Theory of Ratios in Euclid's *Elements* Book V revisited

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1. Why a *general* theory of ratios?

The question needs to be asked since the *Elements* contains a part called the 'arithmetical books' i.e. the books VII to IX. The first of these books, the VIIth contains already a definition of a ratio and what means for four integers m, n, p, q to be or to have the same ratio.
The most important result for us is the proposition 19 which proves what may be (anachronistically) writen:
$m/n = p/q$ is equivalent to $mq = pn$
and for convenience we may consider this last equality as a definition of equality of ratios.

Since it is how we define modern 'ratios', why the need of a 'general' theory. The answer is simple: some centuries ago, the Greek mathematicians found what they called 'irrational magnitudes' ('ἄλογοι μεγέτη'). More precisely they proved (I spoke here on this subject some years ago) there were no 'ratio' ('λόγος') between the magnitude of the diagonal of the square and its side (it is the meaning of 'irrational' ('ἄλογος')). And thereafter they found many other such irrational magnitudes. Hence, everything done until then was useless for this kind of magnitudes. In particular it was not possible to get for these irrational magnitudes an analogous to the fundamental proposition VII.19 on integers since the multiplication has no meaning on magnitudes.

For instance, contrary to what most people, mathematicians included, spontaneously believe, there is no elementary way to prove this seemingly trivial thing that the **area of a rectangle is the product of its sides**.

For instance let us have a look of the famous ‘Pythagoras’ theorem’. There is an extremely elementary proof as the one found in some Chinese and Indian texts (however their dating are dubious) given by the figure 1 below:

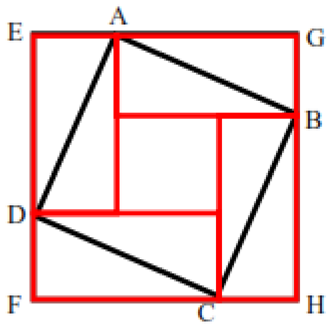


Figure 1

It was certainly not unknown by the ancient Greek geometers since in Plato’s *Meno*, written between one and two centuries before the *Elements*, Socrates uses this very demonstration in the particular case of the square as in figure 2 below:

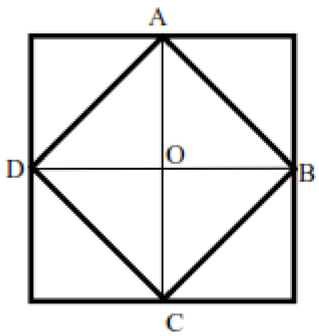


Figure 2

It explains why so many mathematicians think Euclid is out of his mind when he gives his proof of Pythagoras’ theorem through the proposition I.29 using the figures 3-5 below:

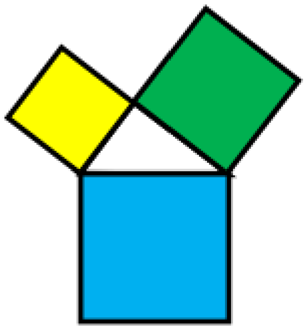


Figure 3

Euclid's proof consists to show the surface of the great square (in blue) in figure 3 is equal to the area of the two other squares (yellow and green).

It is done extremely indirectly using the (non evident) equalities of some triangular and rectangular areas more precisely, he shows the following equalities:

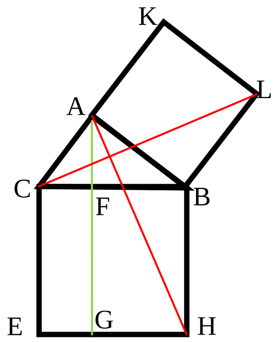


Figure 4

i) The triangles CBL and ABH are equal so that their areas are equal
ii) The areas of the rectangle FBHG is the double of the area of the triangle ABH
iii) The areas of the square KLBA is the double of the area of the triangle LBC

And symmetrically the analogous equalities in the figure 5 below

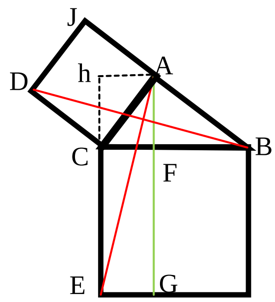


Figure 5

Since the area of the big square CBHE is the sum of the areas of the rectangles CFGE and FBHG, it is also the sum of the areas of the squares KLBA and JACD.

