THE MULTIPLICITIES OF THE EQUIVARIANT INDEX OF TWISTED DIRAC OPERATORS

PAUL-EMILE PARADAN, MICHÈLE VERGNE

Résumé. In this note, we give a geometric expression for the multiplicities of the equivariant index of a Dirac operator twisted by a line bundle.

1. Introduction

This note is an announcement of work whose details will appear later.

Let M be a compact connected manifold. We assume that M is even dimensional and oriented. We consider a spin^c structure on M, and denote by $\mathcal{S}$ the corresponding irreducible Clifford module. Let K be a compact connected Lie group acting on M, and preserving the spin^c structure. We denote by $D : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-)$ the corresponding twisted Dirac operator. The equivariant index of D, denoted $\mathrm{Q}_K^{\mathrm{spin}}(M)$, belongs to the Grothendieck group of representations of K,

$$\mathrm{Q}_K^{\mathrm{spin}}(M) = \sum_{\pi \in \widehat{K}} \mathrm{m}(\pi)\ \pi.$$

An important example is when M is a compact complex manifold, K a compact group of holomorphic transformations of M, and $\mathcal{L}$ any holomorphic K-equivariant line bundle on M (not necessarily ample). Then the Dolbeaut operator twisted by $\mathcal{L}$ can be realized as a twisted Dirac operator D. In this case $\mathrm{Q}_K^{\mathrm{spin}}(M) = \sum_q (-1)^q H^{0,q}(M, \mathcal{L})$.

The aim of this note is to give a geometric description of the multiplicity $\mathrm{m}(\pi)$ in the spirit of the Guillemin-Sternberg phenomenon $[Q, R] = 0$ [3, 7, 8, 11, 9].

Consider the determinant line bundle $\mathbb{L} = \det(\mathcal{S})$ of the spin^c structure. This is a K-equivariant complex line bundle on M. The choice of a K-invariant hermitian metric and of a K-invariant hermitian connection ∇ on $\mathbb{L}$ determines an abstract moment map

$$\Phi_\nabla : M \to \mathfrak{k}^*$$

by the relation $\mathcal{L}(X) - \nabla_{X_M} = \frac{i}{2}\langle \Phi_\nabla, X\rangle$, for all $X \in \mathfrak{k}$. We compute $\mathrm{m}(\pi)$ in term of the reduced "manifolds" $\Phi_\nabla^{-1}(f)/K_f$. This formula extends the result of [10].

However, in this note, we do not assume any hypothesis on the line bundle $\mathbb{L}$, in particular we do not assume that the curvature of the connection ∇ is a symplectic form. In this pre-symplectic setting, a (partial) answer to this question has been obtained by [6, 4, 5, 1] when K is a torus. Our method is based on localization techniques as in [9], [10].

2. Admissible coadjoints orbits

We consider a compact connected Lie group K with Lie algebra $\mathfrak{k}$. Consider an admissible coadjoint orbit $\mathcal{O}$ (as in [2]), oriented by its symplectic structure. Then $\mathcal{O}$ carries a K-equivariant bundle of spinors $\mathcal{S}_{\mathcal{O}}$, such that the associated moment map is the injection $\mathcal{O}$ in $\mathfrak{k}^*$. We denote by $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O})$ the corresponding equivariant index.

Let us describe the admissible coadjoint orbits with their spin^c index.

Let T be a Cartan subgroup of K with Lie algebra $\mathfrak{t}$. Let $\Lambda \subset \mathfrak{t}^*$ be the lattice of weights of T (thus $e^{i\lambda}$ is a character of T). Choose a positive system $\Delta^+ \subset \mathfrak{t}^*$, and let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Let $\mathfrak{t}^*_{\geq 0}$ be the closed Weyl chamber and we denote by $\mathcal{F}$ the set of the relative interiors of the faces of $\mathfrak{t}^*_{\geq 0}$. Thus $\mathfrak{t}^*_{\geq 0} = \coprod_{\sigma\in\mathcal{F}}\sigma$, and we denote $\mathfrak{t}^*_{>0} \in \mathcal{F}$ the interior of $\mathfrak{t}^*_{\geq 0}$.

