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Flexibility of ideal triangle groups in complex hyperbolic geometry

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Abstract

We show that the Teichmüller space of the ideal triangle group in the automorphism group of complex hyperbolic space contains a real four-dimensional ball. This implies the existence of a four-dimensional family of spherical CR structures on the trivial circle bundle over the sphere minus three points. The proof is an explicit construction of fundamental domains whose boundaries are special hypersurfaces foliated by complex geodesics. © 2000 Published by Elsevier Science Ltd. All rights reserved.

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1. Introduction

The discrete embedding of the ideal triangle group in the automorphism group of the complex disc is essentially unique, that is, the Teichmüller space of the ideal triangle group in the automorphism group of the complex disc is a point. In this paper we study its embedding in the automorphism group of the two-dimensional ball, that is $\mathbf{Isom}(H^2_{\mathbb{C}}) = \widehat{PU(2,1)}$ (see Section 2 for definitions). We will start with an embedding fixing a complex disc in $H^2_{\mathbb{C}}$. Rigidity occurs when the action, restricted to that disc, has a compact fundamental domain [7], in the sense that all nearby deformations preserve a complex disc. Here we consider actions with finite volume and we might expect some local rigidity.

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Let Γ be the free product of three involutions $\iota_i(\iota_i^2 = 1)$. We consider discrete embeddings in the space of homomorphisms of Γ into $\widehat{PU(2,1)}$ denoted by $\mathbf{Hom}(\Gamma, \widehat{PU(2,1)})$. An embedding of ι_i can be holomorphic or anti-holomorphic. We deal here with anti-holomorphic embeddings, that is, the subspace $\mathbf{Hom}_p(\Gamma, \widehat{PU(2,1)}) \subset \mathbf{Hom}(\Gamma, \widehat{PU(2,1)})$ of homomorphisms with ι_i anti-holomorphic and $\iota_i \circ \iota_j$ parabolic. We call *ideal deformation space* the quotient space of $\mathbf{Hom}_p(\Gamma, \widehat{PU(2,1)})$ by $\widehat{PU(2,1)}$ acting by conjugation. It is a real four-dimensional space with singularities.

We show the following theorem.

Theorem 1.1. *There exists a neighborhood of the standard embedding in the ideal deformation space containing only discrete embeddings.*

The standard embedding is a certain embedding fixing a complex geodesic (see Definition 5.1). It is associated to a configuration of three **R**-circles (real two-dimensional geodesics, see Section 2). The involutions are reflections on each of the **R**-circles.

This is the opposite phenomenon of local rigidity of a discrete embedding that we call *local flexibility*.

In [11], a one parameter family of discrete deformations in $\mathbf{Hom}_p(\Gamma, \widehat{PU(2,1)})$ is constructed. In our paper we construct fundamental domains using **C**-spheres (definition in Section 2.3, see also [6] where **C**-spheres are used to formulate a general Poincaré's polyhedron theorem and in particular to construct Seifert manifolds) instead of bisectors. The flexibility of **C**-spheres allows one to prove discreteness for all four parameters of the ideal deformation space.

It is interesting to compare this result to the local rigidity theorem of embeddings of cocompact surface groups in the automorphisms group of the complex hyperbolic space which fix a complex geodesic ([7] see also [9,14]). In particular for triangle groups obtained as embeddings fixing a complex geodesic and whose angles in the vertices are non zero there is local rigidity. In our case, as in [11], the initial embedding is not cocompact but of finite volume. Observe that for each embedding the subgroup of holomorphic transformations is an index two free discrete group. It is isomorphic to the fundamental group of the sphere minus three points. The quotient of $H^2_{\mathbf{C}}$ by this family of groups gives rise to a four-dimensional family of complex structures on a trivial disc bundle over the sphere minus three points.

In [10] the embeddings of the involutions are all holomorphic and the natural embedding fixing a complex geodesic is locally rigid, that is, all nearby deformations are non discrete. Moreover, the configuration space has real dimension 1. Their deformations correspond to a different topological component of $\mathbf{Hom}(\Gamma, \widehat{PU(2,1)})$.

It would clearly be interesting to obtain a complete description of the Teichmüller space.

2. The complex hyperbolic space and its boundary

Let $\mathbf{C}^{n+2}$ denote the complex vector space equipped with the Hermitian form

$$b(z, w) = -\bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{n+2} w_{n+2}.$$

Consider the following subspaces in $\mathbf{C}^{n+2}$:

$$V_0 = \{z \in \mathbf{C}^{n+2} : b(z, z) = 0\},$$

$$V = \{z \in \mathbf{C}^{n+2} : b(z, z) < 0\}.$$

Let $P : \mathbf{C}^{n+2} \backslash \{0\} \to \mathbf{C}P^{n+1}$ be the canonical projection onto the complex projective space. Then $H^{n+1}_{\mathbf{C}} = P(V)$ equipped with the Bergman metric is the complex hyperbolic space. The orientation preserving isometry group of $H^{n+1}_{\mathbf{C}}$ is generated by $PU(n+1, 1)$ acting by linear projective transformations and the anti-holomorphic transformations. We denote it by $\widehat{PU(n+1, 1)}$. Also, $PU(n+1, 1)$ is the group of biholomorphic transformations of $H^{n+1}_{\mathbf{C}}$.