Such a proof seems insane while the first one is so simple.

But there is a catch in this first proof: it is based on the ‘trivial’ fact the area of a rectangle is the product of its sides. If it is easy to prove it when the sides are rational, it is simply impossible to get this result with the means available to the early ancient Greek (or non-Greek) mathematicians (let’s say before Socrates’ times).

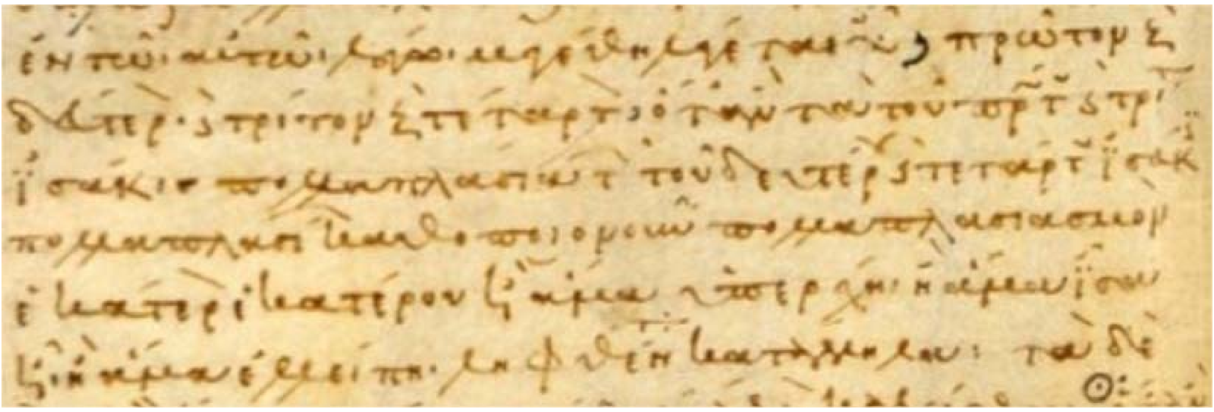
By the way this difficulty does not exist in Socrates’ proof since he makes only use of areas of squares which by definition are the square of the sides, so that the proof is correct.

Therefore it was clear a new theory of ‘ratios’ was needed so that mathematicians could continue using ‘ratios’ between objects having no more ‘ratios’, as for instance the diagonal and the side of the square.

2. Presentation.

We find just this theory in the book V of Euclid’s *Elements*, certainly the most celebrated theory of ancient Greek mathematics. There is a huge literature about its definition 5[1]. It has been intensively studied till nowadays by historians of mathematics since it is one of the most important mathematical statements in the Antiquity. And indeed, according to some mathematicians (from **Lipschitz to Langlands**), it almost contains **Dedekind’s construction of real numbers** written more than 2000 years afterward.

Definition 5 of Book V of Euclid’s Elements



Euclid’s *Elements* definition V.5 in a manuscript dated of the 11th century in the Archiginnasio Library (A18). All the text is written in small-letters and contrary to the propositions, the definitions are not numbered, but written one after the other.

[1] Based on some rather weak testimonies, its construction is commonly attributed to Eudoxus. So in line with Arnold’s among others’ warning about mathematical theorems’ authors, we will not credit anyone for it and only refer to Euclid’s text without implying any claim of authorship. However, the theory was certainly done earlier than Euclid, at least in Plato’s period, which would be consistent with Eudoxus, if not its author, at least a main contributor.

The Text

ε΄. Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καί τρίτου ἰσάκις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκις πολλαπλασίων καθ᾿ ὁποιονοῦν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ᾖ ἢ ἅμα ἐλλείπῃ ληφθέντα κατάλληλα.

Definition 5. *Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.*

The excellent historian of Greek mathematics, Thomas Heath, noted the entire book V of the *Elements* i.e. the whole theory of ratios is founded on this famous definition 5.
This is amazing, since it is almost unknown in mathematics a **definition** to be at the foundation of a global theory.

But there is a pitfall, the difficulty to understand and explain the six lines of the statement.

To ensure the problem does not come from the translator, let us check another one, Thomas Heath's translation (1908).

Definition 5. *Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall* *short of, the latter equimultiples respectively taken in corresponding order.*

3. A modern mathematical interpretation

I do not anyone, mathematician or not, able at first to understand this definition even less to see any connection between it and a definition of ‘ratio’, as known presently.
It is true other statements in the *Elements* are not so simple to understand because they are written in natural language, while we are used in mathematics to a symbolic language. Once translated, they appear clear and sometimes even evident. So let us try to give such a translation.

