# ON QUASI-INVARIANT CURVES

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ABSTRACT. Quasi-invariant curves are used in the study of hedgehog dynamics. Denjoy-Yoccoz lemma is the preliminary step for Yoccoz's complex renormalization techniques for the study of linearization of analytic circle diffeomorphisms. We give a geometric interpretation of Denjoy-Yoccoz lemma using the hyperbolic metric that gives a direct construction of quasi-invariant curves without renormalization.

## 1. INTRODUCTION.

Yoccoz's approach to linearization of analytic circle diffeomorphisms ([20], [21]) is based on complex sectorial renormalizations. These techniques were first used in his celebrated proof of the optimality of the Brjuno condition ([10], [19]).

The sectorial renormalization construction needs enough space around the circle, or, in other words, to have an analytic circle diffeomorphism that extends to an annulus of large modulus. For this, one needs to get first a good control on the real estimates on the Schwarzian derivative and non-linearity for high iterates of smooth circle diffeomorphisms that were developed by M. Herman [7] and J–Ch. Yoccoz [18]. The estimates on the non-linearity, allows to control long orbits outside the circle using a Denjoy type lemma. This Denjoy-Yoccoz lemma is Proposition 4.4 in section 4.4 of [21].

The first application of Denjoy-Yoccoz lemma is to carry-out a sectorial renormalization in order to obtain an analytic circle diffeomorphism which extends in a large annulus containing the circle (section 3.6 of [21]). The analysis of the linearization problem proceeds by successive renormalizations of two types. We have to distinguish when the rotation number is small or large compared to the inverse of the modulus of the annulus. In the first situation with a small rotation number, the lemma is fundamental.

**Non-linearizable dynamics.** Sectorial renormalizations are useful in the non-linearizable situation. They were used by the author to study the dynamics of hedgehogs. Hedgehogs associated to an indifferent irrational non-linearizable fixed point are full non-trivial compact connected sets totally invariant by the dynamics. Hedgehogs were discovered by the author in [15]. They are similar to Birkhoff topological invariant compact connected sets associated to Lyapunov unstable fixed points (see [1]), but they are totally invariant in the holomorphic situation for indifferent fixed points. The topology of hedgehogs is always involved and not completely elucidated (see [2], [3], [12]). Despite this, the dynamics on the hedgehog can be analyzed and exhibits remarkable rigidity properties. In some heuristic sense, the restriction of the dynamics to the hedgehog behaves as a complex automorphism of the disk with a fixed point, that is, as a rigid rotation. So, for example, the $q_n$ iterates of the dynamics, $(q_n)$ being the sequence of denominators of the convergents of the rotation number $\alpha$, converge uniformly to the identity. Thus the dynamics is uniformly recurrent.
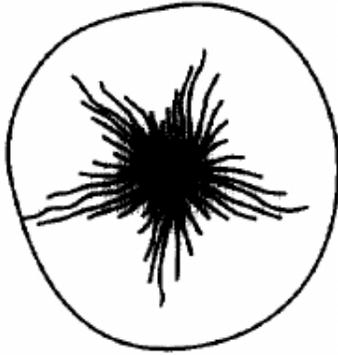


FIGURE 1. A hedgehog and its defining neighborhood.

Hedgehogs and their dynamics are the main ingredient for the analysis of the general non-linearizable dynamics. For example, we have the following Theorem for which no proof is known without using hedgehogs.

**Theorem 1** ([14], [16]). *There is no orbit converging by positive or negative iteration to an indifferent irrational fixed point of an holomorphic map and distinct from the fixed point.*

This Theorem is the key to the solution of some old problems. It solves a question of P. Fatou (1920, [6]), and a question for singularities of differential equations in the complex domain due to É Picard (1896, [9] p.30) and H. Dulac (1904, [5] p. 7, case 2). The original problem was raised by C. Briot and J.-C. Bouquet in 1856 ([4], section 85). For the relation with singularities of holomorphic foliations in $\mathbb{C}^2$ in

the Siegel domain we refer to [17] where we establish with J.-Ch Yoccoz a complete caracterization by the holonomy.

