A SIMPLE DYNAMICAL MODEL LEADING TO PARETO WEALTH DISTRIBUTION AND STABILITY

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ABSTRACT. We propose a simple dynamical model of wealth evolution. The invariant distributions are of Pareto type and are dynamically stable as conjectured by Pareto.

1. INTRODUCTION.

At the end of the XIXth century, in his studies of wealth and income [1] distribution on different countries, Vilfredo Pareto ([7], [8]) discovered the universal power law that governs the upper tail of wealth distribution. It is well known that this is not a good model for the lower part of the curve that is more dependent on specific sociological factors and of log-normal type (see the discusion in [6]). The exponent in the power decay is country dependent and is an indicator of equitative wealth (re)distribution. A larger exponent indicates a more equitative wealth distribution. Pareto's universal assymptotic behaviour appears in distributions from various other contexts, and, as we show, is typical from competitive system where the reward is proportional to the accumulated wealth. The purpose of this article is to provide a simple explanation to Pareto's empirical observation. We propose a natural dynamical model of evolution of wealth where Pareto distributions emerge as invariant dynamically stable [2] distributions of this Dynamical System. The stability of the wealth distribution, which is different from the universality property, was conjectured by Pareto, whose intuition apparently comes from his empirical observations. We can read in [8], chap. VII, point 31, p.393:

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[1]Wealth and income are proxies of each other in first approximation for our purposes.

[2]"Dynamically stable distribution" in the Dynamical System sense not in the probabilistic sense.

Si, par exemple, on enlevait tout leur revenu aux citoyens les plus riches, en sup-primant la queue de la figure des revenus, celle-ci ne conserverait pas cette forme, mais tôt ou tard elle se rétablirait suivant une forme semblable à la première.[3]

There are other classical models and studies of Pareto empirical observation and power laws (like Zipf's law). For the record we cite a few classical ones: H. Simon [11], D.G. Champernowne [2], B. Mandelbrot [6],etc Simon model [11] for Zipf's law is a "genesis model" of the distribution, i.e. it is a model for its creation. Champernowne [2] proposed a general multiplicative stochastic model, and B. Mandelbrot [6] explained Pareto law by the universal limit character of Pareto-Lévy probabilistically stable distributions.

2. The dynamical model.

In this first section, we propose and study a dynamical model of wealth evolution which is a simple first approximation.

1. **Setup.** Let $f(x)$ be the wealth distribution, i.e. $df = f(x) \, dx$ is the number of individuals with wealth in the infinitesimal interval $[x, x + dx[$. The distribution function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, positive and decreasing and $\lim_{x \to +\infty} f(x) = 0$. A distribution is of Pareto type if it presents a power law decay $x^{-\alpha}$ at $+\infty$, that is

$$\lim_{x \to +\infty} -\frac{\log f(x)}{\log x} = \alpha > 0 \ .$$

The exponent $\alpha > 0$ is *Pareto exponent*. A distribution of the form $f(x) = C.x^{-\alpha}$ is called a Pareto distribution. Smaller values of α indicate larger inequalities in wealth distribution. Notice that $\alpha > 1$ is necessary for the distribution to be summable at $+\infty$, i.e. finite wealth at infinite (finitness near 0 is not significant since the model aims to explain the tail behaviour at $+\infty$).

2. **Wealth dynamics.** We focuss on the evolution of individual wealth. We assume that the evolution is based on two main factors: Finantial decisions, that we model as a betting game, and by public redistribution of wealth, that absorbs part of the individual wealth into public wealth.

For the first factor we model the finantial decisions of each individual by a sequence of bets. Each financial decision turns out to be a bet, waging a proportion of his wealth. As a first approximation, we assume that the probability of success is the same for all agents and bets $0 < p < 1$ (this is the average probability). At each

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[3] *"If, for instance, we confiscate all income to the richests citizens, thus erasing the tail of income distribution, this shape will not persist and sooner or later it will evolve to a similar shape of the original."*

round, each agent risks the same percentage of his wealth, a fraction $\gamma > 0$ (that is also an average). If he wins, his wealth is multiplied by the factor $1 + \gamma$ and if he looses his wealth is divided by $1 + \gamma$.

