CIMPA - MAR DEL PLATA - CLUSTER CHARACTERS - EXERCISES 1

Exercise 1. Let Q be the quiver $1 \to 2$, and consider the representations

$$V = \mathbb{C} \xrightarrow{1} \mathbb{C}, \quad W = \mathbb{C} \to 0, \quad X = 0 \to \mathbb{C}, \quad Y = V \oplus W \oplus X = \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \mathbb{C}^2.$$

Compute the F-polynomials of these four representations, and check that $F_Y = F_V F_W F_X$. (Recall that $F_M = \sum_{\mathbf{e} \in \mathbb{N}^{Q_0}} \chi\big(\mathrm{Gr}_{\mathbf{e}}(M)\big) x^{\mathbf{e}}$.)

Exercise 2. Let Q be the Kronecker quiver $1 \overset{\to}{\underset{\to}{}} 2$, having two vertices and two arrows in the same direction. Find an infinite family of non-isomorphic representations of Q that all have the same F-polynomial.

Exercise 3. Let Q be the quiver having only one vertex and one loop ℓ at this vertex, subject to the relation $\ell^2 = 0$.

(1) Show that, up to isomorphism, there are only two indecomposable representations of Q satisfying the relation. (This can be done, for instance, by looking at the Jordan normal form of some matrices). One of them, V_1, is one-dimensional, while the other, V_2, is two-dimensional.
(2) Show that there is an almost-split sequence $0 \to V_1 \to V_2 \to V_1 \to 0$.
(3) Check that $F_{V_1}^2 = F_{V_2} + x$.

Exercise 4. The aim of this exercise is to give a meaning to the remark saying that submodule Grassmannians can be as complicated as possible. We will prove the following statement: *Any complex projective variety can be realized as a submodule Grassmannian.* We follow very closely the proof given by M. Reineke in [2].

Let X be a projective variety in $\mathbb{P}^n$, given by the vanishing of homogeneous polynomials $p_1, \ldots, p_k \in \mathbb{C}[x_0, x_1, \ldots, x_n]$.

(1) Show that we can assume that the p_i's all have the same degree. (Note that we don't require the p_i's to be irreducible polynomials.)
(2) We now assume that the p_i's all have degree d. We will need the d-uple embedding of projective spaces, defined thus: let M be the set of $(n+1)$-tuples $(m_0, \ldots, m_n)$ such that $\sum_{i=0}^n m_i = d$. Define the embedding by

$$\begin{aligned} D : \mathbb{P}^n &\longrightarrow \mathbb{P}^{|M|-1} \\ (x_0 : \ldots : x_n) &\longmapsto (\mathbf{x}^{\mathbf{m}})_{\mathbf{m} \in M}. \end{aligned}$$

We will use that D induces an isomorphism of $\mathbb{P}^n$ onto its image (see, for instance, [1, Exercises I.2.12 and I.3.4]). Show that a point $(\mathbf{x}_{\mathbf{m}})_{\mathbf{m} \in M}$ of $\mathbb{P}^{|M|-1}$ is in the image of D if and only if it satisfies the relations $\mathbf{x}_{\mathbf{m}_1}\mathbf{x}_{\mathbf{m}_2} = \mathbf{x}_{\mathbf{m}_3}\mathbf{x}_{\mathbf{m}_4}$ whenever $\mathbf{m}_1 + \mathbf{m}_2 = \mathbf{m}_3 + \mathbf{m}_4$.

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(3) Show that $D(X)$ is defined in $D(\mathbb{P}^n)$ by imposing the vanishing of homogeneous polynomials $\varphi_1, \dots, \varphi_k$, *all of degree* 1. (See what the p_i's become.)

(4) Show that in step (2), it is sufficient to ask that $\mathbf{x}_{\mathbf{m}+\mathbf{e}_i}\mathbf{x}_{\mathbf{m}'+\mathbf{e}_j} = \mathbf{x}_{\mathbf{m}+\mathbf{e}_j}\mathbf{x}_{\mathbf{m}'+\mathbf{e}_i}$ for all $\mathbf{m}, \mathbf{m}'$ whose entries sum to $d-1$ (call N the set of such vectors), and for all $i, j \in \{0, \dots, n\}$, where $\mathbf{e}_i$ is the vector having 1 in entry i and zero elsewhere. Define a matrix $A(\mathbf{x})$ whose entries are these $\mathbf{x}_{\mathbf{m}+\mathbf{e}_i}$ and such that the above equations are equivalent to saying that $A(\mathbf{x})$ has rank 1.

(5) Let Q be the quiver $1 \overset{k}{\leftarrow} 2 \overset{n+1}{\rightarrow} 3$, where the numbers above the arrows is the number of arrows. Let V be the representation of Q given as follows:$V_1 = \mathbb{C}$, V_2 has basis $(v_\mathbf{m})_{\mathbf{m}\in M}$ and V_3 has basis $(v_\mathbf{n})_{\mathbf{n}\in N}$; moreover, the maps associated to the k arrows from 2 to 1 are the linear forms φ_i, and the map associated to the i-th arrow from 2 to 3 is the map f_i sending $v_\mathbf{m}$ to $v_{\mathbf{m}-\mathbf{e}_i}$ if $m_i > 0$, and to 0 otherwise.

Now let $\mathbf{e} = (0, 1, 1)$. Show that points in $\mathrm{Gr}_\mathbf{e}(V)$ are in bijection with lines in V_2 which are sent to 0 by all the φ_i's and whose sum of images by the f_i's is one-dimensional.

(6) Finally, show that the lines described in step (5) are in bijection with the points in $\mathbb{P}^n$ annihilated by the φ_i's and such that the matrix $A(\mathbf{x})$ of step (4) is of rank 1. Conclude that $\mathrm{Gr}_\mathbf{e}(V)$ is isomorphic to X.

References

[1] Robin Hartshorne, *Algebraic geometry*, Springer, GTM 52, 1977.

[2] Markus Reineke, *Every projective variety is a quiver Grassmannian*, Algebras and Representation Theory, October 2013, Volume 16, Issue 5, pp 1313-1314, arXiv:1204.5730v1 [math.RT].