Let $S^{2n+1} = P(V_0)$. Then S^{2n+1} is the boundary of $H^{n+1}_{\mathbf{C}}$. We may consider $H^{n+1}_{\mathbf{C}}$ and S^{2n+1} as the unit ball and the unit sphere in $\mathbf{C}^{n+1}$.

The complex structure in $\mathbf{C}P^{n+1}$ defines a special distribution on S^{2n+1}, that is, $D = J(TS^{2n+1}) \cap (TS^{2n+1})$. The complex operator J is well defined on D. S^{2n+1} with the special distribution and the operator J on D is a CR-manifold.

The group of CR-automorphisms of S^{2n+1} is $\mathbf{Aut_{CR}}(S^{2n+1}) = PU(n+1, 1)$.

One-dimensional complex manifolds are also two-dimensional real conformal manifolds. In the same way we can see three-dimensional **CR**-manifolds as manifolds with a conformal structure restricted to the distribution. The group of automorphisms of the conformal structure includes $PU(n+1, 1)$, but it also contains the anti-holomorphic transformations.

2.1. The stereographic projection and the Heisenberg group

The mapping

$$C : (z_1, z_2) \mapsto \left(\mathrm{i}\frac{w_1}{1 + w_2}, \mathrm{i}\frac{1 - w_2}{1 + w_2} \right)$$

is usually referred to as the *Cayley transform*. It maps the unit ball

$$B = \{w \in \mathbf{C}^2 : |w_1|^2 + |w_2|^2 < 1\}$$

biholomorphically onto

$$V = \{z \in \mathbf{C}^2 : \mathrm{Im}(z_2) > |z_1|^2\}.$$

The Cayley transform leads to a generalized form of the *stereographic projection*. This mapping $\pi : S^3 \backslash \{-e_2\} \to \mathbf{R}^3$, where $S^3 = \partial B$ and $e_2 = (1, 0) \in \mathbf{C}^2$, is defined as the composition of the Cayley transform restricted to $S^3 \backslash \{-e_2\}$ followed by the projection

$$(z_1, z_2) \mapsto (z_1, \mathrm{Re}(z_2)).$$

The stereographic projection π can be extended to a mapping from S^3 onto the one-point compactification $\bar{\mathbf{R}}^3$ of $\mathbf{R}^3$.

The *Heisenberg group H* is the set of pairs $(z, t) \in \mathbf{C} \times \mathbf{R}$ with the product

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2\mathrm{Im}(z\bar{z}')).$$

Using the stereographic projection, we can identify $S^3\backslash\{-e_2\}$ with H and S^3 with the one-point compactification $\bar{H}$ of H. The inverse function of the stereographic projection is given by

$$\pi^{-1}(z,t)=\left(\frac{-2iz}{1+|z|^2-it},\frac{1-|z|^2+it}{1+|z|^2-it}\right).$$

Observe that the x-axis in the Heisenberg group corresponds to the intersection of S^3 with the real plane $\mathrm{Re}(w_1)=0, \mathrm{Im}(w_2)=0$. Also, the y-axis corresponds to the intersection of S^3 with the real plane $\mathrm{Im}(w_1)=0, \mathrm{Im}(w_2)=0$.

The Heisenberg group acts on itself by left translations. Heisenberg translations by $[0,t]$ for $t\in\mathbf{R}$ are called *vertical translations*.

Positive scalars $\lambda\in\mathbf{R}_+$ act on H by *Heisenberg dilations*

$$d_\lambda:(z,t)\mapsto(\lambda z,\lambda^2 t).$$

If $m\in U(1)$, then m acts on H by

$$m:(z,t)\mapsto(mz,t),$$

where m is called a *Heisenberg rotation*.

The *Heisenberg complex inversion* of H is defined on $H\backslash\{origin\}$ by

$$h:(z,t)\mapsto\left(\frac{-z}{|z|^2-it},-\frac{t}{|z|^4+t^2}\right).$$

Note that $h=\pi\circ j\circ\pi^{-1}$, where j is the involution

$$j:(w_1,w_2)\rightarrow(-w_1,-w_2).$$

The map $\hat{m}$ defined by

$$\hat{m}:(z,t)\mapsto(\bar{z},-t)$$

corresponds to

$$\pi^{-1}\circ\hat{m}\circ\pi(w_1,w_2)=(-\bar{w}_1,\bar{w}_2).$$

All these actions extend trivially to the compactification $\bar{H}$ of H. It is well known that the group G of transformations of $\bar{H}$ generated by all Heisenberg translations, dilations, rotations, and h coincides with $\pi^{-1}\circ PU(2,1)\circ\pi$, and the group $\hat{G}=\langle G,\hat{m}\rangle$ is the group of all conformal transformations of $\bar{H}$ (see [2,12]).

2.2. R-circles and C-circles

There are two kinds of totally geodesic submanifolds of real dimension 2 in $H^2_{\mathbf{C}}$: complex *geodesics* (represented by $H^1_{\mathbf{C}}\subset H^2_{\mathbf{C}}$) and totally real geodesic 2-planes (represented by $H^2_{\mathbf{R}}\subset H^2_{\mathbf{C}}$). Each of these totally geodesic submanifold is a model of the hyperbolic plane.

