# Reading group
# *Étale Cohomology*, J. Milne (1980)

These notes are written after the fact, but only once. So there must be a lot of errors. That is fine anyway because, here I am partially quoting Serre, a mathematician's duty is to

*"remplacer les théorèmes faux par d'autres."*

29th April to — 24

IN PURSUIT OF KNOWLEDGE
IISER
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# 1 *Introduction : The Weil Conjectures after 75 years* (Sudip Pandit, 29th April 24)

In 1900, David Hilbert presented a list of open problems which should pave the road for mathematicians during the 20th century. The 10th problem asks whether there exists an algorithm deciding if a polynomial $f(t_1, \ldots, t_d) = 0$ with coefficients in $\mathbb{Z}$ admits a solution taking integer values ? The answer has in 1970 finally been proven to be "no" **[]**. However, the question remains unanswered for a solution taking values in $\mathbb{Q}$ where $2 \leqslant d \leqslant 10$ [1]. Geometrically, if $X$ is the affine variety associated to $f$ then the question boils down to decide whether $X(\mathbb{Q})$ is empty. One approach to the original question has been to look at solving the equation modulo $p$ a prime number, for if there exists a solution over $\mathbb{Z}$ there should exists a solution over $\mathbb{F}_p$ for all $p$. The number of such an equation is bounded (by $p^n$ in this case) and thus mathematicians were led to the general question of computing the number of points of a variety $X$ defined over a finite field.

To compute the number points of a variety, André Weil generalized the zeta functions studied by Helmut Hasse. Fix $\overline{\mathbb{F}}_q$ a an algebraic closure of $\mathbb{F}_q$ and define the Hasse-Weil zeta function of $X$ to be

$$\mathrm{Z}(X, t) := \exp\left(\sum_{n \geqslant 1} \#X(\mathbb{F}_{q^n}) \frac{t^n}{n}\right) \in \mathbb{Z}[[t]]$$

These are called zeta functions for the following reason. Leonhard Euler showed that the Riemann zeta function can be written as

$$\zeta(s) := \sum_{n \geqslant 1} n^{-s} = \prod_{l \text{ prime}} (1 - l^{-s})^{-1}$$

This Euler product inspired the definition of a zeta function $\zeta_Y$ associated to any scheme $Y$ of finite type over $\mathbb{Z}$

$$\zeta_Y(s) := \prod_{y \text{ closed point}} (1 - N(y)^{-s})^{-1}$$

where $N(y)$ is the number of elements in the residue field of $y$. From this we have first $\zeta = \zeta_{\mathrm{Spec}(\mathbb{Z})}$, for a maximal ideal and a nonzero prime ideal are the same in $\mathbb{Z}$. Furthermore $X$ being also a scheme of finite type over $\mathbb{F}_q$ and therefore of finite type over $\mathbb{Z}$ we have

$$\begin{aligned}
\log \mathrm{Z}(X, q^{-s}) &= \sum_{n \geqslant 1} \#X(\mathbb{F}_{q^n}) \frac{q^{-ns}}{n} \\
&= \sum_{n \geqslant 1} \sum_{d|n} \sum_{N(x)=q^d} \frac{dq^{-ns}}{n} \\
&= \sum_{d \geqslant 1} \sum_{N(x)=q^d} \sum_{m \geqslant 1} \frac{(q^{-ds})^m}{m} \\
&= \sum_{x \text{ cl. pt.}} -\log(1 - N(x)^{-s}) \\
&= \log \zeta_X(s)
\end{aligned}$$

because each closed point of residue field $\mathbb{F}_{q^d}$ is represented by $d$ points in $X(\overline{\mathbb{F}}_q) = \cup_{n \geqslant 1} X(\mathbb{F}_{q^n})$ .

