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A course on cluster tilted algebras

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Contents

Introduction

These notes are an expanded version of a mini-course that will be given in the CIMPA School "Homological methods, representation theory and cluster algebras", due to be held from the 7th to the 18th of March 2016 in Mar del Plata (Argentina). The aim of the course is to introduce the participants to the study of cluster tilted algebras, and their applications in the representation theory of algebras.

Cluster tilted algebras were defined in [40] and also, independently, in [45] for type $\mathbb{A}$. This class of finite dimensional algebras appeared as a byproduct of the now extensive theory of cluster algebras of Fomin and Zelevinsky [49]. They are the endomorphism algebras of tilting objects in the cluster category of [38]. Since their introduction, they have been the subject of several investigations, which we survey in this course.

For reasons of space, it was not possible to be encyclopedic. Thus, we have chosen to concentrate on the representation theoretical aspects and to ignore other aspects of the theory like, for instance, the relations between cluster tilted algebras and cluster algebras arising from surfaces, or the combinatorics of cluster variables. In keeping with the nature of the course, we tried to make these notes as self-contained as we could, providing examples for most results and proofs whenever possible.

The notes are divided into the following sections.

1. Tilting in the cluster category.
2. Cluster tilted algebras.
3. The module category of a cluster tilted algebra.
4. Particular modules over cluster tilted algebras.
5. Hochschild cohomology of cluster tilted algebras.

1 Tilting in the cluster category

1.1 Notation

Throughout these notes, k denotes an algebraically closed field and algebras are, unless otherwise specified, basic and connected finite dimensional k-algebras. For such an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, and by $\operatorname{ind} A$ a full subcategory containing exactly one representative from each isoclass (=isomorphism class) of indecomposable A-modules. When we speak about an A-module or an indecomposable A-module, we always mean implicitly that it belongs to $\operatorname{mod} A$ or to $\operatorname{ind} A$, respectively. Given a module M, we denote by $\operatorname{pd} M$ and $\operatorname{id} M$ its projective and injective dimension, respectively. The global dimension of A is denoted by $\operatorname{gl.dim.} A$. Given an additive category $\mathscr{C}$, we sometimes write $M \in \mathscr{C}_0$ to express that M is an object in $\mathscr{C}$. We denote by $\operatorname{add} M$ the full additive subcategory of $\mathscr{C}$ consisting of the finite direct sums of indecomposable summands of M.

We recall that any algebra A can be written in the form $A \cong \mathsf{k}Q_A / I$ where $\mathsf{k}Q_A$ is the path algebra of the quiver Q_A of A, and I is an ideal generated by finitely many relations. A ***relation*** is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i w_i$ where the λ_i are nonzero scalars and the w_i are paths of length at least two all having the same source and the same target. It is a ***zero-relation*** if it is a path, and a ***commutativity relation*** if it is of the form $\rho = w_1 - w_2$. Following [33], we sometimes consider equivalently an algebra A as a k-category, in which the object class A_0 is a complete set of primitive orthogonal idempotents $\{ e_1, \ldots, e_n \}$ in A and the set of morphisms $A(i, j)$ from e_i to e_j is the vector space $e_i A e_j$. We denote by P_i, I_i and S_i respectively the indecomposable projective, the indecomposable injective and the simple A-module corresponding to e_i.

For unexplained notions and results of representation theory, we refer the reader to [18, 21]. For tilting theory, we refer to [18, 10].

1.2 The derived category of a hereditary algebra

Once an exotic mathematical object, the derived category is now an indispensable tool of homological algebra. For its definition and properties, we refer the reader to [69, 37, 52]. Here we are only interested in derived categories of hereditary algebras.

Let Q be a finite, connected and acyclic quiver. Its path algebra $\mathsf{k}Q$ is hereditary, see [18, 21]. Let $\mathscr{D} = \mathscr{D}^b(\operatorname{mod} \mathsf{k}Q)$ denote the bounded derived category of $\operatorname{mod} \mathsf{k}Q$, that is, the derived category of the category of bounded complexes of finitely generated $\mathsf{k}Q$-modules. It is well-known that $\mathscr{D}$ is a triangulated category with almost split triangles, see [53]. We denote by $[1]_{\mathscr{D}}$ the shift of $\mathscr{D}$ and by $\tau_{\mathscr{D}}$ its Auslander-Reiten translation, or simply by $[1]$ and τ, respectively, if no ambiguity may arise. Both are automorphisms of $\mathscr{D}$.

Because $\mathscr{D}$ is a Krull-Schmidt category, every object in $\mathscr{D}$ decomposes as a finite direct sum of objects having local endomorphisms rings. Let $\operatorname{ind} \mathscr{D}$ denote a full subcategory of $\mathscr{D}$ consisting of exactly one representative from each isoclass of indecomposable objects in $\mathcal{D}$. Because kQ is hereditary, these indecomposable objects have a particularly simple form: they are of the form $M[i]$, that is, stalk complexes with M an indecomposable kQ-module concentrated in degree i, for $i \in \mathbb{Z}$, see [53], p. 49. Morphisms between indecomposable objects in $\mathcal{D}$ are computed according to the rule:

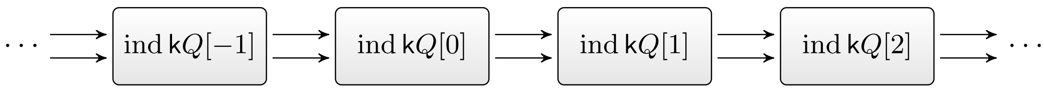
$$\operatorname{Hom}_{\mathcal{D}}(M[i], N[j]) = \begin{cases} \operatorname{Hom}_{kQ}(M,N) & \text{if } j = i \\ \operatorname{Ext}^1_{kQ}(M,N) & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

As is usual when dealing with triangulated categories, we write $\operatorname{Ext}^i_{\mathcal{D}}(M,N) = \operatorname{Hom}_{\mathcal{D}}(M, N[i])$. Denoting by $\mathrm{D} = \operatorname{Hom}_k(-,k)$ the usual vector space duality, the shift and the Auslander-Reiten translation of $\mathcal{D}$ are related by the following bifunctorial isomorphism, known as Serre duality

$$\operatorname{Hom}_{\mathcal{D}}(M, N[1]) \cong \mathrm{D}\operatorname{Hom}_{\mathcal{D}}(N, \tau M)$$

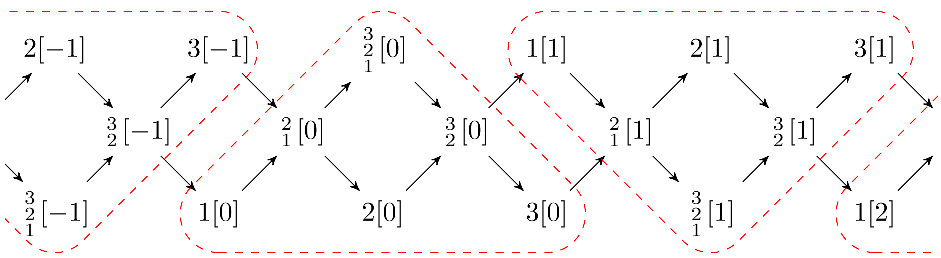
which is the analogue inside $\mathcal{D}$ of the celebrated Auslander-Reiten formula in a module category, see [18], p. 118.

We now describe the Auslander-Reiten quiver $\Gamma(\mathcal{D})$ of $\mathcal{D}$. Let $\Gamma(\operatorname{mod} kQ)$ denote the Auslander-Reiten quiver of kQ. For each $i \in \mathbb{Z}$, denote by Γ_i a copy of $\Gamma(\operatorname{mod} kQ)$. Then $\Gamma(\mathcal{D})$ is the translation quiver obtained from the disjoint union $\coprod_{i\in\mathbb{Z}} \Gamma_i$ by adding, for each $i \in \mathbb{Z}$ and each arrow $x \longrightarrow y$ in Q, an arrow from I_x in Γ_i to P_y in Γ_{i+1} and by setting $\tau P_x[1] = I_x$ for each $x \in Q$, see [53], p. 52. Identifying Γ_i with $\operatorname{ind} kQ[i]$, one can then think of $\operatorname{ind}\mathcal{D}$ as being formed by copies of $\operatorname{ind} kQ$ joined together by extra arrows.

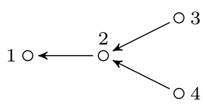


EXAMPLES.(a) Let Q be the quiver $\overset{3}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{1}{\circ}$.

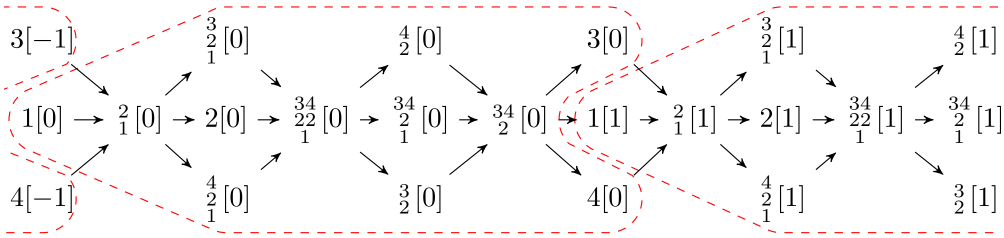
Denoting the indecomposable kQ-modules by their Loewy series, the Auslander-Reiten quiver of $\mathcal{D}^b(\operatorname{mod} kQ)$ is



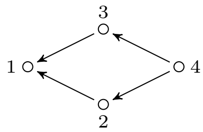
Distinct copies of $\Gamma(\operatorname{mod} kQ)$ are indicated by dotted lines.

(b) Let Q be the quiver .

The Auslander-Reiten quiver of $\mathcal{D}^b(\operatorname{mod} kQ)$ is



where again, distinct copies of $\Gamma(\operatorname{mod} \mathsf{k}Q)$ are indicated by dotted lines.

(c) Let Q be the quiver 

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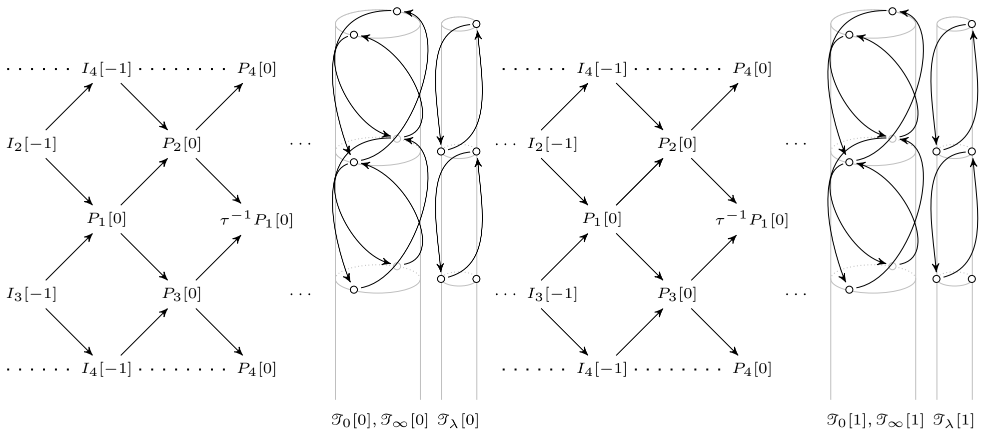
Then $\mathsf{k}Q$ is tame hereditary, $\Gamma(\operatorname{mod} \mathsf{k}Q)$ consists of an infinite postprojective component, an infinite preinjective component, two exceptional tubes of rank two and an infinite family of tubes of rank one. Thus the Auslander-Reiten quiver of $\mathscr{D}^b(\operatorname{mod} \mathsf{k}Q)$ is



where one identifies along the horizontal dotted lines.

This is the general shape: if Q is Dynkin, then $\Gamma(\mathscr{D}^b(\operatorname{mod} \mathsf{k}Q))$ has a unique component of the form $\mathbb{Z}Q$ while, if Q is euclidean or wild, it has infinitely many such components, separated by tubes, or by components of type $\mathbb{Z}\mathbb{A}_\infty$, respectively. The components of the form $\mathbb{Z}Q$ are called ***transjective***, and the others are called ***regular***.

1.3 *The cluster category*

Let Q be a finite, connected and acyclic quiver and $\mathscr{D} = \mathscr{D}^b(\operatorname{mod} \mathsf{k}Q)$ its derived category. Because each of the shift $[1]$ and the Auslander-Reiten translation τ is an automorphism of $\mathscr{D}$, so is the composition $F = \tau^{-1}[1]$. One may thus define the ***orbit category*** $\mathscr{D}/F$. Its objects are the F-orbits of the objects in $\mathscr{D}$. For each $X \in \mathscr{D}_0$, we denote by $\widetilde{X} = (F^i X)_{i \in \mathbb{Z}}$ its F-orbit. Then, for two objects $\widetilde{X}$, $\widetilde{Y}$ in $\mathscr{D}/F$, we define

$$\operatorname{Hom}_{\mathscr{D}/F}(\widetilde{X}, \widetilde{Y}) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathscr{D}}(X, F^i Y).$$

One shows easily that this vector space does not depend on the choices of the objects X and Y in their respective orbits and thus, the Hom-spaces of $\mathscr{D}/F$ are well-defined. We recall that, because $\mathsf{k}Q$ is hereditary, then, for each pair of indecomposable objects X, Y in $\mathscr{D}$, the space $\operatorname{Hom}_{\mathscr{D}}(X, F^i Y)$ is nonzero for at most one value of $i \in \mathbb{Z}$.

Definition. [38] The ***cluster category*** of the quiver Q is the orbit category $\mathscr{D}/F$. It is denoted by $\mathscr{C}_Q$, or simply $\mathscr{C}$, if no ambiguity may arise.

It follows directly from the definition that $\mathscr{C}$ is a k-linear category and that there exists a canonical projection functor $\pi : \mathscr{D} \longrightarrow \mathscr{C}$ which sends each $X \in \mathscr{D}_0$ to its F-orbit $\widetilde{X}$ in $\mathscr{C}$ and acts in the obvious way on morphisms.

The next theorem summarises the elementary properties of $\mathscr{C}$ and π.

Theorem 1 . *[38, 59] With the above notations*

(a) $\mathscr{C}$ is a Krull-Schmidt category and $\pi : \mathscr{D} \longrightarrow \mathscr{C}$ preserves indecomposabiliy,

(b) $\mathscr{C}$ is a triangulated category and $\pi : \mathscr{D} \longrightarrow \mathscr{C}$ is a triangle functor,

(c) $\mathscr{C}$ has almost split triangles and $\pi : \mathscr{D} \longrightarrow \mathscr{C}$ preserves almost split triangles. □

We derive some consequences. Because of (b)(c) above, the shift $[1]_{\mathscr{C}}$ of $\mathscr{C}$ and its Auslander-Reiten translation $\tau_{\mathscr{C}}$ are induced by $[1]_{\mathscr{D}}$ and $\tau_{\mathscr{D}}$, respectively. Thus, for each $X \in \mathscr{D}_0$, we have

$$\widetilde{X}[1]_{\mathscr{C}} = \widetilde{X[1]_{\mathscr{D}}} \qquad \text{and} \qquad \tau_{\mathscr{C}}\widetilde{X} = \widetilde{\tau_{\mathscr{D}}X}.$$

As a direct consequence, we have, for each $\widetilde{X} \in \mathscr{C}_0$,

$$\tau_{\mathscr{C}}\widetilde{X} = \widetilde{X}[1]_{\mathscr{C}}.$$

Indeed, we have $\widetilde{X} = \widetilde{FX} = \widetilde{\tau_{\mathscr{D}}^{-1}X[1]} = \tau_{\mathscr{C}}^{-1}\widetilde{X}[1]$, which establishes our claim.

Again, we denote briefly $[1]_{\mathscr{C}} = \tau_{\mathscr{C}}$ by $[1]$, or by τ, if no ambiguity may arise.

Another easy consequence is that, if Q, Q' are quivers such that there is a triangle equivalence $\mathscr{D}^b(\text{mod}\,\mathsf{k}Q) \cong \mathscr{D}^b(\text{mod}\,\mathsf{k}Q')$, then this equivalence induces another triangle equivalence $\mathscr{C}_Q \cong \mathscr{C}_{Q'}$. This is expressed by saying that $\mathscr{C}_Q$ is invariant under derived equivalence.

Denoting $\text{Ext}^1_{\mathscr{C}}(X,Y) = \text{Hom}_{\mathscr{C}}(X, Y[1])$, we get the following formula.

Lemma . *[38] (1.4) Let $\widetilde{X}, \widetilde{Y} \in \mathscr{C}_0$, then we have a bifunctorial isomorphism*

$$\text{Ext}^1_{\mathscr{C}}(\widetilde{X},\widetilde{Y}) \cong \text{D}\,\text{Ext}^1_{\mathscr{C}}(\widetilde{Y},\widetilde{X}).$$

Proof.

