INTRODUCTORY WORDS TO THE PREPRINT

WIENER ALGEBRAS METHODS FOR LIOUVILLE THEOREMS ON THE STATIONARY NAVIER-STOKES SYSTEM

NICOLAS LERNER

We derive in this paper several new versions of Liouville theorems for the 3D stationary Navier-Stokes system for incompressible fluids, which can be written as

$$\nu\,\mathrm{curl}^2 v + \big((\mathrm{curl}\, v)\times v\big) + \nabla Q = 0, \quad \mathrm{div}\, v = 0, \tag{0.1}$$

where Q is the Bernoulli head pressure. The standard assumptions are that $\mathrm{curl}\, v$ belongs to L^2 and v goes to zero at infinity (the latter notion is clarified in the paper).

• **A regularity result.** A most important result, due to G.P. Galdi, is that v and Q are C^∞ functions going to zero at infinity. We get some improvement on that result, namely that v actually belongs to the Wiener algebra

$$\mathcal{W} = \mathtt{Fourier}\big(L^1(\mathbb{R}^3, \mathbb{R}^3)\big), \tag{0.2}$$

as well as all its derivatives. Since we have with strict inclusions, $\mathcal{W} \subset C^0_{(0)} \subset L^\infty$, where $C^0_{(0)}$ stands for continuous functions with limit 0 at infinity, it means that all derivatives of v actually belong to $\mathcal{W}$; although it looks as a minute improvement, we may point out that the Wiener algebra is a multiplicative algebra which is stable by the action of standard singular integrals, which is not the case of $\cap_{k\ge 0} C^k_{(0)}$ (here $C^k_{(0)}$ stands for C^k functions with limit 0 at infinity as well as their derivatives of order $\le k$). That stability plays an important role in our subsequent arguments, in particular since the Leray projector,

$$\mathbb{P} = I_3 - |D|^{-2}(D\otimes D), \quad \mathbb{P}(\xi) = I_3 - |\xi|^{-2}\big(\xi_j\xi_k\big)_{1\le j,k\le 3}, \tag{0.3}$$

is a matrix singular integral.

• **The Bernoulli head pressure.** Another key result on this topic is due to D. Chae, who proved that Q is a negative function. Along with Sard's Theorem, it provides a way to approach infinity using an increasing sequence of regular relatively compact open sets $(\{x, Q(x) < -\varepsilon_k\})_{k\ge 0}$ whose union is the whole $\mathbb{R}^3$, where $(\varepsilon_k)_{k\ge 0}$ is a decreasing sequence of positive numbers with limit 0. We use extensively this result to provide some theorems involving a condition on $(\Delta Q)_\pm$: if $(\Delta Q)_+$ or $(\Delta Q)_-$ belong to L^1, the vector field v must vanish identically.

• **Spectral localization.** Building on previous theorems due to G.P. Galdi and D. Chae, we are able to weaken the assumptions of classical theorems due to these

2020 *Mathematics Subject Classification.* 76D05, 35B53, 47B90.
Key words and phrases. Navier-Stokes Equation, Liouville Theorem, Wiener Algebra.

authors: in particular we prove essentially that the condition $\mathbf{1}(|D| \le 1)v \in L^{9/2}$ implies that v vanishes identically. Similarly, we prove that the condition

$$\mathbf{1}(|D| \le 1)\Delta v \in L^{6/5},$$

implies the same result. The operator $\mathbf{1}(|D| \le 1)$ must in fact be replaced by a Fourier multiplier $\alpha_0(D)$, where the function α_0 is smooth, equal to 1 near the origin and supported in the unit ball. We provide as well some other results of the same type, involving the Bernoulli head pressure.

• **Comments.** It is clear from the very beginning that the Liouville problem at stake, i.e. obtaining the triviality of v satisfying (0.1) as well as $\operatorname{curl} v \in L^2$ and v has limit 0 at infinity, that this problem is concerned with large values of $|x|$, namely that it is a problem in a neighborhood of infinity in the x variable and in fact a matter of decay with respect to x at infinity. In some sense, this suggests that the decay at infinity of v could be translated in a sense to some regularity result at a finite distance for $\widehat{v}$, the Fourier transform of v. In other words, it is quite natural to expect that the Liouville problem is also a low frequency question. What we do in this article is to give some firm ground to these heuristic remarks by providing some clearcut statements and proofs involving hypotheses bearing only on the low frequencies. Although the details of the proof are not inextricable, they require nevertheless a significant effort to handle a commutator to make sure that the only useful information must come from the low frequency part of the spectrum (the spectrum of v is the support of $\widehat{v}$). It should be said also that the various available elements of pseudo-differential operator theory, do not suffice to handle the problem at stake and the reason for that is due to the presence of a singularity at $\xi = 0$ for $\mathbb{P}(\xi)$ in (0.3). In fact, we had to resort to an explicit expression of the kernels of the operators involved in our commutator problem.

N. Lerner, Institut de Mathématiques de Jussieu, Sorbonne Université , Campus Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris cedex 05, France

E-mail address: nicolas.lerner@imj-prg.fr, nicolas.lerner@sorbonne-universite.fr