Only considering this first factor, one round evolution the distribution transforms into the new distribution

$$\mathcal{W}(f)(x) = \frac{p}{1 + \gamma} \, f(x/(1 + \gamma)) + (1 - p)(1 + \gamma) \, f((1 + \gamma)x) \ .$$

The operator $\mathcal{W}$ is "wealth preserving". In terms of L^1-norm we have

$$||\mathcal{W}(f)||_{L^1} = ||f||_{L^1} \ .$$

The agents will only risk their capital if there is a positive expectation of gain, thus we should assume that $p > 1/2$.

There are other mechanisms that affect wealth evolution that we should consider, as for example inheritances that divide wealth, taxes, etc. Note that public wealth drains individual wealth by the fiscal mechanism. Thus it is natural to consider a broader class of operators $\mathcal{W}$ with a dissipative parameter $\kappa \geq 1$, the *dissipative coefficient*,

$$\mathcal{W}_\kappa(f)(x) = \frac{1}{\kappa}\mathcal{W}(f)(x) = \frac{p}{\kappa(1 + \gamma)}f\left(x/(1 + \gamma)\right) + \frac{(1 - p)(1 + \gamma)}{\kappa}f((1 + \gamma)x) \ ,$$

so that for $\kappa = 1$ the operator is wealth preserving. We name the model for $\kappa = 1$ the "wealth preserving model".

3. **Invariant distributions.** Distributions invariant by the evolution operator $\mathcal{W}_\kappa$ must satisfy the fixed point functional equation $\mathcal{W}_\kappa(f) = f$, that is,

(1) $$f(x) = \frac{p}{\kappa(1 + \gamma)}f\left(x/(1 + \gamma)\right) + \frac{(1 - p)(1 + \gamma)}{\kappa}f((1 + \gamma)x) \ .$$

We solve this equation in the next section.

4. **Solution of the functional equation.** Considering the change of variables $F(x) = f(e^x)$, equation (1) becomes a functional equation for $F : \mathbb{R} \to \mathbb{R}$

(2) $$a \, F(x + \lambda) - F(x) + b \, F(x - \lambda) = 0 \ ,$$

where $\lambda = \log(1 + \gamma) > 0$, $a = (1 - p)(1 + \gamma)/\kappa > 0$ and $b = p/\kappa/(1 + \gamma) > 0$.

We have a general theory of these type of functional equations. L. Schwartz (see [10], and also [5], [3]) studied more general "mean periodic" smooth functions F that satisfy a functional equation of the form

$$\omega \star F = 0 \ ,$$

where ω is a compactly supported distribution. In our case, $\omega = a\,\delta_\lambda - \delta_0 + b\,\delta_{-\lambda}$[4].

In our simplified model we don't need the general theory and the equation can be solved by elementary means. First, the exponential solutions are easy to calculate. A function $F(x) = e^{\rho x}$ is a solution if $e^{\rho\lambda}$ satisfies the following second degree equation:

$$(3) \qquad\qquad a\left(e^{\rho\lambda}\right)^2 - \left(e^{\rho\lambda}\right) + b = 0 \ .$$

Observe that the discriminant $\Delta = 1 - 4ab$ is positive since we have

$$ab = \frac{p(p-1)}{\kappa^2} < \frac{1}{4\kappa^2} \ ,$$

thus $\Delta > 1 - \frac{1}{\kappa^2} > 0$ because $\kappa \geq 1$.

Thus we have two distinct solutions:

$$e^{\rho\lambda} = \frac{1}{2a} \pm \frac{1}{2a}\sqrt{1 - 4ab} \ .$$

Since $a > 0$ and the polynomial $P(x) = ax^2 - x + b$ satisfies $P(0) > 0$ and $P(1) < 0$, we have two real root x_1 and x_2 with $0 < x_1 < 1 < x_2$. Therefore, we have two families of solutions for ρ in two vertical lines in the complex domain, for $k \in \mathbb{Z}$, $j = 0, 1$,

$$\rho_{j,k} = \lambda^{-1}\log x_j + 2\pi i k \lambda^{-1} \ .$$

Note that $\Re\rho_{0,k} < 0 < \Re\rho_{1,k}$. Let $\rho_0 = \rho_{0,0} < 0$ and $\rho_1 = \rho_{1,0} > 0$. Observe that the particular solution $F(x) = C.e^{\rho_0 x}$ leads to the solution $f(x) = F(\log x) = C.x^{\rho_0}$ which is exactly Pareto distribution with Pareto exponent $\alpha = -\rho_0$.