We index the set $\hat{K}$ of classes of finite dimensional irreducible representations of K by the set $(\Lambda+\rho)\cap\mathfrak{t}^*_{>0}$. The irreducible representation π_λ corresponding to $\lambda \in (\Lambda+\rho)\cap\mathfrak{t}^*_{>0}$ is the irreducible representation with infinitesimal character λ. Its highest weight is $\lambda-\rho$.

Let $\sigma \in \mathcal{F}$. The stabilizer K_ξ of a point $\xi\in\sigma$ depends only of σ. We denote it by K_σ, and by $\mathfrak{k}_\sigma$ its Lie algebra. We choose on $\mathfrak{k}_\sigma$ the system of positive roots contained in Δ^+, and let ρ_σ be the corresponding ρ.

When $\mu\in\sigma$, the coadjoint orbit $K\cdot\mu$ is admissible if and only if $\mu-\rho+\rho_\sigma\in\Lambda$. The spin^c equivariant index of the admissible orbits is described in the following lemma.

Lemma 2.1. *Let $K\cdot\mu$ be an admissible orbit : $\mu\in\sigma$ and $\mu-\rho+\rho_\sigma\in\Lambda$. If $\mu+\rho_\sigma$ is regular, then $\mu+\rho_\sigma\in\rho+\overline{\sigma}$. Thus we have*

$$\mathrm{Q}_K^{\mathrm{spin}}(K\cdot\mu) = \begin{cases} 0 & \text{if } \mu+\rho_\sigma \text{ is singular,} \\ \pi_{\mu+\rho_\sigma} & \text{if } \mu+\rho_\sigma \text{ is regular.} \end{cases}$$

In particular, if $\lambda\in(\Lambda+\rho)\cap\mathfrak{t}^*_{>0}$, then $K\cdot\lambda$ is admissible and $\mathrm{Q}_K^{\mathrm{spin}}(K\cdot\lambda)=\pi_\lambda$.

Let $\mathcal{H}_{\mathfrak{k}}$ be the set of conjugacy classes of the reductive algebras $\mathfrak{k}_f, f \in \mathfrak{k}^*$. We denote by $\mathcal{S}_{\mathfrak{k}}$ the set of conjugacy classes of the semi-simple parts $[\mathfrak{h}, \mathfrak{h}]$ of the elements $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$. The map $(\mathfrak{h}) \to ([\mathfrak{h}, \mathfrak{h}])$ induces a bijection between $\mathcal{H}_{\mathfrak{k}}$ and $\mathcal{S}_{\mathfrak{k}}$.

The map $\mathcal{F} \longrightarrow \mathcal{H}_{\mathfrak{k}}$, $\sigma \mapsto (\mathfrak{k}_\sigma)$, is surjective and for $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ we denote by

- $\mathcal{F}(\mathfrak{h})$ the set of $\sigma \in \mathcal{F}$ such that $(\mathfrak{k}_\sigma) = (\mathfrak{h})$,
- $\mathfrak{k}^*_{\mathfrak{h}} \subset \mathfrak{k}^*$ the set of elements $f \in \mathfrak{k}^*$ with infinitesimal stabilizer $\mathfrak{k}_f$ belonging to the conjugacy class $(\mathfrak{h})$.

We have $\mathfrak{k}^*_{\mathfrak{h}} = K\left(\cup_{\sigma\in\mathcal{F}(\mathfrak{h})}\sigma\right)$. In particular all coadjoint orbits contained in $\mathfrak{k}^*_{\mathfrak{h}}$ have the same dimension. We say that such a coadjoint orbit is of type $(\mathfrak{h})$. If $(\mathfrak{h}) = (\mathfrak{t})$, then $\mathfrak{k}^*_{\mathfrak{h}}$ is the open subset of regular elements.