The main tool for studying the dynamics on the hedgehog is the construction of a sequence $(\gamma_n)$ of quasi-invariant Jordan curves that surround and osculate the hedgehog. The Jordan domains $\Omega_n$ bounded by $\gamma_n$ are neighborhoods of the hedgehog. These curves are close to the hedgehog and are almost invariant by high iterates of the dynamics. Moreover, the $q_n$-iterates of the dynamics are close to the identity on these curves. One concludes, using the maximum principle, that the same happens on the hedgehog. Also, orbits near these quasi-invariant curves travel all around and form an $\epsilon_n$-dense orbit. In particular they cannot jump inside the domain $\Omega_n$ without $\epsilon_n$-visiting all points of $\gamma_n$. This is the idea behind the proof of Theorem 1.

Quasi-invariant curves are constructed for analytic circle diffeomorphisms in a complex tubular neighborhood of the circle. The relation between indifferent fixed points and analytic circle diffeomorphisms was elucidated using hedgehogs by a construction presented in [15]. More precisely, for any indifferent irrational non-linearizable fixed point of an holomorphic map $f(z) = e^{2\pi i \alpha} z + \mathcal{O}(z^2)$, $\alpha \in \mathbb{R} - \mathbb{Q}$, in a neighborhood of 0 where $f$ and $f^{-1}$ are well defined, there exists a non-trivial (i.e. larger than the fixed point) full compact connected set $K$ which is totally invariant

$$K = f(K) = f^{-1}(K) .$$
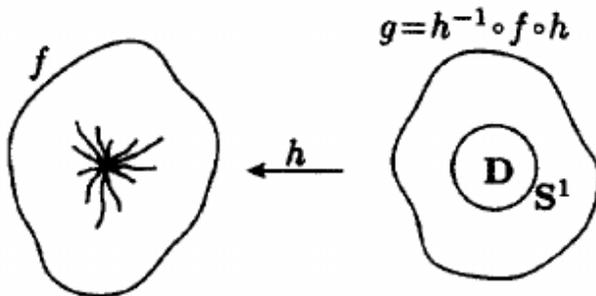
The compact $K$ is a hedgehog.



FIGURE 2. Dictionary between fixed points and circle maps.

We consider a conformal representation $h : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K$ ($\mathbb{D}$ is the unit disk), and we conjugate the dynamics to a univalent map $g$ in an annulus $V$ having the circle $\mathbb{S}^1 = \partial \mathbb{D}$ as the inner boundary,

$$g = h^{-1} \circ f \circ h : V \to \mathbb{C} .$$

One can show ([12], [15]) that $K$ is not locally connected and $h$ does not extend to a continuous correspondence between $\mathbb{S}^1$ and $K$. Nevertheless, it is easy to prove that $f$ extends continuously to Caratheodory's prime-end compactification of $\mathbb{C} - K$. This shows that $g$ extends continuously to $\mathbb{S}^1$ and its Schwarz reflection defines an analytic map of the circle defined on $V \cup \mathbb{S}^1 \cup \bar{V}$, where $\bar{V}$ is the reflected annulus of $V$. Then we can show that $g$ is an analytic circle diffeomorphism with rotation number $\alpha$. Therefore, the dynamics in a complex neighborhood of a hedgehog corresponds to the dynamics of an analytic circle diffeomorphism.