We can now solve the functional equation completely[5]:

Theorem 1. *The general solution of the functional equation (2),*

$$(4) \qquad\qquad a\,F(x+\lambda) - F(x) + b\,F(x-\lambda) = 0 \ ,$$

(with a, b, λ as above) is

$$F(x) = e^{\rho_0 x}L_0(x/\lambda) + e^{\rho_1 x}L_1(x/\lambda)$$

where L_0 and L_1 are $\mathbb{Z}$-periodic functions.

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[4]The way to study these equations is by Fourier transforming it (à la Carleman [1] using hyperfunctions in order to work in sufficient generality). One of the general results by L. Schwartz (see [10] Theorem 10 p.894) is the "spectral synthesis" of solutions: Smooth solutions are uniform limits on compact set of $\mathbb{R}$ of linear combinations of exponential solutions $(e^{\rho x})_\rho$. Also these exponential solutions are not limits of linear combinations of the others, thus the expansion is unique.

[5]We have a strong form of Schwartz spectral theorem.

In order to solve the functional equation (2), we consider $H(x) = F(x + \lambda) - e^{\rho_0 \lambda} F(x)$. Substracting (2) from (3) multiplied by $e^{-\rho_0 \lambda} F(x)$ we get

$$aH(x) - be^{-\rho_0 \lambda} H(x - \lambda) = 0 \ ,$$

or

$$H(x) = \left(\frac{b}{a} \ e^{-\rho_0 \lambda} \right) H(x - \lambda) \ .$$

Considering

$$\hat{H}(x) = \left(\frac{b}{a} \ e^{-\rho_0 \lambda} \right)^{-x/\lambda} H(x) \ ,$$

we have that $\hat{H}(x) = \hat{H}(x - \lambda)$, i.e. there is a $\mathbb{Z}$-periodic function L such that

$$H(x) = \left(\frac{b}{a} \ e^{-\rho_0 \lambda} \right)^{x/\lambda} L(x/\lambda) \ .$$

Therefore we have

$$F(x + \lambda) - e^{\rho_0 \lambda} F(x) = \left(\frac{b}{a} \ e^{-\rho_0 \lambda} \right)^{x/\lambda} L(x/\lambda) \ .$$

Now, put

$$\hat{F}(x) = e^{-\rho_0 x} F(x) \ .$$

Then we need to solve

$$\hat{F}(x + \lambda) - \hat{F}(x) = e^{-\rho_0 \lambda} \left(\frac{b}{a} \right)^{x/\lambda} e^{-2\rho_0 x} L(x/\lambda) \ ,$$

if we write $G(x) = e^{\rho_0 \lambda} \hat{F}(x)$ and $c = -2\rho_0 + \lambda^{-1} \log(b/a)$,

$$G(x + \lambda) - G(x) = e^{cx} L(x/\lambda) \ .$$

We use the following lemma:

Lemma 2. *For $c \in \mathbb{R}$, $\lambda > 0$, and L a $\mathbb{Z}$-periodic function, the solutions of the functional equation*

(5)
$$G(x + \lambda) - G(x) = e^{cx} L(x/\lambda) \ ,$$

are of the form

$$G(x) = G_0(x) + M(x/\lambda) \ ,$$

where M is a $\mathbb{Z}$-periodic function, and for $c \neq 0$,

$$G_0(x) = \frac{e^{cx}}{e^{c\lambda} - 1} \ L(x/\lambda) \ ,$$

and for $c = 0$

$$G_0(x) = \lambda^{-1} x \ L(x/\lambda) \ .$$

Proof. Obviously in both cases G_0 is a particular solution. Then the functional equation is equivalent to $M(x+1) - M(x) = 0$, where $M(x) = G(\lambda x) - G_0(\lambda x)$, i.e. M is $\mathbb{Z}$-periodic. $\qquad \square$