We denote by $A(\mathfrak{h})$ the set of admissible coadjoint orbits of type $(\mathfrak{h})$. This is a discrete subset of orbits in $\mathfrak{k}^*_{\mathfrak{h}}$.

Example 1 : Consider the group $K = SU(3)$ and let $(\mathfrak{h})$ be the conjugacy class such that $\mathfrak{k}^*_{\mathfrak{h}}$ is equal to the set of subregular element $f \in \mathfrak{k}^*$ (the orbit of f is of dimension $\dim(K/T) - 2$). Let ω_1, ω_2 be the two fundamental weights. Let σ_1, σ_2 be the half lines $\mathbb{R}_{>0}\omega_1$, $\mathbb{R}_{>0}\omega_2$. Then $\mathfrak{k}^*_{\mathfrak{h}} \cap \mathfrak{t}^*_{\geq 0} = \sigma_1 \cup \sigma_2$. The set $A(\mathfrak{h})$ is equal to the collection of orbits $K \cdot (\frac{1+2n}{2}\omega_i), n \in \mathbb{Z}_{\geq 0}, i = 1, 2$. The representation $\mathrm{Q}^{\mathrm{spin}}_K(K \cdot (\frac{1+2n}{2}\omega_i))$ is 0 is $n = 0$, otherwise it is the irreducible representation $\pi_{\rho+(n-1)\omega_i}$. In particular, both representations associated to the admissible orbits $\frac{3}{2}\omega_1$ and $\frac{3}{2}\omega_2$ are the trivial representation π_ρ.

3. The theorem

Consider the action of K in M. Let $(\mathfrak{k}_M)$ be the conjugacy class of the generic infinitesimal stabilizer. On a K-invariant open and dense subset of M, the conjugacy class of $\mathfrak{k}_m$ is equal to $(\mathfrak{k}_M)$. Consider the (conjugacy class) $([\mathfrak{k}_M, \mathfrak{k}_M])$.

We start by stating two vanishing lemmas.

Lemma 3.1. *If $([\mathfrak{k}_M, \mathfrak{k}_M])$ does not belong to the set $\mathcal{S}_{\mathfrak{k}}$, then $\mathrm{Q}^{\mathrm{spin}}_K(M) = 0$ for any K-invariant spin^c structure on M.*

If $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ for some $\mathfrak{h} \in \mathcal{H}_{\mathfrak{k}}$, any K-invariant map $\Phi : M \to \mathfrak{k}^*$ is such that $\Phi(M)$ is included in the closure of $\mathfrak{k}^*_{\mathfrak{h}}$.

Lemma 3.2. *Assume that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ with $\mathfrak{h} \in \mathcal{H}_{\mathfrak{k}}$. Let us consider a spin^c structure on M with determinant bundle $\mathbb{L}$. If there exists a K-invariant hermitian connection ∇ on $\mathbb{L}$ such that $\Phi_\nabla(M) \cap \mathfrak{k}^*_{\mathfrak{h}} = \emptyset$, then $\mathrm{Q}^{\mathrm{spin}}_K(M) = 0$.*

Thus from now on, we assume that the action of K on M is such that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ for some $\mathfrak{h} \in \mathcal{H}_\mathfrak{k}$. Let us consider a spin^c structure on M with determinant bundle $\mathbb{L}$ and a K-invariant hermitian connection with moment map $\Phi_\nabla : M \to \mathfrak{k}^*$.

We extend the definition of the index to disconnected even dimensional oriented manifolds by defining $\mathrm{Q}_K^{\mathrm{spin}}(M)$ to be the sum over the connected components of M. If K is the trivial group, $\mathrm{Q}_K^{\mathrm{spin}}(M) \in \mathbb{Z}$ and is denoted simply by $\mathrm{Q}^{\mathrm{spin}}(M)$.