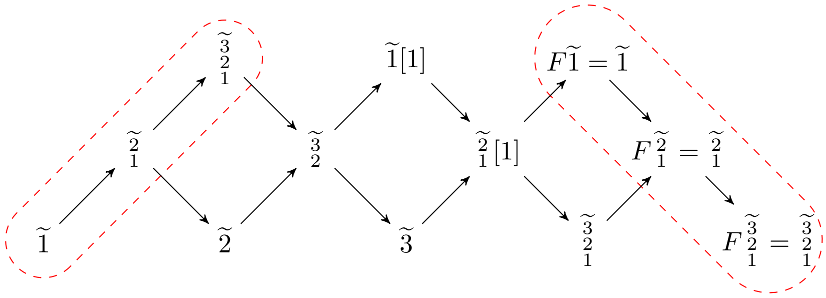
$$\begin{aligned}\text{Ext}^1_{\mathscr{C}}(\widetilde{X},\widetilde{Y}) &= \bigoplus_{i\in\mathbb{Z}} \text{Hom}_{\mathscr{D}}\left(X, F^i[1]_{\mathscr{D}}\right) \cong \bigoplus_{i\in\mathbb{Z}} \text{D}\,\text{Hom}_{\mathscr{D}}\left(F^iY, \tau_{\mathscr{D}}X\right)\\ &\cong \bigoplus_{i\in\mathbb{Z}} \text{D}\,\text{Hom}_{\mathscr{D}}\left(F^iY, F^{-1}X[1]_{\mathscr{D}}\right)\\ &\cong \bigoplus_{i\in\mathbb{Z}} \text{D}\,\text{Hom}_{\mathscr{D}}\left(F^iY, X[1]_{\mathscr{D}}\right) \cong \text{D}\,\text{Ext}^1_{\mathscr{C}}(\widetilde{Y},\widetilde{X}).\end{aligned}$$

□

The previous formula says that $\mathscr{C}$ is what is called a ***2-Calabi Yau*** category. Reading the formula as $\text{Ext}^1_{\mathscr{C}}\left(\widetilde{X},\widetilde{Y}\right) \cong \text{D}\,\text{Hom}_{\mathscr{C}}\left(\widetilde{Y},\tau\widetilde{X}\right)$, we see that it can be interpreted as the Auslander-Reiten formula in $\mathscr{C}$.

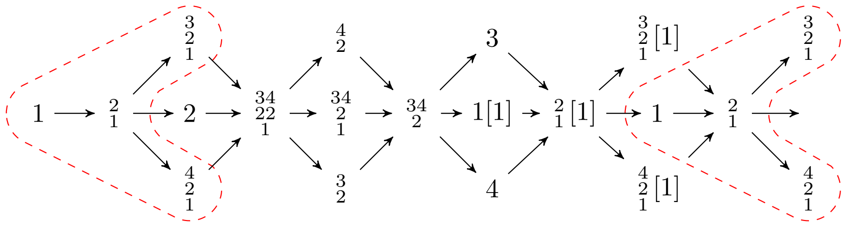
We now show how to compute the Auslander-Reiten quiver $\Gamma(\mathscr{C})$ of $\mathscr{C}$.

EXAMPLES.(a) Let Q be as in example 1.2(a). The cluster category is constructed from the derived category by identifying the objects which lie in the same F-orbit, hence each X with the corresponding $FX = \tau^{-1}X[1]$. Thus $\Gamma(\mathscr{C})$ is obtained by identifying the dotted sections in the figure below, so that $\Gamma(\mathscr{C})$ lies on a Möbius strip.

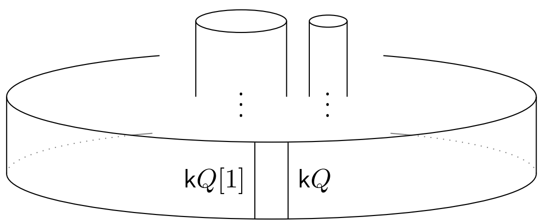


From now on, in examples, we drop the $\sim$ denoting the orbit of an object.

(b) Let Q be as in example 1.2(b). Applying the same recipe, we get that $\Gamma(\mathscr{C})$ lies on the cylinder obtained by identifying the dotted sections.



(c) This procedure is general: in order to construct $\Gamma(\mathscr{C}_Q)$, we must, in $\Gamma(\mathscr{D}^b(\text{mod}\,\mathsf{k}Q))$ identify the sections corresponding to $\mathsf{k}Q_{\mathsf{k}Q}$, that is, the indecomposable projective $\mathsf{k}Q$-modules, and to $\tau^{-1}\mathsf{k}Q[1]$. So, if Q is as in example 1.2(c), then $\Gamma(\mathscr{C}_Q)$ is of the form



Thus, the Auslander-Reiten quiver of the cluster category always admits a transjective component, which is the whole quiver if Q is Dynkin, and is of the form $\mathbb{Z}Q$ otherwise. In this latter case, $\Gamma(\mathscr{C}_Q)$ also admits tubes if Q is euclidean, or components of the form $\mathbb{Z}\mathbb{A}_\infty$ if Q is wild.

1.4 *Tilting objects*

The tilting objects in the cluster category are the analogues of the tilting modules over a hereditary algebra, see [18], Chapter VI. Let Q be a finite, connected and acyclic quiver and $\mathscr{C} = \mathscr{C}_Q$ the corresponding cluster category.

DEFINITION. [38] (3.3) An object T in $\mathscr{C}$ is called ***rigid*** if $\text{Ext}^1_{\mathscr{C}}(T,T) = 0$. It is called ***tilting*** if it is rigid and has a maximal number of isoclasses of indecomposable direct summands.

Actually, the maximality in the definition may be replaced by the following condition, easier to verify.

PROPOSITION 1 . *[38] (3.3) Let T be a rigid object in $\mathscr{C}_Q$. Then T is tilting if and only if it has $|Q_0|$ isoclasses of indecomposable direct summands.* □

EXAMPLES.(a) Let T be a tilting $\mathsf{k}Q$-module. Denoting by

$$i : \operatorname{mod} \mathsf{k}Q \hookrightarrow \mathcal{D} = \mathcal{D}^b(\operatorname{mod} \mathsf{k}Q)$$

the canonical embedding $X \mapsto X[0]$, then the image of T under the composition of functors

$$\operatorname{mod} \mathsf{k}Q \xhookrightarrow{i} \mathcal{D} \xrightarrow{\pi} \mathcal{C}_Q = \mathcal{C}$$

is rigid. It has obviously as many isoclasses of indecomposable summands as T has in $\operatorname{mod} \mathsf{k}Q$. Therefore, it is a tilting object in $\mathcal{C}$. Such an object is said to be ***induced*** from the tilting module T.

For instance, in example 1.3(a), the tilting $\mathsf{k}Q$-module $T = 1 \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus 3$ induces a tilting object in $\mathcal{C}$.

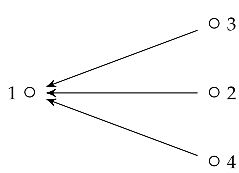
(b) Of course, there exist tilting objects which are not induced from tilting modules. For instance, in the algebra of example 1.3(b), the object

$$T = 2 \oplus \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 1[1]$$

is not induced. We check that it is a tilting object. Because it has obviously $4 = |Q_0|$ isoclasses of indecomposable summands, we just have to check its rigidity. As an example, we check here that $\operatorname{Ext}^1_{\mathcal{C}}(\begin{smallmatrix}4\\2\end{smallmatrix}, 1[1]) = 0$. Because of Lemma (1.3.2), it is equivalent to prove that $\operatorname{Ext}^1_{\mathcal{C}}(1[1], \begin{smallmatrix}4\\2\end{smallmatrix}) = 0$. Now

$$\begin{aligned}
\operatorname{Ext}^1_{\mathcal{C}}(1[1], \begin{smallmatrix}4\\2\end{smallmatrix}) &= \operatorname{Hom}_{\mathcal{C}}(1[1], \begin{smallmatrix}4\\2\end{smallmatrix}[1]) \\
&= \operatorname{Hom}_{\mathcal{C}}(1, \begin{smallmatrix}4\\2\end{smallmatrix}) \\
&= \operatorname{Hom}_{\mathcal{D}}(1, \begin{smallmatrix}4\\2\end{smallmatrix}) \oplus \operatorname{Hom}_{\mathcal{D}}(1, \tau^{-1}\begin{smallmatrix}4\\2\end{smallmatrix}[1]) \\
&= \operatorname{Hom}_{\mathsf{k}Q}(1, \begin{smallmatrix}4\\2\end{smallmatrix}) \oplus \operatorname{Ext}^1_{\mathsf{k}Q}(1, 3) \\
&= 0.
\end{aligned}$$

However, one can look at this example from another point of view. Indeed $\mathsf{k}Q$ is derived equivalent to $\mathsf{k}Q'$ where Q' is the quiver



and, under this triangle equivalence, T corresponds to the tilting $\mathsf{k}Q'$-module $T' = 1 \oplus \begin{smallmatrix}3\,2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\,4\\1\end{smallmatrix} \oplus 2$ that is, T may be considered as induced from a tilting $\mathsf{k}Q'$-module.

This change of quiver is actually always possible.

Proposition 2 . *[38] (3.3) Let T be a tilting object in $\mathcal{C}_Q$. Then there exists a quiver Q' such that $\mathsf{k}Q$ and $\mathsf{k}Q'$ are derived equivalent, and T is induced from a tilting $\mathsf{k}Q'$-module.* □

In one important aspect, tilting objects behave better than tilting modules. Indeed, let A be a hereditary algebra, a rigid A-module T is called an ***almost complete tilting module*** if it has $|Q_0| - 1$ isoclasses of indecomposable summands. Because of Bongartz' lemma, [18] p.196, there always exists an indecomposable module M such that $T \oplus M$ is a tilting module. Such an M is called a ***complement*** of T. It is known that an almost complete tilting module has at most two nonisomorphic complements and it has two if and only if it is sincere, see [57]. We now look at the corresponding result inside the cluster category.

Definition. Let $\mathscr{C}$ be a Krull-Schmidt category, and $X \in \mathscr{C}_0$. For $U \in \mathscr{C}_0$, a morphism

$$f_X : U_X \longrightarrow U$$

with $U_X \in \operatorname{add} X$ is called a ***right*** X***-approximation*** for U if, for every $X' \in \operatorname{add} X$ and morphism $f' : X' \longrightarrow U$, there exists $g : X' \longrightarrow U$ such that $f' = f_X g$

$$\begin{array}{ccc} U_X & \xrightarrow{f_X} & U \\ {\scriptstyle g}\uparrow & \nearrow_{f'} & \\ X' & & \end{array} .$$

Such a right approximation f_X is called ***right minimal*** if, for a morphism $h : U_X \longrightarrow U_X$, the relation $f_X h = f_X$ implies that h is an automorphism

$$\begin{array}{ccc} U_X & \xrightarrow{f_X} & U \\ {\scriptstyle h}\downarrow & & \| \\ U_X & \xrightarrow{f_X} & U \end{array} .$$

One defines dually left X-approximations and left minimal X-approximations.

Let now T be a rigid object in the cluster category $\mathscr{C}$. In analogy with the situation for modules, T is called an ***almost complete tilting object*** if it has $|Q_0| - 1$ isoclasses of indecomposable summands. Again, because of Bongartz' lemma and Proposition 2 above, there exists at least one indecomposable object M in $\mathscr{C}$ such that $T \oplus M$ is a tilting object. Then M is called a ***complement*** to T.

Theorem 3 . *[38] (6.8) An almost complete tilting object T in $\mathscr{C}$ has exactly two isoclasses of indecomposable complements M_1, M_2 and moreover there exist triangles*

$$M_2 \xrightarrow{g_1} T_1 \xrightarrow{f_1} M_1 \longrightarrow M_2[1]$$

and

$$M_1 \xrightarrow{g_2} T_2 \xrightarrow{f_2} M_2 \longrightarrow M_1[1]$$

where f_1, f_2 are right minimal T-approximations and g_1, g_2 are left minimal T-approximations. □

EXAMPLE. In the cluster category of example 1.3(b), the object $T = \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 1[1]$ is almost complete.It has exactly two (isoclasses of) complements, namely $M_1 = 2$ and $M_2 = \begin{smallmatrix}34\\2\end{smallmatrix}$. We also have triangles

$$2 \longrightarrow \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}34\\2\end{smallmatrix} \longrightarrow 2[1]$$

$$\begin{smallmatrix}34\\2\end{smallmatrix} \longrightarrow 1[1] \longrightarrow 2 \longrightarrow \begin{smallmatrix}34\\2\end{smallmatrix}[1]$$

where the morphisms are minimal approximations.

2
Cluster tilted algebras

2.1 The definition and examples

In classical tilting theory, the endomorphism algebra of a tilting module over a hereditary algebra is called a ***tilted algebra***. Due to its proximity with hereditary algebras, this class of algebras was heavily investigated and is by now considered to be well-understood. Moreover, it turned out to play an important rôle in representation theory, see [18, 10]. The corresponding notion in the cluster category is that of cluster tilted algebras.

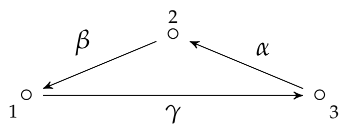
Definition. [40] Let Q be a finite, connected and acyclic quiver. An algebra B is called ***cluster tilted of type*** Q if there exists a tilting object T in the cluster category $\mathscr{C}_Q$ such that $B = \operatorname{End}_{\mathscr{C}_Q} T$.

Because, from the representation theoretic point of view, we may restrict ourselves to basic algebras, we assume, from now on and without loss of generality, that the indecomposable summands of a tilting object are pairwise nonisomorphic. This ensures that the endomorphism algebra is basic. Such a tilting object is then called ***basic***.

Any hereditary algebra A is cluster tilted: let indeed $A = \Bbbk Q$ and consider the tilting object in $\mathscr{C}_Q$ induced by $T = A_A$, its endomorphism algebra in $\mathscr{C}_Q$ is A.

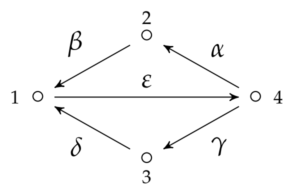
Actually, as we see in 3.2 below, a cluster tilted algebra is either hereditary of it has infinite global dimension.

EXAMPLES.(a) Let Q be as in example 1.3(a) and T be the tilting object in $\mathscr{C}_Q$ induced by the tilting module $1 \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus 3$. Its endomorphism algebra is given by the quiver



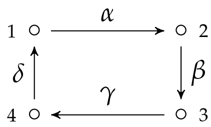
bound by $\alpha\beta = 0$, $\beta\gamma = 0$ and $\gamma\alpha = 0$.

(b) Let Q be as in example 1.3(b), and $T = 2 \oplus \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 1[1]$. Its endomorphism algebra is given by the quiver



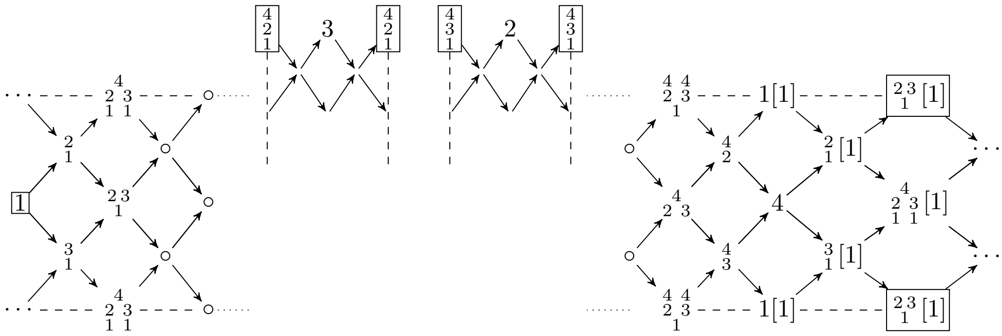
bound by $\alpha\beta = \gamma\delta$, $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$, $\beta\varepsilon = 0$, $\delta\varepsilon = 0$.

(c) For the same Q of example 1.3(b), and $T = 2 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 1[1] \oplus \begin{smallmatrix}4\\2\\1\end{smallmatrix}[1]$, the endomorphism algebra is given by the quiver

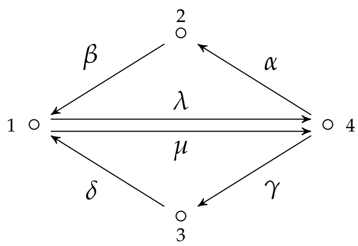


bound by $\alpha\beta\gamma = 0$, $\beta\gamma\delta = 0$, $\gamma\delta\alpha = 0$, $\delta\alpha\beta = 0$.

(d) Let Q be as in example 1.3(c), and $T = 1 \oplus \begin{smallmatrix}4\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\,3\\1\end{smallmatrix}[1]$. Here, $\Gamma(\mathscr{C}_Q)$ is as follows



where we have only drawn the tubes of rank two. One has to identify along the horizontal dotted lines to get the transjective component and along the vertical dotted lines to get the tubes. The direct summands of T are indicated by squares. The endomorphism algebra of T is given by the quiver



bound by $\alpha\beta = 0$, $\beta\lambda = 0$, $\lambda\alpha = 0$, $\gamma\delta = 0$, $\delta\mu = 0$, $\mu\gamma = 0$.

2.2 *Relation with mutations*

Mutation of quivers is an essential tool in the construction of cluster algebras. Let Q be a quiver having neither loops ($\circ \circlearrowleft$) nor 2-cycles ($\circ \rightleftarrows \circ$) and x be a point in Q. The ***mutation*** μ_x at the point x transforms Q into another quiver $Q' = \mu_x Q$ constructed as follows

(a) The points of Q' are the same as those of Q.

(b) If, in Q, there are r_{ij} paths of length two of the form $i \longrightarrow x \longrightarrow j$, then we add r_{ij} arrows from i to j in Q'.

