=0}(1/(z^k(1-e^z)))$.

In view of this example, we call the value "$\psi(t,0)$" of $\psi(t,y)$ at $y=0$ the constant term of the Eisenstein series

$$\sum_{n\in\mathbb{Z}^r} \frac{e^{\langle t, z+2i\pi n\rangle}}{\prod_{j=1}^k \langle \alpha_j, z+2i\pi n\rangle}.$$

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2. Kernel formula. In this section, we briefly recall results of [1] with slightly modified notation. Let V be an r-dimensional complex vector space. Let V^* be the dual vector space, and let $\Delta \subset V^*$ be a finite subset of nonzero linear forms. Each $\alpha \in \Delta$ determines a hyperplane $\{\alpha = 0\}$ in V. Consider the hyperplane arrangement

$$\mathcal{H} = \bigcup_{\alpha\in\Delta} \{\alpha = 0\}.$$

An element $z \in V$ is called *regular* if z is not in $\mathcal{H}$. If S is a subset of V, we write S_{reg} for the set of regular elements in S. The ring R_Δ of rational functions with poles on $\mathcal{H}$ is the ring $\Delta^{-1}S(V^*)$ generated by the ring $S(V^*)$ of polynomial functions on V, together with inverses of the linear functions $\alpha \in \Delta$. The ring R_Δ has a $\mathbb{Z}$-gradation by the homogeneous degree that can be positive or negative. Elements of R_Δ are defined on the open subset V_{reg}. (Our notation differs from [1] in that the roles of V and V^* are interchanged.)

In the one-variable case, the function $1/z$ is the unique function that cannot be obtained as a derivative. There is a similar description of a complement space to the space of derivatives in the ring R_Δ, which we recall now.

A subset σ of Δ is called a *basis of* Δ if the elements $\alpha \in \sigma$ form a basis of V. We denote by $\mathcal{B}(\Delta)$ the set of bases of Δ. An *ordered basis* is a sequence $(\alpha_1, \alpha_2, \dots, \alpha_r)$ of elements of Δ such that the underlying set is a basis. We denote by $O\mathcal{B}(\Delta)$ the set of ordered bases.

For $\sigma \in \mathcal{B}(\Delta)$, set

$$\phi_\sigma(z) := \frac{1}{\prod_{\alpha\in\sigma} \alpha(z)}.$$

We call ϕ_σ a *simple fraction*. Setting $z_j = \langle z, \alpha_j\rangle$, we have

$$\phi_\sigma(z) = \frac{1}{z_1 z_2 \cdots z_r}.$$

Definition 1. The subspace S_Δ of R_Δ spanned by the elements $\phi_\sigma,\ \sigma \in \mathcal{B}(\Delta)$, will be called the space of *simple elements* of R_Δ:

$$S_\Delta = \sum_{\sigma \in \mathcal{B}(\Delta)} \mathbb{C}\phi_\sigma .$$

The space S_Δ consists of homogeneous rational functions of degree $-r$. However, not every homogeneous element of degree $-r$ of R_Δ is in S_Δ (e.g., in the preceding notation, if $r \geq 2$, both functions $1/z_1^r$ and z_2/z_1^{r+1} are not in S_Δ). Furthermore, we must be careful, as the elements ϕ_σ may be linearly dependent. For example, if $V = \mathbb{C}^2$ and $\Delta = \{z_1, z_2, z_1 + z_2\}$, we have

$$S_\Delta = \mathbb{C}\frac{1}{z_1 z_2} + \mathbb{C}\frac{1}{z_1(z_1+z_2)} + \mathbb{C}\frac{1}{z_2(z_1+z_2)}$$

and we have the relation

$$\frac{1}{z_1 z_2} = \frac{1}{z_1(z_1+z_2)} + \frac{1}{z_2(z_1+z_2)}.$$

A description due to Orlik and Solomon of all linear relations between the elements ϕ_σ is given in [1, Proposition 13].

Definition 2. A *basis* B of $\mathcal{B}(\Delta)$ is a subset of $\mathcal{B}(\Delta)$ such that the elements ϕ_σ, $\sigma \in B$, form a basis of S_Δ:

$$S_\Delta = \bigoplus_{\sigma \in B} \mathbb{C}\phi_\sigma .$$

We let elements v of V act on R_Δ by differentiation:

$$\big(\partial(v) f\big)(z) := \frac{d}{d\epsilon} f(z + \epsilon v)|_{\epsilon=0}.$$

Then the following holds (see [1, Proposition 7]).

THEOREM 3. *We have*

$$R_\Delta = \partial(V) R_\Delta \oplus S_\Delta .$$

Thus, we see that only simple fractions cannot be obtained as derivatives.

As a corollary of this decomposition, we can define the projection map

$$\mathrm{Res}_\Delta : R_\Delta \longrightarrow S_\Delta .$$

The projection $\mathrm{Res}_\Delta f(z)$ of a function $f(z)$ is a function of z that we call the *Jeffrey-Kirwan residue* of f. By definition, this function can be expressed as a linear combination of the simple fractions ϕ_σ. The main property of the map Res_Δ is that it vanishes on derivatives, so that for $v \in V,\ f, g \in R_\Delta$,

$$\mathrm{Res}_\Delta \Big(\big(\partial(v) f\big) g\Big) = -\mathrm{Res}_\Delta \Big(f\big(\partial(v) g\big)\Big). \tag{2}$$

If $o\sigma \in O\mathcal{B}(\Delta)$ is an ordered basis, an important functional $\mathrm{Res}^{o\sigma}$ can be defined on R_Δ: the *iterated residue* with respect to the ordered basis $o\sigma$. If we write an element $z \in V$ on the basis $o\sigma = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ as $z = (z_1, \ldots, z_r)$, then

$$\mathrm{Res}^{o\sigma}(f) = \mathrm{Res}_{z_1=0}\left(\mathrm{Res}_{z_2=0}\cdots\left(\mathrm{Res}_{z_r=0} f(z_1, z_2, \ldots, z_r)\right)\cdots\right).$$

The map $\mathrm{Res}^{o\sigma}$ depends on the order $o\sigma$ chosen on σ and not only on the basis σ underlying $o\sigma$. The restriction of the functional $\mathrm{Res}^{o\sigma}$ to S_Δ is called $r^{o\sigma}$. We have

$$\mathrm{Res}^{o\sigma} = r^{o\sigma}\,\mathrm{Res}_\Delta\,. \tag{3}$$

Indeed, we have only to check that $\mathrm{Res}^{o\sigma}$ vanishes on derivatives. If $o\sigma = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $z = (z_1, \ldots, z_r)$, the iterated residue $\mathrm{Res}^{o\sigma}$ vanishes at the step $\mathrm{Res}_{z_j=0}$ on $\partial R_\Delta/\partial z_j$.

Recall the following definition from A. Szenes (see [3, Definition 3.3]).

Definition 4. A *diagonal basis* is a subset OB of $O\mathcal{B}(\Delta)$ such that the following are true.

(1) The set of underlying (unordered) bases forms a basis B of $\mathcal{B}(\Delta)$.

(2) The dual basis to the basis $(\phi_\sigma, o\sigma \in OB)$ is the set of linear forms $(r^{o\sigma}, o\sigma \in OB)$:

$$r^{o\tau}(\phi_\sigma) = \delta_\sigma^\tau.$$

In [3, Proposition 3.4], it is proved that a total order on Δ gives rise to a diagonal basis. (This is proved again in more detail in [1, Proposition 14].)

In the 1-dimensional case, $S_\Delta = \mathbb{C}z^{-1}$, and the space $G = \sum_{k\le -1}\mathbb{C}z^k$ of negative Laurent series is the space obtained from the function $1/z$ by successive derivations. In the case of several variables, we can also characterize the space generated by simple fractions under differentiation.

Let κ be a sequence of (not necessarily distinct) elements of Δ. The sequence κ is called *generating* if the $\alpha \in \kappa$ generate the vector space V^*.

We denote by G_Δ the subspace of R_Δ spanned by the

$$\phi_\kappa := \frac{1}{\prod_{\alpha\in\kappa}\alpha},$$

where κ is a generating sequence. Finally, we denote by $S(V)$ the ring of differential operators on V, with constant coefficients. This ring acts on $S(V^*)$ and on R_Δ.