The properties of quasi-invariant curves for $g$ from [14] and [15] can be formulated using the Poincaré metric of the exterior of the closed unit disk:

**Theorem 2.** *(Quasi-invariant curves) Let $g$ be an analytic circle diffeomorphism with irrational rotation number $\alpha$. Let $(p_n/q_n)_{n \geq 0}$ be the sequence of convergents of $\alpha$ given by the continued fraction algorithm.*

*Given $C_0 > 0$ there is $n_0 \geq 0$ large enough such that there is a sequence of quasi-invariant curves $(\gamma_n)_{n \geq n_0}$ for $g$ which are Jordan curves homotopic to $\mathbb{S}^1$ and exterior to $\overline{\mathbb{D}}$ such that all the iterates $g^j$, $0 \leq j \leq q_n$, are defined on a neighborhood of the closure of the annulus $U_n$ bounded by $\mathbb{S}^1$ and $\gamma_n$, and we have*

$$\mathcal{D}_P(g^j(\gamma_n), \gamma_n) \leq C_0 \ ,$$

*where $\mathcal{D}_P$ denotes the Hausdorff distance between compact sets associated to $d_P$, the Poincaré distance in $\mathbb{C} - \overline{\mathbb{D}}$. We also have for any $z \in \gamma_n$, $d_P(g^{q_n}(z), z) \leq C_0$, that is,*

$$||g^{q_n} - \mathrm{id}||_{C_P^0(\gamma_n)} \leq C_0 \ .$$

The delicate, and useful, part of the construction of quasi-invariant curves is to obtain the estimates for the Poincaré metric, which is much stronger than the estimates for the euclidean metric since the curves $\gamma_n$ are close to $\mathbb{S}^1$. This is also what is needed in order to transport the curves by the conformal representation $h$ and keep the estimates in the dynamical plane with the fixed point for $f$ for the Poincaré metric outside of the hedgehog. We give a new construction of quasi-invariant curves without using renormalization. It builds on the simple observation that Denjoy-Yoccoz Lemma has a natural hyperbolic interpretation. We carry out in this article the simpler construction that is sufficient for the main applications. We assume that the non-linearity of $g$ is small, that is, $||D \log Dg||_{C^0} < \epsilon_0$. The general case is done by carrying out a purely real renormalization as in section 3.6 of [21] that yields a circle map with arbitrarely small non-linearity. Then we transport (only once) the quasi-invariant curves by the purely real renormalization that extends to a sectorial renormalization in a small complex neigborhood of the circle as in [16].

## 2. Analytic circle diffeomorphisms.

2.1. **Notations.** We denote by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the abstract circle, and $\mathbb{S}^1 = E(\mathbb{T})$ its embedding in the complex plane $\mathbb{C}$ given by the exponential mapping $E(x) = e^{2\pi i x}$.

We study analytic diffeomorphisms of the circle, but we prefer to work at the level of the universal covering, the real line, with its standard embedding $\mathbb{R} \subset \mathbb{C}$. We denote by $D^\omega(\mathbb{T})$ the space of non decreasing analytic diffeomorphisms $g$ of the real line such that, for any $x \in \mathbb{R}$, $g(x+1) = g(x) + 1$, which is the commutation to the generator of the deck transformations $T(x) = x + 1$. An element of the space $D^\omega(\mathbb{T})$ has a well defined rotation number $\rho(g) \in \mathbb{R}$. The order preserving diffeomorphism $g$ is conjugated to the rigid translation $T_{\rho(g)} : x \mapsto x + \rho(g)$, by an orientation preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$, such that $h(x+1) = h(x) + 1$.

For $\Delta > 0$, we note $B_\Delta = \{z \in \mathbb{C}; |\Im z| < \Delta\}$, and $A_\Delta = E(B_\Delta)$. The subspace $D^\omega(\mathbb{T}, \Delta) \subset D^\omega(\mathbb{T})$ is composed by the elements of $D^\omega(\mathbb{T})$ which extend analytically to a holomorphic diffeomorphism, denoted again by $g$, such that $g$ and $g^{-1}$ are defined on $B_\Delta$.

2.2. **Real estimates.** We refer to [21] for the results on this section. We assume that the orientation preserving circle diffeomorphism $g$ is $C^3$ and that the rotation number $\alpha = \rho(g)$ is irrational. We consider the convergents $(p_n/q_n)_{n \geq 0}$ of $\alpha$ obtained by the continued fraction algorithm (see [8] for notations and basic properties of continued fractions).