Consider a coadjoint orbit $\mathcal{O} = K \cdot f$. The reduced space $M_\mathcal{O}$ is defined to be the topological space $\Phi_\nabla^{-1}(\mathcal{O})/K = \Phi_\nabla^{-1}(f)/K_f$. We also denote it by M_f. This space might not be connected.

In the next section, we define a $\mathbb{Z}$-valued function $\mathcal{O} \mapsto \mathrm{Q}^{\mathrm{spin}}(M_\mathcal{O})$ on the set $A(\mathfrak{h})$ of admissible orbits of type $(\mathfrak{h})$. We call it the reduced index :

- if $M_\mathcal{O} = \emptyset$, then $\mathrm{Q}^{\mathrm{spin}}(M_\mathcal{O}) = 0$,
- when $M_\mathcal{O}$ is an orbifold, the reduced index $\mathrm{Q}^{\mathrm{spin}}(M_\mathcal{O})$ is defined as an index of a Dirac operator associated to a natural "reduced" spin^c structure on $M_\mathcal{O}$.

Otherwise, it is defined via a limit procedure. Postponing this definition, we have the following theorem.

Theorem 3.3. *Assume that* $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ *with* $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$. *Then*

$$\mathrm{Q}_K^{\mathrm{spin}}(M) = \sum_{\mathcal{O} \in A(\mathfrak{h})} \mathrm{Q}^{\mathrm{spin}}(M_\mathcal{O})\ \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O}).$$

In the expression above, when $\mathfrak{h}$ is not abelian, $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O})$ can be 0, and several orbits $\mathcal{O} \in A(\mathfrak{h})$ can give the same representation.

Theorem 3.3 is in the spirit of the $[Q, R] = 0$ theorem. However it has some radically new features. First, as Φ_∇ is not the moment map of a Hamiltonian structure, the definition of the reduced space requires more care. For example, the fibers of Φ_∇ might not be connected, and the Kirwan set $\Phi_\nabla(M) \cap \mathfrak{t}^*_{\geq 0}$ is not a convex polytope. Furthermore, this Kirwan set depends of the choice of connection ∇. Second, the map $\mathcal{O} \in A(\mathfrak{h}) \to \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O})$ is not injective, when $\mathfrak{h}$ is not abelian. Thus the multiplicities m_λ of the representation π_λ in $\mathrm{Q}_K^{\mathrm{spin}}(M)$ will be eventually obtained as a sum of reduced indices involving several reduced spaces.

We explicit this last point.

Theorem 3.4. *Assume that* $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ *with* $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$. *Let* $\mathrm{m}_\lambda \in \mathbb{Z}$ *be the multiplicity of the representation* π_λ *in* $\mathrm{Q}_K^{\mathrm{spin}}(M)$. *We have*

$$\mathrm{m}_\lambda = \sum_{\substack{\sigma \in \mathcal{F}(\mathfrak{h}) \\ \lambda - \rho_\sigma \in \sigma}} \mathrm{Q}^{\mathrm{spin}}(M_{\lambda - \rho_\sigma}). \tag{1}$$

More explicitly, the sum is taken over the (relative interiors of) faces σ of the Weyl chamber such that

$$([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]), \qquad \Phi_\nabla(M) \cap \sigma \neq \emptyset, \qquad \lambda \in \{\sigma + \rho_\sigma\}. \tag{2}$$

If $\mathfrak{k}_M$ is abelian, we have simply $\mathrm{m}_\lambda = \mathrm{Q}^{\mathrm{spin}}(\Phi_\nabla^{-1}(\lambda)/T)$. In particular, if the group K is the circle group, and λ is a regular value of the moment map Φ_∇, this result was obtained in [1].

If $\mathfrak{k}_M$ is not abelian, and the curvature of the connection ∇ is symplectic, Kirwan convexity theorem implies that the image $\Phi_\nabla(M) \cap \mathfrak{t}^*_{\geq 0}$ is contained in the closure of one single σ. Thus there is a unique σ satisfying Conditions (2). In this setting Theorem 3.4 is obtained in [10].