Proposition 5 [1, Theorem 1]. *The space G_Δ is the $S(V)$-submodule of R_Δ generated by S_Δ.*

For example, if $\Delta = \{z_1, z_2, z_1+z_2\}$, we have

$$\frac{1}{z_1z_2(z_1+z_2)} = -\frac{\partial}{\partial z_1}\left(\frac{1}{z_1z_2}\right) + \left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}\right)\left(\frac{1}{z_1(z_1+z_2)}\right).$$

In particular, every element of G_Δ can be expressed as a linear combination of elements
$$\frac{1}{\prod_{\alpha\in\sigma}\alpha^{n_\alpha}},$$
where σ is a basis and the n_α are positive integers.

For example, the above equality is equivalent to
$$\frac{1}{z_1z_2(z_1+z_2)}=\frac{1}{z_1^2z_2}-\frac{1}{z_1^2(z_1+z_2)}.$$

The ring $S(V^*)$ operates by multiplication on R_Δ. It is also useful to consider the action of the ring $\mathcal{D}(V)$ of differential operators with polynomial coefficients, generated by $S(V)$ and $S(V^*)$. The following lemma is an obvious corollary of the description of G_Δ.

LEMMA 6. *The space R_Δ is generated by G_Δ as an $S(V^*)$-module. It is generated by S_Δ as a $\mathcal{D}(V)$-module.*

Consider now the space $\mathcal{O}$ of holomorphic functions on V defined in a neighborhood of zero. Let $\mathcal{O}_\Delta=\Delta^{-1}\mathcal{O}$ be the space of meromorphic functions in a neighborhood of zero, with products of elements of Δ as denominators. The space $\mathcal{O}_\Delta$ is a module for the action of differential operators with constant coefficients. Via the Taylor series at the origin of elements of $\mathcal{O}$, the residue $\operatorname{Res}_\Delta f(z)$ still has a meaning if $f(z)\in\mathcal{O}_\Delta$; indeed, $\operatorname{Res}_\Delta f(z)=0$ if $f\in R_\Delta$ is homogeneous of degree not equal to $-r$.

If $y\in V$ is sufficiently near zero and $f\in\mathcal{O}_\Delta$, the function
$$\big(\mathcal{T}(y)f\big)(z):=f(z-y)$$
is still an element of $\mathcal{O}_\Delta$. Moreover, if y is regular, then $f(z-y)$ is defined for $z=0$ and thus is an element of $\mathcal{O}$.

If $f\in R_\Delta$, we denote by $m(f)$ the operator of multiplication by f:
$$\big(m(f)\phi\big)(z):=f(z)\phi(z).$$
It operates on $\mathcal{O}_\Delta$. Finally, we denote by C the operator
$$(Cf)(z):=f(-z)$$
on $\mathcal{O}_\Delta$.

THEOREM 7 (Kernel theorem). *Let $A:R_\Delta\to\mathcal{O}_\Delta$ be an operator commuting with the action of differential operators with constant coefficients. For $y\in V$ regular, sufficiently near zero, and for $f\in G_\Delta$, we have the formula*
$$(Af)(y)=\operatorname{Tr}_{S_\Delta}\big(\operatorname{Res}_\Delta m(f)C\mathcal{T}(y)A\operatorname{Res}_\Delta\big).$$

More explicitly, choose a basis B of $\mathcal{B}(\Delta)$, and let $(\phi^\sigma, \sigma \in B)$ be the basis of S_Δ^* dual to the basis $(\phi_\sigma, \sigma \in B)$ of S_Δ. Then we have the kernel formula

$$(Af)(y) = \sum_{\sigma \in B} \left\langle \phi^\sigma, \operatorname{Res}_\Delta \left(f(z) A_\sigma (y-z) \right) \right\rangle,$$

where $A_\sigma(z) = A(\phi_\sigma)(z)$.

Concretely, this formula has the following meaning. Let f be homogeneous of degree d. We fix y regular and small. The function $z \mapsto A_\sigma(y-z)$ is defined near $z = 0$. The Jeffrey-Kirwan residue $\operatorname{Res}_\Delta$ of the function $z \mapsto f(z)A_\sigma(y-z)$ is a function of z belonging to the space S_Δ. We pair it with the linear form ϕ^σ on S_Δ, and we obtain a certain complex number depending on y. More precisely, consider the Taylor expansion

$$A_\sigma(y-z) = A_\sigma(y) + \sum_{j=1}^{\infty} A_\sigma^j(y,z),$$

where $A_\sigma^j(y,z)$ is the part of the Taylor expansion at zero of the holomorphic function $z \mapsto A_\sigma(y-z)$, which is homogeneous of degree j in z. We have

$$A_\sigma^j(y,z) = (-1)^j \sum_{(k),|(k)|=j} A_\sigma^{(k)}(y) \frac{z^{(k)}}{(k)!},$$

where $(k) = (k_1, \dots, k_r)$ is a multi-index, and $A_\sigma^{(k)}(y) = ((\partial/\partial y)^{(k)} A_\sigma)(y)$. Then, as the Jeffrey-Kirwan residue vanishes on homogeneous terms of degree not equal to $-r$, we obtain

$$\begin{aligned} \operatorname{Res}_\Delta \left(f(z) A_\sigma(y-z) \right) &= \operatorname{Res}_\Delta \left(f(z) A_\sigma^{-d-r}(y,z) \right) \\ &= (-1)^{d+r} \sum_{(k),|(k)|=-d-r} A_\sigma^{(k)}(y) \operatorname{Res}_\Delta \left(f(z) \frac{z^{(k)}}{(k)!} \right). \end{aligned}$$

Thus, $\langle \phi^\sigma, \operatorname{Res}_\Delta(f(z)A_\sigma(y-z)) \rangle$ is equal to

$$(-1)^{d+r} \sum_{(k),|(k)|=-d-r} A_\sigma^{(k)}(y) \left\langle \phi^\sigma, \operatorname{Res}_\Delta \left(f(z) \frac{z^{(k)}}{(k)!} \right) \right\rangle.$$

Set $c_\sigma^{(k)}(f) = \langle \phi^\sigma, \operatorname{Res}_\Delta(f(z)(z^{(k)}/(k)!)) \rangle$. Let $P_\sigma^f(\partial/\partial y)$ be the differential operator with constant coefficients defined by

$$P_\sigma^f \left(\frac{\partial}{\partial y} \right) = (-1)^{d+r} \sum_{(k),|(k)|=-d-r} c_\sigma^{(k)}(f) \left(\frac{\partial}{\partial y} \right)^{(k)}.$$

Then P_σ^f depends linearly on f, and

$$\left\langle \phi^\sigma, \operatorname{Res}_\Delta \left(f(z) A_\sigma(y-z) \right) \right\rangle = \left(P_\sigma^f \left(\frac{\partial}{\partial y} \right) A_\sigma \right)(y).$$

The claim of the theorem is that

$$(Af)(y) = \sum_{\sigma \in B} P_\sigma^f \left(\frac{\partial}{\partial y} \right) \cdot A_\sigma(y).$$

We now prove this theorem.

Proof. Define an operator $A' : R_\Delta \to \mathbb{O}_\Delta$ by

$$(A'f)(y) = \sum_{\sigma \in B} \left\langle \phi^\sigma, \operatorname{Res}_\Delta \left(f(z) A_\sigma(y-z) \right) \right\rangle.$$

We first check that A' commutes with the action of differential operators with constant coefficients. Using the equation

$$\left(\partial_y(v) \phi \right)(y-z) = -\left(\partial_z(v) \phi \right)(y-z)$$

and the main property (2) of $\operatorname{Res}_\Delta$, we obtain

$$\begin{aligned} \partial_y(v) \cdot \left\langle \phi^\sigma, \operatorname{Res}_\Delta \left(f(z) A_\sigma(y-z) \right) \right\rangle &= \left\langle \phi^\sigma, \operatorname{Res}_\Delta \left(f(z) \left(\partial_y(v) \cdot A_\sigma(y-z) \right) \right) \right\rangle \\ &= -\left\langle \phi^\sigma, \operatorname{Res}_\Delta \left(f(z) \left(\partial_z(v) \cdot A_\sigma(y-z) \right) \right) \right\rangle \\ &= \left\langle \phi^\sigma, \operatorname{Res}_\Delta \left(\left(\partial_z(v) \cdot f \right) A_\sigma(y-z) \right) \right\rangle. \end{aligned}$$

It remains to see that A and A' coincide on S_Δ. For this, we use the following formula. If P is a polynomial and ϕ a simple fraction, then

$$\operatorname{Res}_\Delta(P\phi) = P(0)\phi. \tag{4}$$

To see this, recall that the function ϕ is homogeneous of degree $-r$. As $P \in S(V^*)$, $P - P(0)$ is a sum of homogeneous terms of positive degree. Thus, for homogeneity reasons, $\operatorname{Res}_\Delta((P - P(0))\phi) = 0$.