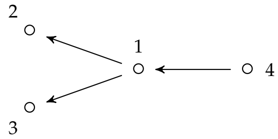
(c) We reverse the direction of all arrows incident to x.

(d) All other arrows remain the same.

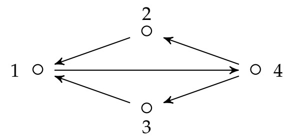
(e) We successively delete all pairs of 2-cycles thus obtained until Q' has no more 2-cycles.

It is well-known and easy to prove that mutation is an involutive process, that is, μ_x^2 is the identity transformation on Q.

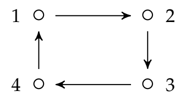
EXAMPLE.(a) Let Q be the Dynkin quiver



then $Q' = \mu_1 Q$ is the quiver



which is the quiver of the cluster tilted algebra of example 2.1(b). Repeating, and mutating this time at 2, we get $Q'' = \mu_2\mu_1 Q$ which is the quiver



of example 2.1(c).

Now, recall from Theorem (1.4.3) that any almost complete tilting object T_0 of the cluster category $\mathscr{C}$ has exactly two nonisomorphic complements M_1 and M_2, giving rise to two tilting objects $T_1 = T_0 \oplus M_1$ and $T_2 = T_0 \oplus M_2$. To these correspond in turn two cluster tilted algebras $B_1 = \operatorname{End}_{\mathscr{C}} T_1$ and $B_2 = \operatorname{End}_{\mathscr{C}} T_2$ with respective quivers Q_{B_1} and Q_{B_2}. It turns out that one can pass from one to the other using mutation.

Theorem 1 . *[41] With the previous notation, let x be the point in Q_{B_1} corresponding to the summand M_1 of T_1, then $Q_{B_2} = \mu_x Q_{B_1}$.* □

EXAMPLE.(b) As seen in example 1.4(c), the almost complete tilting object $T_0 = \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 1[1]$ in the cluster category of example 1.3(c) has exactly two complements, $M_1 = 2$ and $M_2 = \begin{smallmatrix}34\\2\end{smallmatrix}$, The endomorphism algebra of $T_2 = T_0 \oplus M_1$ is the algebra of example 2.1(b), while that of $T_2 = T_0 \oplus M_2$ is that of example 2.1(c). We have just seen in example (a) that mutating the quiver of End T_1 gives the quiver of End T_2.

Because mutation creates neither loops nor 2-cycles, we deduce the following corollary.

Corollary 2 . *The quiver of a cluster tilted algebra contains neither loops nor 2-cycles.* □

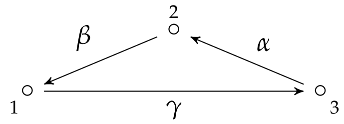
Moreover, we can obtain all the quivers of cluster tilted algebras of type Q by repeatedly mutating the quiver Q itself. This indeed follows from the fact that, if T, T' are two tilting objects in $\mathscr{C}_Q$, then there exists a sequence $T = T_0, T_1, \ldots, T_n = T'$ such that, for each i with $0 \leqslant i < n$, we have that T_i and T_{i+1} are as in Theorem (1.4.3), that is, they have all but one indecomposable summand in common. This is sometimes expressed by saying that the exchange graph is connected, see [38](3.5).

Corollary 3 . *Let Q be a finite, connected and acyclic quiver. The class of quivers obtained from Q by successive mutations coincides with the class of quivers of cluster tilted algebras of type Q.* □

2.3 Relation extensions algebras

So far, in order to know whether a given algebra is cluster tilted or not, we need to identify a tilting object in some cluster category and verify whether the given algebra is its endomorphism algebra or not. This is clearly a difficult process in general (however, in the representation-finite case, we refer the reader to [30]). It is thus reasonable to ask for an intrinsic characterisation of cluster tilted algebras.

In order to motivate the next definition, let us consider the cluster tilted algebra B of example 2.1(a). It is given by the quiver



bound by $\alpha\beta = 0$, $\beta\gamma = 0$ and $\gamma\alpha = 0$. Deleting the arrow γ, we get the quiver

$$\overset{1}{\circ} \xleftarrow{\;\beta\;} \overset{2}{\circ} \xleftarrow{\;\alpha\;} \overset{3}{\circ}$$

bound by $\alpha\beta = 0$: this is the bound quiver of a tilted algebra, which we call C. The two-sided ideal $E = B\gamma B$ has a natural structure of C-C-bimodule and $C = B/E$. As a k-vector space, $B = C \oplus E$. This is actually a classical construction.

Definition. Let C be an algebra, and E a C-C-bimodule. The ***trivial extension*** $B = C \ltimes E$ is the k-vector space

$$B = C \oplus E = \{ (c,e) \mid c \in C, e \in E \}$$

with the multiplication induced from the bimodule structure of E, that is

$$(c,e)(c',e') = (cc', ce' + ec')$$

for $c, c' \in C$ and $e, e' \in E$.

Equivalently, we may describe B as being the algebra of 2×2-matrices

$$B = \left\{ \begin{pmatrix} c & 0 \\ e & c \end{pmatrix} \mid c \in C, e \in E \right\}$$

with the usual matrix addition and the multiplication induced from the bimodule structure of E.

If $B = C \ltimes E$, then there exists a short exact sequence of C-C-bimodules

$$0 \longrightarrow E \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

where $i : e \mapsto (0,e)$ (for $e \in E$) is the canonical inclusion and the projection $p : (c,e) \mapsto c$ (for $(c,e) \in B$) is an algebra morphism with section $q : c \mapsto (c,0)$ (for $c \in C$). Thus, this sequence splits as a sequence of C-C-bimodules. Moreover, $E^2 = 0$, so that $E \subseteq \operatorname{rad} B$. This implies that $\operatorname{rad} B = \operatorname{rad} C \oplus E$, as vector spaces. We now show how to compute the quiver of a trivial extension.

Lemma 1 . *[3] Let C be an algebra, and E a C-C-bimodule. The quiver Q_B of $B = C \ltimes E$ is constructed as follows:*

(a) $(Q_B)_0 = (Q_C)_0$,

(b) for $x, y \in (Q_C)_0$, the set of arrows in Q_B from x to y equals the set of arrows in Q_C from x to y plus

$$\dim_k \frac{e_x E e_y}{e_x (\operatorname{rad} C) E e_y + e_x E (\operatorname{rad} C) e_y}$$

additional arrows.

Proof. (a) This follows from the fact that $E \subseteq \operatorname{rad} B$.

(b) The arrows in Q_B from x to y are in bijection with a basis of $e_x\left(\frac{\operatorname{rad} B}{\operatorname{rad}^2 B}\right)e_y$. Now $\operatorname{rad} B = \operatorname{rad} C \oplus E$ and $E^2 = 0$ imply that

$$\operatorname{rad}^2 B = \operatorname{rad}^2 C \oplus \Big((\operatorname{rad} C)E + E(\operatorname{rad} C)\Big).$$

The statement follows from the facts that $\operatorname{rad}^2 C \subseteq \operatorname{rad} C$ and $(\operatorname{rad} C)E + E(\operatorname{rad} C) \subseteq E$. □

Recall that an algebra whose quiver is acyclic is called ***triangular***.

Definition. [3] Let C be a triangular algebra of global dimension at most two. Its ***relation extension*** is the algebra $\widetilde{C} = C \ltimes E$, where $E = \operatorname{Ext}^2_C(\mathrm{D}\,C, C)$ is considered as a C-C-bimodule with the natural actions.

If C is hereditary, then $E = 0$ and $\widetilde{C} = C$ is its own relation extension. On the other hand, if $\operatorname{gl.dim.} C = 2$, then there exist simple C-modules S, S' such that $\operatorname{Ext}^2_C(S, S') \neq 0$. Let I be the injective envelope of S and P' the projective cover of S', then the short exact sequences $0 \longrightarrow \operatorname{rad} P' \longrightarrow P' \longrightarrow S' \longrightarrow 0$ and $0 \longrightarrow S \longrightarrow I \longrightarrow I/S \longrightarrow 0$ induce an epimorphism $\operatorname{Ext}^2_{\mathscr{C}}(I, P') \longrightarrow Ext^2_{\mathscr{C}}(S, S')$. Therefore $\operatorname{Ext}^2_{\mathscr{C}}(\mathrm{D}\,C, C) \neq 0$.

Following [34], we define a ***system of relations*** for an algebra $C = \mathsf{k}Q_C/I$ to be a subset R of $\bigcup_{x,y\in(Q_C)_0} e_x I e_y$ such that R, but no proper subset of R, generates I as a two-sided ideal.

Theorem 2 . *[3] (2.6) Let $C = \mathsf{k}Q_C/I$ be a triangular algebra of global dimension at most two,and R be a system of relations for C. The quiver of the relation extension $\widetilde{C}$ is constructed as follows*

(a) $\left(Q_{\widetilde{C}}\right)_0 = (Q_C)_0$

(b) For $x, y \in (Q_C)_0$, the set of arrows in $Q_{\widetilde{C}}$ from x to y equals the set of arrows in Q_C from x to y plus $|R \cap (e_y I e_x)|$ additional arrows.

"Proof". Let S be the direct sum of a complete set of representatives of the isoclasses of simple C-modules. Because C is basic, we have $S = \operatorname{top} C_C = \operatorname{soc}(\mathrm{D}\,C)_C$. Because of [34](1.2), the relations in R correspond to a k-basis of $\operatorname{Ext}^2_{\mathscr{C}}(S, S)$. Because of [3](2.4), $\operatorname{Ext}^2_{\mathscr{C}}(S, S) \cong \operatorname{top}\operatorname{Ext}^2_{\mathscr{C}}(\mathrm{D}\,C, C)$. Lemma 1 implies that the number of additional arrows is $\dim_{\mathsf{k}} e_x \operatorname{Ext}^2_{\mathscr{C}}(S, S)e_y = \dim_{\mathsf{k}} \operatorname{Ext}^2_{\mathscr{C}}(S_y, S_x)$, hence the result. □

In view of the theorem, we sometimes refer to the arrows of Q_C as the "old" arrows in $Q_{\widetilde{C}}$, the remaining being called the "new" arrows

Note that the quiver of a nonhereditary relation extension always has oriented cycles. We still have to describe the relations occurring in the quiver of a relation extension algebra. This is done in the next subsection. For the time being we establish the relation between cluster tilted algebras and relation extensions.

Theorem 3 . *[3] (3.4) An algebra B is cluster tilted of type Q if and only if there exists a tilted algebra C of type Q such that $B = \widetilde{C}$.*

Proof. Assume that B is cluster tilted of type Q. Then there exists a tilting object T in the cluster category $\mathscr{C}_Q$ such that $B = \operatorname{End}_{\mathscr{C}_Q} T$. Because of Proposition (1.4.2), we may assume that T is induced from a tilting $\mathsf{k}Q$-module. Let $\mathscr{D} = \mathscr{D}^b(\operatorname{mod} \mathsf{k}Q)$. We have

$$B = \operatorname{End}_{\mathscr{C}_Q} T = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathscr{D}}(T, F^i T)$$

as k-spaces. Because T is a $\mathsf{k}Q$-module, $\operatorname{Hom}_{\mathscr{D}}(T, F^i T) = 0$ for $i \geqslant 2$. Moreover $C = \operatorname{End} T_{\mathsf{k}Q}$ is tilted and, as k-vector spaces

$$B \cong \operatorname{End} T_{\mathsf{k}Q} \oplus \operatorname{Hom}_{\mathscr{D}}(T, FT) = C \oplus \operatorname{Hom}_{\mathscr{D}}(T, FT).$$

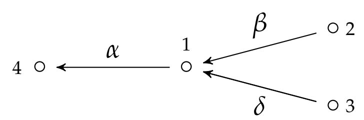
Because of Happel's theorem, see [53] p.109, $\mathscr{D}$ is triangle equivalent to $\mathscr{D}' = \mathscr{D}^b(\text{mod}\, C)$. Setting $F' = \tau^{-1}[1]$ in $\mathscr{D}'$, we have

$$\begin{aligned}\text{Hom}_{\mathscr{D}}(T, FT) &\cong \text{Hom}_{\mathscr{D}'}(C, F'C)\\ &\cong \text{Hom}_{\mathscr{D}'}(\tau C[1], C[2])\\ &\cong \text{Hom}_{\mathscr{D}'}(\text{D}\,C, C[2])\\ &\cong \text{Ext}^2_C(\text{D}\,C, C).\end{aligned}$$

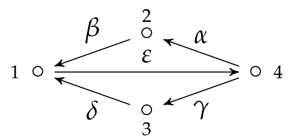
We leave to the reader the verification that the multiplicative structure of B is the same as that of $\widetilde{C}$. This proves the necessity. The sufficiency is proved in the same way. □

Thus, there exists a surjective map from the class of tilted algebras to the class of cluster tilted algebras, given by $C \mapsto \widetilde{C}$. However, this map is not injective as we shall see in example (a) below. It is therefore an interesting question to find all the tilted algebras which lie in the fibre of a given cluster tilted algebra. We return to this question in 3.4 below.

EXAMPLES.(a) Let C be the algebra given by the quiver



bound by $\beta\varepsilon = 0$, $\delta\varepsilon = 0$. It is tilted of type $\mathbb{D}_4$. Because of Theorem 2, the quiver of its relation extension is



with α, γ new arrows. In order to compute a system of relations, we use the following observation. Let P_x, $\widetilde{P}_x$ denote respectively the indecomposable projective C and $\widetilde{C}$-modules corresponding to x. The short exact sequence $0 \longrightarrow E \longrightarrow B \longrightarrow C \longrightarrow 0$ induces another exact sequence

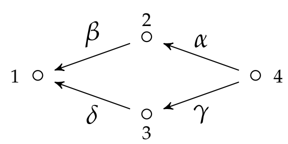
$$0 \longrightarrow \text{Ext}^2_C(\text{D}\,C, P_x) \longrightarrow \widetilde{P}_x \xrightarrow{p_x} P_x \longrightarrow 0$$

where p_x is a projective cover. Now, in this example, it is easily seen that $\text{Ext}^2_C(I_2, P_4) \cong \text{Ext}^2_C(I_3, P_4) \cong \text{Ext}^2_C(I_1, P_4) \cong \mathsf{k}$ and all other $\text{Ext}^2_C(I_i, P_j) = 0$. Thus

$$\widetilde{C}_{\widetilde{C}} = \begin{smallmatrix}1\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 2 & & 3\\ & 1 & \end{smallmatrix}$$

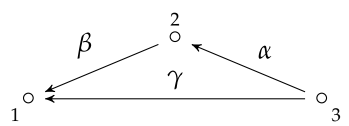
and so the quiver above is bound by $\alpha\beta = \gamma\delta$, $\beta\varepsilon = 0$, $\varepsilon\gamma = 0$, $\varepsilon\delta = 0$, $\alpha\varepsilon = 0$. This is the bound quiver of example 2.1(b)

Now let C' be the tilted algebra of type $\mathbb{D}_4$ given by the quiver

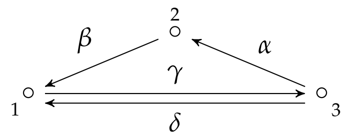


bound by $\alpha\beta = \gamma\delta$, then a similar calculation yields $\widetilde{C'} \cong \widetilde{C}$. This shows that the mapping $C \mapsto \widetilde{C}$ is not injective.

(b) Let C be the triangular algebra of global dimension 2 given by



bound by $\alpha\beta = 0$. Here, C is not tilted. Applying Theorem 2 and a similar calculation as that of example (a) above show that $\widetilde{C}$ is given by



bound by $\alpha\beta = 0$, $\beta\delta = 0$, $\delta\alpha = 0$, $\delta\gamma\delta = 0$. We see that $\widetilde{C}$ is not cluster-tilted because its quiver contains a 2-cycle, contrary to Corollary (2.2.2)

2.4 The relations on a cluster tilted algebra

Starting from a tilted algebra C, Theorem (2.3.2) allows to construct easily the quiver of its relation extension $\widetilde{C}$ which is cluster tilted, thanks to Theorem (2.3.3). Now we show how to compute as easily a system of relations for $\widetilde{C}$.

Let $C = \mathsf{k}Q_C/I$ be a triangular algebra of global dimension at most 2, and $R = \{\rho_1, \dots, \rho_t\}$ be a system of relations for C. To the relation ρ_i from x_i to y_i, say, there corresponds in $\widetilde{C}$ a new arrow $\alpha_i : y_i \longrightarrow x_i$, as in Theorem (2.3.2). The ***Keller potential*** on $\widetilde{C}$ is the element

$$w = \sum_{i=1}^{t} \rho_i \alpha_i$$

of $\mathsf{k}Q_{\widetilde{C}}$. This element is considered up to cyclic equivalence: two potentials are called ***cyclically equivalent*** if their difference is a linear combination of elements of the form $\gamma_1\gamma_2\dots\gamma_m - \gamma_m\gamma_1\dots\gamma_{m-1}$, where $\gamma_1\gamma_2\dots\gamma_m$ is a cycle in the quiver. For a given arrow γ, the ***cyclic partial derivative*** of this cycle with respect to γ is defined to be

$$\partial_\gamma(\gamma_1 \dots \gamma_m) = \sum_{\gamma_i = \gamma} \gamma_{i+1} \dots \gamma_m \gamma_1 \dots \gamma_{i-1}.$$

In particular, the cyclic partial derivative is invariant under cyclic permutations. The ***Jacobian algebra*** $\mathcal{J}(Q_{\widetilde{C}}, w)$ is the quotient of $\mathsf{k}Q_{\widetilde{C}}$ by the ideal generated by all cyclic partial derivatives $\partial_\gamma w$ of the Keller potential w with respect to all the arrows γ in $Q_{\widetilde{C}}$, see [60].

