CIMPA - MAR DEL PLATA - CLUSTER CHARACTERS - EXERCISES 2

For the next exercises, $\mathcal{C}$ is a Hom-finite, Krull–Schmidt, 2-Calabi–Yau triangulated category, with basic cluster-tilting object $T = T_1 \oplus \ldots \oplus T_n$.

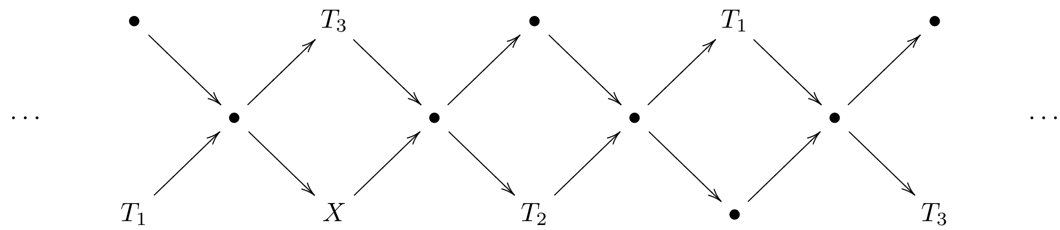
Exercise 1. Show that for any two objects X and Y of $\mathcal{C}$, $\mathrm{ind}_T(X \oplus Y) = \mathrm{ind}_T(X) + \mathrm{ind}_T(Y)$.

Exercise 2. Let $f : T^X \to X$ be a right $\mathrm{add}\,(T)$-approximation of X, that is, T^X is in $\mathrm{add}\,(T)$, and if R is in $\mathrm{add}\,(T)$ and $g : R \to X$ is a morphism, then there exists a morphism $h : R \to T^X$ such that $f \circ h = g$.

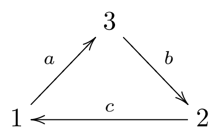
Complete f into a triangle $S \to T^X \xrightarrow{f} X \to S[1]$ in $\mathcal{C}$. Show that S is in $\mathrm{add}\,(T)$. (Hint: apply the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$ to the triangle, and use the induced exact sequence to show that $\mathrm{Hom}_{\mathcal{C}}(T, S[1])$ is zero).

Exercise 3. Let R be in $\mathrm{add}\,(T)$, and let $R \xrightarrow{f} R \to X \to R[1]$ be a triangle. Assume that f is not an isomorphism. Show that X is not rigid. (Hint: apply the functor $\mathrm{Hom}_{\mathcal{C}}(-, X[1])$ to the triangle, and prove that the map induced by $f[1]$ is not injective, which in this case implies that it is not surjective. Deduce that $\mathrm{Hom}_{\mathcal{C}}(X, X[1])$ is non-zero).

For the next exercises, we take $\mathcal{C}$ to be the cluster category in Dynkin type A_3. We put $T = T_1 \oplus T_2 \oplus T_3$, where the T_i's are as pictured in the Auslander-Reiten quiver of $\mathcal{C}$ below.



Exercise 4. Show that T is a cluster-tilting object of $\mathcal{C}$, and that its endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$ is isomorphic to the path algebra of the quiver



with relations $ba = cb = ac = 0$. (It is a cluster-tilted algebra).

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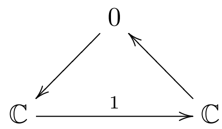
Exercise 5. Let X be as in the picture of the Auslander-Reiten quiver of $\mathcal{C}$. Show that the $\mathrm{End}_{\mathcal{C}}(T)$-module $\mathrm{Hom}_{\mathcal{C}}(T, X[1])$ is given by the representation



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Show that $\mathrm{ind}_T(X) = (0, -1, 0)$, and compute $CC(X)$, the value of the cluster character applied to X.

Exercise 6. (1) Mutate T at T_2.

(2) Find a sequence of mutations relating T and $T[1]$.