Let us give an example where several σ contribute to the multiplicity of a representation π_λ.

We take the notations of Example 1. We label ω_1, ω_2 so that $\mathfrak{k}_{\omega_1}$ is the group $S(U(2) \times U(1))$ stabilizing the line $\mathbb{C}e_3$ in the fundamental representation of $SU(3)$ in $\mathbb{C}^3 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$.

Let $P = \{0 \subset L_2 \subset L_3 \subset \mathbb{C}^4\}$ be the partial flag manifold with L_2 a subspace of $\mathbb{C}^4$ of dimension 2 and L_3 a subspace of $\mathbb{C}^4$ of dimension 3. Denote by $\mathcal{L}_1, \mathcal{L}_2$ the equivariant line bundles on P with fiber at (L_2, L_3) the one-dimensional spaces $\wedge^2 L_2$ and L_3/L_2 respectively. Let M be the subset of P where L_2 is assumed to be a subspace of $\mathbb{C}^3$. Thus M is fibered over $P_2(\mathbb{C})$ with fiber $P_1(\mathbb{C})$. The group $SU(3)$ acts naturally on M, and the generic stabilizer of the action is $SU(2)$. We denote by $\mathcal{L}_{a,b}$ the line bundle $\mathcal{L}_1^a \otimes \mathcal{L}_2^b$ restricted to M. This line bundle is equipped with a natural holomorphic and hermitian connection ∇. Consider the spin^c structure with determinant bundle $\mathbb{L} = \mathcal{L}_{2a+1,2b+1}$, where a, b are positive integers. If $a \geq b$, the curvature of the line bundle $\mathbb{L}$ is non degenerate, and we are in the symplectic case. Let us consider $b > a$. It is easy to see that, in this case, the Kirwan set $\Phi_\nabla(M) \cap \mathfrak{t}^*_{\geq 0}$ is the non convex set $[0, b-a]\omega_1 \cup [0, a+1]\omega_2$. We compute the character of the representation $\mathrm{Q}_K^{\mathrm{spin}}(M)$ by the Atiyah-Bott fixed point formula, and find

$$\mathrm{Q}_K^{\mathrm{spin}}(M) = \sum_{j=0}^{b-a-2} \pi_{\rho + j\omega_1} \oplus \sum_{j=0}^{a-1} \pi_{\rho + j\omega_2}.$$

In particular the multiplicity of π_ρ (the trivial representation) is equal to 2. We use now Theorem 3.3 and the discussion of Example 1, and obtain (reduced multiplicities are equal to 1)

$$\mathrm{Q}_K^{\mathrm{spin}}(M) = \sum_{j=0}^{b-a-1} \mathrm{Q}_K^{\mathrm{spin}}(K \cdot (\frac{1+2j}{2}\omega_1)) \oplus \sum_{j=0}^{a} \mathrm{Q}_K^{\mathrm{spin}}(K \cdot (\frac{1+2j}{2}\omega_2)).$$

Using the formulae for $\mathrm{Q}_K^{\mathrm{spin}}(K \cdot (\frac{1+2n}{2}\omega_i))$ given in Example 1, these two formulae (fortunately) coincide. Furthermore we see that both faces σ_1, σ_2 give a non zero contribution to the multiplicity of the trivial representation.

4. Definition of the reduced index

We start by defining the reduced index for the action of an abelian torus H on a connected manifold Y. Denote by Λ the lattice of weights of H. We do not assume Y compact, but we assume that the set of stabilizers H_m of points in Y is finite. Let $\mathfrak{h}_Y$ be the generic infinitesimal stabilizer of the action H on Y, and H_Y be the connected subgroup of H with Lie algebra $\mathfrak{h}_Y$. Thus H_Y acts trivially on Y. Let us consider a spin^c structure on Y with determinant bundle $\mathbb{L}$, and a H invariant connection ∇ on $\mathbb{L}$. The image $\Phi_\Delta(Y)$ spans an affine space I_Y parallel to $\mathfrak{h}_Y^\perp$. We assume that the fibers of the map Φ_Δ are compact. We can easily prove that there exists a finite collection of hyperplanes $W^1, \ldots, W^p$ in I_Y such that the group H/H_Y acts locally freely on $\Phi_\Delta^{-1}(f)$, when f is in $\Phi_\nabla(Y)$, but not on any of the hyperplanes W^i.

