A Note on the Value of Zero-Sum Sequential Repeated Games with Incomplete Information

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Abstract: We consider repeated two-person zero-sum games with lack of information on both sides. If the one shot game is played sequentially, it is proved that the sequence \( v_n \) is monotonic, \( v_n \) being the value of the \( n \) shot game. Moreover the speed of convergence is bounded by \( K/n \), and this is the best bound.

Introduction

The class of games considered in this note are those introduced and studied by Aumann/Maschler [1966, 1967, 1968]. Later it was proved by Mertens/Zamir [1971] that, if \( v_n \) is the value of an \( n \) repeated zero-sum game with lack of information on both sides, \( \lim v_n \) exists and \( d_n = \lim v_n - \lim v_n \) is bounded by \( K/\sqrt{n} \). Then Zamir [1972] has proved that it is the best bound. Now, if we consider the "independent" case, where moreover the moves are made sequentially, we can compute \( v_n \) by using an explicit formula proved by Ponssard [1975] for "games with almost perfect information". Then we prove that the sequence \( v_n \) is monotonic and that \( d_n \) is bounded by \( K/n \). We also give an example of a game with \( d_n = 1/2n \).

1. The Game

Let \((A^r_s), r \in \{1, \ldots, R\}, s \in \{1, \ldots, S\}\), be \( m \times n \) matrices viewed as payoff matrices of two-person zero-sum games, with elements \( a_{ij}^rs, i \in I = \{1, \ldots, m\}\), \( j \in J = \{1, \ldots, n\}\). Let \( P \) (resp. \( Q \)) be the simplex of \( \mathbb{R}^R \) (resp. \( \mathbb{R}^S \)).

For each \( p \in P \), \( q \in Q \), \( n \in \mathbb{N} \), \( G_n(p, q) \) is the \( n \)-times repeated game played as follows.

Stage 0. Chance chooses some \( r \) (resp. \( s \)) according to the probability distribution \( p \) (resp. \( q \)). Then player I is informed of \( r \) and player II of \( s \). (All of the above description is common knowledge.)

Stage 1. Player I chooses \( i_1 \in I \), player II is told which \( i_1 \) was choosen and chooses \( j_1 \in J \); then player I is informed of \( j_1 \).

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Now this stage is repeated again and again. After the \( n \)-stage player I receives from player II the following amount:

\[
\frac{1}{n} \sum_{h=1}^{n} a_{i_h, i_{h+1}},
\]

where \((i_h, i_{h+1})\) are the strategies used at the \( h \)-th stage. We denote by \( v_n(p, q) \) the value of \( G_n(p, q) \).

Let us define, for each \( p \in P \), \( q \in Q \):

\[
A(p, q) = \sum_{r=1}^{R} \sum_{s=1}^{S} p^r q^s A_{rs} \quad \text{and let } u(p, q) \text{ be the value of the zero-sum sequential game with matrix } A(p, q). (\text{The game where both players play non revealing strategies.})
\]

For each real function \( f \) on \( P \times Q \) we denote by \( Cf \) the smallest real function \( g \) on \( P \times Q \) such that:

\[
g(\cdot, q) \text{ is concave on } P \text{ for each } q \in Q
\]

\[
g(p, q) \geq f(p, q) \text{ on } P \times Q.
\]

Similarly we define \( Vf = \max_{Q} f \).

We can now state the main result of Mertens/Zamir [1971]:

\[
\lim_{n \rightarrow \infty} v_n(p, q) \text{ exists and is the only solution of the system:}
\]

\[
\begin{align*}
x(p, q) &= V \max \{u(p, q), x(p, q)\} \\
x(p, q) &= C \min \{u(p, q), x(p, q)\}.
\end{align*}
\]

In order to symplify the notations we shall denote by:

\[
\begin{align*}
M &\text{ the maximum over } i, i \in I, \\
m &\text{ the minimum over } j, j \in J, \\
\Sigma &\text{ the expression } \sum_{r=1}^{R} \sum_{s=1}^{S} p^r q^s A_{rs}. (\text{In fact } \Sigma \text{ is some } \Sigma(i, j) \text { but no confusion will result.)}
\end{align*}
\]

2. The Results

**Proposition 1:** For each \( p \in P, q \in Q \), the sequence \( v_n(p, q) \) is increasing.

**Proof:** Using Theorem 1 of Ponsard [1975] we have:

\[
nv_n(p, q) = CMVm(\Sigma + (n - 1)v_{n-1}(p, q))
\]

for all \( n \geq 1 \), where \( v_0(p, q) = 0 \) on \( P \times Q \). Then

\[
2v_2(p, q) = CMVm(\Sigma + v_1(p, q)) \geq CM(Vm \Sigma + v_1(p, q))
\]
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since \( \text{Vex} \ (a + b) \geq \text{Vex} \ (a) + \text{Vex} \ (b) \) and \( \nu_n \ (p, q) \) is convex w.r.t. \( q \), for all \( n \). Now we have

\[
2 \nu_2 \ (p, q) \geq C \left( MVm \ \Sigma + \nu_1 \ (p, q) \right).
\]

Let us denote by \( g_1 \ (p, q) \) the function \( MVm \ \Sigma \), then

\[
\nu_1 \ (p, q) = Cg_1 \ (p, q).
\]

But \( \text{Cav} \ (a + \text{Cav} \ (a)) = 2 \text{Cav} \ (a) \), so that

\[
2 \nu_2 \ (p, q) \geq 2 \nu_1 \ (p, q).
\]

Let us assume now that

\[
\nu_n \ (p, q) \geq \nu_{n-1} \ (p, q) \text{ on } P \times Q.
\]