Let y be regular, and let $\sigma, \tau \in B$. As the function $z \to A_\sigma(y-z)$ is an element of $\mathbb{O}$, by formula (4) we obtain

$$\operatorname{Res}_\Delta \left(\phi_\tau(z) A_\sigma(y-z) \right) = A_\sigma(y) \phi_\tau(z).$$

Thus,

$$\begin{aligned} A'(\phi_\tau)(y) &= \sum_{\sigma \in B} \left\langle \phi^\sigma, \operatorname{Res}_\Delta \left(\phi_\tau(z) A_\sigma(y-z) \right) \right\rangle \\ &= \sum_{\sigma \in B} \langle \phi^\sigma, \phi_\tau \rangle A_\sigma(y) = \sum_{\sigma \in B} \delta_\sigma^\tau A_\sigma(y) = A_\tau(y) = A(\phi_\tau)(y). \end{aligned}$$ □

Choosing a diagonal basis *OB* and using equation (3), we obtain an iterated residue formula for $(Af)(y)$.

COROLLARY 8. *For any diagonal basis OB of $\mathcal{B}(\Delta)$, we have, for $f \in G_\Delta$,*

$$(Af)(y) = \sum_{o\sigma \in OB} \mathrm{Res}^{o\sigma}\big(f(z)A_\sigma(y-z)\big),$$

where $A_\sigma(z) = A(\phi_\sigma)(z)$.

Corollary 8 applies to the identity operator $A : R_\Delta \to R_\Delta$. If $f \in G_\Delta$, we obtain $f(y) = \sum_{o\sigma \in OB} \mathrm{Res}^{o\sigma}(f(z)\phi_\sigma(y-z))$. But if $f \in NG_\Delta$, then clearly $\mathrm{Res}^{o\sigma}(f(z)\phi_\sigma(y-z)) = 0$, as the Taylor series of $f(z)\phi_\sigma(y-z)$ at $z = 0$ is also in NG_Δ. As a consequence, we obtain a formula for the Jeffrey-Kirwan residue as a function of iterated residues.

LEMMA 9. *For any $f \in R_\Delta$, we have*

$$(\mathrm{Res}_\Delta f)(y) = \sum_{o\sigma \in OB} \mathrm{Res}^{o\sigma}(f)\phi_\sigma(y).$$

Similarly, if $Z : R_\Delta \to \mathbb{O}$ is an operator commuting with the action of differential operators with constant coefficients, the formula

$$Z(f)(y) = \mathrm{Tr}_{S_\Delta}\big(\mathrm{Res}_\Delta\, m(f) C \mathcal{T}(y) Z\, \mathrm{Res}_\Delta\big)$$

is valid for *all* elements $y \in V$ sufficiently near zero and for all $f \in G_\Delta$. In particular, we have the following proposition.

PROPOSITION 10. *Let $Z : R_\Delta \to \mathbb{O}$ be an operator commuting with the action of differential operators with constant coefficients. Then we have, for $f \in G_\Delta$,*

$$Z(f)(0) = \mathrm{Tr}_{S_\Delta}\big(\mathrm{Res}_\Delta\, m(f) C Z\, \mathrm{Res}_\Delta\big),$$

where $(CZ)(\phi)(z) = Z(\phi)(-z)$.

Choosing a diagonal basis of $O\mathcal{B}(\Delta)$, we can express the preceding formula as a residue formula in several variables:

$$Z(f)(0) = \sum_{o\sigma \in OB} \mathrm{Res}^{o\sigma}\big(f(z)Z_\sigma(-z)\big),$$

with $Z_\sigma(z) = Z(\phi_\sigma)(z)$.

For later use, we prove a vanishing property of the linear form $\mathrm{Res}^{o\sigma}$. Let $o\sigma$ be an ordered basis. We write $o\sigma = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $z = (z_1, z_2, \ldots, z_r)$. Set $o\sigma' = (\alpha_2, \ldots, \alpha_r)$ and $z' = (z_2, \ldots, z_r)$; then $z = (z_1, z')$. Let $\psi(z')$ in $\mathbb{O}_{\Delta'}$ be a meromorphic function with a product of linear forms $\alpha(z')$, where $\alpha \in \Delta$ is not a multiple of α_1, as a denominator.

LEMMA 11. *For any $f \in G_\Delta$ and for any $\psi \in \mathbb{O}_{\Delta'}$,*

$$\mathrm{Res}^{o\sigma}\left(\frac{1}{z_1} f(z_1, z')\psi(z')\right) = 0.$$

Proof. We have

$$\operatorname{Res}^{o\sigma}\left(\frac{1}{z_1} f(z_1,z')\psi(z')\right) = \operatorname{Res}_{z_1=0}\left(\frac{1}{z_1}\operatorname{Res}^{o\sigma'}\big(f(z_1,z')\psi(z')\big)\right).$$

In computing $\operatorname{Res}^{o\sigma'}(f(z_1,z')\psi(z'))$, the variable z_1 is fixed to a nonzero value. The result $\operatorname{Res}^{o\sigma'}(f(z_1,z')\psi(z'))$ is a meromorphic function of z_1. It is thus sufficient to prove that $\operatorname{Res}^{o\sigma'}(f(z_1,z')\psi(z'))$ belongs to the space $G=\sum_{k\le -1}\mathbb{C}z_1^k$.

We check this for $f=\phi_\kappa$, where

$$\phi_\kappa(z)=\frac{1}{\prod_{\alpha\in\kappa}\langle\alpha,z\rangle}$$

and κ is a generating sequence. Let

$$\kappa_1 := \big\{\alpha\in\kappa,\ \langle\alpha,(z_1,0)\rangle\neq 0\big\}$$

and

$$\kappa' = \big\{\alpha\in\kappa,\ \langle\alpha,(z_1,0)\rangle = 0\big\}.$$

As κ is generating, the set κ_1 is nonempty. We fix $z_1\neq 0$. We have

$$\phi_\kappa(z_1,z')\psi(z')=\phi_{\kappa_1}(z_1,z')\phi_{\kappa'}(z')\psi(z')$$

and $\phi_{\kappa'}\in\mathbb{O}_{\Delta'}$. For $\alpha\in\kappa_1$, we set $\langle\alpha,(z_1,z')\rangle = c_\alpha z_1+\langle\beta,z'\rangle$, with $c_\alpha\neq 0$. We consider the Taylor expansion at $z'=0$ of the holomorphic function of z':

$$\frac{1}{\langle\alpha,(z_1,z')\rangle}=\frac{1}{c_\alpha z_1+\langle\beta,z'\rangle}=\frac{1}{c_\alpha z_1\big(1+\langle\beta,z'\rangle/(c_\alpha z_1)\big)}.$$

This is of the form

$$\sum_{k=1}^{\infty} z_1^{-k}P_{k-1}(z'),$$

where $P_{k-1}(z')$ is homogeneous of degree $k-1$ in z'. Let $n=|\kappa_1|$; then $n\ge 1$. We see that the function

$$z'\longmapsto \phi_{\kappa_1}(z_1,z')=\frac{1}{\prod_{\alpha\in\kappa_1}\langle\alpha,(z_1,z')\rangle}$$

has a Taylor expansion of the form

$$\sum_{k\ge n} z_1^{-k}Q_{k-1}(z'),$$

where $Q_{k-1}(z')$ is homogeneous of degree $k-1$ in z'. Thus

$$\operatorname{Res}^{o\sigma'}\big(\phi_{\kappa_1}(z_1,z')\phi_{\kappa'}(z')\psi(z')\big)=\sum_{k\ge n} z_1^{-k}\operatorname{Res}^{o\sigma'}\big(Q_{k-1}(z')\phi_{\kappa'}(z')\psi(z')\big).$$

Via the Taylor series at $z'=0$, the function $\phi_{\kappa'}(z')\psi(z')$ can be expressed as an infinite sum of homogeneous elements with finitely many negative degrees.As the iterated residue $\operatorname{Res}^{o\sigma'}$ vanishes on elements ofdegree not equal to $-(r-1)$ and as

$Q_{k-1}(z')$ is homogeneous of degree $k-1$, we see that the sum is finite and that $\mathrm{Res}^{o\sigma'}(\phi_{\kappa_1}(z_1,z')\phi_{\kappa'}(z')\psi(z'))$ is in the space G as claimed. □

3. Eisenstein series. Results of Section 2 are used for a complex vector space that is the complexification of a real vector space. Thus, we slightly change the notation in this section.