Proposition 4.1. • *When $\mu \in I_Y \cap \Lambda$ is a regular value of $\Phi_\nabla : Y \to I_Y$, the reduced space Y_μ is an oriented orbifold equipped with an induced spin^c structure : we denote $\mathrm{Q}^{\mathrm{spin}}(Y_\mu)$ the corresponding spin^c index.*

• *For any connected component $\mathcal{C}$ of $I_Y \setminus \cup_{k=1}^p W^k$, we can associate a periodic polynomial function $q^{\mathcal{C}} : \Lambda \cap I_Y \to \mathbb{Z}$ such that*

$$q^{\mathcal{C}}(\mu) = \mathrm{Q}^{\mathrm{spin}}(Y_\mu)$$

for any element $\mu \in \Lambda \cap \mathcal{C}$ which is a regular value of $\Phi : Y \to I_Y$.

• *If $\mu \in \Lambda$ belongs to the closure of two connected components $\mathcal{C}_1$ and $\mathcal{C}_2$ of $I_Y \setminus \cup_{k=1}^p W^k$, we have*

$$q^{\mathcal{C}_1}(\mu) = q^{\mathcal{C}_2}(\mu).$$

We can now state the definition of the "reduced" index on Λ :

• $\mathrm{Q}^{\mathrm{spin}}(Y_\mu) = 0$ if $\mu \notin \Lambda \cap I_Y$,

• for any $\mu \in \Lambda \cap I_Y$, we define $\mathrm{Q}^{\mathrm{spin}}(Y_\mu)$ as being equal to $q^{\mathcal{C}}(\mu)$ where $\mathcal{C}$ is any connected component containing μ in its closure. In fact $\mathrm{Q}^{\mathrm{spin}}(Y_\mu)$ is computed as an index of a particular spin^c structure on the orbifold $\Phi_\nabla^{-1}(\mu+\epsilon)/H$ for any ϵ small and such that $\mu+\epsilon$ is a regular value of Φ_∇.

If Y is not connected, we define the reduced index at a point $\mu \in \Lambda$ as the sum of reduced indices over all connected components of Y.

More generally, let H be a compact connected group acting on Y and such that $[H,H]$ acts trivially on Y. Let $\mathcal{S}_Y$ be an equivariant spin^c structure on Y with determinant bundle $\mathbb{L}$. For any $\mu \in \mathfrak{h}^*$ such that $\mu([\mathfrak{h},\mathfrak{h}])=0$, and admissible for H, it is then possible to define $\mathrm{Q}^{\mathrm{spin}}(Y_\mu)$. Indeed eventually passing to a double cover of the torus $H/[H,H]$ and translating by the square root of the action of $H/[H,H]$ on the fiber of $\mathbb{L}$, we are reduced to the preceding case of the action of the torus $H/[H,H]$, and a $H/[H,H]$-equivariant spin^c structure on Y.

Consider now the action of a connected compact group K on M. Let σ be a (relative interior) of a face of $\mathfrak{t}^*_{\geq 0}$ which satisfies the following conditions

$$([\mathfrak{k}_M,\mathfrak{k}_M]) = ([\mathfrak{k}_\sigma,\mathfrak{k}_\sigma]), \qquad \Phi_\nabla^{-1}(\sigma) \neq \emptyset. \tag{3}$$

Let us explain how to compute the "reduced" index map $\mu \to \mathrm{Q}^{\mathrm{spin}}(M_\mu)$ on the set $\sigma \cap \{\Lambda+\rho-\rho_\sigma\}$ that parameterizes the admissible orbits intersecting σ. We work with the "slice" Y defined by σ. The set $U_\sigma := K_\sigma(\cup_{\sigma\subset\overline{\tau}}\tau)$ is an open neighborhood of σ in $\mathfrak{k}^*_\sigma$ such that the open subset $KU_\sigma \subset \mathfrak{k}^*$ is isomorphic to $K\times_{K_\sigma} U_\sigma$. We consider the K_σ-invariant subset $Y=\Phi_\nabla^{-1}(U_\sigma)$. The following lemma allows us to reduce the problem to the abelian case.