Generalising the result already obtained by H. Hasse for quadratic extensions of $\mathbb{Q}$, and his own results on smooth projective irreducible curves over a finite field, A. Weil conjectured the following

[1]"yes" for $d = 1$ and "no" for $d \geqslant 11$ **[]**

**Conjecture 1.** *(Weil Conjectures)*

*(I)* Rationality. *$\mathrm{Z}(X,t) \in \mathbb{Z}(t)$ is a rational function and there exists $2d+1$ polynomials $P_0(t) = 1-t, P_1(t), \ldots, P_{2d-1}(t), P_{2d}(t) = 1-q^d t \in \mathbb{Z}[t]$ such that*

$$\mathrm{Z}(X,t) = \prod_{i=0}^{2d} P_i(X,t)^{(-1)^{i+1}} = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)\cdots P_{2d}(t)}$$

*(II)* Functional equation. *If $\chi(X) = \sum_{i=0}^{2d}(-1)^i \deg P_i$ then*

$$\mathrm{Z}(X, q^{-d}t^{-1}) = \pm q^{\chi d/2} t^{\chi} \mathrm{Z}(X,t)$$

*(III)* Betti numbers. *If $X$ comes from a variety defined over a number field $F$ via the reduction modulo a prime $\mathfrak{p}$ in the ring of the integers of $F$, then*

$$\deg P_i(t) = \dim_{\mathbb{C}} H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{C})$$

*and therefore $\chi(X) = \chi(X(\mathbb{C}))$.*

*(IV)* Riemann Hypothesis. *For all $0 < i < 2d$, the roots of $P_i(t)$ in $\mathbb{C}$ have absolute value $q^{i/2}$.*

A. Weil had the geometric intuition that there should exist a cohomology theory that we can define on $X$ even if it does not come from a reduction modulo a prime. This is what we call now a Weil cohomology.

**Definition 1.** *Let $K$ a field of characteristic zero. A* ***Weil cohomology*** *over $K$ is a contravariant functor $H^i(-,K)$ from the category of geometrically irreductible projective smooth varieties defined over $\mathbb{F}_q$ with values in the category of finite dimensional $K$–vector space such that*

*(a) If $X$ is a variety of dimension $d$ then for all $i > 2d$, $H^i(X,K) = 0$.*

*(b) The Künneth formula holds (the isomorphism is given by a natural transformation)*

$$H^m(X \times Y, K) \simeq \bigoplus_{i+j=m} H^i(X,K) \otimes_K H^j(Y,K)$$

*(c) If $X$ is of dimension $d$ then $H^{2d}(X,K) \simeq K$ and the cup-product gives a perfect pairing*

$$H^i(X,K) \times H^{2d-i}(X,K) \longrightarrow H^{2d}(X,K)$$

*(d) If $\varphi : X \longrightarrow X$ is a regular morphism then we have a map $\varphi : X(\overline{\mathbb{F}}_q) \longrightarrow X(\overline{\mathbb{F}}_q)$ whose set of fixed points $X^{\varphi}$ has cardinality*

$$\# X^{\varphi} = \sum_{i=0}^{2d} (-1)^i \mathrm{Tr}(\varphi^* : H^i)$$

*(e) If $K \hookrightarrow \mathbb{C}$ then $H^i(X,K) \otimes_K \mathbb{C} \simeq H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{C})$.*

Let's see how the existence of a Weil cohomology (and actually some other properties...) enable us to prove (I),(II),(III). Let us recall the following lemmas :

**Lemma 1.** *Let $F$ be an endomorphism of a finite-dimensional $K$–vector space. Then*

$$\det{}_{K(t)}(1-tF)^{-1} = \exp\left(\sum_{n\geqslant 1} \mathrm{Tr}(F^n)\frac{t^n}{n}\right)$$

*Proof.* The proof relies on the fact that if $K$ is algebraically closed then there exists a basis of $V$ such that the relative matrix $A$ of $F$ is triangular, this implies that $\mathrm{Tr}(F^n) = \sum \alpha_i^n$ where $\alpha_i$ are the diagonal elements of $A$, and that $\det{}_{K(t)}(1-tF) = (1-\alpha_1 t)\cdots(1-\alpha_n t)$. The lemma follows then from a computation regarding usual power series. □