This is not so easy since they are many different possible translations in symbolic writing, and moreover, in his following proofs Euclid uses the definition in different ways.
However the most usual modern translation is as follows:

Definition V.5 (modern interpretation). Let a, b, c, d be magnitudes. We say $a/b = c/d$ (i.e. we **define** their equality) if for any (positive) integers m and n we have simultaneously:

$na > mb$ and $nc > md$

or

$na = mb$ and $nc = md$

or

$na < mb$ and $nc < md$

4. Some commentaries

Though it is written in perfect modern symbolic language, it is not much easier to understand, let alone to make the connection with anything related to the notion of ‘*ratios*’.

For instance, even the mathematician D. Fowler who worked many years on this question had to admit that, even for mathematicians, the 'Definition 5 (…) as a description, is almost impenetrable (…) though its latent power and scope are enormous.'[2]

It is not a modern claim since as Heath wrote in his own translation of the *Elements*: 'From the revival of learning in Europe onwards the Euclidean definition of proportion was the subject of much criticism.'
It is not actually right since long before, in the eastern world, already from the 10th century mathematicians complained against this very definition (for instance in the 11th century the famous Persian mathematician Al Khayyam wrote a complete book critical of Euclid's *Elements*, especially of the definition V.5).

5. The strange statute of the 'ratio of magnitudes'

Even before trying to explain this definition, let us consider some difficulties hidden by the translation.

Firstly Euclid as we saw in his statement (in plain language) does not use the term '**equal**' ('ἴσος'), because it would imply 'ratios' are some quantities while it is specifically told it is a **relation:**

Definition V.3. *A ratio is a certain type of relation with respect to size of two magnitudes of the same kind*. ('λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ *πηλικότητά* ποια σχέσις.') (T. Heath's translation)

Such a definition of a ratio is certainly not very clarifying and for a good reason.

Euclid indeed does not usually consider '**a**' ratio as a subject, as we do and will continue to do for convenience. He says four magnitudes or

[2] David Fowler, Ratio in Early Greek Mathematics, *Bull. Amer. Math. Soc.*, 1, 6, 1979, p. 813.

integers either ‘**to be in**’ (‘ἐν τῷ αὐτῷ λόγῳ’) or ‘**to have**’ the same ratio or to form a ‘proportion’ (‘ανἀλόγον’ i.e. to be ‘according to a ratio’).
The meaning is a ‘ratio’ is not directly a mathematical object but something connecting mathematical objects as magnitudes and numbers.

As already indicated by the ‘ποια σχέσις’ (‘**a sort of relation**’) the vocabulary emphasizes a fundamental point in this theory: the **extreme relative** meaning of ‘ratio’.

Moreover the problem is not only we do not actually know what ‘ratio’ means, but we do not even know the meaning of ‘**magnitude**’. And indeed Euclid nowhere defines such an object as ‘a magnitude’.
Hence when speaking of ‘ratio of magnitudes’, we know nothing: neither a useful meaning of ratio, nor of what this unknown term put in relation. It is purely an empty shell.

If Berkeley was living in Euclid’ times, he may be forgiven to make the same commentaries on the theory of ratios as he did on the ‘infinitesimals’: this theory is more irrational than any religious superstition.

6. An (absurdly) anachronistic interpretation

Since there are so many difficulties to understand this short text (6 handwritten lines in Bologna's manuscript), let us consider some desperate try: the recourse to modern mathematical tools.
So for a moment, let us forget any historical and chronological concerns for the sake of understanding, and let us reason according the modern way.

i) First let us recall the definition for the ratios a/b and c/d to be the same. For any integers m and n we have:

$na > mb$ and $nc > md$
or
$na = mb$ and $nc = md$
or
$na < mb$ and $nc < md$.

Then we make the following transformations: for any (positive) integers m and n we have simultaneously:
$a/b > m/n$ and $c/d > m/n$
or
$a/b = m/n$ and $c/d = m/n$
or
$a/b < m/n$ and $c/d < m/n$

It can be translated in: for any (positive) 'rational number' α, we have: ($\alpha > a/b$ and $\alpha > c/d$) or ($\alpha = a/b$ and $\alpha = c/d$) or ($\alpha < a/b$ and $\alpha < c/d$).