For $n \geq 0$, we define the map $g_n(x) = g^{q_n}(x) - p_n$ and the intervals $I_n(x) = [x, g_n(x)]$, $J_n(x) = I_n(x) \cup I_n(g_n^{-1}(x)) = [g_n^{-1}(x), g_n(x)]$. Let $m_n(x) = g^{q_n}(x) - x - p_n = \pm|I_n(x)|$, $M_n = \sup_{\mathbb{R}} |m_n(x)|$, and $m_n = \min_{\mathbb{R}} |m_n(x)|$. Topological linearization is equivalent to $\lim_{n \to +\infty} M_n = 0$. This is always true for analytic diffeomorphisms by Denjoy's Theorem, that holds for $C^1$ diffeomorphisms such that $\log Dg$ has bounded variation.

Since $g$ is topologically linearizable, combinatorics of the irrational translation (or the continued fration algorithm) shows:

**Lemma 3.** *Let $x \in \mathbb{R}$, $0 \leq j < q_{n+1}$ and $k \in \mathbb{Z}$ the intervals $g^j \circ T^k(I_n(x))$ have disjoint interiors, and the intervals $g^j \circ T^k(J_n(x))$ cover $\mathbb{R}$ at most twice.*

We have the following estimates on the Schwarzian derivatives of the iterates of $f$, for $0 \leq j \leq q_{n+1}$,

$$\left| Sg^j(x) \right| \leq \frac{M_n e^{2V} S}{|I_n(x)|^2} \ ,$$

with $S = ||Sg||_{C^0(\mathbb{R})}$ and $V = \operatorname{Var} \log Dg$.

These estimates imply a control of the non-linearity of the iterates (Corollary 3.18 in [21]):

**Proposition 4.** *For* $0 \leq j \leq 2q_{n+1}$, $c = \sqrt{2S}e^V$,

$$||D \log Dg^j||_{C^0(\mathbb{R})} \leq c \, \frac{M_n^{1/2}}{m_n} \; .$$

These give estimates on $g_n$. More precisely we have (Corollary 3.20 in [21]):

**Proposition 5.** *For some constant* $C > 0$, *we have*

$$|| \log Dg_n||_{C^0(\mathbb{R})} \leq C M_n^{1/2} \; .$$

**Corollary 6.** *For any* $\epsilon > 0$, *there exists* $n_0 \geq 1$ *such that for* $n \geq n_0$, *we have*

$$||Dg_n - 1||_{C^0(\mathbb{R})} \leq \epsilon \; .$$

*Proof.* Take $n_0 \geq 1$ large enough so that for $n \geq n_0$, $CM_n^{1/2} < \min(\frac{2}{3}\epsilon, \frac{1}{2})$, then use Proposition 5 and $|e^w - 1| \leq \frac{3}{2}|w|$ for $|w| < 1/2$. □

**Corollary 7.** *For any* $\epsilon > 0$, *there exists* $n_0 \geq 1$ *such that for* $n \geq n_0$, *for any* $x \in \mathbb{R}$ *and* $y \in I_n(x)$ *we have*

$$1 - \epsilon \leq \frac{m_n(y)}{m_n(x)} \leq 1 + \epsilon \; .$$

*Proof.* We have $Dm_n(x) = Dg_n(x) - 1$, and

$$|m_n(y) - m_n(x)| \leq ||Dm_n||_{C^0(\mathbb{R})}|y - x| \leq ||Dg_n - 1||_{C^0(\mathbb{R})}|m_n(x)| \; .$$

We conclude using Lemma 6. □

## 3. Denjoy-Yoccoz Lemma.

Once we have these real estimates, and, more precisely, a control on the non-linearity, we can use them in a complex neighborhood. Using the notations introduced in the previous section, the raw form of the Denjoy-Yoccoz Lemma (see [19]) is the following:

