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Logic and Complexity

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Introduction

The book presents some of the fundamental ideas of Logic for Computer Science. It introduces classical notions of mathematical logic to Computer Scientists and the new ideas brought by the theory of complexity. Given a problem, it is important to know if there is an algorithmic solution, i.e. if the problem can be solved by an automatic procedure based on a finite set of instructions. The classical notions of mathematical logic, such as decidability, completeness and incompleteness are used to answer this question. When an algorithmic solution exists, it is then important to know if there exists an efficient solution, i.e. an automatic procedure which uses few resources such as time or space. Complexity theory introduces notions such as NP-completeness, reductions, approximations which are used to answer this refined question. We present these concepts from the viewpoint of logic, called *descriptive complexity* and emphasize the roles of randomness and approximation. The presentation is divided into three parts:

Part 1. Model theory and recursive functions. We introduce the basic model theory of propositional, 1st order, inductive definitions and 2nd order logic. Then, we define recursive functions, show their equivalence with Turing computable functions, prove the completeness of first-order logic and the incompleteness theorems.

Part 2. Descriptive complexity. The functions computable in polynomial time are called effectively computable and can be defined as global recursive functions on classes of finite ordered structures. Decision problems of other complexity classes such as AC, NC, LOGSPACE, NLOGSPACE, NP, PH, have similar logical characterizations and allow to view complexity questions as definability problems in logic. This approach is called descriptive as it is independent of any model of computation.

Part 3. Approximation. Randomized algorithms in polynomial time define the class BPP for decision problems which generalizes the class P and the class IP (Interactive proofs) for verification problems which generalizes the class NP. Other models of verification include PCP (probabilistic checkable proofs) and property testing. Some optimization problems and counting problems can also be approximated according to their logical form. The PCP techniques allow to prove non-approximability results.

The design of programming languages, query languages and of specification languages follows many logical principles presented in the first part of the book. The second part explains the relationship between the definitions of algorithms, queries, properties of programs and their computational complexity. For many classical tasks such as the verification of programs, optimization problems or counting problems, it is essential to first study the complexity of these problems. When the degree of unfeasibility is too high, approximation techniques presented in the third part of the book can be useful.

This book is based on two books of the same authors, in French: *Logique et Fondements de l'Informatique*, published in 1993 and *Logique et Complexité*, published in 1996, (Editions Hermès, *http://www.editions-hermes.fr*). The first part is the translation of some of the chapters of the first book, whereas the second and third parts of the present book are translations of the second book with additional new subjects. We will maintain: *http://www.lri.fr/~mdr/logicbook/* for the corrections of some of the exercises, remarks and updates.

Most of the presented material appears in the books: *Computational complexity* by C. Papadimitriou [**Pap94**], *Descriptive Complexity* by N. Immerman [**Imm99**], and *Finite Model Theory* by H. D. Ebbinghaus and J. Flum [**EF91**].

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