Proposition 2.1. (Cartan [3], Chen and Greenberg [4]). *Let M be a totally geodesic submanifold in $H^2_{\mathbf{C}}$ and let $I(M)$ be the stabilizer of M in $PU(2,1)$. Then we have the following.*

(i) *If $M = H^1_{\mathbf{C}}$ then $I(M)$ is isomorphic to* $P(U(1)\times U(1,1))$
(ii) *If $M = H^2_{\mathbf{R}}$ then $I(M)$ is isomorphic to* $PSO(2,1)$.

Consider the complex hyperbolic space $H^2_{\mathbf{C}}$ and its boundary $\partial H^2_{\mathbf{C}} = S^3$. We will call **C**-*circles* the intersections of S^3 with the boundaries of totally geodesic complex submanifolds $H^1_{\mathbf{C}}$ in $H^2_{\mathbf{C}}$. Analogously, we call **R**-*circles* the intersections of S^3 with the boundaries of totally geodesic totally real submanifolds $H^2_{\mathbf{R}}$ in $H^2_{\mathbf{C}}$. The **R**-circles are always tangent to the special distribution, that implies that the angle between them is well defined at a point of intersection.

Proposition 2.2. (see Goldman [8]) *In the Heisenberg model*, **C**-*circles are either vertical lines or ellipses, whose projection on the z-plane is a circle, determined by a center and a radius.*

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two circles of centers $(a_1, b_1, c_1), (a_2, b_2, c_2)$ and radii R_1 and R_2. Let d, h and S be

$$d = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2},\ h = c_2 - c_1,\ S = 1/2(a_1 b_2 - a_2 b_1).$$

Proposition 2.3. (Linking of **C**-circles) *The* **C**-*circles* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are linked if and only if*

$$\begin{aligned}&(d^2 - (R_1 + R_2)^2)(d^2 - (R_1 - R_2)^2) + (h + 4S)^2\\ &\quad = (d^2 - (R_2^2 - R_1^2))^2 + (h + 4S)^2 - 4d^2 R_1^2 < 0.\end{aligned}$$

Proof. Suppose that projections of $\mathbf{C}_1$ and $\mathbf{C}_2$ intersect in two points. They are projections of two points N and M of $\mathbf{C}_2$. $\mathbf{C}_1$ and $\mathbf{C}_2$ are linked if these points are not in the same side of the plane defining $\mathbf{C}_1$, that is the announced result. $\square$

Observe that $\mathbf{C}_1$ and $\mathbf{C}_2$ are not linked if their projections are not, that is

$$(d^2 - (R_1 + R_2)^2)(d^2 - (R_1 - R_2)^2) > 0$$

or if

$$4d^2 R_1^2 < (h + 4S)^2$$

that is, $\mathbf{C}_1$ does not intersect the plane defining $\mathbf{C}_2$.

Definition 2.1. An inversion on an **R**-circle R is a non-trivial conformal transformation which fixes pointwise R.

Observe that an inversion has invariant **R**-circles. One of them is pointwise fixed. Moreover, an **R**-circle defines a unique inversion. For instance the transformation $\hat{m}(z,t) = (\bar{z}, -t)$ on the Heisenberg group is the inversion that fixes pointwise the **R**-circle $\mathrm{Im}(z) = 0$.

2.3. C-spheres

In the following definition we allow a point to be a (degenerate) **C**-circle.

Definition 2.2. A **C**-sphere around an **R**-circle is a disjoint union of invariant **C**-circles under the inversion on the **R**-circle, which is homeomorphic to a sphere. We call axis of the **C**-sphere the set of centers of these invariant **C**-circles.

In particular a **C**-sphere contains two degenerate **C**-circles and its axis has starting point and end point in the **R**-circle. See also [6].

Definition 2.3. The surface of centers of an **R**-circle is the set of centers of invariant **C**-circles under the inversion on the **R**-circle. Such **C**-circles have two points in common with the **R**-circle. Observe that for finite **R**-circles this is a two dimensional surface but for an infinite **R**-circle this coincides with the **R**-circle.

Definition 2.4. An axis of a finite **R**-circle is a simple curve in the surface of centers with starting and end points in the **R**-circle.

Observe that for an infinite **R**-circle a center does not determine a **C**-circle and that a radius should also be specified. On the other hand for a finite **R**-circle the center completely determines the **C**-circle (Proposition 2.4).

An axis determines a surface containing the **R**-circle obtained by constructing the union of all **C**-circles defined by the centers. But that surface might have self-intersections.

We call a *good axis* an axis whose associated surface is homeomorphic to the two-dimensional sphere.

2.4. Standard R-circle

Consider the following transformation on the Heisenberg group:

$$I = \hat{m} \circ h : (z, t) \mapsto \left(\frac{-\bar{z}}{|z|^2 + \mathrm{i}t}, \frac{t}{|z|^4 + t^2} \right).$$

which corresponds to

$$\pi^{-1} \circ I \circ \pi(w_1, w_2) = \pi^{-1} \circ \hat{m} \circ h \circ \pi(w_1, w_2) = (\bar{w}_1, -\bar{w}_2).$$

I is the inversion that leaves invariant the circles $|z|^2 = \sqrt{1 - t^2}$. It leaves also invariant their union, the **C**-sphere $S_0 = \{|z|^4 + t^2 = 1\}$ and $I(\mathrm{int}\, S_0) = \mathrm{ext}\, S_0$. We call its axis *vertical axis*, that is, the segment $[(0, 0, -1), (0, 0, 1)]$.