Let V be a *real* vector space of dimension r equipped with a lattice N. The complex vector space $V_{\mathbb{C}}$ is the space to which we apply the results of Section 2.

We consider the dual lattice $M = N^*$ to N. We consider the compact torus $T = iV/(2i\pi N)$ and its complexification $T_{\mathbb{C}} = V_{\mathbb{C}}/(2i\pi N)$. The projection map $V_{\mathbb{C}} \to T_{\mathbb{C}}$ is denoted by the exponential notation $v \to e^v$. If $\{e^1, e^2, \ldots, e^r\}$ is a $\mathbb{Z}$-basis of N, we write an element of $V_{\mathbb{C}}$ as $z = z_1e^1 + z_2e^2 + \cdots + z_re^r$ with $z_j \in \mathbb{C}$. We can identify $T_{\mathbb{C}}$ with $\mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ by $z \mapsto (e^{z_1}, e^{z_2}, \ldots, e^{z_r})$.

If $m \in M$, we denote by e^m the character of T defined by $\langle e^m, e^v\rangle = e^{\langle m,v\rangle}$. We extend e^m to a holomorphic character of the complex torus $T_{\mathbb{C}}$. The ring of holomorphic functions on $T_{\mathbb{C}}$ generated by the functions e^m is denoted by $R(T)$. A quotient of two elements of $R(T)$ is called a *rational function* on the complex torus $T_{\mathbb{C}}$. Via the exponential map $V_{\mathbb{C}} \to T_{\mathbb{C}}$, a function on $T_{\mathbb{C}}$ is sometimes identified with a function on $V_{\mathbb{C}}$, invariant under translation by the lattice $2i\pi N$. If $\{e^1, e^2, \ldots, e^r\}$ is a $\mathbb{Z}$-basis of N, a rational function on $T_{\mathbb{C}}$ written in exponential coordinates is a rational function of $e^{z_1}, e^{z_2}, \ldots, e^{z_r}$. We briefly say that it is a rational function of e^z.

Let us consider a finite set Δ of nontrivial characters of T. We identify Δ with a subset of M; for $\alpha \in \Delta$, we denote by e^α the corresponding character of $T_{\mathbb{C}}$.

Definition 12. We denote by $R(T)_\Delta$ the subring of rational functions on T generated by $R(T)$ and the inverses of the functions $1-e^{-\alpha}$ with $\alpha \in \Delta$.

Observe that R_Δ is left unchanged when each element of Δ is replaced by a nonzero scalar multiple, but that $R(T)_\Delta$ strictly increases when (say) each $\alpha \in \Delta$ is replaced by 2α. We assume from now on that all elements of Δ are indivisible in the lattice M.

Via the exponential map, we consider elements of $R(T)_\Delta$ as periodic meromorphic functions on $V_{\mathbb{C}}$. On $V_{\mathbb{C}}$, the function

$$\frac{\langle\alpha, z\rangle}{1-e^{-\langle\alpha,z\rangle}}$$

is defined at $z = 0$, so it is an element of $\mathbb{O}$. Writing

$$\frac{1}{1-e^{-\langle\alpha,z\rangle}} = \frac{1}{\langle\alpha, z\rangle}\frac{\langle\alpha, z\rangle}{1-e^{-\langle\alpha,z\rangle}},$$

we see that $R(T)_\Delta$ is contained in $\mathbb{O}_\Delta$. We see furthermore from the formula

$$\frac{d}{dz}\frac{1}{1-e^{-z}} = \frac{1}{(1-e^z)(1-e^{-z})} = \frac{-e^{-z}}{(1-e^{-z})^2}$$

that $R(T)_\Delta \subset \mathbb{O}_\Delta$ is stable under differentiation.

Our aim is to find a natural map from R_Δ to $R(T)_\Delta$ commuting with the action of differential operators with constant coefficients. In particular, we want to force a rational function of $z \in V_{\mathbb{C}}$ to become periodic, so that it is natural to define the Eisenstein series

$$E(f)(z) = \sum_{n\in N} f(z+2i\pi n).$$

We need to be more careful, as the sum is usually not convergent for an arbitrary $f \in R_\Delta$. We introduce an oscillating factor $e^{\langle t,2i\pi n\rangle}$ with $t \in V^*$ in front of each term of this infinite sum.

Let

$$U_\Delta = \left\{z \in V_{\mathbb{C}}, \langle \alpha, z+2i\pi n\rangle \neq 0 \text{ for all } n \in N \text{ and for all } \alpha \in \Delta\right\}.$$

Then $R(T)_\Delta$ consists of periodic holomorphic functions on U_Δ.

Let $f \in R_\Delta$; then $f(z+2i\pi n)$ is defined for each $n \in N$ if $z \in U_\Delta$. For $z \in U_\Delta$, we consider the function on V^* defined by

$$t \longmapsto \sum_{n\in N} e^{\langle t,z+2i\pi n\rangle} f(z+2i\pi n).$$

If $n \mapsto f(z+2i\pi n)$ is sufficiently decreasing at infinity, the series is absolutely convergent and sums up to a continuous function of t with value at $t=0$ equal to

$$\sum_{n\in N} f(z+2i\pi n).$$

In any case, it is easy to see that this series of functions of t converges to a generalized function of t.

Proposition 13. *For each $f \in R_\Delta$ and $z \in U_\Delta$, the function on V^* defined by*

$$t \longmapsto \sum_{n\in N} e^{\langle t,z+2i\pi n\rangle} f(z+2i\pi n)$$

is well defined as a generalized function of t, which depends holomorphically on z for z in the open set U_Δ.

Proof. Indeed, if $s(t)$ is a smooth function on V^* with compact support, consider the series

$$\sum_{n\in N} f(z+2i\pi n) \int_{V^*} e^{\langle t,z+2i\pi n\rangle} s(t)\,dt = \sum_{n\in N} c(z,n) f(z+2i\pi n).$$

The coefficient

$$c(z,n) = \int_{V^*} e^{2i\pi\langle t,n\rangle} e^{\langle t,z\rangle} s(t)\,dt$$

is rapidly decreasing in n, as the function $t \mapsto e^{\langle t,z\rangle} s(t)$ is smooth and compactly supported. Thus, $c(z,n)f(z+2i\pi n)$ is also a rapidly decreasing function of n.

Furthermore, $c(z,n)f(z+2i\pi n)$ depends holomorphically on $z \in U_\Delta$. So the result of the summation

$$\sum_{n\in N} c(z,n) f(z+2i\pi n)$$

exists and is a holomorphic function of z. □

We write

$$E(f)(t,z) = \sum_{n\in N} e^{\langle t,z+2i\pi n\rangle} f(z+2i\pi n)$$

for this generalized function of t depending holomorphically on z. We analyze this function of (t,z), $t \in V^*$, $z \in U_\Delta$.

We first summarize some of the obvious properties of $E(f)(t,z)$.

PROPOSITION 14. *The following equations are satisfied.*

(1) *For every* $P \in S(V^*)$ *and* $f \in R_\Delta$,

$$E(Pf)(t,z) = P(\partial_t)E(f)(t,z).$$

(2) *For every* $v \in V$ *and* $f \in R_\Delta$,

$$E\big(\partial(v)f\big)(t,z) = \partial_z(v)E(f)(t,z) - \langle t,v\rangle E(f)(t,z).$$

(3) *For every* $m \in M$ *and* $z \in U_\Delta$,

$$E(f)(t+m,z) = e^{\langle m,z\rangle} E(f)(t,z).$$

As R_Δ is generated by S_Δ under the action of $S(V)$ and $S(V^*)$, we see that the operator E is completely determined by the functions $E(\phi_\sigma)(t,z)$ ($\sigma \in \mathcal{B}(\Delta)$).

A *wall* of Δ is a hyperplane of V^* generated by $r-1$ linearly independent vectors of Δ. We consider the system of affine hyperplanes generated by the walls of Δ together with their translates by M (the dual lattice of N). We denote by $V^*_{\Delta,\text{areg}}$ the complement of the union of these affine hyperplanes. A connected component of $V^*_{\Delta,\text{areg}}$ is called an *alcove* and is denoted by $\mathfrak{a}$.