Proposition 1 . *[15] (5.2) Let C be a triangular algebra of global dimension at mots two, and w be the Keller potential on $\widetilde{C}$. Then*

$$\widetilde{C} \cong \mathcal{J}(Q_{\widetilde{C}}, w)/J$$

where J is the square of the ideal generated by the new arrows.

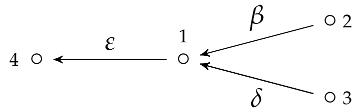
"Proof". It was shown in [60](6.12a) that $\mathcal{J}(Q_{\widetilde{C}}, w)$ is isomorphic to the endomorphism algebra of the tilting object C in Amiot's generalised cluster category associated with C. Because of [1](1.7), this endomorphism algebra is isomorphic to the tensor algebra of the bimodule ${}_CE_C$ and its quiver is isomorphic to $Q_{\widetilde{C}}$, which is also the quiver of $\mathcal{J}(Q_{\widetilde{C}}, w)$. Taking the quotient of the tensor algebra by the ideal J generated by all tensor powers $E^{\otimes_C i}$ with $i \geqslant 2$, we get exactly $\widetilde{C}$. But now J is the square of the ideal generated by the new arrows. □

The next result is proven, for instance, in [23](4.22) or [2] p.17.

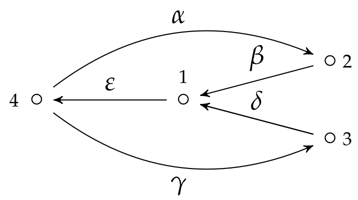
Proposition 2 . *Let C be a tilted algebra, and w be the Keller potential on $\widetilde{C}$, then $\widetilde{C} = \mathcal{J}(Q_C, w)$.* □

That is, if C is tilted, then the square J of the ideal generated by the new arrows is contained in the ideal generated by all cyclic partial derivatives of the Keller potential. This gives a system of relations on a cluster tilted algebra.

EXAMPLES.(a) Let C be the tilted algebra of example (2.3)(a), given by the quiver



bound by $\beta\varepsilon = 0$, $\delta\varepsilon = 0$. Applying Theorem (2.3.2) yields the quiver

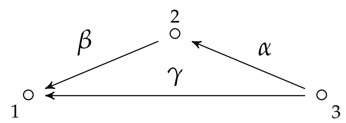


with new arrows α, γ. The Keller potential is then $w = \beta\varepsilon\alpha + \delta\varepsilon\gamma$. We compute its cyclic partial derivatives.

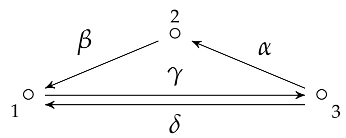
$$\partial_\alpha(w) = \beta\varepsilon,\ \ \partial_\beta(w) = \varepsilon\alpha,\ \ \partial_\gamma(w) = \delta\varepsilon,\ \ \partial_\delta(w) = \varepsilon\gamma,\ \ \partial_\varepsilon(w) = \alpha\beta + \gamma\delta.$$

Thus, besides the "old" relations $\beta\varepsilon = 0$, $\delta\varepsilon = 0$, we also have "new" relations $\varepsilon\alpha = \varepsilon\gamma = 0$ and $\alpha\beta + \gamma\delta = 0$. Moreover $J = \langle\alpha, \gamma\rangle^2 = 0$ so that we get the cluster tilted algebra of example (2.1)(b).

(b) Let C be the (non tilted) triangular algebra of global dimension two given by the quiver



bound by $\alpha\beta = 0$. Here, $\widetilde{C}$ is given by the quiver



and $w = \alpha\beta\delta$. Thus, the Jacobian algebra $\mathcal{J}(Q_{\widetilde{C}}, w)$ is given by the previous quiver bound by $\alpha\beta = 0$, $\beta\delta = 0$, $\delta\alpha = 0$. Here, $J = \langle\delta\rangle^2 = \langle\delta\gamma\delta\rangle$ is nonzero. Therefore $\widetilde{C}$ is given by the above quiver bound by $\alpha\beta = 0$, $\beta\delta = 0$, $\delta\alpha = 0$, $\delta\gamma\delta = 0$.

The set of relations given by the cyclic partial derivatives of the Keller potential is generally not a system of minimal relations. Following [39], we say that a relation ρ is ***minimal*** if, whenever $\rho = \sum_i \beta_i\rho_i\gamma_i$, where ρ_i is a relation for each i, then there is an index i such that both β_i and γ_i are scalars, that is, a minimal relation in a bound quiver (Q, I) is any element of I not lying in $(\mathrm{k}Q^+)I + I(\mathrm{k}Q^+)$, where $\mathrm{k}Q^+$ denotes the two-sided ideal generated by all the arrows of Q. There is however one particular case in which we have minimal relations. We need the following definitions.

DEFINITION. Let Q be a quiver with neither loops nor 2-cycles.

(a) [24] A full subquiver of Q is a ***chordless cycle*** if it is induced by a set of points $\{ x_1, x_2, \ldots, x_p \}$ which is topologically a cycle, that is, the edges on it are precisely the edges $x_i \;—\; x_{i+1}$ (where we set $x_{p+1} = x_1$).

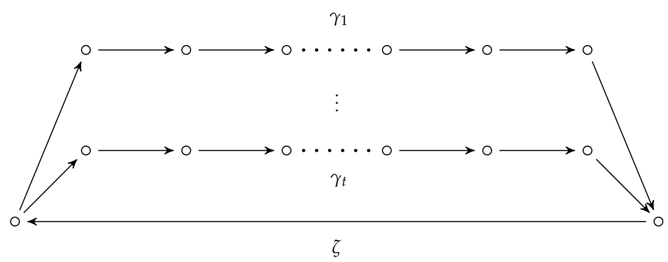
(b) [25] The quiver Q is called ***cyclically oriented*** of each chordless cycle is an oriented cycle.

For instance, any tree is trivially cyclically oriented. The easiest nontrivial cyclically oriented quiver is a single oriented cycle. Note that the definition of cyclically oriented excludes the existence of multiple arrows. It is also easy to see that the quiver of a cluster tilted algebra of Dynkin type is cyclically oriented.

Theorem 3 . *[25] Let B be cluster tilted with a cyclically oriented quiver. Then:*

(a) The arrows in Q_B which occur in some chordless cycle are in bijection with the minimal relations in any presentation of B.

(b) Let $\zeta \in (Q_B)_1$ occur in a chordless cycle, and $\gamma_1 \ldots \gamma_t$ be all the shortest paths antiparallel to ζ. Then the minimal relation corresponding to ζ is of the form $\sum_{i=1}^{t} \lambda_i \gamma_i$, where the λ_i are nonzero scalars. Also, the quiver restricted to the γ_i is of the form

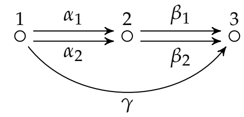


In particular, the γ_i only share their endpoints. □

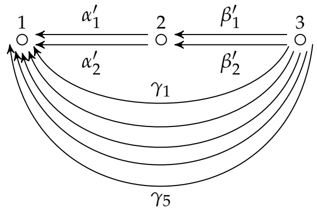
In particular, let B be a representation-finite cluster tilted algebra. As we see in (3.1) below, B is of Dynkin type, therefore its quiver is cyclically oriented and the previous theorem yields a system of minimal relations for B. Actually, in this case, for any arrow ζ, the number t of shortest antiparallel paths is 1 or 2. If there is one shortest path γ, we choose γ as a generator and, if there are 2, γ_1 and γ_2, we choose $\gamma_1 - \gamma_2$ as a generator. Then the ideal generated by these relations is a system of minimal relations [39].

If Q_B is not cyclically oriented, then the assertion of the theorem does not necessarily hold true, as we now see.

EXAMPLE.(c) [25] Let A be the path algebra of the quiver



Mutating at 2 yields the quiver



All four paths from 3 to 1, namely the $\beta'_i \alpha'_j$ are zero. Hence there are 4 relations from 3 to 1, but there are 5 arrows antiparallel to them.

Besides representation-finite cluster tilted algebras, minimal relations are only known for cluster tilted algebras of type $\widetilde{\mathbb{A}}$, see (2.5) below. We may formulate the following problem.

Problem. Give systems of minimal relations for any cluster tilted algebra.

2.5 Gentle cluster tilted algebras

There were several attempts to classify classes of cluster tilted algebras, see, for instance, [45, 63, 22, 36, 51], or to classify algebras derived equivalent to certain cluster tilted algebras, see, for instance [42, 32, 26, 27, 28]. We refrain from quoting all these results and concentrate rather on gentle algebras, introduced in [19].

DEFINITION. An algebra B is ***gentle*** if there exists a presentation $B \cong \mathrm{k}Q/I$ such that

(a) every point of Q is the source, or the target, of at most two arrows;

(b) I is generated by paths of length 2;

(c) for every $\alpha \in Q_1$, there is at most one $\beta \in Q_1$ such that $\alpha\beta \notin I$ and at most one $\gamma \in Q_1$ such that $\gamma\alpha \notin I$;

(d) for every $\alpha \in Q_1$, there is at most one $\xi \in Q_1$ such that $\alpha\xi \in I$ and at most one $\zeta \in Q_1$ such that $\zeta\alpha \in I$.

Gentle algebras are string algebras [44], so we can describe all their indecomposable modules and all their almost split sequences. Gentle algebras are also tame and this class is stable under tilting [66]. We characterise gentle cluster tilted algebras.

THEOREM 1 . *[4] Let C be a tilted algebra, the following conditions are equivalent*

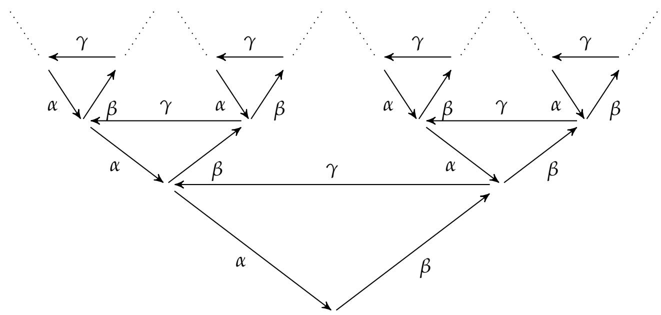
(a) C is gentle

(b) $\widetilde{C}$ is gentle

(c) C is of Dynkin type $\mathbb{A}$ or of euclidean type $\widetilde{\mathbb{A}}$. □

Moreover, the set of relations induced from the Keller potential is a system of minimal relations in these two cases.

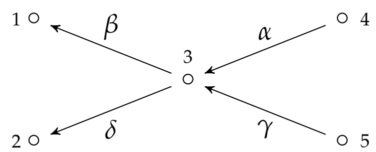
Cluster tilted algebras of type $\mathbb{A}$ are particularly easy to describe. Their quiver are full connected subquivers of the following infinite quiver



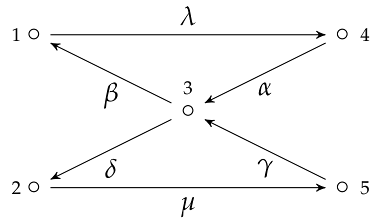
bound by all possible relations of the forms $\alpha\beta = 0$, $\beta\gamma = 0$, $\gamma\alpha = 0$.

EXAMPLES. Clearly, the algebra of example (2.1)(a) is gentle of type $\mathbb{A}_3$. We give two more examples.

(a) Let C be the tilted algebra of type $\mathbb{A}_5$ given by the quiver

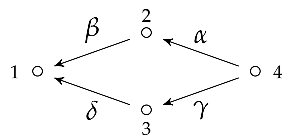


bound by $\alpha\beta = 0$, $\gamma\delta = 0$. Its relation extension $\widetilde{C}$ is given by the quiver



bound by $\alpha\beta = 0$, $\beta\lambda = 0$, $\lambda\alpha = 0$, $\gamma\delta = 0$, $\delta\mu = 0$ and $\mu\gamma = 0$.

(b) The cluster tilted algebra of example (2.1)(d) is gentle and is the relation extension of the tilted algebra C of type $\widetilde{\mathbb{A}}_5$ given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$, which is also gentle. Note that, while C is representation-finite, $\widetilde{C}$ is representation-infinite.

3
The module category of a cluster tilted algebra

3.1 Recovering the module category from the cluster category

Let T be a tilting object in a cluster category $\mathscr{C}$ and $B = \operatorname{End}_{\mathscr{C}} T$ be the corresponding cluster tilted algebra. Then there is an obvious functor

$$\operatorname{Hom}_{\mathscr{C}}(T,-) : \mathscr{C} \longrightarrow \operatorname{mod} B$$

which projectivises T, that is, which induces an equivalence between add T and the full subcategory of mod B consisting of the projective B-modules, see [21] p. 32. We claim that $\operatorname{Hom}_{\mathscr{C}}(T,-)$ is full and dense.

Indeed, let M be a B-module, and take a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

in mod B. Because P_0, P_1 are projective, there exist T_0, T_1 in add T and a morphism $g : T_1 \longrightarrow T_0$ such that $\operatorname{Hom}_{\mathscr{C}}(T,T_i) \cong P_i$ for $i = 0,1$ and $\operatorname{Hom}_{\mathscr{C}}(T,g) = f$. Then there exists a triangle

$$T_1 \xrightarrow{g} T_0 \longrightarrow X \longrightarrow T_1[1]$$

in $\mathscr{C}$. Applying $\operatorname{Hom}_{\mathscr{C}}(T,-)$ yields an exact sequence

$$\operatorname{Hom}_{\mathscr{C}}(T,T_1) \xrightarrow{\operatorname{Hom}_{\mathscr{C}}(T,g)} \operatorname{Hom}_{\mathscr{C}}(T,T_0) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(T,X) \longrightarrow 0$$

because $\operatorname{Hom}_{\mathscr{C}}(T,T_1[1]) = \operatorname{Ext}^1_{\mathscr{C}}(T,T_1) = 0$. Therefore $M \cong \operatorname{Hom}_{\mathscr{C}}(T,X)$ and our functor is dense. One proves its fullness in exactly the same way.

On the other hand, it is certainly not faithful, because

$$\operatorname{Hom}_{\mathscr{C}}(T,\tau T) = \operatorname{Ext}^1_{\mathscr{C}}(T,T) = 0$$

and hence the image of any object in add τT is zero.

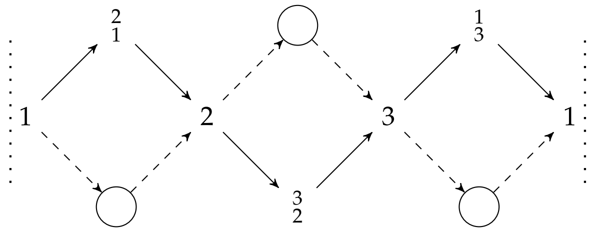
Let $\mathscr{C}/\langle \operatorname{add} \tau T\rangle$ denote the quotient of $\mathscr{C}$ by the ideal $\langle \operatorname{add} \tau T\rangle$ consisting of all the morphisms which factor through an object in add τT. The objects in this quotient category are the same as those of $\mathscr{C}$ and the set of morphisms from X to Y, say, equals $\operatorname{Hom}_{\mathscr{C}}(X,Y)$ modulo the subspace consisting of those lying in $\langle \operatorname{add} \tau T\rangle$. In $\mathscr{C}/\langle \operatorname{add} \tau T\rangle$, the objects of add τT are isomorphic to zero.

Because $\operatorname{Hom}_{\mathscr{C}}(T,-) : \mathscr{C} \longrightarrow \operatorname{mod} B$ is a full and dense functor which vanishes on $\langle \operatorname{add} \tau T\rangle$, it induces a full and dense functor from $\mathscr{C}/\langle \operatorname{add} \tau T\rangle$ to mod B. It turns out that this induced functor is also faithful.

Theorem 1 . *[40](2.2) The functor* $\operatorname{Hom}_{\mathscr{C}}(T,-) : \mathscr{C} \longrightarrow \operatorname{mod} B$ *induces an equivalence between* $\mathscr{C}/\langle \operatorname{add} \tau T\rangle$ *and* mod B. □

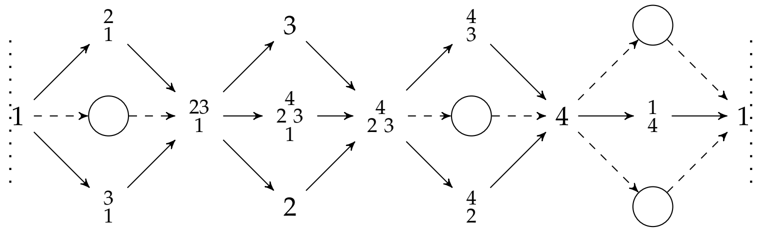
An immediate consequence is the shape of the Auslander-Reiten quiver of B. Indeed, starting from $\Gamma(\mathcal{C})$, the theorem says that one gets $\Gamma(\text{mod}\, B)$ by setting equal to zero all the indecomposable summands of τT, thus by deleting the corresponding points from $\Gamma(\mathcal{C})$. In particular, $\Gamma(\text{mod}\, B)$ has the same type of components as $\Gamma(\mathcal{C})$, that is, transjective and regular, from which are deleted each time finitely many points.