So, in the non-degenerate case $(c \neq 0)$, absorbing the multiplicative constants into L and M, the general solutions of (2) are of the form

$$F(x) = e^{(-\rho_0 + \lambda^{-1} \log(b/a))x} L(x/\lambda) + e^{\rho_0 x} M(x/\lambda) \ .$$

And coming back to the second degree equation (3) we have

$$e^{-\rho_0 \lambda} \frac{b}{a} = e^{\rho_1 \lambda} \ ,$$

so

$$F(x) = e^{\rho_1 x} L(x/\lambda) + e^{\rho_0 x} M(x/\lambda) \ .$$

Indeed the degenerate case never happens:

Lemma 3. *We have* $c \neq 0$.

Proof. If $c = 0$ then $e^{2\rho_0 \lambda} = b/a = e^{\rho_0 \lambda} . e^{\rho_1 \lambda}$ and $e^{\rho_0 \lambda} = e^{\rho_1 \lambda}$, the root of the second degree equation would be double and the discriminant would be $\Delta = 0$ but we have seen that $\Delta > 0$. $\qquad \square$

If we request that $F > 0$ and $F(x) \to 0$ for $x \to +\infty$ (the only sound solutions) then $L = 0$ and $M > 0$,

$$F(x) = e^{\rho_0 x} M(x/\lambda) \ .$$

Finally we have

$$f(x) = x^{\rho_0} M(\lambda^{-1} \log x) \ .$$

If we look for continuous solutions, then M must be continuous and bounded since it is $\mathbb{Z}$-periodic, thus f satisfies Pareto assymptotics

$$\lim_{x \to +\infty} -\frac{\log f(x)}{\log x} = -\rho_0 = \alpha > 0 \ .$$

5. **Pareto exponent.** It is interesting that we can compute an explicit expression of the Pareto exponent in terms of the parameters κ, γ and p,

Corollary 4. *The Pareto exponent is given by*

$$\alpha = -\rho_0 = -\lambda^{-1} \log \left(\frac{1 - \sqrt{1 - 4ab}}{2a} \right)$$

or

$$\alpha = 1 - \frac{\log \left(\frac{\kappa - \sqrt{\kappa^2 - 4p(1-p)}}{2(1-p)} \right)}{\log(1 + \gamma)} \ .$$

It is interesting to note that the Pareto exponent α decreases when γ increases. This means that a more risky finantial behaviour, or more active economy, favours unequal distribution. Fortunes are created and lost more often. Ruin is more common. Indeed we know by the Kelly criterion [4] that ruin is almost sure in the long run if γ is larger than a certain threshold. With a slightly modified model we can explain Pareto's theory of "Circulation of Elites". Indeed this circulation occurs at all level of social status when the agents are not enough conservative to satisfy Kelly criterion. We will discuss these questions in a companion article [9].

The Pareto exponent also increases with κ since

$$\frac{d\alpha}{d\kappa} = \frac{1}{\log(1+\gamma)} \frac{\kappa - \sqrt{\kappa^2 - 4p(1-p)}}{\sqrt{\kappa^2 - 4p(1-p)}\left(\kappa - \sqrt{\kappa^2 - 4p(1-p)}\right)} \ ,$$

is positive. This is natural since a larger κ means a larger demographic and fiscal pressure and thus we expect a better redistribution of wealth and a larger Pareto exponent.

6. A remarkable solution in the wealth preserving model.
A Pareto exponent $\alpha > 1$ is necessary for summability of the tail of the distribution and is always observed in experimental studies. It is remarkable that in the wealth preserving model with the critical value of the dissipative coefficient $\kappa = 1$, the Pareto exponent is exactly $\alpha = 1$.