PROPOSITION 15. *The function* $E(f)(t,z)$ *is smooth when* t *varies on* $V^*_{\Delta,\text{areg}}$ *and when* $z \in U_\Delta$. *More precisely, let* $\mathfrak{a}$ *be an alcove. Assume that* f *is homogeneous of degree* d. *Then, on the open set* $\mathfrak{a} \times U_\Delta$, *the function* $E(f)(t,z)$ *is a polynomial in* t *of degree at most* $-d-r$, *with coefficients in* $R(T)_\Delta$.

Proof. Consider first the one-variable case. The set $V^*_{\Delta,\text{areg}}$ is $\mathbb{R}-\mathbb{Z}$. Let $[t]$ be the integral part of t. Fix $z \in \mathbb{C} - 2i\pi\mathbb{Z}$. Consider the locally constant function of $t \in \mathbb{R}-\mathbb{Z}$ defined by

$$t \longmapsto \frac{e^{[t]z}}{1-e^{-z}}.$$

We extend this function as a locally L^1-function on $\mathbb{R}$ (defined except on the set $\mathbb{Z}$ of measure zero).

LEMMA 16. *We have the equality of generalized functions of t:*

$$\sum_{n\in\mathbb{Z}} \frac{e^{t(z+2i\pi n)}}{z+2i\pi n} = \frac{e^{[t]z}}{1-e^{-z}}.$$

Proof. We compute the derivative in t of the left-hand side. It is equal to

$$\sum_{n\in\mathbb{Z}} e^{t(z+2i\pi n)} = e^{tz}\delta_{\mathbb{Z}}(t),$$

where $\delta_{\mathbb{Z}}$ is the delta function of the set of integers.

We compute the derivative in t of the right-hand side. This function of t is constant on each interval $(n, n+1)$. The jump at the integer n is

$$\frac{e^{nz}}{1-e^{-z}} - \frac{e^{(n-1)z}}{1-e^{-z}} = e^{nz}.$$

It follows that the derivative in t of the right-hand side is also equal to $e^{tz}\delta_{\mathbb{Z}}(t)$. Thus,

$$\sum_{n\in\mathbb{Z}} \frac{e^{t(z+2i\pi n)}}{z+2i\pi n} = c(z) + \frac{e^{[t]z}}{1-e^{-z}},$$

where $c(z)$ is a constant. We verify that $c(z)$ is equal to zero by using periodicity properties in t. It is clear that

$$e^{-tz}\sum_{n\in\mathbb{Z}} \frac{e^{t(z+2i\pi n)}}{z+2i\pi n} = \sum_{n\in\mathbb{Z}} \frac{e^{2i\pi nt}}{z+2i\pi n}$$

is a periodic function of t as is

$$e^{-tz}\frac{e^{[t]z}}{1-e^{-z}} = \frac{e^{([t]-t)z}}{1-e^{-z}}.$$

It follows that $e^{-tz}c(z)$ is also a periodic function of t. This implies $c(z) = 0$. □

Consider now, for $k \in \mathbb{Z}$,

$$E_k(t,z) = \sum_{n\in\mathbb{Z}} e^{t(z+2i\pi n)}(z+2i\pi n)^k.$$

We just saw that

$$E_{-1}(t,z) = \frac{e^{[t]z}}{1-e^{-z}}.$$

To determine $E_k(t,z)$ for $k \le -1$, we use the differential equation in z,

$$\partial_z E_k(t,z) = tE_k(t,z) + kE_{k-1}(t,z).$$

Using decreasing induction over k, we see that $E_k(t,z)$ is an L^1-function of t, equal to a polynomial function of t of degree $-k-1$ on each interval $(n,n+1)$ and with rational functions of e^z as coefficients. For example, we obtain the value of the convergent series

$$\sum_n \frac{e^{t(z+2i\pi n)}}{(z+2i\pi n)^2} = (t-[t])\frac{e^{[t]z}}{1-e^{-z}} - \frac{e^{[t]z}}{(1-e^{-z})(1-e^z)}.$$

When $k \geq 0$, we use the differential equation

$$\partial_t E_k(t,z) = E_{k+1}(t,z)$$

so that, as we have already used,

$$E_0(t,z) = \sum_{n\in\mathbb{Z}} e^{t(z+2i\pi n)} = e^{tz}\delta_{\mathbb{Z}}(t).$$

More generally, $E_k(t,z) = (\partial_t)^k(e^{tz}\delta_{\mathbb{Z}}(t))$ is supported on $\mathbb{Z}$; in particular, it is identically zero on $\mathbb{R}-\mathbb{Z}$.

We return to the proof of Proposition 15. For a simple fraction ϕ, consider the function

$$t \longmapsto E(\phi)(t,z).$$

We first prove that it is a locally L^1-function, which is constant when t varies in an alcove.

Let $\sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a basis of Δ. Let $t \in V^*$. If $t = \sum_j t_j\alpha_j$ is the decomposition of t on the basis σ, set $[t]_\sigma = \sum_j [t_j]\alpha_j$. The function $t \mapsto [t]_\sigma$ is constant when t varies in an alcove. Consider the sublattice

$$M_\sigma = \bigoplus_{\alpha\in\sigma} \mathbb{Z}\alpha \subseteq M.$$

We say that σ is a $\mathbb{Z}$-basis if $M_\sigma = M$. In general, the quotient M/M_σ is a finite set; let $\mathcal{R}$ be a set of representatives of this quotient. We can choose $\mathcal{R}$ in the following standard way. We consider the box

$$Q_\sigma = \bigoplus_{\alpha\in\sigma}[0,1)\alpha = \big\{u \in V^*, [u]_\sigma = 0\big\}.$$

Then we can take

$$\mathcal{R} = Q_\sigma \cap M = \big\{u \in M, [u]_\sigma = 0\big\}.$$

Define

$$\mathcal{R}(t,\sigma) = (t - Q_\sigma)\cap M = \big\{u \in M, [t-u]_\sigma = 0\big\}.$$

The set $\mathcal{R}(t,\sigma)$ is also a set of representatives of M/M_σ. If σ is a $\mathbb{Z}$-basis of M, this set is reduced to the single element $[t]_\sigma$. Remark that the set $\mathcal{R}(t,\sigma)$ is constant when t varies in an alcove $\mathfrak{a}$. We denote it by $\mathcal{R}(\mathfrak{a},\sigma)$.

Definition 17. If $\mathfrak{a}$ is an alcove and if σ is a basis of Δ, we set

$$F_\sigma^{\mathfrak{a}} = \left|\frac{M}{M_\sigma}\right|^{-1} \frac{\sum_{m\in\mathcal{R}(\mathfrak{a},\sigma)} e^m}{\prod_{\alpha\in\sigma}(1-e^{-\alpha})}.$$

Thus, an alcove $\mathfrak{a}$ together with a basis $\sigma \in \mathcal{B}(\Delta)$ produces a particular element $F_\sigma^{\mathfrak{a}}$ of $R(T)_\Delta$.

Consider on the set $V^*_{\Delta,\mathrm{areg}}$ the locally constant function of t defined by $F_\sigma(t,z) = F_\sigma^{\mathfrak{a}}(z)$ when t is in the alcove $\mathfrak{a}$. This defines a locally L^1-function of t, still denoted by $F_\sigma(t,z)$, defined except on the set $V^* - V^*_{\Delta,\mathrm{areg}}$ of measure zero. This locally L^1-function of t defines a generalized function of t which depends holomorphically on z.