**Lemma 2.** *Let* Frob *be the Frobenius map $X \to X$ given by the change of scalars $\mathbb{F}_q \to \mathbb{F}_q,\ a \mapsto a^q$. Then*

$$\#X^{\mathrm{Frob}^n} = \#X(\mathbb{F}_{q^n})$$

Alexandre Grothendieck and Michael Artin achieved to define a Weil cohomology over $\mathbb{Q}_l$ where $l \neq p$. This is called the $l$–adic étale cohomology and noted $H^i_{\mathrm{et}}(-,\mathbb{Q}_l)$. Combining this fact with the two lemmas, it is a straightforward computation to check (I), (II), (III) by using the additional fact that $\mathrm{Frob}^* : H^i_{\mathrm{et}}(X,\mathbb{Q}_l) \longrightarrow H^i_{\mathrm{et}}(X,\mathbb{Q}_l)$ has a characteristic polynomial in $\mathbb{Z}[t]$ (and does not depend on $l$ !). Alexandre Grothendieck proposed a list of conjectures, called standard conjectures, which imply the Riemann Hypothesis (IV). However we still don't know if they hold, but (IV) has been proved independently by his student Pierre Deligne for which he got the Fields Medal in 1978.

## 2 Étale morphisms (Ayush Khare, 2nd - 16th May 24)

**Definition 2.** *Let $f$ be an affine map $X \longrightarrow Y$. If $O_Y(f^{-1}(U))$ is a finite (resp. finitely generated) $O_X(U)$–algebra then $f$ is said to be* ***finite*** *(resp.* ***of finite type****). Furthermore, $f$ is said to be* ***quasi-finite*** *if it is of finite type and with finite fibers.*

In everything that follows $k$ is a field and $\overline{k}$ is an algebraic closure of $k$. If $A$ is a $k$–algebra then $A := A \underset{k}{\otimes} \overline{k}$. More generally if $X$ is a scheme over $\operatorname{Spec} k$ then $\overline{X} := X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} \overline{k}$.

**Proposition 3.** *Let $f : X \longrightarrow \operatorname{Spec} k$. The following are equivalent.*

1. *$X$ is affine and $O_X(X)$ is an Artin ring.*
2. *$X$ is finite and discrete.*
3. *$X$ is discrete.*
4. *$f$ is finite.*

**Definition 3.** *Let $A$ be a $k$–algebra. $A$ is said to be* ***separable*** *over $k$ if the Jacobson ideal of $\overline{A}$ is reduced to $0$.*

**Proposition 4.** *Let $A$ be a finite $k$–algebra, so in particular an Artin ring. The following are equivalent.*

1. *$A$ is separable over $k$.*
2. *$\overline{A}$ is isomorphic to $\overline{k}^N$, where $N$ is an integer.*
3. *$A$ is isomorphic to $L_1 \times \cdots \times L_N$, where $L_i|k$ are finite separable extension.*

4. *The trace* $\mathrm{Tr}_{A|k} : A \times A \longrightarrow k$ *is a non-degenerate bilinear form.*

*Proof.*
$(1 \implies 2)$ : Let $\mathfrak{m}_1, \dots, \mathfrak{m}_N$ the maximal ideals of $\overline{A}$, which is also an Artin ring. Then $\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_N = \{0\}$ and therefore by the chinese remainders theorem

$$\overline{A} \simeq \overline{A}/\mathfrak{m}_1 \times \dots \times \overline{A}/\mathfrak{m}_N$$

Finally the conlcusion is reached by observing that $\overline{A}/\mathfrak{m}_1|\overline{k}$ is a finite separable extension.

$(2 \implies 3)$ : Let $I$ be the Jacobson ideal of $A$ and $\mathfrak{m}_1, \dots, \mathfrak{m}_N$ the maximal ideals of $A$. Then

$$A/I \simeq A/\mathfrak{m}_1 \times \dots \times A/\mathfrak{m}_N$$

Let $L_i$ denote the finite extension $A/\mathfrak{m}_i|k$ which is decomposed in $L_i|L_i^s|k$ with $L_i^s|k$ being separable. Tensoring by $\overline{k}$, the extension becomes separable

$(3 \implies 4)$ : $\mathrm{disc}(A) = \mathrm{disc}(L_1) \cdots \mathrm{disc}(L_N)$ is non-zero as a product of non-zero.