EXAMPLES.(a) If B is as in example (2.1)(a), then $\Gamma(\text{mod}\, B)$ is



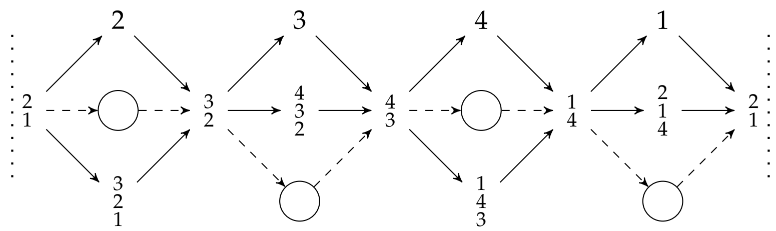
where we identify along the vertical dotted lines. If we add the indecomposable summands of τT, denoted by ◯, we get exactly $\Gamma(\mathcal{C})$.

(b) Let B be as in example (2.1)(b), then $\Gamma(\text{mod}\, B)$ is



where we identify along the vertical dotted lines. Adding the points denoted by ◯, we get again $\Gamma(\mathcal{C})$.

(c) Let B be as in example (2.1)(c), then $\Gamma(\text{mod}\, B)$ is



where we identify along the vertical dotted lines.

Corollary 2 . *[40](2.4) A cluster tilted algebra B of type Q is representation-finite if and only if Q is a Dynkin quiver. In this case, the numbers of isoclasses of indecomposable B-modules and $\mathsf{k}Q$-modules are equal.*

Proof. The first statement follows easily from Theorem 1. Let $n = |Q_0|$ and m be the number of isoclasses of indecomposable $\mathsf{k}Q$-modules. The cluster category $\mathcal{C}_Q$ has exactly $n + m$ isoclasses of indecomposable objects. To get the number of isoclasses of indecomposable B-modules, we subtract the number n of indecomposable summands of τT, getting $(n + m) - n = m$, as required. □

The examples also show that the Auslander-Reiten translation is preserved by the equivalence of Theorem 1.

Proposition 3 . *[40](3.2) The almost split sequences in* mod B *are induced from the almost split triangles of $\mathcal{C}$.* □

3.2 Global dimension

As an easy application of Theorem (3.1.1), we compute the global dimension of a cluster tilted algebra. Let B be cluster tilted of type Q and T a tilting object in $\mathscr{C} = \mathscr{C}_Q$ such that $B = \operatorname{End}_{\mathscr{C}} T$. For $x \in (Q_B)_0$ we denote by $\widetilde{P}_x$, $\widetilde{I}_x$ respectively the corresponding indecomposable projective and injective B-modules. It follows easily from [21] p.33 that $\widetilde{P}_x = \operatorname{Hom}_{\mathscr{C}}(T, T_x)$, where T_x is the summand of T corresponding to x. We now compute $\widetilde{I}_x$.

Lemma 1 . *With this notation,* $\widetilde{I}_x = \operatorname{Hom}_{\mathscr{C}}(T, \tau^2 T_x)$.

Proof. Because of [21] p.33, we have $\widetilde{I}_x = \operatorname{D}\operatorname{Hom}_{\mathscr{C}}(T_x, T)$. Setting $\mathscr{D} = \mathscr{D}^b(\operatorname{mod} \Bbbk Q)$ we have functorial isomorphisms

$$\begin{aligned}
\widetilde{I}_x &= \operatorname{D}\operatorname{Hom}_{\mathscr{D}}(T_x, T) \oplus \operatorname{D}\operatorname{Hom}_{\mathscr{D}}(T_x, \tau^{-1}T[1]) \\
&= \operatorname{Ext}^1_{\mathscr{D}}(T, \tau T_x) \oplus \operatorname{D}\operatorname{Ext}^1_{\mathscr{D}}(T_x, \tau^{-1}T) \\
&\cong \operatorname{Hom}_{\mathscr{D}}(T, \tau T_x[1]) \oplus \operatorname{Hom}_{\mathscr{D}}(T, \tau^2 T_x) \\
&\cong \operatorname{Hom}_{\mathscr{C}}(T, \tau^2 T_x).
\end{aligned}$$

□

Recall from [20] that an algebra B is ***Gorenstein*** if both $\operatorname{id} B_B < \infty$ and $\operatorname{pd}(\operatorname{D} B))_B < \infty$. Actually, if both dimensions are finite then they are equal. Letting $d = \operatorname{id} B_B = \operatorname{pd}(\operatorname{D} B)_B$, we then say that B is ***d-Gorenstein***.

Theorem 2 . *[61] Any cluster tilted algebra B is 1-Gorenstein. In particular* $\operatorname{gl.dim.} B \in \{1, \infty\}$.

Proof. Let, as above, $B = \operatorname{End}_{\mathscr{C}} T$, with T a tilting object in the cluster category $\mathscr{C}$. In order to prove that $\operatorname{pd}(\operatorname{D} B)_B < \infty$, we must show that, for any injective B-module $\widetilde{I}$, we have

$$\operatorname{Hom}_B(\operatorname{D} B, \tau_B \widetilde{I}) = 0.$$

Because of Lemma 1 above, we have $\widetilde{I} = \operatorname{Hom}_{\mathscr{C}}(T, \tau^2 T_0)$, for some T_0 in $\operatorname{add} T$. Because $\operatorname{Hom}_B(-, ?)$ is a quotient of $\operatorname{Hom}_{\mathscr{C}}(-, ?)$, it suffices to prove that $\operatorname{Hom}_{\mathscr{C}}(\tau^2 T, \tau^3 T_0) = 0$. But this follows from $\operatorname{Hom}_{\mathscr{C}}(\tau^2 T, \tau^3 T_0) \cong \operatorname{Hom}_{\mathscr{C}}(T, \tau T_0) \cong \operatorname{Ext}^1_{\mathscr{C}}(T, T_0) = 0$. Thus, $\operatorname{pd}(\operatorname{D} B)_B < \infty$. Similarly, $\operatorname{id} B_B < \infty$.

We now prove that, for any B-module M, the finiteness of $\operatorname{id} M$ implies $\operatorname{pd} M \leqslant 1$. Indeed, if $\operatorname{id} M = m < \infty$, then we have a minimal injective coresolution

$$\begin{array}{ccccccccccccccccc}
0 \to M \to I^0 & & \longrightarrow & & I^1 & & \longrightarrow & & I^2 \to \cdots & \longrightarrow & & I^{m-1} \to I^m \to 0 \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
& & L^0 & & & & L^1 & & & & L^{m-2} & \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
0 & & & & 0 & & & & 0 \quad 0 & & & & 0
\end{array}$$

The short exact sequence $0 \longrightarrow L^{m-2} \longrightarrow I^{m-1} \longrightarrow I^m \longrightarrow 0$ and the argument above yield $\operatorname{pd} L^{m-2} \leqslant 1$. An easy induction gives $\operatorname{pd} M \leqslant 1$.

Thus, if $\operatorname{gl.dim.} B > 1$, then there exists a module M such that $\operatorname{pd} M > 1$. But then $\operatorname{id} M = \infty$ and so $\operatorname{gl.dim.} B = \infty$. This proves the second statement. □

3.3 *The cluster repetitive algebra*

Let C be an algebra and E a C-C-bimodule. We construct a Galois covering of the trivial extension $C \ltimes E$. Consider the following locally finite dimensional algebra without identity

$$\check{C} = \begin{bmatrix} \ddots & & & & \\ \ddots & C_{-1} & & \mathbf{0} & \\ & E_0 & C_0 & & \\ \mathbf{0} & & E_1 & C_1 & \\ & & & \ddots & \ddots \end{bmatrix}$$

where matrices have only finitely many nonzero coefficients, $C_i = C$ are all terms on the main diagonal, $E_i = E$ are all terms below it, for all $i \in \mathbb{Z}$, and all the remaining coefficients are zero. Addition is the usual matrix addition, while multiplication is induced from the bimodule structure of E and the map $E \otimes_C E \longrightarrow 0$. The identity maps $C_i \longrightarrow C_{i-1}$, $E_i \longrightarrow E_{i-1}$, induce an automorphism φ of $\check{C}$. The orbit category $\check{C}/\langle\varphi\rangle$ inherits from $\check{C}$ an algebra structure, which is easily seen to be isomorphic to that of $C \ltimes E$. The projection functor $G : \check{C} \longrightarrow C \ltimes E$ induces a Galois covering with the infinite cyclic group generated by φ, see [50]. In view of Theorem (2.3.3), we are mostly interested in the case where C is tilted and $E = \operatorname{Ext}^2_C(\mathrm{D}\,C, C)$ so that $C \ltimes E = \widetilde{C}$. In this case, $\check{C}$ is called the ***cluster repetitive algebra***. Its quiver follows easily from Theorem (2.3.2)

Lemma 1 . *[6](1.3) Let $C = \mathrm{k}Q_C/I$ be a tilted algebra and R a system of relations for I. The quiver of $\check{C}$ is constructed as follows.*

(a) $\left(Q_{\check{C}}\right)_0 = (Q_C)_0 \times \mathbb{Z} = \{\, (x,i) \mid x \in (Q_C)_0, i \in \mathbb{Z}\,\}$

(b) for $(x,i), (y,j) \in \left(Q_{\check{C}}\right)_0$, the set of arrows in $\left(Q_{\check{C}}\right)_0$ from (x,i) to (y,j) equals:

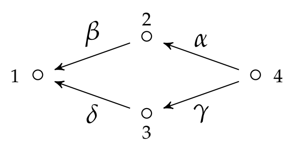
i) the set of arrows from x to y if $i = j$, or

ii) $\left|R \cap e_y I e_x\right|$ new arrows if $i = j+1$

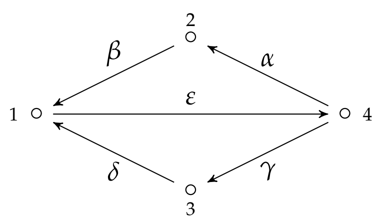
and is empty otherwise. □

Because the relations are just lifted from those of $\widetilde{C}$, this allows to compute without difficulty the bound quiver of $\check{C}$.

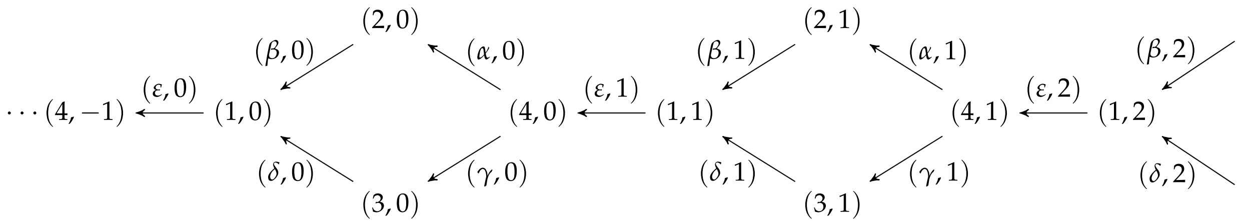
EXAMPLES.(a) Let B be the cluster tilted algebra of example (2.1)(b), that is, let C be given by the quiver



bound by $\alpha\beta = \gamma\delta$. Then $B = \widetilde{C}$ is given by the quiver

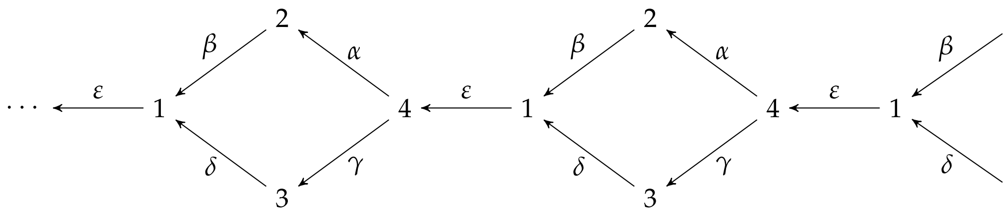


bound by $\alpha\beta = \gamma\delta$, $\beta\varepsilon = 0$, $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$, $\delta\varepsilon = 0$. The quiver of the cluster repetitive algebra $\check{C}$ is the infinite quiver



bound by all the lifted relations: $(\alpha,i)(\beta,i) = (\gamma,i)(\delta,i)$, $(\beta,i)(\varepsilon,i) = 0$, $(\delta,i)(\varepsilon,i) = 0$, $(\varepsilon,i+1)(\alpha,i) = 0$, $(\varepsilon,i+1)(\gamma,i) = 0$ for all $i \in \mathbb{Z}$.

In practice, one drops the index $i \in \mathbb{Z}$ so that the quiver of $\check{C}$ looks as follows

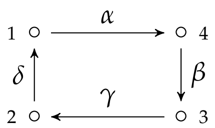


and the relations read exactly as those of $\widetilde{C}$.

(b) Let B be the cluster tilted algebra of example (2.1)(c), that is, let C be given by the quiver

$$\overset{1}{\circ} \xleftarrow{\ \delta\ } \overset{2}{\circ} \xleftarrow{\ \gamma\ } \overset{3}{\circ} \xleftarrow{\ \beta\ } \overset{4}{\circ}$$

bound by $\beta\gamma\delta = 0$. Then $B = \widetilde{C}$ is given by the quiver



bound by $\alpha\beta\gamma = 0$, $\beta\gamma\delta = 0$, $\gamma\delta\alpha = 0$, $\delta\alpha\beta = 0$. Then $\check{C}$ is given by the quiver

$$\cdots \xleftarrow{\delta} \circ \xleftarrow{\gamma} \circ \xleftarrow{\beta} \circ \xleftarrow{\alpha} \circ \xleftarrow{\delta} \circ \xleftarrow{\gamma} \circ \xleftarrow{\beta} \circ \xleftarrow{\alpha} \circ \xleftarrow{\delta} \cdots$$

bound by all possible relations of the forms $\alpha\beta\gamma = 0$, $\beta\gamma\delta = 0$, $\gamma\delta\alpha = 0$, $\delta\alpha\beta = 0$.

Assume that C is a tilted algebra of type Q and that T is a tilting object in $\mathscr{C}_Q$ such that $\widetilde{C} = \mathrm{End}_{\mathscr{C}_Q} T$. Because of Proposition (1.3.2), we may assume without loss of generality that T is a tilting $\mathsf{k}Q$-module so that $C = \mathrm{End}\, T_{\mathsf{k}Q}$.

Theorem 2 . *[6](1.2)(2.1) Let T be a tilting $\mathsf{k}Q$-module and $C = \mathrm{End}\, T_{\mathsf{k}Q}$. Then we have*

(a) $\check{C} = \mathrm{End}_{\mathscr{D}^b(\mathrm{mod}\,\mathsf{k}Q)} \left(\bigoplus_{i\in\mathbb{Z}} F^i T\right)$

(b) $\mathrm{Hom}_{\mathscr{D}^b(\mathrm{mod}\,\mathsf{k}Q)} \left(\bigoplus_{i\in\mathbb{Z}} F^i T, -\right) : \mathscr{D}^b(\mathrm{mod}\,\mathsf{k}Q) \longrightarrow \mathrm{mod}\,\check{C}$ *induces an equivalence* $\mathscr{D}^b(\mathrm{mod}\,\mathsf{k}Q)/\langle \mathrm{add}\left(\bigoplus_{i\in\mathbb{Z}} \tau F^i(T)\right)\rangle$ $\mathrm{mod}\,\check{C}$.

Proof. (a) Set $\mathscr{D} = \mathscr{D}^b(\mathrm{mod}\,\mathsf{k}Q)$. As k-vector spaces, we have

$$\mathrm{End}_{\mathscr{D}}\left(\bigoplus_{i\in\mathbb{Z}} F^i T\right) \cong \bigoplus_{i,j} \mathrm{Hom}_{\mathscr{D}}\left(F^i T, F^j T\right).$$

Because T is a $\mathsf{k}Q$-module, all the summands on the right hand side vanish except when $j \in \{\, i, i+1 \,\}$. If $j = i$, then the corresponding summand is $\mathrm{Hom}_{\mathscr{D}}(T,T) = \mathrm{Hom}_{\mathsf{k}Q}(T,T) \cong C$, while, if $j = i+1$, it is $\mathrm{Hom}_{\mathscr{D}}(T, FT) \cong \mathrm{Ext}^2_C(\mathrm{D}C, C)$ as seen in the proof of Theorem (2.3.3).

□

Associated to the Galois covering $G : \check{C} \longrightarrow \widetilde{C}$, there is a pushdown functor $G_\lambda : \mathrm{mod}\,\check{C} \longrightarrow \mathrm{mod}\,\widetilde{C}$ defined on the objects by

$$G_\lambda \check{M}(a) = \bigoplus_{x \in G^{-1}(a)} \check{M}(x)$$

where $\check{M}$ is a $\check{C}$-module and $a \in \left(Q_{\widetilde{C}}\right)_0$, see [50]. We now state the main result of this subsection

Theorem 3 . *[6](2.4) There is a commutative diagram of dense functors*

$$\begin{array}{ccc} \mathcal{D}^b(\mathrm{mod}\,kQ) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,kQ)}\left(\bigoplus_{i\in\mathbb{Z}} F^iT,-\right)} & \mathrm{mod}\,\check{C} \\ \big\downarrow \pi & & \big\downarrow G_\lambda \\ \mathcal{C}_Q & \xrightarrow{\mathrm{Hom}_{\mathcal{C}_Q}(\pi T,-)} & \mathrm{mod}\,\widetilde{C}\ . \end{array} \quad \square$$

As an immediate consequence of the density of G_λ and [50](3.6), we have the following corollary.