LEMMA 18. *We have the equality of generalized functions of $t \in V^*$:*

$$E(\phi_\sigma)(t,z) = F_\sigma(t,z).$$

Proof. If σ is a $\mathbb{Z}$-basis of M, this follows from the formula in dimension 1. In general, we consider $M_\sigma \subseteq M$ and the dual lattice $N_\sigma = M_\sigma^*$. Then $N \subseteq N_\sigma$. We set

$$E_\sigma(\phi_\sigma)(t,z) := \sum_{\ell\in N_\sigma} e^{\langle t,z+2i\pi\ell\rangle}\phi_\sigma(z+2i\pi\ell).$$

For any set of representatives $\mathcal{R}$ of M/M_σ, we have $\sum_{u\in\mathcal{R}} e^{-\langle u,2i\pi\ell\rangle} = 0$ if $\ell \in N_\sigma$ is not in N, while this sum equals $|M/M_\sigma|$ if $n \in N$. Thus,

$$\begin{aligned}
E(\phi_\sigma)(t,z) &= \sum_{n\in N}\phi_\sigma(z+2i\pi n)e^{\langle t,z+2i\pi n\rangle}\\
&= \sum_{\ell\in N_\sigma}\phi_\sigma(z+2i\pi\ell)e^{\langle t,z+2i\pi\ell\rangle}\left(\left|\frac{M}{M_\sigma}\right|^{-1}\sum_{u\in\mathcal{R}} e^{-\langle u,2i\pi\ell\rangle}\right)\\
&= \left|\frac{M}{M_\sigma}\right|^{-1}\sum_{u\in\mathcal{R}}\sum_{\ell\in N_\sigma}\phi_\sigma(z+2i\pi\ell)e^{\langle t-u,z+2i\pi\ell\rangle}e^{\langle u,z\rangle}\\
&= \left|\frac{M}{M_\sigma}\right|^{-1}\sum_{u\in\mathcal{R}} e^{\langle u,z\rangle}E_\sigma(\phi_\sigma)(t-u,z).
\end{aligned}$$

This holds as an equality of generalized functions of t. Further, we have the following, by the 1-dimensional case:

$$E_\sigma(\phi_\sigma)(t,z) = \frac{e^{\langle [t]_\sigma,z\rangle}}{\prod_{\alpha\in\sigma}\left(1-e^{-\langle\alpha,z\rangle}\right)}.$$

It follows that $E(\phi_\sigma)(t,z)$ is a locally L^1-function of t, as is $E_\sigma(\phi_\sigma)$. It remains to determine the value of this function when t is in an alcove. For $m \in M_\sigma$, we have

$$E_\sigma(\phi_\sigma)(t+m,z) = e^{\langle m,z\rangle} E_\sigma(\phi_\sigma)(t,z),$$

so that the sum $\sum_{u\in\mathcal{R}} e^{\langle u,z\rangle} E_\sigma(\phi_\sigma)(t-u,z)$ is independent of the choice of the system of representatives $\mathcal{R}$ of M/M_σ. We choose $\mathcal{R} = \mathcal{R}(t,\sigma)$. Then

$$E(\phi_\sigma)(t,z) = \left|\frac{M}{M_\sigma}\right|^{-1} \frac{\sum_{u\in\mathcal{R}(t,\sigma)} e^{\langle u,z\rangle}}{\prod_{\alpha\in\sigma}\left(1-e^{-\langle\alpha,z\rangle}\right)}$$

because $[t-u]_\sigma = 0$ for all $u \in \mathcal{R}(t,\sigma)$. □

Every function $f \in R_\Delta$, homogeneous of degree d, is obtained from an element of S_Δ by the action of a differential operator with polynomial coefficients. This operator is of degree $d+r$, if multiplication by z_j is given degree 1, while derivation $\partial/\partial z_j$ is given degree -1. Using Proposition 14, we see that Proposition 15 follows from the fact that the function $t \mapsto E(\phi_\sigma)(t,z)$ is constant on each alcove. □

From Proposition 15, we see that there exist functions $\phi^{\mathfrak{a}}_{(k)}(z) \in R(T)_\Delta$ such that we have the equality for t in the alcove $\mathfrak{a}$:

$$E(f)(t,z) = \sum_{n\in N} e^{\langle t,z+2i\pi n\rangle} f(z+2i\pi n) = \sum_{(k)} t^{(k)} \phi^{\mathfrak{a}}_{(k)}(z),$$

where the sum is over a finite number of multi-indices (k). This defines an operator

$$E^t : R_\Delta \longrightarrow R(T)_\Delta, \qquad f \longmapsto E(f)(t,z)$$

obtained by fixing the regular value t.

The operator E^t satisfies the following relation, which is just relation (2) in Proposition 14: For $v \in V$ and $f \in R_\Delta$,

$$E^t\big(\partial(v)f\big)(z) = \partial_z(v)E^t(f)(z) - \langle t,v\rangle E^t(f)(z).$$

Let B be a basis of $\mathcal{B}(\Delta)$. Let $(\phi_\sigma,\ \sigma\in B)$ be the corresponding basis of S_Δ, and let $(\phi^\sigma,\ \sigma \in B)$ be the dual basis of S^*_Δ. For $\sigma \in B$ and an alcove $\mathfrak{a}$, consider the element $F^{\mathfrak{a}}_\sigma$ of $R(T)_\Delta \subset \mathcal{O}_\Delta$ associated to $\sigma,\mathfrak{a}$. We obtain a kernel formula for the operator E^t.

THEOREM 19. *Let $f \in G_\Delta$. For $y \in U_\Delta$ and $t \in \mathfrak{a}$, we have*

$$\begin{aligned} E^t(f)(y) &= \operatorname{Tr}_{S_\Delta}\left(\operatorname{Res}_\Delta m\big(e^{\langle t,\cdot\rangle} f\big) C\mathcal{T}(y) E^t \operatorname{Res}_\Delta\right) \\ &= \sum_{\sigma\in B} \Big\langle \phi^\sigma, \operatorname{Res}_\Delta\left(e^{\langle t,z\rangle} f(z) F^{\mathfrak{a}}_\sigma(y-z)\right)\Big\rangle, \end{aligned}$$

where $F^{\mathfrak{a}}_{\sigma}$ is given by Definition 17. Moreover, if B is the underlying basis of a diagonal basis OB, then

$$E^t(f)(y) = \sum_{o\sigma\in OB} \operatorname{Res}^{o\sigma}\left(e^{\langle t,z\rangle} f(z) F^{\mathfrak{a}}_{\sigma}(y-z)\right).$$

Proof. By a method entirely similar to the proof of Theorem 1, we see that the operator

$$A^t(f)(y) = \sum_{\sigma\in B} \left\langle \phi^{\sigma}, \operatorname{Res}_{\Delta}\left(e^{\langle t,z\rangle} f(z) F^{\mathfrak{a}}_{\sigma}(y-z)\right)\right\rangle$$

satisfies the relation

$$A^t\big(\partial(v) f\big)(z) = \partial_z(v) A^t(f)(z) - \langle t, v\rangle A^t(f)(z)$$

for $v \in V$, $f \in R_{\Delta}$. Thus, to prove that $E^t = A^t$ on G_{Δ}, it is sufficient to prove that they coincide for $f = \phi_{\tau}$. In this case, we obtain

$$A^t(\phi_{\tau})(y) = \sum_{\sigma\in B} \left\langle \phi^{\sigma}, \phi_{\tau}(z)\right\rangle F^{\mathfrak{a}}_{\sigma}(y) = F^{\mathfrak{a}}_{\tau}(y) = E^t(\phi_{\tau})(y).$$

□

In view of the kernel formula for the Eisenstein series E^t, it is natural to introduce the following definition.

Definition 20. The *constant term* of the Eisenstein series E^t is the linear form $f \to \mathrm{CT}(f)(t)$ defined for $f \in R_{\Delta}$ and t in the alcove $\mathfrak{a}$ by

$$\mathrm{CT}(f)(t) = \operatorname{Tr}_{S_{\Delta}}\left(\operatorname{Res}_{\Delta} m\big(e^{\langle t,\cdot\rangle} f\big) C E^t \operatorname{Res}_{\Delta}\right).$$

More explicitly, if OB is a diagonal basis of $\mathcal{B}(\Delta)$, then

$$\mathrm{CT}(f)(t) = \sum_{o\sigma\in OB} \operatorname{Res}^{o\sigma}\left(e^{\langle t,z\rangle} f(z) F^{\mathfrak{a}}_{\sigma}(-z)\right).$$

4. Partial Eisenstein series. Let $N_{\mathrm{reg}} = N \cap V_{\mathrm{reg}}$ be the set of regular elements of N. The aim of this section is to prove that the function

$$E_{N_{\mathrm{reg}}}(f)(t,z) = \sum_{n\in N_{\mathrm{reg}}} e^{\langle t, z+2i\pi n\rangle} f(z+2i\pi n)$$

is analytic in (t, z) when t is in an alcove and $z \in V_{\mathbb{C}}$ is close to zero. In the next section we prove the Szenes residue formula for

$$E_{N_{\mathrm{reg}}}(f)(t,0) = \sum_{n\in N_{\mathrm{reg}}} e^{\langle t, 2i\pi n\rangle} f(2i\pi n).$$

Let Γ be a subset of N. We can define, for $f \in R_{\Delta}$, the generalized function of t,

$$E_{\Gamma}(f)(t,z) = \sum_{n\in\Gamma} e^{\langle t, z+2i\pi n\rangle} f(z+2i\pi n).$$

Introduce the set

$$U_{\Delta,\Gamma} = \{z \in V_{\mathbb{C}}, \langle \alpha, z+2i\pi n\rangle \neq 0 \text{ for all } \alpha \in \Delta \text{ and } n \in \Gamma\}.$$

The generalized function $E_\Gamma(f)(t,z)$ depends holomorphically on z, when $z \in U_{\Delta,\Gamma}$.