I leaves pointwise fixed the standard **R**-circle $\mathbf{R}_0$ (see [8] for details)

$$r^2 + \mathrm{i}t = -\mathrm{e}^{-2\mathrm{i}\theta},$$

where $z = re^{i\theta}$. In cylindrical coordinates $\mathbf{R}_0$ is given by

$$r = \sqrt{-\cos(2\theta)}, \quad z = \sin(2\theta).$$

Using the change of coordinate $\tan(\theta) = (1 + t^2)/2t$ (see [7,8]) we obtain

$$r^2 = \frac{(1-t^2)^2}{(1+t^2)^2 + 4t^2}, \quad x = -2\frac{(t^2-1)t}{6t^2+t^4+1}, \quad y = -\frac{(t^2-1)(t^2+1)}{6t^2+t^4+1}, \quad z = 4\frac{(t^2+1)t}{6t^2+t^4+1}.$$

It is well known that two points M_1 and M_2 determine a **C**-circle passing by them. If the two points $M(t_1)$, $M(t_2)$ lie on the standard **R**-circle we obtain the **C**-circle of center

$$X = -\frac{(p-1)s}{(p+1)^2+s^2}, \quad Y = -\frac{(p-1)(p+1)}{(p+1)^2+s^2}, \quad Z = 2\frac{s(p+1)}{(p+1)^2+s^2}.$$

and radius $R^2 = (s^2 - 4p)/((p+1)^2 + s^2)$. Here we put $s = t_1 + t_2, p = t_1 t_2$. Observe that the **C**-circle is defined for $0 \neq (1+p)^2 + s^2$, that is, if the parameters are not 1 and -1. In that case the **C**-circle is the vertical line.

From this parameterization we then obtain

Proposition 2.4. *The surface of centers of the standard* **R**-*circle is given, in cylindrical coordinates, by* $(r, \theta, \sin(2\theta))$ *with* $r^2 + \cos(2\theta) \geqslant 0$. *The radius of a* **C**-*circle with coordinates* r, θ *is* $\sqrt{r^2 + \cos(2\theta)}$.

Proof. From the above parameterization we see that the surface of centers satisfies the equation

$$Z = 2\frac{XY}{X^2+Y^2}$$

which can be parameterized as

$$X = r\cos(\theta), Y = r\sin(\theta), Z = \sin(2\theta).$$

The radius of the **C**-circle is then $R^2 = r^2 + \cos(2\theta)$. We obtain the standard **R**-circle for $R = 0$. Conversely, for given r and θ, one obtains

$$s = 2\frac{\cos(\theta)}{r+\sin(\theta)}, \quad p = \frac{\sin(\theta) - r}{\sin(\theta) + r}.$$

We can solve, then, for distinct t_1 and t_2 if the condition $s^2 - 4p > 0$ is verified. This is simply $r^2 + \cos(2\theta) > 0$. $\square$

Observe that the surface of centers is foliated by horizontal infinite segments emanating from the **R**-circle or from the vertical axis. It projects into the exterior of the region delimited by the lemniscate. Parker also noticed that it is an embedding of the Möbius strip on the compactification of the Heisenberg group.

3. Poincaré's polyhedron theorem

To prove discreteness of a subgroup we will use a simple version of Poincaré's polyhedron theorem.

A general version was proved in [6] but, there, parabolic points were not considered for the sake of simplicity. Here we state a simplified version which is sufficient for our purposes.

Theorem 3.1. (Poincaré polyhedron). *Let $\{R_i\}$ be a finite collection of finite* **R***-circles and $\{S_i\}$ be a collection of* **C***-spheres around each of them. Suppose that, pairwise, the* **R***-circles intersect at most at one point, and then the corresponding* **C***-spheres intersect tangentially. Suppose furthermore that the closure of the unbounded component of the complement of each sphere contains all others. Then the group generated by inversions on each* **R***-circle is discrete and a fundamental domain is the unbounded component of the complement of the union of all* **C***-spheres.*

For simplicity, we do not take into account the possibility of more complicated side pairings and cycles. Observe that in our formulation the condition on the intersection of **R**-circles implies that the composition of two inversions on two intersecting **R**-circles is automatically parabolic. The fact that the **C**-surfaces are unions of **C**-circles implies that one can extend those surfaces canonically as hypersurfaces in the complex hyperbolic space where they define a "polyhedron". The proof of the theorem then goes along as the classical Poincaré's theorem. See, for instance, [1,5,13].

The originality of this formulation is due to the flexibility of **C**-spheres. They allow more liberty than other formulations of Poincaré's theorem, essentially based on Mostow's bisectors. See [6], where fundamental domains for Seifert manifolds are constructed using **C**-spheres.

4. Groups generated by three inversions

An inversion fixes pointwise a unique **R**-circle. **Hom**$(\Gamma,\widehat{PU(2,1)})$ is described by all triples of **R**-circles. The quotient of that space by equivalence under conjugation under $\widehat{PU(2,1)}$ has dimension 7. We consider the subspace of **R**-circles intersecting pairwise at only one point.