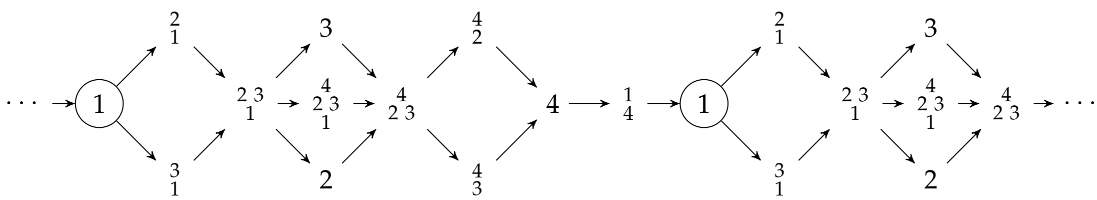
Corollary 4 . *[6](2.5)*

(a) The pushdown of an almost split sequence in $\mathrm{mod}\,\check{C}$ *is an almost split sequence in* $\mathrm{mod}\,\widetilde{C}$.

(b) The pushdown functor G_λ *induces an isomorphism between the orbit quiver* $\Gamma(\mathrm{mod}\,\check{C})/\mathbb{Z}$ *of* $\Gamma(\mathrm{mod}\,\check{C})$ *under the action of* $\mathbb{Z} \cong \langle\varphi\rangle$, *and* $\Gamma(\mathrm{mod}\,\widetilde{C})$. □

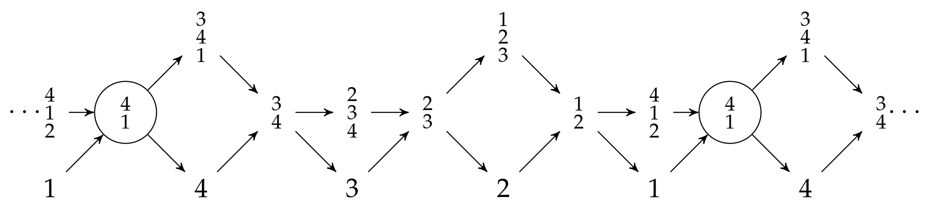
Thus, in order to construct $\Gamma(\mathrm{mod}\,\widetilde{C})$, it suffices to compute $\Gamma(\mathrm{mod}\,\check{C})$ and then do the identifications required by passing to the orbit quiver.

EXAMPLES. (c) Let C, $\widetilde{C}$, $\check{C}$ be as in example (a) above, then $\Gamma(\mathrm{mod}\,\check{C})$ is



In order to get $\Gamma(\mathrm{mod}\,\widetilde{C})$, it suffices to identify the two encircled copies of 1.

(d) Let C, $\widetilde{C}$, $\check{C}$ be as in example (b) above, then $\Gamma(\mathrm{mod}\,\check{C})$ is



In order to get $\Gamma(\mathrm{mod}\,\widetilde{C})$, it suffices to identify the two encircled copies of $\begin{smallmatrix}4\\1\end{smallmatrix}$.

3.4 Cluster tilted algebras and slices

We recall that the map $C \mapsto \widetilde{C}$ from tilted algebras to cluster tilted is surjective, but generally not injective. We then ask, given a cluster tilted algebra B, how to find all the tilted algebras C such that $B = \widetilde{C}$. We answer this question by means of slices. Indeed, tilted algebras are characterised by the existence of complete slices, see, for instance [18] p.320. The corresponding notion in our situation is the following.

Definition. Let B be an algebra. A ***local slice*** in $\Gamma(\mathrm{mod}\,B)$ is a full connected subquiver Σ of a component Γ of $\Gamma(\mathrm{mod}\,B)$ such that:

(a) Σ is a presection, that is, if $X \longrightarrow Y$ is an arrow in Γ, then

(i) $X \in \Sigma_0$ implies either $Y \in \Sigma_0$ or $\tau Y \in \Sigma_0$,

(ii) $Y \in \Sigma_0$ implies either $X \in \Sigma_0$ or $\tau^{-1} X \in \Sigma_0$.

(b) Σ is sectionally convex, that is, if $X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t = Y$ is a sectional path of irreducible morphisms between indecomposable modules, then $X, Y \in \Sigma_0$ implies that $X_i \in \Sigma_0$ for all i.

(c) $|\Sigma_0| = \operatorname{rk} K_0(C)$ (that is, equals the number of isoclasses of simple C-modules).

For instance, if C is a tilted algebra, then it is easily seen that any complete slice in $\Gamma(\operatorname{mod} C)$ is a local slice. For cluster tilted algebras, in examples (3.1)(a), (b) and (c), the sets $\left\{ \begin{smallmatrix}2\\1\end{smallmatrix}, 2, \begin{smallmatrix}3\\2\end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}2&3\\1&\end{smallmatrix}, \begin{smallmatrix}&4&\\2&&3\\&1&\end{smallmatrix}, 2 \right\}$ and $\left\{ 2, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\end{smallmatrix} \right\}$ are local slices respectively. We shall now see that cluster tilted algebras always have (a lot of) local slices. Assume that C is a tilted algebra, and Σ a complete slice in $\Gamma(\operatorname{mod} C)$. Then there exist a hereditary algebra A and a tilting module T_A such that $C = \operatorname{End} T_A$ and $\Sigma = \operatorname{add} \operatorname{Hom}_A(T, \mathrm{D}\, A)$, see [18] p.320. On the other hand $\widetilde{C} = C \ltimes \operatorname{Ext}^2_C(\mathrm{D}\, C, C)$ is cluster tilted, and the surjective algebra morphism $p : \widetilde{C} \longrightarrow C$ of (2.3) induces an embedding $\operatorname{mod} C \hookrightarrow \operatorname{mod} \widetilde{C}$.

Proposition 1 . *[5] With the above notation, Σ embeds in $\Gamma(\operatorname{mod} \widetilde{C})$ as a local slice in the transjective component. Moreover, every local slice in $\Gamma(\operatorname{mod} \widetilde{C})$ occurs in this way.* □

The above embedding turns out to preserve the Auslander-Reiten translates.

Lemma 2 . *[5] With the above notation, if $M \in \Sigma_0$ then*

(a) $\tau_C M \cong \tau_{\widetilde{C}} M$ and

(b) $\tau_C^{-1} M \cong \tau_{\widetilde{C}}^{-1} M$. □

Consider Σ as embedded in $\Gamma(\operatorname{mod} \widetilde{C})$. Its ***annihilator*** $\operatorname{Ann}_{\widetilde{C}} \Sigma$, namely, the intersection of all the annihilators $\bigcap_{M \in \Sigma_0} \operatorname{Ann}_{\widetilde{C}} M$ of the modules $M \in \Sigma_0$, is equal to $\operatorname{Ext}^2_C(\mathrm{D}\, C, C)$. This is the main step in the proof of the main theorem of this subsection, which answers the question asked at its beginning.

Theorem 3 . *[5] Let B be a cluster tilted algebra. Then there exists a tilted algebra C such that $B = \widetilde{C}$ if and only if there exists a local slice Σ in $\Gamma(\operatorname{mod} B)$ such that $C = B / \operatorname{Ann}_B \Sigma$.* □

Cluster tilted algebras have usually a lot of local slices.

Proposition 4 . *[5] Let B be cluster tilted of tree type and M be an indecomposable B-module lying in its transjective component.Then there exists a local slice Σ such that $M \in \Sigma_0$.* □

In particular, if B is representation-finite, then any indecomposable B-module lies on some local slice.

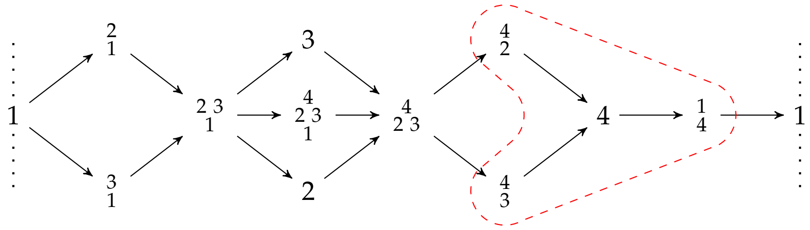
The following remark, which is an immediate consequence of [13](1.3), is particularly useful in calculations.

Proposition 5 . *[5] Let B be a cluster tilted algebra, and Σ be a local slice in $\Gamma(\operatorname{mod} B)$. Then $\operatorname{Ann}_B \Sigma$ is generated, as a two-sided ideal, by arrows in the quiver of B.* □

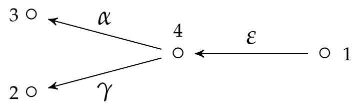
For another approach to find all tilted algebras whose relation extension is a given cluster tilted algebra, we refer the reader to [31].

EXAMPLES.(a) Let B be the cluster tilted algebra of example (2.1)(b). We illustrate a local slice Σ

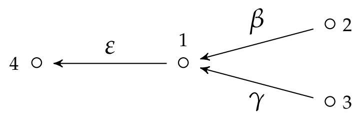
in $\Gamma(\text{mod } B)$ by a dotted line.



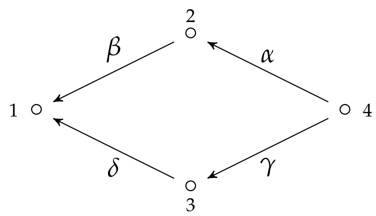
where we identify the two copies of 1. Here, $\text{Ann}_B \Sigma = \langle \beta, \gamma \rangle$ so that C is the quiver containing the remaining arrows



bound by $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$. There are only two other algebras which arise in this way from local slices. Namely, the algebra C_1 given by the quiver

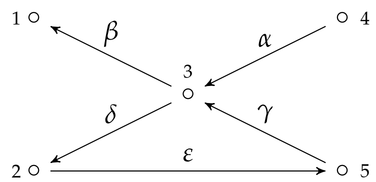


bound by $\beta\varepsilon = 0$, $\gamma\varepsilon = 0$, and C_2 given by the quiver

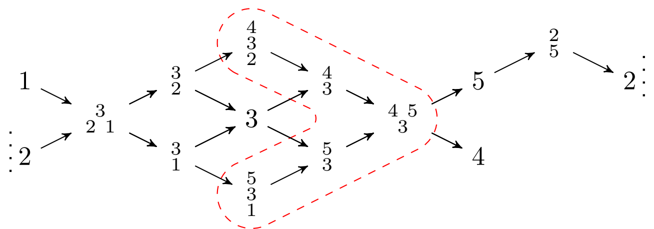


bound by $\alpha\beta = \gamma\delta$. Thus, we have $\widetilde{C} = \widetilde{C_1} = \widetilde{C_2}$.

(b) In contrast to tilted algebras, local slices do not characterise cluster tilted algebras. We give an example of an algebra which is not cluster tilted but has a local slice. Let A be given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\delta\varepsilon = 0$, $\varepsilon\gamma = 0$. We show a local slice in $\Gamma(\text{mod } A)$



where we identify the two copies of 2.

In view of example (b), we may formulate the following problem.

Problem. Identify the class of algebras having local slices.

A partial solution is presented in [8].

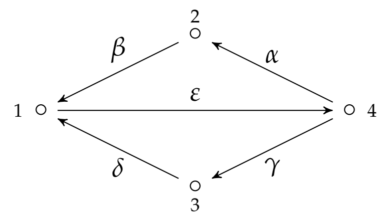
3.5 *Smaller and larger cluster tilted algebras*

It is known that any full subcategory of a tilted algebra is itself tilted. That is, if C is tilted and $e \in C$ is an idempotent, then eCe is tilted, see [53] p.146. This is not the case for cluster tilted algebras. On the other hand, factoring out the two-sided ideal generated by an idempotent, we obtain a smaller cluster tilted algebra.

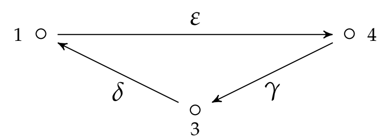
Theorem 1 . *[41] Let B be a cluster tilted algebra, and $e \in B$ an idempotent. Then B/BeB is cluster tilted.* □

If B is given as a bound quiver algebra and e is the sum of primitive idempotents corresponding to points in the quiver, then the bound quiver of B/BeB is obtained from that of B by deleting the points appearing in e, and all arrows incident to these points, with the inherited relations.

EXAMPLES.(a) If B is as in example (2.1)(b), thus given by the quiver

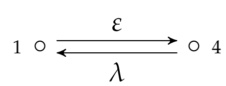


bound by $\alpha\beta = \gamma\delta$, $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$, $\delta\varepsilon = 0$, $\beta\varepsilon = 0$, and e_2 is the primitive idempotent corresponding to the point 2, then B/Be_2B is given by the quiver



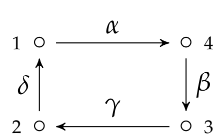
bound by $\varepsilon\gamma = 0$, $\delta\varepsilon = 0$ and also $\gamma\delta = 0$ (because in B, we have $\gamma\delta = \alpha\beta$ and both α, β are set equal to zero when passing to the quotient B/Be_2B). This is the algebra in example (2.1)(a).

We also give an example of a full subcategory of a cluster tilted algebra which is not cluster tilted. In the previous example, let $e = e_1 + e_4$, then eBe is given by the quiver

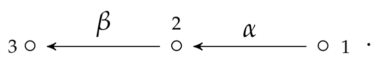


bound by $\varepsilon\lambda = 0$, $\lambda\varepsilon = 0$. This is not a cluster tilted algebra because its quiver contains a 2-cycle, see Corollary (2.2.2).

(b) If B is as in example (2.1)(c), given by the quiver



bound by $\alpha\beta\gamma = 0$, $\beta\gamma\delta = 0$, $\gamma\delta\alpha = 0$, $\delta\alpha\beta = 0$, then B/Be_4B is hereditary with quiver



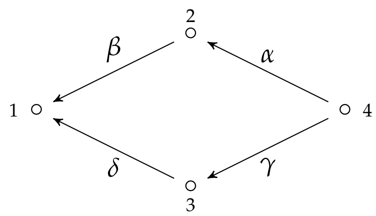
We may ask whether the above procedure can be reversed, that is, given a cluster tilted algebra B, whether there exists a (larger) cluster tilted algebra B' and an idempotent $e' \in B'$ such that $B \cong B'/B'e'B'$.

Let B be cluster tilted, and Σ be a local slice in $\Gamma(\text{mod}\, B)$. Let $C = B/\operatorname{Ann}_B \Sigma$. Then Σ embeds in $\Gamma(\text{mod}\, C)$ as a complete slice, because of Proposition (3.4.1). Let M be a, not necessarily indecomposable, B-module all of whose indecomposable summands lie on Σ. In particular, M is a C-module. It is then known, and easy to prove, that the one-point extension $C' = C[M]$ is tilted. Let $B' = C' \ltimes \operatorname{Ext}^2_{C'}(\mathrm{D}\, C', C')$ be the relation extension of C'. We have the following theorem.

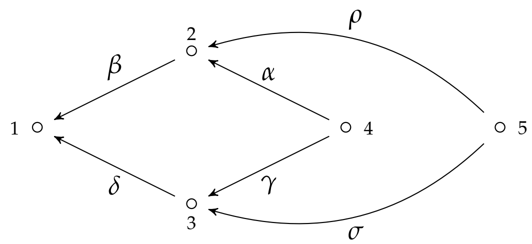
Theorem 2 . *[62] With the above notation, B' is cluster tilted and, if e' is the primitive idempotent corresponding to the extension point, then*

$$B'/B'e'B' \cong B.$$

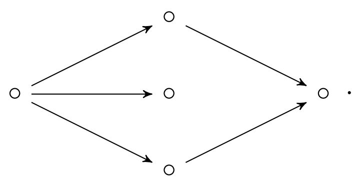
EXAMPLES.(c) Let B be as in example (a), with C given by the quiver



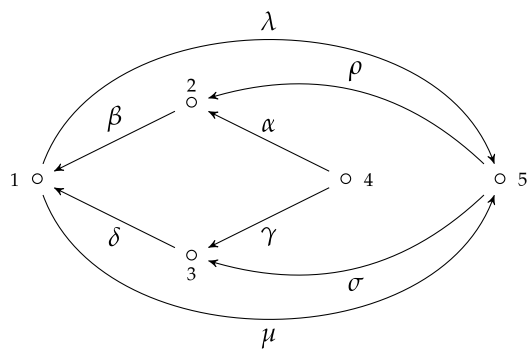
bound by $\alpha\beta = \gamma\delta$. Let $M = 2 \oplus 3$. Then both summands of M lie on a complete slice and $C' = C[M]$ is the tilted algebra given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\rho\beta = 0$, $\sigma\delta = 0$. It is of wild type



The relation extension B' of C' is given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$, $\beta\varepsilon = 0$, $\delta\varepsilon = 0$, $\lambda\rho = 0$, $\rho\beta = 0$, $\beta\lambda = 0$, $\mu\sigma = 0$, $\sigma\delta = 0$, $\delta\mu = 0$.

According to Theorem 2, B' is cluster tilted and moreover $B'/B'e_5B' = B$.