Lemma 4.2. • *Y is a non-empty submanifold of M such that KY is an open susbset of M isomorphic to $K\times_{K_\sigma} Y$.*

• *The Clifford module $\mathcal{S}_M$ on M determines a Clifford module $\mathcal{S}_Y$ on Y with determinant line bundle $\mathbb{L}_Y = \mathbb{L}_M|_Y \otimes \mathbb{C}_{-2(\rho-\rho_\sigma)}$. The corresponding moment map is $\Phi_\nabla|_Y - \rho + \rho_\sigma$.*

• *The group $[K_\sigma,K_\sigma]$ acts trivially on Y and on the bundle of spinors $\mathcal{S}_Y$.*

We thus consider Y with action of K_σ, and Clifford bundle $\mathcal{S}_Y$. If $\mu\in\sigma$ is admissible for K, then $\mu-\rho+\rho_\sigma \in \Lambda$ is admissible for K_σ. The reduced space $M_\mu = \Phi_\nabla^{-1}(\mu)/K_\sigma$ is equal to the reduced space $Y_{\mu-\rho+\rho_\sigma}$. As $[K_\sigma,K_\sigma]$ acts trivially on $(Y,\mathcal{S}_Y)$, we are in the abelian case, and we define $\mathrm{Q}^{\mathrm{spin}}(M_\mu) := \mathrm{Q}^{\mathrm{spin}}(Y_{\mu-\rho+\rho_\sigma})$.

Acknowledgments

We wish to thank the Research in Pairs program at Mathematisches Forschungsinstitut Oberwolfach (January 2014), which gave us the opportunity to work on these questions.

Références

[1] A. Cannas da Silva, Y. Karshon and S. Tolman, Quantization of presymplectic manifolds and circle actions, *Trans. Amer. Math. Soc.* **352** (2000), 525-552.

[2] M. Duflo, *Construction de représentations unitaires d'un groupe de Lie*, CIME, Cortona (1980).

[3] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), 515–538.

[4] M. Grossberg and Y. Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke Mathematical Journal **76** (1994), 23-58.

[5] M. Grossberg and Y. Karshon, *Equivariant index and the moment map for completely integrable torus actions*, Advances in Mathematics **133** (1998), 185-223.

[6] Y. Karshon and S. Tolman, *The moment map and line bundles over presymplectic toric manifolds*, J. Differential Geom **38** (1993), 465-484.

[7] E. Meinrenken, *Symplectic surgery and the Spin^c-Dirac operator*, Advances in Math. **134** (1998), 240-277.

[8] E. Meinrenken and R. Sjamaar, *Singular reduction and quantization*, Topology **38** (1999), 699-763.

[9] P.-E. Paradan, *Localization of the Riemann-Roch character*, J. Functional Analysis **187** (2001), 442–509.

[10] P.-E. Paradan, *Spin-quantization commutes with reduction*, J. Symplectic Geometry **10** (2012), 389-422.

[11] Y. Tian and W. Zhang, *An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg*, Invent. Math. **132** (1998), 229–259.

Institut de Mathématiques et de Modélisation de Montpellier (I3M), UMR CNRS 5149, Université Montpellier 2

E-mail address: Paul-Emile.Paradan@math.univ-montp2.fr

Institut de Mathématiques de Jussieu, UMR CNRS 7586, Université Paris-Diderot paris 7

E-mail address: Vergne@math.jussieu.fr