**Lemma 8** (Denjoy-Yoccoz Lemma). *Let $\Delta > 0$ and $g \in D^\omega(\mathbb{T}, \Delta)$ holomorphic and continuous on $\overline{B_\Delta}$. We assume that*

$$\tau = \|D \ \log \ Dg\|_{C^0(\overline{B_\Delta})} < \frac{1}{16},$$

*and that for $n \geq 0$,*

$$M_n \leq \frac{\Delta}{2D_0},$$

*where $4 < D < \frac{1}{4\tau}$.*

*Let $z \in \mathbb{C}$, we write $z_0 = x_0 + im_n(x_0)y_0$, $y_0 \in \mathbb{C}$, and we assume that $|y_0| \leq D_0$. Then for $0 \leq j \leq q_{n+1}$, we have*

$$g^j(z_0) = f^j(x_0) + im_n(g^j(x_0)) \ y_j,$$

*with*

$$|y_j - y_0| \leq 3D_0\tau|y_0|.$$

*Proof.* Let $z_j = g^j(z_0)$ and $x_j = g^j(x_0)$. We prove the Lemma by induction on $j \geq 0$. For $j = 0$ the result is obvious. Assume the result for $i \leq j - 1$, $|y_i| \leq 7/4 \ |y_0| \leq 2D_0 - 1$, since $3D_0\tau \leq 3/4$ and $D_0 > 4$.

By the chain rule we have

$$\log Dg^j(z_0) = \sum_{l=0}^{j-1} \log Dg(z_l) \ ,$$

so

$$\left| \log Dg^j(z_0) - \log Dg^j(x_0) \right| \leq \sum_{l=0}^{j-1} |\log Dg(z_l) - \log Dg(x_l)|$$

$$\leq \tau \sum_{l=0}^{j-1} |z_l - x_l|$$

$$\leq \tau(2D_0 - 1) \sum_{l=0}^{j-1} |m_n(x_l)| \ .$$

Considering the $j$-iterate of $g$ on the interval $]x_0, g^{q_n}(x_0) - p_n[$, we obtain a point $\zeta \in ]x_0, g^{q_n}(x_0) - p_n[$ such that,

$$Dg^j(\zeta) = \frac{m_n(x_j)}{m_n(x_0)},$$

and

$$\left| \log Dg^j(\zeta) - \log Dg^j(x_0) \right| \leq \tau |m_n(x_0)| \leq \tau \sum_{l=0}^{j-1} |m_n(x_l)|.$$

Adding the two previous inequalities, we have

$$\left| \log Dg^j(z_0) - \log \frac{m_n(x_j)}{m_n(x_0)} \right| \leq 2D_0\tau \sum_{l=0}^{j-1} |m_n(x_l)|.$$

The intervals $]x_l, g^{q_n}(x_l) - p_n[$, $0 \leq l < q_{n+1}$ being disjoint modulo 1, we have

$$\sum_{l=0}^{q_{n+1}-1} |m_n(x_l)| < 1 \ .$$

So we obtain

$$\left| \log Dg^j(z_0) - \log \frac{m_n(x_j)}{m_n(x_0)} \right| \leq 2D_0\tau,$$

and taking the exponential (using $|e^w - 1| \leq 3/2|w|$, for $|w| < 1/2$, since $2D\tau < 1/2$),

$$\left| Dg^j(z_0) - \frac{m_n(x_j)}{m_n(x_0)} \right| \leq 3D_0\tau \frac{m_n(x_j)}{m_n(x_0)}.$$

This last estimate holds for any point $z_t$ in the rectilinear segment $[x_0, z_0]$. Integrating along this segment we get the definitive estimate,

$$\left| g^j(z_0) - g^j(x_0) - iy_0 m_n(x_j) \right| \leq 3D_0\tau|y_0||m_n(x_j)|.$$

$\square$

## 4. Hyperbolic Denjoy-Yoccoz Lemma.

4.1. **Flow interpolation in $\mathbb{R}$.** Since $g$ is analytic, from Denjoy's Theorem we know that $g_{/\mathbb{R}}$ is topologically linearizable, i.e. there exists an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$, such that for $x \in \mathbb{R}$, $h(x+1) = h(x) + 1$, and

$$h^{-1} \circ g \circ h = T_\alpha \ ,$$

where $T_\alpha : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + \alpha$.