Theorem 5. *In the wealth preserving model, $\kappa = 1$, the Pareto exponent is exactly equal to $\alpha = 1$.*

Proof. For $\kappa = 1$ we have

$$\kappa^2 - 4p(1-p) = (2p-1)^2 \ .$$

Therefore

$$\kappa - \sqrt{\kappa^2 - 4p(1-p)} = 2(1-p) \ .$$

And the formula in the previous section gives $\alpha = 1$. $\square$

This result is natural and to be expected: For $\kappa < 1$ the wealth in increasing without limit and the invariant distributions could not be summable at $+\infty$, and for $\kappa > 1$ we have finite wealth at $+\infty$. From the form of the invariant solutions, we have:

Theorem 6. *For an invariant solution, the following conditions are equivalent:*

(1) *The tail wealth is summable, $W(f, x_0) < +\infty$.*
(2) *The Pareto exponent α is larger than 1, $\alpha > 1$.*
(3) *The model is wealth dissipative, that is $\kappa > \kappa_0$.*

It has been observed that the Pareto exponent of the wealthiest fraction of the population has a Pareto exponent which is much closer to 1 than expected (or to the rest of the medium class, whatever this means). So for this class of the population the dissipative coefficient is closer to the critical one κ_0, this means that the wealthiest part of the population is able to avoid the mechanisms of fiscal redistribution of wealth.

7. **Stability of invariant solutions.** We now study the Pareto problem of stability of the Pareto distribution.

Since $\kappa > 1$, we can observe that for the L^1-norm the operator $\mathcal{W}_\kappa$ is contracting:

Lemma 7. *Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+$ be measurable functions , with $f - g \in L^1(\mathbb{R}_+^*)$, then*

$$||\mathcal{W}_\kappa(f) - \mathcal{W}_\kappa(g)||_{L^1} \leq \kappa^{-1}||f - g||_{L^1} \ .$$

Proof. We have

$$|\mathcal{W}_\kappa(f)(x) - \mathcal{W}_\kappa(g)(x)| \leq \frac{p}{\kappa(1+\gamma)} |f(x/(1+\gamma)) - g(x/(1+\gamma))|$$
$$+ \frac{(1-p)(1+\gamma)}{\kappa} |f(x(1+\gamma)) - g(x(1+\gamma))|$$

and the result follows integrating over $\mathbb{R}_+^*$. $\square$

Obviously this lemma is only interesting when $||f-g||_{L^1}$ is finite. For each invariant solution f_0 it is natural to consider the space of measurable bounded perturbations of f_0 for the L^1-norm, $\mathcal{M}(\mathbb{R}_+^*, \mathbb{R})$ denotes the space of Borel measurable functions,

$$\mathcal{S}_{f_0} = \{g \in \mathcal{M}(\mathbb{R}_+^*, \mathbb{R}); ||g - f_0||_{L^1} < +\infty\} \ .$$

Then the fixed point f_0 is a global attractor in $\mathcal{S}_{f_0}$ and we have:

Theorem 8. *For any $g \in \mathcal{S}_{f_0}$, we have that $\mathcal{W}_\kappa^n(g) \to f_0$ for the L^1-norm at a geometric rate.*

This proves the Pareto stability conjecture, exactly as stated by Pareto (see the citation in the introduction): If we remove all wealth larger than some value x from the invariant solution, then the perturbation thus obtained is L^1 bounded because of summability of the tail, hence the stability.

3. Other more refined models.

With the same ideas, we can build more sophisticated models that will be studied in the future. The main difference with the model presented here is that the invariant solutions cannot be computed explicitely in general, nor we can give close formulas

for the Pareto exponents. But this does not prevent numerical studies of the invariant solutions.

We may more realistically assume that there are different sorts of individuals with different skills for finantial investment (different p's), and different risk profiles (different γ's). If we assume that each class of individuals is equally represented accross wealth classes (which is not true, the more skilled ones should be more numerous in the upper classes), then we end with a general wealth operator of the form

$$\mathcal{W}_\kappa(f) = \sum_i \frac{p_i(1+\gamma_i)}{\kappa} f(x/(1+\gamma_i)) + \frac{q_i(1+\gamma_i)}{\kappa} f(x(1+\gamma_i)) \ ,$$

with

$$\sum_i p_i + \sum_i q_i = 1 \ .$$

The exponentials of the Pareto exponents appear then as roots of a Dirichlet polynomial. One can prove, using results from [10] that the invariant solutions obey Pareto law.

A more realistic model consists in allowing the dissipative coefficient κ to be non constant and make it dependent on x. In principle, $x \mapsto \kappa(x)$ should be increasing. Then the search for invariant solutions leads to a functional equation with non-constant coefficients whose possible explicit resolution depends on the form of the function $x \mapsto \kappa(x)$.

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