We have

\[
(n + 1) \nu_{n+1} \ (p, q) = CMV \ (m \ \Sigma + n \nu_n \ (p, q)) \\
\geq CMV \ (m \ \Sigma + (n - 1) \nu_{n-1} \ (p, q) + \nu_n \ (p, q)).
\]

But the right expression is greater than

\[
CM \left( V \ (m \ \Sigma + (n - 1) \nu_{n-1} \ (p, q)) + \nu_n \ (p, q) \right).
\]

Denoting by \( g_n \ (p, q) \) the function \( MV \ (m \ \Sigma + (n - 1) \nu_{n-1} \ (p, q)) \), we obtain

\[
(n + 1) \nu_{n+1} \ (p, q) \geq C \ (g_n \ (p, q) + \frac{1}{n} Cg_n \ (p, q))
\]

and the right side is

\[
\frac{n + 1}{n} Cg_n \ (p, q) = (n + 1) \nu_n \ (p, q).
\]

If we denote \( \lim \nu_n \ (p, q) \) by \( \nu \ (p, q) \) we get obviously.

**Corollary 1:**

\[
\nu_n \ (p, q) \leq \nu \ (p, q) \text{ on } P \times Q \text{ for all } n.
\]

**Proposition 2 (The Error Term):**

\[
| \nu \ (p, q) - \nu_n \ (p, q) | \leq K/n \text{ on } P \times Q, \text{ for some } K \in \mathbb{R}, \text{ and this is the best bound.}
\]
**Proof:** We still suppose that player I is the maximizer. We shall write, for \( i \in I \),

\[
f_i(p, q) = -m \left( \sum_{r=1}^{R} \sum_{s=1}^{S} p^r q^s d_{ij}^{rs} \right).
\]

Note that the \( f_i \) are convex and piecewise linear in both variables.

Let \( f(p, q) = \sum_{i=1}^{m} f_i(p, q) - L \) where \( L \in \mathbb{R} \) is chosen such that

\[
v_1(p, q) \geq v(p, q) + f(p, q) \text{ on } P \times Q. \text{ (} v_1 \text{ and } v \text{ are bounded on } P \times Q. \)
\]

Let us suppose now that:

\[
n v_n(p, q) \geq n v(p, q) + f(p, q).
\]

Using (2) we have

\[
(n + 1) v_{n+1}(p, q) \geq CM V m (S + f(p, q) + n v(p, q)) \geq CM (V (m S + f(p, q)) + n v(p, q))
\]

since \( v \) is convex w.r.t. \( q \).

But, by construction, \( m S + f(p, q) \) is convex w.r.t. \( q \), for each \( i \in I \), so that

\[
(n + 1) v_{n+1}(p, q) \geq C (u(p, q) + f(p, q) + n v(p, q)).
\]

Now \( \text{Cav}(a+b) \leq \text{Cav}(a) + \text{Cav}(b) \) so by letting \( a = u(p, q) + f(p, q) + n v(p, q) \) and \( b = -f(p, q) \), the right member is greater than

\[
C (u(p, q) + n v(p, q)) - C (-f(p, q)).
\]

Now \( -f \) is concave w.r.t. \( p \) so we obtain

\[
(n + 1) v_{n+1}(p, q) \geq C (u(p, q) + n v(p, q)) + f(p, q) \geq C ((n + 1) \min(u(p, q), v(p, q))) + f(p, q).
\]

Using (1), the fact that \( f \) is bounded on \( P \times Q \) and \( v_n \leq v \) for all \( n \) (Cor. 1), we arrive at the proof.

**Example:**

The following example shows that it is the best bound. Assume that \( R = 1 \) and \( S = 2 \) (there is lack of information on one side but player I is uninformed). The payoff matrices are given by:
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\[ A^{11} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \quad A^{12} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \]

The functions \( u(q) \), \( v_n(q) \), \( v(q) \), are given in the diagrams below. We note that \( v(1/2) - v_n(1/2) = 1/2n \).

Remarks: If there is lack of information on one side, the informed player maximizing and moving first, we obviously have \( v_1(p) \geq \text{Cav} u(p) = v(p) \) so that Prop. 1 and Cor. 1 imply \( v_n(p) = v(p) \) for all \( n \), a result which was already proved by Ponssard/Zamir [1973].

In the general case of game with lack of information on one side, the sequence \( v_n \) is monotonic, but decreases if the informed player is the maximizer, as already mentioned by Aumann/Maschler [1968]. This can be seen immediately if one writes the recursion formula [Zamir] in the following manner:

\[
(n + 1) v_{n+1}(p) = \max_s \left\{ \min_t \sum_k s^k A^k t + n \sum_i \bar{s}_i v_n(p_i) \right\}
\]

where \( s = (s^1, \ldots, s^k, \ldots, s^r) \), \( s^k \) is a probability vector over \( I \) for all \( k \), \( t \) is a probability vector over \( J \), \( \bar{s}_i = \sum_k s^k p^k \), and \( p_i \) is the conditional probability over \( K \) given \( i \).

Assuming \( v_n(p) \leq v_{n-1}(p) \) we have

\[
(n + 1) v_{n+1}(p) \leq \max_s \left\{ \min_t \sum_k s^k A^k t + (n - 1) \sum_i \bar{s}_i v_{n-1}(p_i) + \right. \\
\left. + \sum_i \bar{s}_i v_n(p_i) \right\}
\]

and since \( v_n \) is concave it follows that

\[
(n + 1) v_{n+1}(p) \leq \max_s \left\{ \min_t \sum_k s^k A^k t + (n - 1) \sum_i \bar{s}_i v_{n-1}(p_i) \right\} + v_n(p).
\]

Hence

\[
(n + 1) v_{n+1}(p) \leq (n + 1) v_n(p).
\]

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References


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