Proposition 4.1. *The space of configurations up to conjugation under $\widehat{PU(2,1)}$ of three* **R***-circles with pairwise at least one point of intersection is diffeomorphic to a quotient of $S^1 \times S^1 \times S^1 \times [0, \pi/2]$.*

Proof. The three points of contact are parameterized by the absolute value of Cartan's invariant which has values in the interval $[0, \pi/2]$. Moreover each **R**-circle is defined by two points and a tangent vector at one of the points. Observe that if the three points are contained in a **R**-circle, that is, Cartan's invariant is 0, there are degenerate configurations corresponding to two of the **R**-circles or the three coinciding. On the other hand given two special angles the two **R**-circles determined by them will touch elsewhere on a fourth point. Those configurations correspond to diagonals in the parameters. Finally, permuting the three points will make more identifications on the configuration space. $\square$

Another description of the configuration space goes as follows. We start with three **R**-circles touching pairwise at one point. We chose one of the points to be ∞. We have now two infinite **R**-circles and a finite one. Using conjugation on $\widehat{PU(2,1)}$ we can suppose that the finite **R**-circle is the standard one. We still are allowed to impose that one of the points of intersection has angle parameters between $\pi/4$ and $\pi/2$. This is done using reflections on the x-axis or y-axis. At each intersection point the infinite **R**-circle is determined by an angle. But observe that, as long as the contact points are not aligned vertically, for each angle chosen at one point there will be an angle at the other contact point such that the two infinite **R**-circles intersect (if the contact points are aligned horizontally there is a further degeneracy which will not concern us here). Also there are angles such that the **R**-circles intersect twice the finite **R**-circle. We obtain $S^1 \times S^1 \times S^1 \times [\pi/4, \pi/2] - S^1 \times S^1 \times D$, where $D = \{(x, x), x \in [\pi/4, \pi/2]\}$. As the three points of contact can be permuted one can obtain the configuration space as a quotient by the permutation group of the space described (when the **R**-circles intersect at more than one point there is a further degeneracy).

Observe that the permutation group fixes the three points if they are aligned vertically or horizontally. Those configurations will be singular points in the configuration space.

We are interested in a neighborhood of the *standard configuration*. We will understand it to be the configuration of three vertically aligned contact points. One infinite **R**-circle touching the north pole of the standard **R**-circle at an angle of $-\pi/4$ and the other infinite **R**-circle touching the south pole at an angle of $\pi/4$.

It is interesting to interpret the case of the three **R**-circles coinciding at only one point as a limit case of the space of configurations. This case corresponds to crystallographic groups in the Heisenberg group.

5. Proof of the theorem

Consider the following configuration, composed of two infinite **R**-circles $\mathbf{R}_1$ and $\mathbf{R}_2$ touching the standard **R**-circle $\mathbf{R}_0$ at the points p_1 and p_2, respectively. $\mathbf{R}_1$ is determined by the angle from the x-axis to the projection of the line, we call it α_1. Analogously, $\mathbf{R}_2$ is determined by the angle α_2. We will prove that for p_1 near $(0, 0, 1)$, p_2 near $(0, 0, -1)$, $\alpha_1 + \pi/4$ and $\alpha_2 - \pi/4$ small enough, the group generated by the inversions in all three **R**-circles is discrete.

Definition 5.1. (*Standard configuration*) We call standard configuration the case where $p_1 = (0, 0, 1)$, $p_2 = (0, 0, -1)$, $\alpha_1 = -\pi/4$, $\alpha_2 = \pi/4$.

In this case a fundamental domain is bounded by the two horizontal planes containing p_1 and p_2 respectively, and the Heisenberg sphere $S_0 = \{|z|^4 + t^2 = 1\}$. See Fig. 4.

In order to prove discreteness we need to find three surfaces invariant by each corresponding inversion which touch each other precisely at the unique point of intersection.

Locally, at each intersection point, this is always possible. We find in this way 6 surfaces which are disjoint, two at each of the three intersection points. The problem is to interpolate between these surfaces using nonintersecting invariant surfaces.

5.1. Strategy for the axes

Let $q_1 \in \mathbf{R}_1$ and $q_2 \in \mathbf{R}_2$ whose projections are the intersection of projections of $\mathbf{R}_1$ and $\mathbf{R}_2$ on the z-plane. For $\mathbf{R}_1$ we will take centers in the segment $[p_1, q_1]$ with appropriate radii. Then we will complete the surface by a union of $\mathbf{C}$-circles of centers q_1. We proceed analogously with $\mathbf{R}_2$.

The invariant surface for $\mathbf{R}_0$ will be given by its axis $r(\theta)$ (see Fig. 2). The upper side S_1 and the lower side S_2 of the $\mathbf{C}$-sphere of $\mathbf{R}_0$ are separated by the z-plane. Using Proposition 2.3, this is obtained when the axis satisfies $r^2(\theta) \leqslant \sin^2(\theta)$. Observe that in this case $R^2(\theta) \leqslant 1$.

At the beginning, we follow the projection of $\mathbf{R}_1$ on the surface of centers of $\mathbf{R}_0$ down to $\pi/6$. Then we will take $r(\theta) = a\theta$ from $\pi/6$ down to 0. We thus obtain S_1 and analogously S_2.

5.2. Technical steps

In polar coordinates p_1 is determined by an angle θ_1, that is $p_1 = (\sqrt{-\cos(2\theta_1)}, \theta_1, \sin(2\theta_1))$. Analogously, p_2 is determined by an angle θ_2 and $\mathbf{R}_2$ is determined by an angle α_2.