We can embed $g$ into a topological flow on the real line $(\varphi_t)_{t \in \mathbb{R}}$ defined, for $t \in \mathbb{R}$, $\varphi_t = h \circ T_{t\alpha} \circ h^{-1}$. In general, when $g$ is not analytically linearizable (i.e. $h$ is not analytic), the maps $\varphi_t$ are only homeomorphism of the real line, although for $t \in \mathbb{Z}$, $\varphi_t$ is analytic since for these values they are iterates of $g$. In some cases for other values of $t$, $\varphi_t$ is an analytic diffeomorphism in the analytic centralizer of $g$ (see [13] for more information on analytic centralizers). Now $(\varphi_t)_{t \in [0,1]}$ is an isotopy from the identity to $g$. The flow $(\varphi_t)_{t \in \mathbb{R}}$ is a one parameter subgroup of homeomorphisms of the real line commuting to the translation by 1.

4.2. **Flow interpolation in $\mathbb{C}$.** There are different complex extensions of the flow $(\varphi_t)_{t \in \mathbb{R}}$ suitable for our purposes. For $n \geq 0$, we can extend this topological flow to a topological flow in $\mathbb{C}$ by defining, for $z_0 = x_0 + i \left| m_n(x_0) \right| y_0 \in \mathbb{C}$, with $x_0, y_0 \in \mathbb{R}$,

$$\varphi_t^{(n)}(z_0) = z_0(t) = \varphi_t(x_0) + i \left| m_n(\varphi_t(x_0)) \right| y_0 \ .$$

We denote $\Phi_{z_0}^{(n)}$ the flow line passing through $z_0$,

$$\Phi_{z_0}^{(n)} = (\varphi_t^{(n)}(z_0))_{t \in \mathbb{R}}.$$

4.3. **Hyperbolic Denjoy-Yoccoz Lemma.** We are now ready to give a geometric version of Denjoy-Yoccoz Lemma. We denote by $d_P$ the Poincaré distance in the upper half plane.

**Lemma 9** (Hyperbolic Denjoy-Yoccoz Lemma). *There exists $\epsilon_0 > 0$ small enough universal constant such that the following holds. Let $4 < D_0 < \frac{1}{4\epsilon_0}$. Let $\Delta > 0$ and $g \in D^\omega(\mathbb{T}, \Delta)$ holomorphic and continuous on $\overline{B_\Delta}$ such that $\|D \log Dg\|_{C^0(\overline{B_\Delta})} < \epsilon_0$. Then there exists $n_0 \geq 1$ such that for $n > n_0$, we have for $z_0 \in B_\Delta$, $\Im z_0 > 0$, $z_0 = x_0 + im_n(x_0)y_0$, with $0 < y_0 < D_0$, $0 \leq j \leq q_{n+1}$,*

$$d_P(g^j(z_0), \varphi_j^{(n)}(z_0)) \leq C_0 \ ,$$

*for some constant $C_0 > 0$.*

*Proof.* Since $M_n \to 0$, we choose $n_0 \geq 1$ big enough so that for $n \geq n_0$ we have

$$\frac{\Delta}{2M_n} \leq D_0 < \frac{1}{4\epsilon_0} \ ,$$

so we can use the Denjoy-Yoccoz lemma in the previous section. The Poincaré metric in the upper half plane is given by

