Exercises Sheet 5

1. Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by $(Tx)_n = \frac{1}{n+1}x_{n+1}$.
   Show that T is bounded and $\mathrm{sp}(T) = \{0\}$.
2. Let H be a Hilbert space and $A, B \in \mathcal{B}(H)$.
   (a) Show that AB and BA have the same spectral radius.
   (b) Show that $\mathrm{sp}(AB) \cup \{0\} = \mathrm{sp}(BA) \cup \{0\}$.
   (c) Give an example for which $\mathrm{sp}(AB) \neq \mathrm{sp}(BA)$.
3. Let H be a Hilbert space and $\mathcal{K}(H) \subset \mathcal{B}(H)$ be the set of compact operators.
   (a) Show that $\mathcal{K}(H)$ is a two-sided ideal in $\mathcal{B}(H)$.
   (b) Suppose that H is infinite dimensional. Show that if there exists $P \in \mathbb{C}[X]$ such that $P(0) \neq 0$ and $P(T) = 0$ then T is not compact.
4. Let T be the operator defined on $l^2(\mathbb{N})$ by $Te_n = \frac{1}{n+1}e_{n+1}$. Show that T is compact, compute the norm and the spectral radius. Compute the spectrum, the point spectrum and the dimensions of each eigenspace of the operators TT^* and T^*T.
5. Let A be the operator defined on $L^2(\mathbb{R}, \lambda)$ (λ is the Lebesgue measure) by
$$(Af)(x) = \sin(x)f(-x).$$
   Is A compact ?
6. Let (X, μ) be a σ-finite measure space. Define $K_\varphi : L^2(X, \mu) \to L^2(X, \mu)$ by
$$(K_\varphi\xi)(x) = \int_X \varphi(x, y)\xi(y)d\mu(y) \quad \text{where} \quad \varphi \in L^2(X \times X, \mu \times \mu).$$
   (a) Show that K_φ is bounded and compute the adjoint.
   (b) Show that K_φ is compact.

   Such an operator is called an *integral* operator, φ is called the *kernel*.
7. Let H be the Hilbert space $L^2([0, \pi/2], \mu)$, where μ is the Lebesgue measure on $[0, \pi/2]$. for all $f \in H$, define Tf on $[0, \pi/2]$ by
$$\forall x \in [0, \pi/2], \quad (Tf)(x) = \sin(x)\int_0^x \cos(t)f(t)dt + \cos(x)\int_x^{\pi/2} \sin(t)f(t)dt.$$
   (a) Show that T is a self-adjoint and compact operator.
   (b) Show that if f is continuous then Tf is C^2 on $[0, \pi/2]$ and the function $G = Tf$ satisfies the differential equation $G'' + G = f$ with the boundary conditions $G'(0) = G'(\pi/2) = 0$.

(c) Compute the spectrum of T.

8. Let V be the *Volterra* operator defined on $\mathrm{L}^2([0,1],\lambda))$ by $(Vf)(x) = \int_0^x f(t)dt$.
   (a) Show that V is a compact operator.
   (b) Compute V^* and V^*V.
   (c) Compute the point spectrum and the spectrum of V^*V.
   (d) Deduce the value of $\|V\|$.
9. Let φ be a continuous complex valued function defined on $[0,1]^2$ and a, b continuous functions from $[0,1]$ to itself. Define, for $f \in C([0,1])$,

$$(T_\varphi f)(x) = \int_{a(x)}^{b(x)} \varphi(x,y)f(y)dy.$$

   Show that T_φ is a compact operator on $C([0,1])$.