The middle identity has a special significance as the commensurable case, so that it may be put apart and we get:

- the ratios (a/b) and (c/d) are equal to α is simultaneously true or false (*)
- **and** restating negatively the first and third conditions and leaving aside the cases of equality settled by (*), we get: there is no rational number α such that:

($a/b > \alpha$ or $c/d > \alpha$) and ($a/b < \alpha$ or $c/d < \alpha$).

By distributivity of the operator 'and' with respect to 'or', we finally obtain:

($a/b > \alpha$ and $c/d < \alpha$) or ($c/d > \alpha$ and $a/b < \alpha$) (**).

From (*) and (**) we obtain the equality of the ratios $a/b = c/d$ means:

i) a/b and c/d are both equal to the same rational number α such that:

$a/b = \alpha$ and $c/d = \alpha$

ii) there is no rational number separating a/b and c/d.

Since between two different rational numbers there is always another one separating them, the first condition is a particular case of the second. And we finally obtain:

Two ratios are equal if there is no rational numbers separating them.

7. **Return to Euclid**

Now we reformulate the formulas in the previous paragraph to be consistent with ancient Greek mathematics.
It is clear we have to change the ‘rational numbers’ into ‘ratios of integers’. Let us immediately remark thanks to proposition VII.19, the results concerning the ratios of integers does not depend of the integers having the same ratio (in modern terms the representative in the class of a rational number).
So that the definition V.5 means simply:

Two ratios are equal if there are no ratios of integers separating them.

Now let us recall Lipschitz’s remark to Dedekind about his construction of the real numbers: it was already in Euclid’s book V.

And according to T. Heath, Euclid’s ‘definition of equal ratios corresponds exactly to the modern theory of irrationals due to Dedekind.’[3]

Reading the definition V.5 it may seem at the least a strange point of view. Nevertheless the above interpretation makes clear its connection (which was not the one pointed by Lipschitz or Heath) to the set of real numbers.

Indeed, the contrapose of this definition gives: two ratios of magnitudes are different when there are no ratios of integers between them. Since a ratio of integers is a particular ratio, the process can be iterated and we obtain two sequences of ratios of integers, each of them converging to one of these two ratios.
In (modern topological) terms the meaning is as follows:

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[3] Thomas Heath, *A History of Greek Mathematics*, Clarendon Press, 1921, p. 327.

the rational numbers is a dense subset inside the ‘ratios of magnitudes’

In ancient Greek mathematics, the meaning was any ratios can be approximated, as precisely as we want, by rational integers.

And since of fundamental property of the real set is the density of the subset of the rational numbers, neither Lipschitz, nor Heath or Langlands were absolutely wrong.

8. **The consistence problems with the theory(ies) of ratio**

As we will see there are some difficulties because there are three different constructions for ‘ratio’.

- The first is the ‘general theory’ given in book V (essentially through definitions 5 and 7) for ‘general ratios’ i.e. (homogeneous) magnitudes.
- The second, as we said at the beginning of the lecture, is done the book VII.
- The last was introduced above to understand the ‘general theory’ relating ‘general ratios’ and ‘ratios of integers’ in a ‘mixed theory.

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It is not difficult to show, using essentially the fundamental ‘common notions’ (‘κοιναὶ ἔννοιαι’) at the beginning of book I, all these theories are consistent.

9. **Conclusion**

Since I have to conclude I will do it on two points.

A. The first is purely mathematical. Euclid’s construction as we noted has nothing to do with the construction of real numbers, since we do not know what a ‘magnitude’ is, even less what a ‘ratio of homogeneous magnitudes’ is. On the one hand it does not contain such a construction, but on the other hand it gives a much larger theory. As an empty shell, it can contain many different things, in modern terms compact sets, non-connected sets, sets with partial order and so on.

B. The second point is more philosophical and historical. The amazing change consequence of the general theory of ratios was to move the centre of mathematics from the study of things to the study of relations. Its consequences were felt through the whole field of science. The new theory of ratios, which has its origins in the irrational magnitudes, did not only shook the whole mathematics. It shook the whole science and even the ancient Greek way of thinking in his totality as we can see in the hunt of the ‘irrationality’ (‘ἄλογος’) through the reason (‘λόγος’) in Plato’s *Theaetetus*. But this is another story.