$$|ds| = \frac{|d\xi|}{\Im\xi} \ .$$

Therefore

$$d_P(z_j, \varphi_j^{(n)}(z_0)) \leq \int_{[z_j, \varphi_j^{(n)}(z_0)]} \frac{|d\xi|}{\Im\xi} \leq |m_n(x_j)|\,|y_j - y_0| \frac{1}{\inf_{\xi \in [z_j, \varphi_j^{(n)}(z_0)]} \Im\xi}$$

$$\leq |m_n(x_j)|\,|y_j - y_0| \frac{4}{|m_n(x_j)|\,y_0}$$

$$\leq 4\frac{|y_j - y_0|}{y_0} \leq 3 = C_0$$

where in the first line we used that $\Re y_j \geq \frac{1}{4}y_0$ which follow from $|y_j - y_0| \leq \frac{3}{4}y_0$ that we also used in the last inequality. $\qquad\square$

## 5. Quasi-invariant curves.

We prove now that the flow lines $\Phi_{z_0}^{(n)}$, with $y_0 > 1/2$ and $n \geq n_0$ for $n_0 \geq 1$ large enough, are quasi-invariant curves. These flow lines are graphs over $\mathbb{R}$. Given an interval $I \subset \mathbb{R}$, we label $\tilde{I}^{(n)}$ the piece of $\Phi_{z_0}^{(n)}$ over $I$.
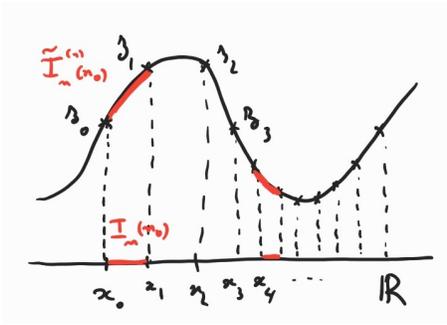


FIGURE 3. A quasi-invariant curve.

**Lemma 10.** *There is $n_0 \geq 1$ such that for $n \geq n_0$ and for any $x \in \mathbb{R}$, the piece $\tilde{I}_n^{(n)}(x)$ has bounded Poincaré diameter.*

*Proof.* Let $z = x + i |m_n(x)| y_0$ be the current point in $\tilde{I}_n^{(n)}(x)$. We have

$$dz = (1 \pm i (Dg_n(x) - 1)y_0) \, dx \, .$$

For any $\epsilon_0 > 0$, choosing $n_0 \geq 1$ large enough, for $n \geq n_0$, according to Lemma 6 we have

$$\left| \frac{dz}{dx} - 1 \right| \leq \epsilon_0 \, .$$

Therefore, we have

$$l_P(\tilde{I}_n^{(n)}(x_0)) = \int_{\tilde{I}_n^{(n)}(x_0)} \frac{1}{|m_n(x)| \, y_0} |dz| \leq \int_{I_n(x_0)} \frac{1}{|m_n(x)| \, y_0} (1 + \epsilon_0) \, dx \, .$$

Now using Lemma 7 with $\epsilon = \epsilon_0$ and increasing $n_0$ if necessary, we have

$$l_P(\tilde{I}_n^{(n)}(x)) \leq \int_{I_n(x_0)} \frac{1}{|m_n(x_0)| \, y_0} \frac{1 + \epsilon_0}{1 - \epsilon_0} \, dx \leq \frac{1}{y_0} \frac{1 + \epsilon_0}{1 - \epsilon_0} \leq 2 \frac{1 + \epsilon_0}{1 - \epsilon_0} \leq C \, .$$

$\square$

We assume $n \geq n_0$ from now on.