The $\mathbf{C}$-sphere S_1 around p_1 will be the union of $\mathbf{C}$-circles determined by the curve on the surface of centers defined by the projection of $\mathbf{R}_1$. It is

$$r_1(\theta) = r_1 \sin(\theta_1 - \alpha_1)/\sin(\theta - \alpha_1),\ r_1 = \sqrt{-\cos(2\theta_1)}.$$

Projections of the $\mathbf{C}$-circles will not have intersections if their radii grow faster than the distance to p_1. That will insure that the $\mathbf{C}$-circles themselves are not linked.

Lemma 5.1. *The family of* $\mathbf{C}$-*circles defined by the segment* $r_1(\theta)$ *from* θ_1 *to* θ *has neither self-intersections nor linking if*

$$\sin(2\theta) - \cos(2\theta_1)\sin(\theta_1 - \alpha_1)\cos(\theta_1 - \alpha_1)\frac{1}{\sin^2(\theta - \alpha_1)} > 0$$

Proof. A sufficient condition for no self intersection nor linking is that the function $R - d$, where R is the radius of a $\mathbf{C}$-circle and d is the distance of its center to p_1 be increasing as we go from $r_1(\theta_1)$ to $r_1(\theta)$. In that case the projection of the $\mathbf{C}$-circles are disjoint. We compute the derivative of $R^2 - d^2$ with respect to θ. As θ is decreasing as we go along $r_1(\theta)$, observe that the derivative should be negative so that $R - d$ be increasing. Now $R^2 = (r_1(\theta))^2 + \cos(2\theta)$ and $d^2 = r_1^2 + (r_1(\theta))^2 - 2r_1 r_1(\theta)\cos(\theta_1 - \theta)$. So $R^2 - d^2 = \cos(2\theta) - r_1^2 + 2r_1 r_1(\theta)\cos(\theta_1 - \theta)$. Differentiating the expression above we obtain

$$\frac{\mathrm{d}(R^2 - \mathrm{d}^2)}{\mathrm{d}\theta} = -2\sin(2\theta) + 2\cos(2\theta_1)\sin(\theta_1 - \alpha_1)\cos(\theta_1 - \alpha_1)\frac{1}{\sin^2(\theta - \alpha_1)}.$$

There are neither self-intersections nor linking if this derivative is negative and that proves the lemma. □

Corollary 5.1. *If $\theta_1 - \pi/4 \geqslant 0$ and $\alpha_1 + \pi/4$ are sufficiently small, the family of* **C**-*circles defined by the segment $r_1(\theta)$ from θ_1 down to $\pi/6$ has neither self intersections nor linking and does not intersect the z-plane.*

Proof. Observe that in the formula of the Lemma 5.1 the second term contains $\cos(\theta_1 - \alpha_1)$ which can be made arbitrarily small with our hypothesis. If r_1 is small enough then $r_1(\theta) < \sin(\theta)$. □

Observe that we could have instead of $\pi/6$ chosen any other angle smaller than θ_1. That angle serves as a reference for an interpolation with another part of the **C**-sphere. It is important, though, that it be smaller than $\pi/4$. If $\theta_1 = 3\pi/11$ and $\alpha_1 = -\pi/4$ the corollary can be applied as a simple verification of the inequality shows. The corresponding line $r_1(\theta)$ is shown in Fig. 1.

In order to verify whether **C**-circles are linked or not we will need the following:

Proposition 5.1. *A sufficient condition for a curve $r(\theta)$, in the surface of centers, to define a union of disjoint non-linked* **C**-*circles is that $(\mathrm{d}r/\mathrm{d}\theta)^2 \leqslant \cos^2(2\theta)/R^2 - r^2$, where R is an upper bound to the radii of the* **C**-*circles. In this case for $\theta < \theta_0$, the family of* **C**-*circles defined by the segment $r(\theta)$ is below the plane defined by $r(\theta_0)$.*

Proof. From the Proposition 2.3, if one verifies the inequality $2\mathrm{d}R_1(\theta) \leqslant t_1 - t_2 + 4S$ between any two points, the proposition will be proved. Observe that, as $S \geqslant 0$, it is sufficient to prove $2dR_1(\theta) \leqslant t_1 - t_2$. But if we assume that $R_1(\theta) \leqslant R$ for all θ this inequality is transitive, so we are

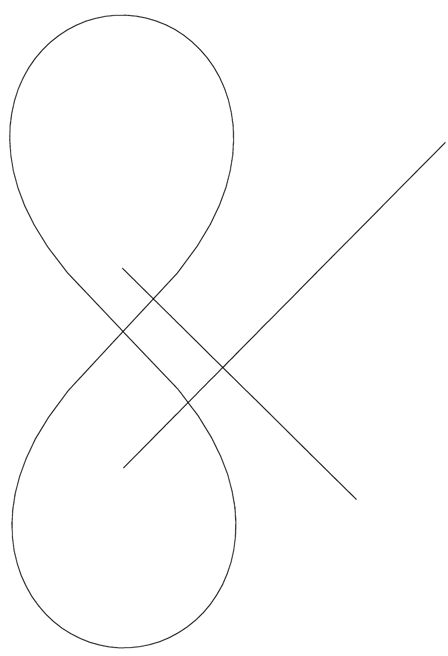

Fig. 1. Configuration: top view.

allowed to verify it infinitesimally. The inequality is obtained making the two points arbitrarily near each other.

Observe that the curve $r(\theta) = a\theta \leqslant \sin(\theta)$ from 0 to $\pi/6$ satisfies the condition above with $R = 1$ if a is small enough.