4
Particular modules over cluster tilted algebras

4.1 The left part of a cluster tilted algebra

Because tilting theory lies at the heart of the study of cluster tilted algebras, it is natural to ask what are the tilting modules over these algebras. We first see that they correspond to tilting objects in the cluster category

Theorem 1 . *[68] Let Q be a finite acyclic quiver, $\mathscr{C}_Q$ be the corresponding cluster category, T be a tilting object in $\mathscr{C}_Q$ and $B = \operatorname{End}_{\mathscr{C}_Q} T$. Then:*

(a) any partial tilting B-module lifts to a rigid object in $\mathscr{C}_Q$;

(b) any tilting B-module lifts to a tilting object in $\mathscr{C}_Q$. □

An immediate consequence of this theorem and Theorem (3.1.1) is that, if U is a tilting B-module with lift $\overline{U}$, then $\operatorname{End}_B U$ is a quotient of $\operatorname{End}_{\mathscr{C}_Q} \overline{U}$.

Recall that, for an algebra A, the ***left part*** $\mathscr{L}_A$ of $\operatorname{mod} A$ is the full subcategory of $\operatorname{ind} A$ consisting of all the M such that, for any L in $\operatorname{ind} A$ such that there exists a path of nonzero morphisms between indecomposables $L = L_0 \longrightarrow L_1 \longrightarrow \cdots \longrightarrow L_t = M$ we have $\operatorname{pd} L \leqslant 1$. The ***right part*** $\mathscr{R}_A$ is defined dually, see [56]. We want to study the left and right parts of a cluster tilted algebra. We need one lemma.

Lemma 2 . *[68](5.1) Let B be a nonhereditary cluster tilted algebra. Then any connected component of $\Gamma(\operatorname{mod} B)$ either contains no projectives and no injectives, or it contains both projectives and injectives.*

Proof. Let P be an indecomposable projective lying in a component Γ of $\Gamma(\operatorname{mod} B)$. Let Σ be the maximal full, connected convex subquiver of Γ containing only indecomposable projectives, including P. Because B is not hereditary, the number of points of Σ is strictly less than the number of τ-orbits in Γ. Therefore there exist $P' \in \Sigma_0$ and $M \notin \Sigma_0$ such that there is an irreducible morphism $M \longrightarrow P'$: indeed, if this is not the case, then there is an irreducible morphism $P' \longrightarrow N$ with $N \notin \Sigma_0$ and N projective, a contradiction. Let T' be the indecomposable summand of the tilting object T in $\mathscr{C}_Q$ corresponding to P'. Because M is nonprojective, there is in $\mathscr{C}_Q$ an arrow $\tau^2 T' \longrightarrow \overline{M}$, where $\overline{M}$ denotes the lift of M. This corresponds in Γ to an irreducible morphism from an indecomposable injective B-module to τM. Hence Γ contains at least one injective. Dually, if Γ contains an injective, then it also contains a projective. □

Proposition 3 . *[68](5.2) Let B be a nonhereditary cluster tilted algebra.Then $\mathscr{L}_B$ and $\mathscr{R}_B$ are finite.*

Proof. Assume $\mathscr{L}_B \neq \varnothing$. Because $\mathscr{L}_B$ is closed under predecessors in $\operatorname{ind} B$, it contains at least one indecomposable projective B-module P. Because of [47](1.1) and Lemma 2 above, there exists $m \geqslant 0$ such that $\tau^{-m} P$ is a successor of an injective module. We may assume m to be minimal for this property. Because of [11](1.6), we have $\tau^{-m}P \notin \mathcal{L}_B$ and so $\tau^{-m}P$ is Ext-injective in $\mathcal{L}_B$. Because this holds for any indecomposable projective in $\mathcal{L}_B$, it follows from [12](5.4) that $\mathcal{L}_B$ is finite. Dually, $\mathcal{R}_B$ is finite. □

As easy consequences, any cluster tilted algebra is left and right supported in the sense of [12], and it is laura [11] if and only if it is hereditary or representation-finite.

Given an algebra A, its ***left support*** A_λ is the endomorphism algebra of the direct sum of all indecomposable projective A-modules lying in $\mathcal{L}_A$. The dual notion is the ***right support*** algebra A_ρ. It is shown in [12](2.3) that A_λ, A_ρ are always products of quasi-tilted algebras. We show that it is, for cluster tilted algebras, a product of hereditary algebras.

Proposition 4 . *[68](5.4) Let B be 1-Gorenstein, then B_λ, B_ρ are direct products of hereditary algebras.*

Proof. Because $\mathcal{L}_B \subseteq \operatorname{ind} B_\lambda$, see [12], it suffices to prove that, if P is a projective indecomposable B-module lying in $\mathcal{L}_B$ and $M \longrightarrow P$ is an irreducible morphism with M indecomposable, then M is projective. Assume not, then $\tau M \neq 0$ and $\mathrm{Hom}_B(\tau^{-1}(\tau M), P) \neq 0$ implies $\mathrm{id}\,(\tau M)_B > 1$, because of [18] p.115. Because B is 1-Gorenstein, we infer that $\mathrm{pd}\,(\tau M) > 1$, contradicting the fact that $\tau M \in \mathcal{L}_B$. Therefore M is projective. This shows that B_λ is a direct product of hereditary algebras. Dually, B_ρ is also a direct product of hereditary algebras. □

Actually, one can show, see [68](5.5), that $\mathcal{L}_B$ contains no indecomposable injective B-module. Therefore $\mathcal{L}_B$ can be characterised as the set of those indecomposable modules which are not successors of an injective (by a sequence of nonzero morphisms between indecomposable modules). Dually, $\mathcal{R}_B$ consists of those indecomposable which are not predecessors of a projective.

We also refer the reader to [29] for modules of projective dimension one over cluster tilted algebras.

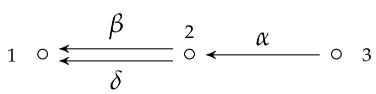
4.2 Modules determined by their composition factors

It is a standard question in representation theory to identify those indecomposable modules over a given algebra which are uniquely determined by their composition factors or, equivalently, by their dimension vectors. We have seen in (3.1) that the Auslander-Reiten quiver of a cluster tilted algebra contains a unique transjective component, and this is the only component containing local slices, see (3.4). If the cluster tilted algebra is of Dynkin type, then the transjective component is the whole Auslander-Reiten quiver. We have the following theorem.

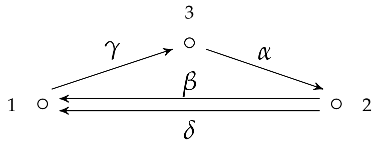
Theorem 1 . *[14] Let B be a cluster tilted algebra and M, N be indecomposable B-modules lying in the transjective component.Then $M \cong N$ if and only if M and N have the same composition factors.* □

As a consequence, over a representation-finite cluster tilted algebra, all indecomposables are uniquely determined by their composition factors.

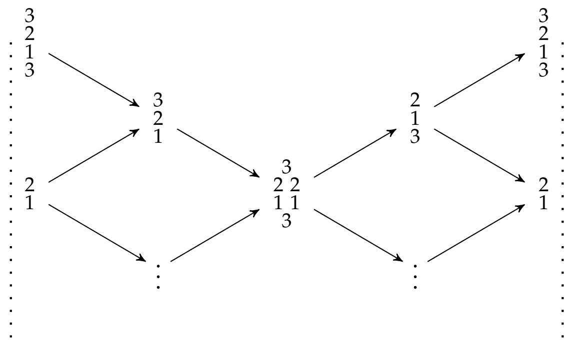
EXAMPLE. The statement of the theorem does not hold true if M, N are not transjective. Let indeed C be the tilted algebra of type $\widetilde{\mathbb{A}}_2$ given by the quiver



bound by $\alpha\beta = 0$. Its relation extension $B = \widetilde{C}$ is given by the quiver



bound by $\alpha\beta = 0$, $\beta\gamma = 0$, $\gamma\alpha = 0$. Then $\Gamma(\text{mod}\, B)$ contains exactly one tube of rank 2, all others being of rank 1. This tube is of the form



where we identify along the vertical dotted lines. Clearly the modules $\text{rad}\, P_3 = \begin{smallmatrix} 2 \\ 1 \\ 3 \end{smallmatrix}$ and $P_3 / \text{soc}\, P_3 = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$ are nonisomorphic but have the same composition factors.

The situation is slightly better for cluster concealed algebras of euclidean type, see [64]: these are the relation extensions of concealed algebras, that is, of tilted algebras which are endomorphism algebras of a postprojective (or a preinjective) tilting module over a hereditary algebra.

Proposition 2 . *[14] Let B be a cluster concealed algebra of euclidean type, and M, N be two rigid indecomposable modules. Then $M \cong N$ if and only if M and N have the same composition factors.* □

If B is cluster concealed of wild type and M, N are not only rigid but also lift to rigid objects in the cluster category, then the statement holds true: $M \cong N$ if and only if M and N have the same composition factors.

4.3 Induced and coinduced modules

Another successful approach for studying modules over cluster tilted algebras is by considering them as induced or coinduced from modules over an underlying tilted algebra. A similar approach is used extensively in the representation theory of finite groups. Indeed, let C be tilted, $E = \text{Ext}^2_C(D\,C, C)$ and $B = C \ltimes E$ be its relation extension. There are two change of rings functors allowing to pass from mod C to mod B, these are:

i) the ***induction functor*** $- \otimes_C B_C : \text{mod}\, C \longrightarrow \text{mod}\, B$, and

ii) the ***coinduction functor*** $\text{Hom}_B({}_B B_C, -) : \text{mod}\, C \longrightarrow \text{mod}\, B$.

A B-module is said to be ***induced*** (or ***coinduced***) if it lies in the image of the induction functor (or the coinduction functor, respectively).

Lemma 1 . *[65](4.2) Let M be a C-module, then*

(a) $\text{id}\, M_C \leqslant 1$ *if and only if* $M \otimes_C B \cong M$,

(b) $\text{pd}\, M_C \leqslant 1$ *if and only if* $\text{Hom}_C(B, M) \cong M$.

Proof. We only prove (a), the proof of (b) is similar. Recall that, as left C-modules, we have ${}_C B \cong {}_C C \oplus {}_C E$. Therefore $M \otimes_C B \cong M \oplus (M \otimes_C E)$. Thus, $M \otimes_C B \cong M$ if and only if $M \otimes_C E = 0$. Now we have

$$E = \operatorname{Ext}^2_C(\mathrm{D}\,C, C) \cong \operatorname{Ext}^1_C(\Omega\,\mathrm{D}\,C, C) \cong \mathrm{D}\operatorname{Hom}_C(C, \tau\Omega\,\mathrm{D}\,C) \cong \mathrm{D}(\tau\Omega\,\mathrm{D}\,C)$$

where we used that $\operatorname{pd}(\Omega\,\mathrm{D}\,C) \leqslant 1$ because $\operatorname{gl.dim.} C \leqslant 2$. Therefore

$$\begin{aligned} M \otimes_C E &\cong M \otimes_C \mathrm{D}(\tau\Omega\,\mathrm{D}\,C) \cong \mathrm{D}\operatorname{Hom}_C(M, \tau\Omega\,\mathrm{D}\,C) \\ &\cong \operatorname{Ext}^1_C(\Omega\,\mathrm{D}\,C, M) \cong \operatorname{Ext}^2_C(\mathrm{D}\,C, M) \\ &\cong \operatorname{Ext}^1_C(\mathrm{D}\,C, \Omega^{-1}M) \cong \mathrm{D}\operatorname{Hom}_C(\tau^{-1}\Omega^{-1}M, \mathrm{D}\,C) \\ &\cong \tau^{-1}\Omega^{-1}M \end{aligned}$$

where we used that $\operatorname{pd}(\Omega\,\mathrm{D}\,C) \leqslant 1$ and also that $\operatorname{id}(\Omega^{-1}M) \leqslant 1$. Now, $\tau^{-1}\Omega^{-1}M$ vanishes if and only if $\Omega^{-1}M$ is injective, that is, if and only if $\operatorname{id} M_C \leqslant 1$. □

We recall some notation associated with the tilting theorem, see [18] p.205. Let $\mathrm{k}Q$ be the path algebra of a quiver Q, $T_{\mathrm{k}Q}$ be a tilting module and $C = \operatorname{End} T_{\mathrm{k}Q}$. Then every indecomposable C-module belongs to one of the classes

$$\mathcal{X}(T) = \{\, M \mid M \otimes_C T = 0 \,\}$$

and

$$\mathcal{Y}(T) = \left\{\, M \mid \operatorname{Tor}^C_1(M, T) = 0 \,\right\}.$$

Let $\mathcal{C}_Q$ denote the cluster category.

Lemma 2 . *[65](6.2)(6.4) Let M be an indecomposable C-module, then*

(a) $M \otimes_C B \cong \begin{cases} \operatorname{Hom}_{\mathcal{C}_Q}(T, M \otimes_C T) & \text{if } M \in \mathcal{Y}(T) \\ M & \text{if } M \in \mathcal{X}(T), \end{cases}$

(b) $\operatorname{Hom}_C(B, M) \cong \begin{cases} \operatorname{Ext}^1_{\mathcal{C}_Q}(T, \operatorname{Tor}^C_1(M, T)) & \text{if } M \in \mathcal{X}(T) \\ M & \text{if } M \in \mathcal{Y}(T). \end{cases}$

Proof. We only sketch the proof of (a). We know that M either lies in $\mathcal{X}(T)$ or in $\mathcal{Y}(T)$. If $M \in \mathcal{X}(T)$, then $\operatorname{id} M_C \leqslant 1$ (see [18](VIII.3)). Because of Lemma 1, we have $M \otimes_C B \cong M$. If, on the other hand, $M \in \mathcal{Y}(T)$, then, because of the tilting theorem, we have $M \cong \operatorname{Hom}_{\mathrm{k}Q}(T, M \otimes_C T)$. One can then prove that $M \otimes_C B \cong \operatorname{Hom}_{\mathrm{k}Q}(T, M \otimes_C T) \otimes_C B \cong \operatorname{Hom}_{\mathcal{C}_Q}(T, M \otimes_C T)$, see [65](6.1). □

We can now state the main result of this subsection.

Theorem 3 . *[65](7.2)(7.4)(7.5) Let B be a cluster tilted algebra.*

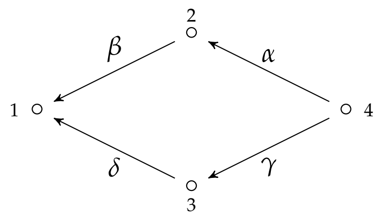
(a) If B is representation-finite and M is an indecomposable B-module, then there exists a tilted algebra C such that M is both induced and coinduced from a C-module.

(b) If B is arbitrary, and M is an indecomposable B -module lying in the transjective component, then there exists a tilted algebra C such that $B = \widetilde{C}$ and M is a C-module. In particular, M is induced or coinduced from a C-module.

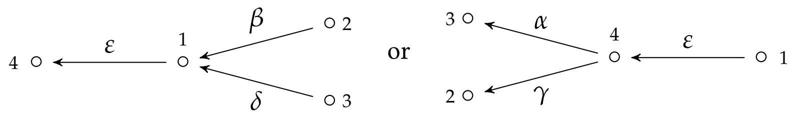
(c) If B is cluster concealed, and M is an indecomposable B-module, then there exists a tilted algebra C such that $B \cong \widetilde{C}$ and M is a C-module. In particular, M is induced or coinduced from a C-module.

Proof of (a). Because B is representation-finite, C is of tree type. Because of Proposition (3.4.4), there exists a local slice Σ in $\Gamma(\operatorname{mod} B)$ on which M lies. Let $C = B/\operatorname{Ann}_B \Sigma$. Then B is the relation extension of the tilted algebra C and M lies on the complete slice Σ in $\Gamma(\operatorname{mod} C)$. But then $\operatorname{id} M_C \leqslant 1$ and $\operatorname{pd} M_C \leqslant 1$. Because of Lemmata 1 and 2 above, we have both $M \cong M \otimes_C B$ and $M \cong \operatorname{Hom}_C(B, M)$. This completes the proof. □

It is important to observe that the tilted algebra C depends essentially on the choice of M.
EXAMPLE. Let B be given by the bound quiver of example (2.1)(b).Choosing $M = \begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix}$, we get that C is given by the quiver



bound by $\alpha\beta = \gamma\delta$. Then $M \cong M \otimes_C B \cong \operatorname{Hom}_C(B, M)$, so is induced and coinduced. On the other hand, if we choose $M' = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix}$, we get that C is given by one of the quivers



bound respectively by $\beta\varepsilon = 0$, $\delta\varepsilon = 0$ and $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$. This indeed depends on the chosen local slice containing M'. In each case, M' is again both induced and coinduced.

5

Hochschild cohomology of cluster tilted algebras

5.1 The Hochschild projection morphism

The Hochschild cohomology groups were introduced by G. Hochschild in 1945, see [46]. Let C be an algebra and ${}_C E_C$ a bimodule. Denoting by $C^{\otimes_k i}$ the i^{th} tensor power of C over k, we have a complex

$$0 \longrightarrow E \xrightarrow{b^1} \operatorname{Hom}_k(C, E) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_k(C^{\otimes_k i}, E) \xrightarrow{b^{i+1}} \operatorname{Hom}_k(C^{\otimes_k i+1}, E) \longrightarrow \cdots$$

where $b^1 : E \longrightarrow \operatorname{Hom}_k(C, E)$ is defined for $x \in E$, $c \in C$ by

$$b^1(x)(c) = cx - xc$$

while $b^{i+1} : \operatorname{Hom}_k(C^{\otimes_k i}, E) \longrightarrow \operatorname{Hom}_k(C^{\otimes_k i+1}, E)$ maps $f : C^{\otimes_k i} \longrightarrow E$ to $b^{i+1}(f) : C^{\otimes_k i+1} \longrightarrow E$ defined on the generators by

$$\begin{aligned} b^{i+1}(f)(c_1 \otimes \cdots \otimes c_{i+1}) = {} & c_1 f(c_2 \otimes \cdots \otimes c_{i+1}) \\ & + \sum_{j=1}^{i} (-1)^j f(c_1 \otimes \cdots \otimes c_j c_{j+1} \otimes \cdots \otimes c_{i+1}) \\ & + (-1)^{i+1} f(c_1 \otimes \cdots c_i) c_{i+1} \end{aligned}$$

where all $c_j \in C$.