**Lemma 11.** *For $0 \leq j < q_{n+1}$ and any $x \in \mathbb{R}$, the pieces $(g^j \circ T^k(\tilde{J}_n^{(n)}(x)))_{0 \leq j \leq q_{n+1}, k \in \mathbb{Z}}$ have bounded Poincaré diameter and cover $\Phi_{z_0}^{(n)}$.*

*Proof.* From Lemma 10 any $\tilde{I}_n^{(n)}(x)$ has bounded Poincaré diameter, thus also any $\tilde{J}_n^{(n)}(x) = \tilde{I}_n^{(n)}(x) \cup \tilde{I}_n^{(n)}(g_n^{-1}(x))$. Moreover, we have $g^j \circ T^k(J_n(x)) = J_n(g^j \circ T^k(x))$, and all $\tilde{J}_n^{(n)}(g^j \circ T^k(x))$ have also bounded Poincaré diameter. From Lemma 3 these pieces cover $\Phi_{z_0}^{(n)}$. $\square$

**Corollary 12.** *For some $C_0 > 0$, the flow orbit $(\varphi_{j,k}^{(n)}(z_0))_{0 \leq j < q_{n+1}, k \in \mathbb{Z}}$ is $C_0$-dense in $\Phi_{z_0}^{(n)}$ for the Poincaré metric.*

We prove the first property stated in Theorem 2:

**Proposition 13.** *Let $\gamma_n = \Phi_{z_0}^{(n-1)}$ for some $z_0$ from the previous lemma, then we have, for $0 \leq j \leq q_n$,*

$$\mathcal{D}_P(g^j(\gamma_n), \gamma_n) \leq 2C_0$$

*Proof.* We prove this Proposition for $n + 1$ instead of $n$ (the proposition is stated to match $n$ in Theorem 2). It follows from the hyperbolic Denjoy-Yoccoz Lemma that the orbit $(g^j \circ T^k(z_0))_{0 \leq j < q_{n+1}, k \in \mathbb{Z}}$ is $C_0$-close to flow orbit $(\varphi_{j,k}^{(n)}(z_0))_{0 \leq j < q_{n+1}, k \in \mathbb{Z}}$, and from Corollary 12 we have that a $2C_0$-neighborhood of $g^j(\gamma_{n+1})$ contains $\gamma_{n+1}$. Conversely, since we can chooose any $z_0 \in \gamma_{n+1}$, we also have that $g^j(\gamma_{n+1})$ is in a $C_0$-neighborhood of $\gamma_{n+1}$. $\square$

We prove the second property of Theorem 2. We observe that $g^{q_{n+1}}(z_0) \in \tilde{J}_n^{(n)}(x_0)$, that $z_0 \in \tilde{J}_n^{(n)}(x_0)$, and that $\tilde{J}_n^{(n)}(x_0)$ has a bounded Poincaré diameter by Lemma 11. Thus we get (taking a larger $C_0 > 0$ if necessary):

**Proposition 14.** *For any $z_0 \in \Phi^{(n)}$ , we have*

$$d_P(z_0, g^{q_{n+1}}(z_0)) \le C_0 \ .$$

From Lemma 11 we also get the property that the hyperbolic balls $B_P(\varphi_{t+k}^{(n)}(z_0), C_0)$ cover $\Phi_{z_0}^{(n)}$.

**Proposition 15.** *We have that*

$$U_n = \bigcup_{0 \le j < q_{n+1}, k \in \mathbb{Z}} B_P(\varphi_{t+k}^{(n)}(z_0), C_0)$$

*is a neighborhood of the flow line $\Phi_{z_0}^{(n)}$*

Enlarging the constant $C_0$, we can construct a "transient annulus", i.e. any orbit from the outside of $\gamma_n$ that has iterates in between $\gamma_n$ and the circle $\mathbb{S}^1$ must visit $C_0$-close any point of $\gamma_n$.

**Proposition 16.** *Let $(g^j(z))_{0 \le j \le q_{n+1}}$ be an orbit that starts on a point $z$ exterior to $\gamma_n$ and has some iterate in between $\mathbb{S}^1$ and $\gamma_n$. Then for any $w \in \gamma_n$, there is an iterate $g^j(z)$ such that*

$$d_P(g^j(z), w) \le C_0 \ .$$

This property is used in the proof of Theorem 1.

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