$(4 \implies 1)$ : It amounts to show that the only nilpotent is zero. First, $\mathrm{Tr}_{\overline{A}|\overline{k}} = \mathrm{Tr}_{A|k} \underset{k}{\otimes} \overline{k}$ which is therefore also non-degenerate. Let $x$ be a nilpotent in $\overline{A}$. Then for all $a \in \overline{A}$, $xa$ is also nilpotent and therefore left-multiplication by $xa$ is of trace $0$. Thus for all $a \in \overline{A}$, $\mathrm{Tr}_{\overline{A}|\overline{k}}(xa) = 0$. Hence $x = 0$. □

**Definition 4.** *Let $f : A \longrightarrow B$ be a finite morphism of rings, and $\mathfrak{p}$ a prime ideal of $A$. $f$ is said* ***unramified at*** *$\mathfrak{p}$ if for all prime ideal $\mathfrak{q}$ of $B$ over $\mathfrak{p}$, the extension $K(\mathfrak{q})|K(\mathfrak{p})$ is finite and separable. A locally finite type morphism of schemes $f : X \longrightarrow S$ is said to be* ***unramified at*** *$s$ if for all $x \in X_s$, the map $O_{S,s} \longrightarrow O_{X,x}$ is unramified at $\mathfrak{m}_s$ ; and said* ***unramified*** *if it is at each point.*

**Proposition 5.** *Let $f : X \longrightarrow S$ be a map between* $\mathrm{Spec}\, k$*–schemes locally of finite type. The following are equivalent.*

1. *For all $s \in S$, $f$ is unramified at $s$.*
2. *For all $s \in S$, $f_{|X_s} : X_s \longrightarrow \{s\}$ is unramified.*
3. *For all $s \in \overline{S}$, the induced map $\overline{f} : \overline{X}_s \longrightarrow \{S\}$ is unramified.*
4. *For all $s \in S$, $X_s$ has an open covering by spectrum of finite separable $K(s)$–algebra.*

**Definition 5.** *A morphism $f : X \longrightarrow S$ is said to be* ***flat*** *if for all $x \in X, s \in X_s$, the map $O_{S,s} \longrightarrow O_{X,x}$ induces a structure of a flat $O_{S,s}$–module on $O_{X,x}$.*

**Definition 6.** *A morphism $f : X \longrightarrow S$ is said to be* ***étale*** *if it is flat and unramified.*

**Proposition 6.** *Open immersions are étale. The property of being étale is stable by composition and base change.*

*Proof.* An open immersion induces isomorphism between the local rings. Therefore, first : one local ring is free over the other and in particular flat ; second : the extensions of residue fields are trivial. □

**Proposition 7.** *Let $f : X \longrightarrow S$ be a locally of finite type. The following are equivalent.*

1. *$f$ is unramified.*
2. *$\Omega^1_{X|S} = 0$.*
3. *The diagonal map $\Delta : X \longrightarrow X \underset{S}{\times} X$ is an open immersion.*

*Proof.*
$(2 \implies 3)$ : $\Delta$ is a locally closed map, so there exists an open subset $W \subset X \underset{S}{\times} X$ such that $\Delta(Y)$ is a closed subset of $W$. Call $\mathscr{I}$ the sheaf. □

# 3 Étale fundamental group (Prakash, 17$^{\text{th}}$ May 24)

Let us recall the construction of the fundamental group of a pointed connected topological space $(X, x)$.