Let W be a rational subspace of V. Then $N \cap W$ is a lattice in W. Consider, for $f \in R_\Delta$,

$$E_{N\cap W}(f)(t,z) = \sum_{n \in N\cap W} e^{\langle t, z+2i\pi n\rangle} f(z+2i\pi n).$$

We analyze the singularities in (t,z) of $E_{N\cap W}(f)(t,z)$. If W is zero, then $E_{\{0\}}(f)(t,z) = e^{\langle t,z\rangle} f(z)$ is analytic in (t,z) when z is regular in $V_{\mathbb{C}}$. Assume that W is nonzero and consider the subspace $W^\perp$ of V^*. Notice that if $u \in M + W^\perp$, we have the relation

$$E_{N\cap W}(f)(t+u,z) = e^{\langle u,z\rangle} E_{N\cap W}(f)(t,z).$$

It is clear that the singular set of $E_{N\cap W}(f)(t,z)$ is stable by translation by $M+W^\perp$. Define a (W,Δ)-*wall* in V^* as a hyperplane generated by $W^\perp$ together with $\dim W - 1$ vectors of Δ. We introduce the set $\mathcal{H}^*_{W,\Delta,M}$ consisting of the union of all (W,Δ)-walls and of their translates by elements of M. We define $V^*_{W,\Delta,\mathrm{areg}}$ as the complement of $\mathcal{H}^*_{W,\Delta,M}$ in V^*. This set $V^*_{W,\Delta,\mathrm{areg}}$ is invariant by translation by $M+W^\perp$.

LEMMA 21. *For $f \in R_\Delta$, the function $E_{N\cap W}(f)(t,z)$ is analytic in (t,z) when t varies on $V^*_{W,\Delta,\mathrm{areg}}$ and $z \in U_{\Delta,N\cap W}$. Furthermore, if $t \in V^*_{W,\Delta,\mathrm{areg}}$ and z is near zero, the function $z \mapsto E_{N\cap W}(f)(t,z)$ defines an element of $\mathcal{O}_\Delta$.*

Proof. Let σ be a basis of Δ. Although we are not able to give a nice formula for the function $E_{N\cap W}(\phi_\sigma)(t,z)$, we can still obtain an inductive expression that suffices to give some information on it. Consider the set $V^*_{W,\sigma,\mathrm{areg}}$, that is, the complement of the union of (W,σ)-walls together with their translates by M. Let $U_{\sigma,N\cap W}$ be the set of all $z \in V_{\mathbb{C}}$ such that $\langle \alpha, z+2i\pi n\rangle \neq 0$ for all $\alpha \in \sigma$ and $n \in N\cap W$. The intersection of this set with a small neighborhood of zero is contained in the complement of the union of the complex hyperplanes $\{z \in V_{\mathbb{C}}, \langle \alpha, z\rangle = 0\}$, for $\alpha \in \sigma$.

LEMMA 22. *The function $E_{N\cap W}(\phi_\sigma)(t,z)$ is analytic in $t \in V^*_{W,\sigma,\mathrm{areg}}$ and $z \in U_{\sigma,N\cap W}$. Furthermore, when $t \in V^*_{W,\sigma,\mathrm{areg}}$, the function*

$$z \longmapsto \left(\prod_{\alpha\in\sigma} \langle \alpha, z\rangle\right) E_{N\cap W}(\phi_\sigma)(t,z)$$

is holomorphic at $z = 0$.

We prove this by induction on the codimension of W. If $W = V$, this follows from the explicit formula for $E(\phi_\sigma)(t,z)$. Let α be an indivisible element of M such that W is contained in the real hyperplane

$$H_\alpha = \{y \in V,\ \langle \alpha, y\rangle = 0\}.$$

We assume first that α is an element of σ. We number it the first vector α_1 of the basis σ. We set $\sigma' = (\alpha_2, \dots, \alpha_r)$, $z' = (z_2, \dots, z_r)$, and so on; then $z = (z_1, z')$. Our subspace W is contained in $V' = V \cap \{z_1 = 0\}$. Thus, we have

$$E_{N\cap W}(\phi_\sigma)(t,z) = \sum_{n\in N\cap W} e^{\langle t, z+2i\pi n\rangle}\phi_\sigma(z+2i\pi n) = \frac{e^{t_1 z_1}}{z_1} E_{N'\cap W}(\phi_{\sigma'})(t', z').$$

By induction, $E_{N'\cap W}(\phi_{\sigma'})(t', z')$ is analytic in (t', z') for $z' \in U_{\sigma', N'}$, except if there exist $m' \in M'$ such that $t' + m'$ is in a hyperplane generated by $W^{\perp'}$ (the orthogonal of W in V') and some vectors of σ'. As $W^\perp = W^{\perp'} \oplus \mathbb{R}\alpha_1$, we see that the singular set of $E_{N\cap W}(\phi_\sigma)(t,z)$ is contained in $\mathcal{H}^*_{W,\sigma,M}$. Furthermore, the function

$$z_1 z_2 \cdots z_r E_{N\cap W}(\phi_\sigma)(t,z) = e^{t_1 z_1} z_2 \cdots z_r E_{N'\cap W}(\phi_{\sigma'})(t', z')$$

is holomorphic in z near $z = 0$.

Assume now that α is not an element of σ. We add it to the system Δ if α is not an element of Δ. Writing $\alpha = \sum_j c_j \alpha_j$, we obtain one of the Orlik-Solomon relations of the system $\Delta \cup \{\alpha\}$,

$$\phi_\sigma = \sum_j c_j \phi_{\sigma^j},$$

where $\sigma^j = \sigma \cup \{\alpha\} - \{\alpha_j\}$. A (W, σ^j)-wall is a hyperplane of V^* generated by $W^\perp$ and $\dim W - 1$ vectors of σ^j; then these vectors are distinct from α, because $\alpha \in W^\perp$. Thus, all W-walls for the basis σ^j are also W-walls for the basis σ. By our first calculation, it follows that $E_{N\cap W}(\phi_{\sigma^j})(t,z)$ is analytic when t is not on a translate of a (W,σ)-wall. Moreover, we have

$$E_{N\cap W}(\phi_\sigma)(t,z) = \sum_j c_j E_{N\cap W}(\phi_{\sigma^j})(t,z),$$

so that the function

$$z \longmapsto \langle \alpha, z\rangle \left(\prod_{j=1}^r \langle \alpha_j, z\rangle\right) E_{N\cap W}(\phi_\sigma)(t,z)$$

is holomorphic in z in a neighborhood of zero.

By the induction hypothesis applied to $W \subseteq V' = \{\alpha = 0\}$, the function $z \mapsto E_{N\cap W}(\phi_\sigma)(t,z)$ is holomorphic on a nonempty open subset of $V'_\mathbb{C}$. So this function, considered as a function of $z \in V_\mathbb{C}$, has no pole along $\alpha = 0$. This proves Lemma 22 and, hence, Lemma 21 when f is a simple fraction. The operator $E_{N\cap W}$ satisfies also the commutation relation of Proposition 14. Thus, using differential operators with polynomial coefficients, we obtain the statement of Lemma 21 when f is any element in R_Δ. □

Let I be a subset of Δ, and let $W_I = \cap_{i\in I} H_{\alpha_i}$. This is a rational subspace of V, and the (W_I, Δ)-walls are some of the walls of Δ. Then it follows from Lemma 21 that $E_{N\cap W_I}(f)(t,z)$ is *a fortiori* analytic when $t \in V^*_{\text{areg}}$ and $z \in U_\Delta$.

Definition 23. A subset Γ of N is *admissible* if the characteristic function of Γ is a linear combination of characteristic functions of sets $N \cap W_I$, where I ranges over subsets of Δ.

Then we have the following by Lemma 21.