Proposition 5.2. *If $\theta_1 - \pi/4 \geqslant 0$ and $\alpha_1 + \pi/4$ are sufficiently small, the family of* **C***-circles defined by the segment $r_1(\theta)$ from θ_1* to *$\pi/6$ is above the plane defined by $r_1(\pi/6)$.*

Proof. Using the inequality $2dR_1 \leqslant t_1 - t_2 + 4S$, with the second point being $r_1(\pi/6)$ we obtain $2dR1 \leqslant \sin(2\theta) - \sin(\pi/3) + 4S$. One should first observe that if d is small enough and $R_1 \leqslant 1$ then the inequality is true from θ_1 to an angle slightly greater than $\pi/4$ and smaller than $\pi/3$. From that angle until $\pi/6$ we then observe that the sufficient condition of the proposition above holds when the hypothesis are satisfied. $\square$

Now we can describe a good axis. It consists of the segment $r_1(\theta)$ from θ_1 to $\pi/6$, the curve $r(\theta) = a\theta$ from $\pi/6$ to 0, with $a = r_1(\pi/6)/\pi/6$. Analogously, we construct the curve starting with the segment $r_2(\theta)$. In that case we have $\theta_2 - 7\pi/4 \leqslant 0$ and $\alpha_2 - \pi/4$ small or $\theta_2 - 3\pi/4 \leqslant 0$ and $\alpha_2 - \pi/4$ small (see Fig. 2). The propositions above guarantee that this is a good axis. The corresponding **C**-sphere has projection on the z-plane that is included in a finite circle centered at the origin with radius R_0.

We still have to construct invariant surfaces containing $\mathbf{R}_1$ and $\mathbf{R}_2$.

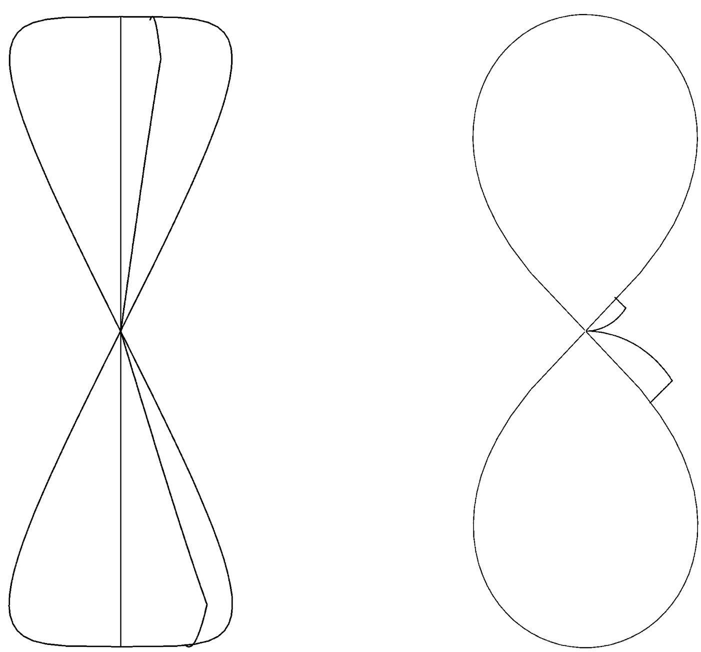

Fig. 2. Axis: lateral view (left) and top view (right) with the vertical axis.

Proposition 5.3. *Consider the* **C**-*circles whose centers are on* $\mathbf{R}_1$ *with radii equal to the corresponding* **C**-*circles whose centers are determined by the segment* $r_1(\theta)$, *from* θ_1 *to* $\pi/6$, *in the surface of centers. Then their union is disjoint and non-linked with the* **C**-*sphere determined by the axis defined above.*

Proof. Observe that the projection of the **C**-circles in the family coincides with the projection of the **C**-circles determined by the axis above the segment. The only possible intersection is for coincident **C**-circles. But each **C**-circle with center on the line $\mathbf{R}_1$ is above the one with center in the surface of centers. Furthermore all these **C**-circles are above the plane defined by $r(\pi/6)$ which is above all the **C**-circles of $\mathbf{R}_0$, for θ from 0 to $\pi/6$. □

To complete the construction in $\mathbf{R}_1$, we consider the concentric family of **C**-circles centered at the point above $r_1(\pi/6)$ with radii from $R_1(\pi/6)$ to $R_0 + r_1(\pi/6)$. The projection of this **C**-circle contains, in its interior, the projection of the **C**-sphere S_1.

Finally the projection of the point q_1 of $\mathbf{R}_1$ is near the origin if $\theta_1 - \pi/4 \geqslant 0$, $\alpha_1 + \pi/4$ and $\theta_2 - 3\pi/4 \leqslant 0$, $\alpha_2 - \pi/4$ or $\theta_2 - 7\pi/4 \leqslant 0$ and $\alpha_2 - \pi/4$ are sufficiently small. We can also suppose that $R_0 + r_1(\pi/6) < 2$ and that the **C**-circle of radius 2 centered at q_1 does not intersect the z-plane.

We can always construct a family of **C**-circles whose centers move in such a way that their projection move from the point determined by $\pi/6$ to q_1 and its radii from the radius at $R_0 + r_1(\pi/6)$ to 2.

An analogous construction is done for $\mathbf{R}_2$.