The fundamental group $\pi_{1,\text{loops}}(X, x)$ is the set of all loops based at $x$ up to homotopy, i.e. equivalence classes of continuous maps $\gamma : [0, 1] \to X$ such that $\gamma(0) = x = \gamma(1)$ (called **loops** based at $x$) where $\gamma \sim \gamma'$ if there is a homotopy from $\gamma$ to $\gamma'$ via loops based at $x$. This construction is at first glance irrelevant in the context of algebraic geometry, since the Zariski topology is too coarse to allow non-trivial loops. There is however another approach, namely using unramified covers. If $X$ is semi-locally simply connected then there exists a map $\widetilde{\pi} : (\widetilde{X}, \widetilde{x}) \to (X, x)$, called a **universal cover** of $(X, x)$, such that

$$\pi_{1,\text{loops}}(X, x) \simeq \operatorname{Aut}(\widetilde{\pi})$$

When $Y \longrightarrow X$ is a finite unramified cover, the fundamental group $\pi_{1,\text{loops}}(X, x)$ acts on the fibre $Y_x$. You can check that it induces a functor

$$\begin{aligned} F: \quad X-\mathsf{Fin.Un.Cov.} &\longrightarrow \pi_{1,\text{loops}(X,x)}-\mathsf{Fin.Sets} \\ (Y \to X) &\longmapsto Y_x \end{aligned}$$

It happens that this functor is an equivalence of category, and with values in Sets it is **representable** by an object in the category of pointed $(X, x)$–topological spaces precisely by $\widetilde{\pi} : (\widetilde{X}, \widetilde{x}) \to (X, x)$, i.e. there is a natural transformation inducing bijections for all $\pi : Y \to X$

$$F(\pi) \simeq \operatorname{Hom}_{X\text{–}\mathsf{Top}}(\widetilde{\pi}\ ,\ \pi)$$

which concretely means that the following diagram commutes

$$\begin{array}{ccc} \widetilde{X} & \dashrightarrow & Y \\ & {\scriptstyle\widetilde{\pi}}\searrow & \downarrow{\scriptstyle\pi} \\ & & X \end{array}$$

Furthermore the action of $\pi_{1,\text{loops}}(X, x)$ is completely natural if identified with $\operatorname{Aut}(\widetilde{\pi})$. Finally even without considering the universal cover we can still say that $F$ is **pro-representable**, i.e. there exists a family of finite unramified Galois cover $(\pi_i : Y_i \to X)_i$ such that there is a natural equivalence

$$F(-) \simeq \varinjlim_i \operatorname{Hom}_{X\text{–}\mathsf{Top}}(\pi_i, -)$$

and an isomorphism between groups

$$\operatorname{Aut}(\widetilde{\pi}) \simeq \varprojlim_i \operatorname{Aut}(\pi_i)$$

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In the context of algebraic geometry, we will replace *finite unramified coverings* by *finite étale maps*. In the following, $X$ will be a connected scheme ($\neq \varnothing$ ...).

**Proposition 8.** *Let $\pi : Y \longrightarrow X$ be a finite étale map (with $Y \neq \varnothing$...). Then $\pi$ is surjective.*

*Proof.* A finite (resp. étale) map is in particular a closed (resp. open) map. Hence the result since $X$ is the only non-empty clopen subset. ☐

Let $\mathsf{F{-}Et}$ be the category of finite étale maps between connected schemes. As usual, $\mathsf{F-Et}/X$ denote the category of finite étale maps $Y \to X$, and maps $Y' \to Y$ such that the following commutes

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ & \searrow & \downarrow \\ & & X \end{array}$$

By proposition ???, $Y' \to Y$ is also a finite étale map.

**Definition 7.** *A **geometric point** is a map $\overline{x} \to X$ such that $\overline{x}$ is the spectrum of a separably algebraic closed field.*

Let $\overline{x} \to X$ be a geometric point. Consider the fibre functor

$$\begin{aligned} F: \quad \mathsf{F{-}Et}/X &\longrightarrow Sets \\ (Y \to X) &\longmapsto \mathrm{Hom}_{\mathsf{Schemes}/X}(\overline{x}, Y) \end{aligned}$$

Again, we say that $F$ is **pro-representable** if there exists a family $(Y_i \to X)_i$ of étale finite maps such that there is a natural equivalence

$$F(-) \simeq \varprojlim_i \mathrm{Hom}_{\mathsf{Schemes}/X}(Y_i, -)$$

In this case, we define

$$\pi_{1,et}(X, \overline{x}) := \varprojlim_i \mathrm{Aut}_{\mathsf{Schemes}/X}(X_i)$$

Don't forget that this definition depends on the family chosen. However, the universal property verified by two such families will make the two groups isomorphic up to unique isomorphism commuting with all the relevant maps.