LEMMA 24. *If Γ is an admissible subset of N, the function $(t,z) \mapsto E_\Gamma(f)(t,z)$ is analytic when $t \in V^*_{\Delta,\mathrm{areg}}$ and $z \in U_{\Delta,\Gamma}$. Furthermore, when z is near zero and $t \in V^*_{\Delta,\mathrm{areg}}$, the function $z \mapsto E_\Gamma(f)(t,z)$ defines an element of $\mathcal{O}_\Delta$.*

If Γ is an admissible subset of N, we can take the value at t of the generalized function

$$E_\Gamma(f)(t,z) = \sum_{n\in\Gamma} e^{\langle t, z+2i\pi n\rangle} f(z+2i\pi n)$$

provided that t is in an alcove $\mathfrak{a}$. Thus, for $t \in \mathfrak{a}$, we can define the operator E^t_Γ : $R_\Delta \to \mathcal{O}_\Delta$, $f \mapsto E_\Gamma(f)(t,z)$. Now the argument of Theorem 19 proves the following proposition.

PROPOSITION 25. *For $f \in G_\Delta$, $t \in V^*_{\Delta,\mathrm{areg}}$, and $y \in U_{\Delta,\Gamma}$, we have*

$$E^t_\Gamma(f)(y) = \mathrm{Tr}_{S_\Delta}\left(\mathrm{Res}_\Delta\, m\big(e^{\langle t,\cdot\rangle} f\big) C\mathcal{T}(y) E^t_\Gamma\, \mathrm{Res}_\Delta\right).$$

More explicitly, if we choose a diagonal basis OB, then

$$E^t_\Gamma(f)(y) = \sum_{o\sigma\in OB} \mathrm{Res}^{o\sigma}\left(f(z)e^{\langle t,z\rangle} F^t_{\Gamma,\sigma}(y-z)\right),$$

where $F^t_{\Gamma,\sigma}(z) = E_\Gamma(\phi_\sigma)(t,z)$.

5. Witten series and the Szenes formula. For $f \in R_\Delta$, let us form the series

$$Z(f)(t,z) = \sum_{n\in N_{\mathrm{reg}}} e^{\langle t, z+2i\pi n\rangle} f(z+2i\pi n),$$

where N_{reg} is the set of regular elements of N. Then $Z(f)(t,z)$ is defined as a generalized function of t. As n varies in N_{reg}, this generalized function of t depends holomorphically on z when z varies in a neighborhood of zero. As N_{reg} is an admissible subset of N, we obtain the following from Lemma 24.

PROPOSITION 26. *For any alcove $\mathfrak{a}$, $Z(f)(t,z)$ is an analytic function of (t,z) when $t \in \mathfrak{a}$ and z is in a neighborhood of zero.*

We have

$$Z(f)(t,0) = \sum_{n\in N_{\mathrm{reg}}} e^{\langle t, 2i\pi n\rangle} f(2i\pi n).$$

This is well defined as a generalized function of t when t is in an alcove. If $n \mapsto f(2i\pi n)$ is sufficiently decreasing, then $Z(f)(t,0)$ is a continuous function of t; it generalizes the Bernoulli polynomial

$$B_k(t) = \sum_{n \neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^k},$$

where $0 < t < 1$.

We reformulate the Szenes formula as an equality between $Z(f)(t,0)$ and the constant term of the Eisenstein series $E(f)(t,z)$.

THEOREM 27. *For any $f \in R_\Delta$ and t in an alcove $\mathfrak{a}$, we have*

$$Z(f)(t,0) = \mathrm{CT}(f)(t) = \mathrm{Tr}_{S_\Delta}\left(\mathrm{Res}_\Delta\, m\big(e^{\langle t,\cdot\rangle} f\big) C E^t\, \mathrm{Res}_\Delta\right).$$

In particular, $Z(f)(t,0)$ is a polynomial function of t when t varies in an alcove $\mathfrak{a}$.

As a consequence, if OB is a diagonal basis, then we recover the following residue formula (see [3, Theorem 4.4]):

$$\sum_{n \in N_{\mathrm{reg}}} e^{\langle t, 2i\pi n\rangle} f(2i\pi n) = \sum_{o\sigma \in OB} \mathrm{Res}^{o\sigma}\left(e^{\langle t,z\rangle} f(z) F_\sigma^{\mathfrak{a}}(-z)\right).$$

Thus, when

$$f = \frac{1}{\prod_{j=1}^k \alpha_j}$$

is sufficiently decreasing, this formula expresses the series

$$\sum_{n \in \mathbb{Z}^r,\, \langle \alpha_j, n\rangle \neq 0} \frac{1}{\prod_{j=1}^k \langle \alpha_j, 2i\pi n\rangle}$$

as an explicit rational number.

Proof. From the definitions of $Z(f)(t,z)$ and $\mathrm{CT}(f)(t)$, we obtain, for any $P \in S(V^*)$,

$$P(\partial_t) Z(f)(t,0) = Z(Pf)(t,0), \qquad P(\partial_t)\mathrm{CT}(f)(t) = \mathrm{CT}(Pf)(t).$$

Thus, it is enough to prove that $Z(f)(t,0) = \mathrm{CT}(f)(t)$ for $f \in G_\Delta$, because G_Δ generates R_Δ as a $S(V^*)$-module by Lemma 6.

For t in an alcove $\mathfrak{a}$, we can define the operator $Z^t : R_\Delta \to \mathbb{O}$ by

$$Z^t(f)(z) = \sum_{n \in N_{\mathrm{reg}}} e^{\langle t, z+2i\pi n\rangle} f(z + 2i\pi n).$$

The kernel formula holds for the operator Z^t. In particular, we obtain, for $f \in G_\Delta$,

$$Z^t(f)(0) = \mathrm{Tr}_{S_\Delta}\left(\mathrm{Res}_\Delta m\big(e^{\langle t,\cdot\rangle} f\big) C Z^t \,\mathrm{Res}_\Delta\right).$$

We thus need to prove that, for $f \in G_\Delta$,

$$\mathrm{Tr}_{S_\Delta}\left(\mathrm{Res}_\Delta m\big(e^{\langle t,\cdot\rangle} f\big) C\big(E^t - Z^t\big)\mathrm{Res}_\Delta\right) = 0.$$

But E^t is given by a sum over the full lattice N, while Z^t is only over the regular elements of N. Thus, we can write (in many ways) $E^t - Z^t$ as a linear combination of operators $E^t_{\Gamma_\alpha}$, where each Γ_α is an admissible subset of N contained in the real hyperplane H_α. Now the Szenes formula follows from the next proposition.

PROPOSITION 28. *Let Γ be an admissible subset of N contained in the real hyperplane H_α. Then, for $f \in G_\Delta$,*

$$\mathrm{Tr}_{S_\Delta}\left(\mathrm{Res}_\Delta m\big(e^{\langle t,\cdot\rangle} f\big) C E^t_\Gamma \,\mathrm{Res}_\Delta\right) = 0.$$

Proof. It suffices to prove that

$$\sum_{o\sigma \in OB} \mathrm{Res}^{o\sigma}\left(e^{\langle t,z\rangle} f(z) E^t_\Gamma(\phi_\sigma)(-z)\right) = 0$$

for some diagonal basis *OB*.

A total order on Δ provides us with a special diagonal basis *OB* of $O\mathcal{B}(\Delta)$ (see, for example, [1, Proposition 14]). We choose this order such that α is minimal. In this case, every element of *OB* is of the form $o\sigma = (\alpha_1, \alpha_2, \dots, \alpha_r)$ with $\alpha_1 = \alpha$. We claim that for each $o\sigma \in OB$,

$$\mathrm{Res}^{o\sigma}\left(e^{\langle t,z\rangle} f(z) E^t_\Gamma(\phi_\sigma)(-z)\right) = 0.$$

Indeed, we use the notation of Lemma 11 and write $V' = H_\alpha$. Then our set Γ is contained in V'. Thus,

$$E^t_\Gamma(\phi_\sigma)(z_1, z') = \frac{e^{t_1 z_1}}{z_1} \sum_{\gamma\in\Gamma} \frac{e^{\langle t', z' + 2i\pi\gamma\rangle}}{\prod_{j=2}^r \langle \alpha_j, z' + 2i\pi\gamma\rangle}.$$

We see that for t fixed and regular,

$$e^{\langle t,z\rangle} f(z) E^t_\Gamma(\phi_\sigma)(-z) = \frac{1}{z_1} f(z_1, z')\psi(z'),$$

where $f \in G_\Delta$ and $\psi(z')$ has poles at most on the complex hyperplanes $\alpha_j = 0$ for $j = 2, \dots, r$. Thus the claim follows from Lemma 11. Therefore, both Theorem 27 and Proposition 28 are proved. □

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Brion: Institut Fourier, Boîte Postale 74, F-38402 Saint-Martin d'Hères Cedex, France

Vergne: Centre de Mathématiques, École Polytechnique, F-91128 Palaiseau Cedex, France