From the two points q_1 and q_2 in the lines whose projection is p we define **C**-circles whose centers are those points and increasing radii. The two families are clearly parallel and they do not intersect

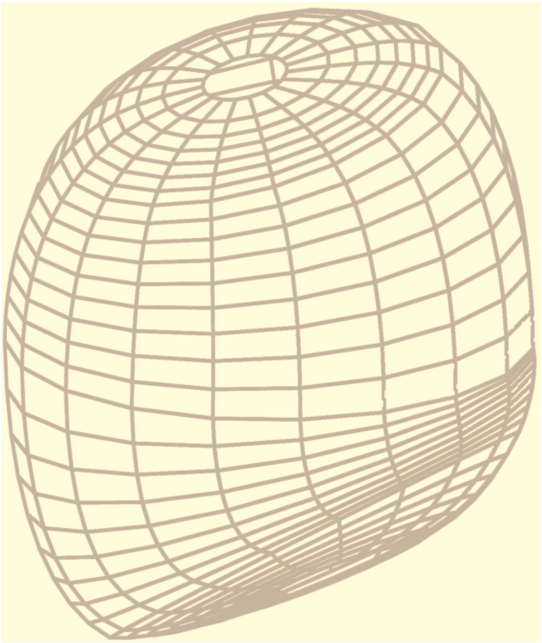

Fig. 3. C-sphere determined by the axis.

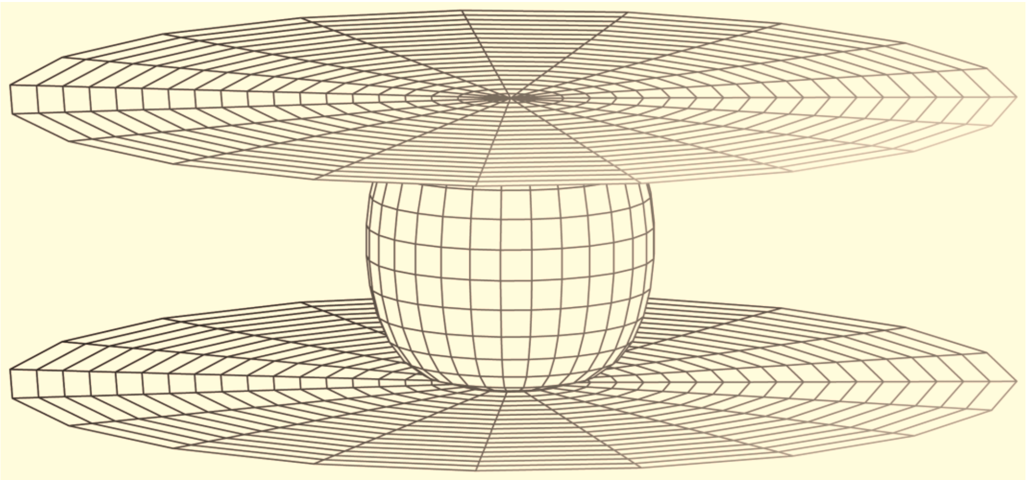

Fig. 4. A fundamental domain for the standard embedding.

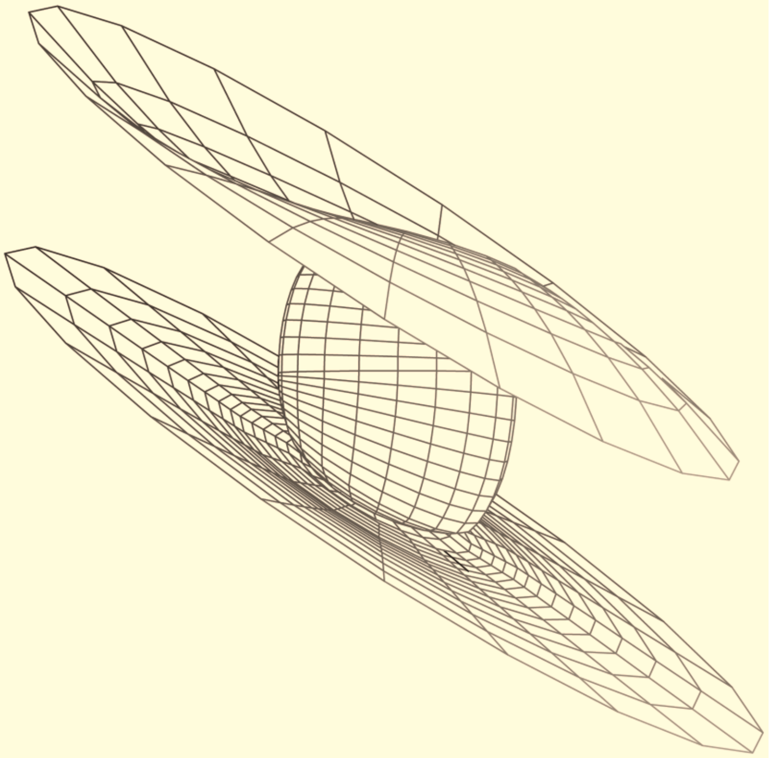

Fig. 5. A fundamental domain.

the **C**-sphere because their projections do not intersect any projection of **C**-circle on the **C**-sphere, their radii being greater than 2. In Figs. 3 and 5 we show the **C**-sphere and a fundamental domain for the following values; $\theta_1 = 3\pi/11, \theta_2 = -3\pi/10, \alpha_1 = -\pi/4, \alpha_2 = \pi/4$. One can follow the proof above for those values of the parameters (Figs. 3–5).

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