**Definition 8.** *A finite étale map $Y \to X$ of degree $d$ is said to be **Galois** if $\mathrm{Aut}_{\mathsf{Schemes}/X}(Y)$ has $d$ elements.*

**Proposition 9.** *Let $Y \to X$ be a finite étale map. It is Galois if and only if $\mathrm{Aut}_{\mathsf{Schemes}/X}(Y)$ acts freely transtively on $F(Y)$.*

Let us look at some cases where we can construct the universal family.

**Proposition 10.** *Let $K$ be a field along with a separable algebraic closure $\overline{K}$ (which is nothing but the data of a geometric point $\mathrm{Spec}\,\overline{K} \longrightarrow \mathrm{Spec}\,K$). Then the family $(\mathrm{Spec}\,L_i)_i$ of finite Galois extensions $L_i = K[T]/(P_i(T))$ pro-represents the fibre functor.*

*Proof.* □

This implies that

$$\pi_{1,et}(\mathrm{Spec}\,K, \mathrm{Spec}\,\overline{K}) = \varprojlim_i \mathrm{Aut}(\mathrm{Spec}\,L_i) \simeq \varprojlim_i \mathrm{Gal}(L_i|K) = \mathrm{Gal}(\overline{K}|K)$$

The étale fundamental group on a point is exactly the absolute Galois group of its residue field !

**Theorem 1.** *Assume that $X$ is a scheme over $\mathrm{Spec}\,\mathbb{C}$. Recall that $X(\mathrm{Spec}\,\mathbb{C})$ is by definition $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}, X)$, the set of $\mathbb{C}$–points of $X$, and let $x$ denote the base $\mathbb{C}$–point $\mathrm{Spec}\,\mathbb{C} \to \overline{x} \to X$. Then $\pi_{1,et}(X, \overline{x})$ is isomorphic to the profinite completion of $\pi_{1,top}(X(\mathbb{C}), x)$.*

# 4 Étale site (Salim Alloun, 28$^{\text{th}}$ May 24)

## General aspects of schemes

When we say that a property (P) is **stable by base change** it means that if (P) holds for $X \longrightarrow S$ then for all $S' \longrightarrow S$, (P) also holds for the induced map $X \underset{S}{\times} S' \longrightarrow S'$.

**Definition 9.**

1. *If $A$ is a ring then $\operatorname{Spec} A$ denotes the* ***spectrum*** *of $A$ as a scheme. A scheme is said* ***affine*** *if it is isomorphic to the spectrum of a ring.*

2. *If $X$ is a scheme then $O_X$ is the* ***structure sheaf*** *of $X$.*

3. *If $x$ is a point of a scheme $X$ then $O_{X,x}$ is the local ring at $x$, of maximal ideal $\mathfrak{m}_x$ and of residue field $K(x) := O_{X,x}/\mathfrak{m}_x$.*

4. *A morphism of schemes $f : X \longrightarrow Y$ is said to be*

   - ***affine*** *if for all open affine $U \subset X$, $f^{-1}(U)$ is open affine in $X$.*
   - *an* ***open immersion*** *if the induced map $X \longrightarrow f(X)$ is an isomorphism of schemes and $f(X)$ is open in $Y$.*
   - ***proper*** *if it is separated, of finite type and universally closed.*

**Proposition 11.**

1. *A closed immersion is finite.*

2. *Finite morphisms are stable by composition and finite products.*

3. *A finite morphisme is proper.*

# Bibliography.

**[1] :** *Étale cohomology*, J. Milne (1980)