The i^{th} ***Hochschild cohomology group*** is the i^{th} cohomology group of this complex

$$\mathrm{H}^i(C, E) = \frac{\operatorname{Ker} b^{i+1}}{\operatorname{Im} b^i}.$$

If $E = {}_C C_C$, then we write

$$\mathrm{H}^i(C, C) = \mathrm{HH}^i(C).$$

The lower index groups have concrete interpretations. For instance,

$$\mathrm{H}^0(C, E) = \{ c \in C \mid cx = xc \text{ for all } x \in E \}.$$

In particular, $\mathrm{HH}^0(C)$ is the centre of the algebra C. For the first group $\mathrm{H}^1(C, E)$, let $\operatorname{Der}(C, E)$ denote the subspace of $\operatorname{Hom}_k(C, E)$ consisting of all $d : C \longrightarrow E$ such that

$$d(cc') = d(c)\, c' + c\, d(c')$$

for all $c, c' \in C$. Such maps are called ***derivations***. For instance, to each $x \in E$ corresponds a derivation d_x defined by $d_x(c) = cx - xc$ (for $c \in C$). The d_x are called ***inner (or interior) derivations***, and we denote their set by $\operatorname{IDer}(C, E)$. Then, clearly

$$\mathrm{H}^1(C, E) = \frac{\operatorname{Der}(C, E)}{\operatorname{IDer}(C, E)}.$$

The Hochschild groups are not only invariants of the algebra, they are also derived invariants, that is, if $\mathcal{D}^b(\operatorname{mod} C) \cong \mathcal{D}^b(\operatorname{mod} C')$ is a triangle equivalence, then $\operatorname{HH}^i(C) \cong \operatorname{HH}^i(C')$ for all i, see [54, 58].

Moreover $\operatorname{HH}^*(C) = \bigoplus_{i \geqslant 0} \operatorname{HH}^i(C)$ carries a natural ring structure with the so-called cup product if $\xi = [f] \in \operatorname{HH}^i(C)$ and $\zeta = [g] \in \operatorname{HH}^j(C)$, then we define $f \times g : C^{\otimes_k i} \otimes_k C^{\otimes_k j} \longrightarrow C$ by

$$(f \times g)(c_1 \otimes \cdots \otimes c_i \otimes c_{i+1} \otimes \cdots \otimes c_j) = f(c_1 \otimes \cdots \otimes c_i) g(c_{i+1} \otimes \cdots \otimes c_j)$$

where all $c_k \in C$. One verifies that this defines unambiguously a product. We set $\xi \cup \zeta = [f \times g]$ and call it the ***cup product*** of ξ and ζ. With this product $\operatorname{HH}^*(C)$ is a graded commutative ring, that is, if ξ, ζ are as above, then

$$\xi \cup \zeta = (-1)^{ij} \zeta \cup \xi.$$

We now let C be triangular of global dimension two and $E = \operatorname{Ext}^2_C(\mathrm{D}C, C)$. Denoting by $B = C \ltimes E$ the relation extension of C, we have a short exact sequence

$$0 \longrightarrow E \xrightarrow{\ i\ } B \underset{q}{\overset{p}{\rightleftarrows}} C \longrightarrow 0$$

as in (2.3). Let $[f] \in \operatorname{HH}^i(B)$, then we have a diagram

$$\begin{array}{ccc} B^{\otimes_k i} & \xrightarrow{\ f\ } & B \\ {\scriptstyle q^{\otimes i}} \uparrow & & \downarrow {\scriptstyle p} \\ C^{\otimes_k i} & \xrightarrow{\ pfq^{\otimes i}\ } & C. \end{array}$$

We set $\varphi^i[f] = [pfq^{\otimes i}]$. It is easily checked that this gives rise to a well-defined k-linear map $\varphi^i : \operatorname{HH}^i(B) \longrightarrow \operatorname{HH}^i(C)$, which we call the i^{th} ***Hochschild projection morphism***, see [15](2.2).

Theorem 1 . *[15](2.3) Considering $\operatorname{HH}^*(B)$ and $\operatorname{HH}^*(C)$ as associative algebras with the cup product, the φ^i induce an algebra morphism*

$$\varphi^* : \operatorname{HH}^*(B) \longrightarrow \operatorname{HH}^*(C). \quad \square$$

Note that φ^* is only a morphism of associative algebras: the Hochschild cohomology ring also carries a natural Lie algebra structure, but φ^* is not in general a morphism of Lie algebras. For a counterexample, see [15](2.5).

Consider the short exact sequence of B-B-bimodules

$$0 \longrightarrow E \longrightarrow B \longrightarrow C \longrightarrow 0$$

and apply to it the functor $\operatorname{Hom}_{B-B}(B, -)$ (we denote by Hom_{B-B} the morphisms of B-B-bimodules). We get a long exact cohomology sequence

$$0 \longrightarrow \mathrm{H}^0(B, E) \longrightarrow \operatorname{HH}^0(B) \longrightarrow \mathrm{H}^0(B, C) \xrightarrow{\delta^0} \mathrm{H}^1(B, E) \longrightarrow \operatorname{HH}^1(B) \longrightarrow \mathrm{H}(B, C) \xrightarrow{\delta^1} \cdots$$

where δ^i denotes the i^{th} connecting morphism.

It is easy to prove that $\mathrm{H}^0(B, C) \cong \operatorname{HH}^0(C)$, see, for instance [15](2.7), and thus the composition of this isomorphism with the map $\operatorname{HH}^0(B) \longrightarrow \mathrm{H}^0(B, C)$ of the previous long exact sequence is just $\varphi^0 : \operatorname{HH}^0(B) \longrightarrow \operatorname{HH}^0(C)$. Now C is triangular, and $\operatorname{HH}^0(C)$ is its centre, hence $\operatorname{HH}^0(C) = k$. On the other hand $\varphi^0 \neq 0$ because it maps the identity of B to that of C. Therefore, we have a short exact sequence

$$0 \longrightarrow \mathrm{H}^0(B, E) \longrightarrow \operatorname{HH}^0(B) \xrightarrow{\varphi^0} \operatorname{HH}^0(C) \longrightarrow 0.$$

Theorem 2 . *[15](5.7) Let C be triangular of global dimension at most two, and B be its relation extension. Then we have a short exact sequence*

$$0 \longrightarrow \mathrm{H}^1(B,E) \longrightarrow \mathrm{HH}^1(B) \xrightarrow{\varphi^1} \mathrm{HH}^1(C) \longrightarrow 0. \quad \square$$

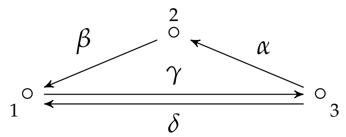
If, in particular, C is tilted so that B is cluster tilted, then φ^1 is surjective.

In actual computations, one uses the fact that, for C triangular of global dimension two, one can prove that

$$\mathrm{H}^1(B,E) \cong \mathrm{H}^1(C,E) \oplus \mathrm{End}_{C-C}\, E$$

see [15](5.9).

EXAMPLE. Let B be the algebra of example (2.3)(b), given by the quiver



bound by $\alpha\beta = 0$, $\beta\delta = 0$, $\delta\alpha = 0$, $\delta\gamma\delta = 0$. This is the relation extension of the (non tilted) algebra $C = B/B\delta B$. Then one can prove that $\mathrm{H}^1(C,E) = 0$ while $\mathrm{End}_{C-C}\, E = \mathsf{k}$ (indeed, E has simple top generated by the arrow δ). Because $\mathrm{HH}^1(C) = \mathsf{k}$, we get that $\mathrm{HH}^1(B) = \mathsf{k}^2$.

In this example, the higher φ^i are not surjective: indeed, one can prove that $\varphi^2 = 0$, while $\mathrm{HH}^2(C) \neq 0$, see [15](5.12).

Corollary 3 . *[15](5.8) Let B be cluster tilted and C be tilted such that $B = \widetilde{C}$, then there is a short exact sequence*

$$0 \longrightarrow \mathrm{H}^0(B,E) \oplus \mathrm{H}^1(B,E) \longrightarrow \mathrm{HH}^*(B) \xrightarrow{\varphi^*} \mathrm{HH}^*(C) \longrightarrow 0.$$

Proof. Because C is tilted, it follows from [55] that $\mathrm{HH}^i(C) = 0$ for all $i \geqslant 2$. $\square$

5.2 *The tame and representation-finite cases*

Now we consider cluster tilted algebras of Dynkin or euclidean type. Let C be tilted and $B = \widetilde{C}$. We need to define an invariant $n_{B,C}$ depending on the choice of C.

Let $\rho = \sum_{i=1}^{m} \lambda_i w_i$ be a relation in a bound quiver (Q, I), where each w_i is a path of length at least two from x to y, say, and each λ_i is a nonzero scalar. Then ρ is called ***strongly minimal*** if, for every nonempty proper subset J of $\{1, 2, \ldots, m\}$ and every family $(\mu_j)_{j\in J}$ of nonzero scalars, we have $\sum_{j\in J} \mu_j w_j \notin I$. It is proved in [17](2.2) that, if B is cluster tilted, then it has a presentation consisting of strongly minimal relations.

Let now $C = \mathsf{k}Q/I$ be a tilted algebra and $B = \widetilde{C} = \mathsf{k}\widetilde{Q}/\widetilde{I}$ be its relation extension, where $\widetilde{I}$ is generated by the partial derivatives of the Keller potential, see (2.4). Let $\rho = \sum_{i=1}^{m} \lambda_i w_i$ be a strongly minimal relation in $\widetilde{I}$, then either ρ is a relation in I, or there exist exactly m new arrows $\alpha_1, \ldots, \alpha_m$ such that $w_i = u_i \alpha_i v_i$, with u_i, v_i paths consisting entirely of old arrows [17](3.1). Moreover, each new arrow α_i must appear in this way.

We define a relation $\sim$ on the set $\widetilde{Q}_1 \setminus Q_1$ of new arrows. For every $\alpha \in \widetilde{Q}_1 \setminus Q_1$, we set $\alpha \sim \alpha$. If $\rho = \sum_{i=1}^{m} \lambda_i w_i$ is a strongly minimal relation in $\widetilde{I}$ and the α_i are as above, then we set $\alpha_i \sim \alpha_j$ for all i, j such that $1 \leqslant i, j \leqslant m$.

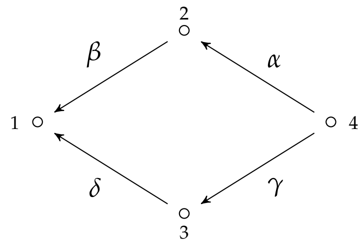
One can show that $\sim$ is unambiguously defined. It is clearly reflexive and symmetric. We let $\approx$ be the least equivalence relation on $\widetilde{Q}_1 \setminus Q_1$ such that $\alpha \sim \beta$ implies $\alpha \approx \beta$ (that is, $\approx$ is the transitive closure of $\sim$).

We define the ***relation invariant*** of B, relative to C, to be the number $n_{B,C}$ of equivalence classes of new arrows under the relation $\approx$.

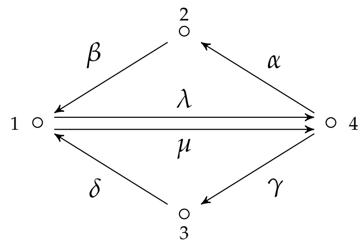
This equivalence is related to the direct sum decomposition of the C-C-bimodule $E = \mathrm{Ext}^2_C(DC, C)$. Indeed, E is generated, as C-C-bimodule, by the new arrows. If two new arrows occur in a strongly minimal relation, this means that they are somehow yoked together in E. It is shown in [9](4.3) that E decomposes, as C-C-bimodule, into the direct sum of $n_{B,C}$ summands.

Theorem 1 . *[17](5.3) Let $B = C \ltimes E$ be a cluster tilted algebra. If B is of Dynkin or of euclidean type, then* $\mathrm{HH}^1(B) = \mathrm{HH}^1(C) \oplus \mathsf{k}^{n_{B,C}}$. □

EXAMPLE. Let C be the (representation-finite) tilted algebra of euclidean type $\widetilde{\mathbb{A}}_3$ given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$. Its relation extension B is as in example (2.1)(d), that is, given by the quiver

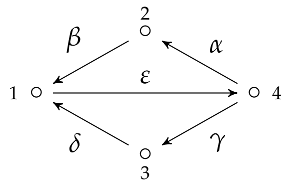


bound by $\alpha\beta = 0$, $\beta\lambda = 0$, $\lambda\alpha = 0$, $\gamma\delta = 0$, $\delta\mu = 0$, $\mu\gamma = 0$. There are two equivalence classes of new arrows, namely $\{\lambda\}$ and $\{\mu\}$. Therefore $n_{B,C} = 2$. Because of the theorem, we have $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(C) \oplus \mathsf{k}^2 \cong \mathsf{k}^3$.

There is a better result in the representation-finite case. If B is representation-finite, then C is tilted of Dynkin type. Because Dynkin quivers are trees, and the Hochschild groups are invariant under tilting (see (5.1) above), we have $\mathrm{HH}^1(C) = 0$. Therefore $\mathrm{HH}^1(B) = \mathsf{k}^{n_{B,C}}$ and so the invariant $n_{B,C}$ doesn not depend on the choice of C. We therefore denote it by n and give an easy way to compute it. We recall that we have defined chordless cycles in (2.4). An arrow in the quiver of a cluster tilted algebra is called ***inner*** if it belongs to two chordless cycles.

Theorem 2 . *[17](6.4) If B is a representation-finite cluster tilted algebra, then the dimension n of $\mathrm{HH}^1(B)$ equals the number of chordless cycles minus the number of inner arrows in the quiver of B.* □

EXAMPLE. Let B be as in example (2.1)(b) given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\varepsilon\alpha = 0$, $\beta\varepsilon = 0$, $\varepsilon\gamma = 0$, $\delta\varepsilon = 0$. There are two chordless cycles and just one inner arrow so $n = 2 - 1 = 1$ and $\mathrm{HH}^1(B) = \mathsf{k}$.

We get a characterisation of the fundamental group of B.

Corollary 3 . *[16](4.1) If $B = \mathsf{k}\widetilde{Q}/\widetilde{I}$ is a representation-finite cluster tilted algebra, then $\pi_1(\widetilde{Q}, \widetilde{I})$ is free on n generators.* □

For instance, in the above example, $\pi_1(\widetilde{Q}, \widetilde{I}) \cong \mathbb{Z}$. This is a particular case of the following problem.

Problem. Let $B = \mathsf{k}\widetilde{Q}/\widetilde{I}$ be a cluster tilted algebra, with the presentation induced from the Keller potential. Prove that $\pi_1(\widetilde{Q}, \widetilde{I})$ is free.

Finally, we refer the reader to [43] for the study of the Hochschild groups as derived invariants of (an overclass of) cluster tilted algebras of type $\mathbb{A}$.

5.3 *Simply connected cluster tilted algebras*

In [67], Skowroński asked for which algebras the vanishing of the first Hochschild cohomology group is equivalent to simple connectedness. We prove that this is the case for cluster tilted algebras.

Theorem 1 . *[17](5.11) Let B be cluster tilted. The following conditions are equivalent:*

(a) $\mathrm{HH}^1(B) = 0$

(b) *B is simply connected*

(c) *B is hereditary and its quiver is a tree.*

Proof. *(b) implies (c).* If B is simply connected, then it is triangular and hence it is hereditary. Moreover its quiver must be a tree.

(c) implies (a). This is trivial, see [54].

(a) implies (c). If B is not hereditary, and C is tilted such that $B = \widetilde{C}$, then because of Lemma (3.3.1), we have a connected Galois covering $\check{C} \longrightarrow \widetilde{C} = B$ with group $\mathbb{Z}$. The universal property of Galois coverings yields a group epimorphism $\pi_1(\widetilde{Q}, \widetilde{I}) \longrightarrow \mathbb{Z}$ where $B = \mathsf{k}\widetilde{Q}/\widetilde{I}$. This epimorphism induces a monomorphism of abelian groups $\mathrm{Hom}(\mathbb{Z}, \mathsf{k}^+) \longrightarrow \mathrm{Hom}\big(\pi_1(\widetilde{Q}, \widetilde{I}), \mathsf{k}^+\big)$. Because of a well-known result of [48], we have a monomorphism

$$\mathrm{Hom}\big(\pi_1(\widetilde{Q}, \widetilde{I}), \mathsf{k}^+\big) \longrightarrow \mathrm{HH}^1(B).$$

Therefore the composed monomorphism $\mathrm{Hom}(\mathbb{Z}, \mathsf{k}^+) \longrightarrow \mathrm{HH}^1(B)$ gives $\mathrm{HH}^1(B) \neq 0$. Thus $\mathrm{HH}^1(B) = 0$ implies that B is hereditary. Applying [54] we get that $\widetilde{Q}$ is a tree. □

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