NEW ENERGY-CAPACITY-TYPE INEQUALITIES AND UNIQUENESS OF CONTINUOUS HAMILTONIANS

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Abstract. We prove a new variant of the energy-capacity inequality for closed rational symplectic manifolds (as well as certain open manifolds such as $\mathbb{R}^{2n}$, cotangent bundle of closed manifolds...) and we derive some consequences to C^0–symplectic topology. Namely, we prove that a continuous function which is a uniform limit of smooth normalized Hamiltonians whose flows converge to the identity for the spectral (or Hofer's) distance must vanish. This gives a new proof of uniqueness of continuous generating Hamiltonian for hameomorphisms. This also allows us to improve a result by Cardin and Viterbo on the C^0–rigidity of the Poisson bracket.

1. Introduction and results

Let (M, ω) denote a closed and connected symplectic manifold. It is said to be rational if $\omega(\pi_2(M)) = \Omega\mathbb{Z}$ for a non-negative $\Omega \in \mathbb{R}$. A rational symplectic manifold is called monotone if there exists $\lambda \in \mathbb{R}$ such that $[\omega] = \lambda c_1$ on $\pi_2(M)$, where c_1 denotes the first Chern class of (M, ω). We say that M is positively monotone if $\lambda \geqslant 0$ and negatively monotone if $\lambda < 0$.

Recall that, because ω is non-degenerate, a smooth Hamiltonian, that is, a smooth map $H : \mathbb{S}^1 \times M \to \mathbb{R}$, generates a family of Hamiltonian vector fields defined by $dH_t = \omega(X_H^t, \cdot\,)$ and which in turn generates a 1–parameter family of diffeomorphisms ϕ_H^t such that ϕ_H^0 is the identity and $\partial_t \phi_H^t = X_H^t(\phi_H^t)$.

The time–1 diffeomorphisms obtained as the end of a Hamiltonian flow form a group called the Hamiltonian diffeomorphism group and usually denoted $\mathrm{Ham}(M, \omega)$. Its universal cover, $\widetilde{\mathrm{Ham}}(M, \omega)$, is naturally isomorphic to the set of equivalence classes of normalized Hamiltonians. Recall that (on compact manifolds), a Hamiltonian is said to be normalized if for all t, $\int_M H_t\, \omega^n = 0$ and that two normalized Hamiltonians H and K are equivalent if there exists a homotopy running from H to K, consisting of normalized Hamiltonians whose flows have fixed ends, namely Id and $\phi := \phi_H^1 = \phi_K^1$.

The universal cover $\widetilde{\mathrm{Ham}}(M, \omega)$ admits two natural "(pseudo-)norms". The first one was introduced by Hofer in [11] (and is now called Hofer's

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norm). It is defined by

$$\|\tilde{\phi}\| = \inf_K \int_0^1 \left(\max_{x\in M} K(t,x) - \min_{x\in M} K(t,x) \right) dt$$

where the infimum is taken over all Hamiltonians K whose flow is a representative of the homotopy class $\tilde{\phi}$.

The second one arises as a consequence of the theory of spectral invariants. One can associate to every smooth Hamiltonian a real number called the spectral invariant of H; it is usually denoted by $c(1,H)$. This is, roughly speaking, the action level at which the neutral element $1 \in QH^*(M)$ appears in the Floer homology of H. These invariants were introduced by Viterbo, Schwarz and Oh (See [30], [26], the lecture notes [22] and references therein). They have been extensively studied and have had many interesting applications to symplectic topology. For example, they were used by Entov and Polterovich in their construction of Calabi quasimorphisms [5], and by Ginzburg in his proof of the Conley conjecture [10].

Note that, even though the unit of the quantum cohomology ring is not necessarily the only class to which one can associate such invariants, it is the only one used in this article and thus $c(1,H)$ will be denoted $c(H)$.

Spectral invariants lead to a "spectral pseudo-norm" which is defined for an element $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$ generated by a Hamiltonian H as

$$\gamma(\tilde{\phi}) = c(H) + c(\bar{H})$$

($\bar{H}$ is explicitly defined in Section 2, it generates the Hamiltonian isotopy $(\phi_H^t)^{-1}$.) One quite remarkable fact is that the spectral pseudo-norm is bounded from above by Hofer's norm (see Section 2).

In this article, we are interested in limits of Hamiltonian flows for these (pseudo-)norms; this is a central theme of what is now called "C^0–symplectic topology". This terminology refers to a family of problems in symplectic topology that tries to define and study continuous analogs of the classical smooth objects of the symplectic world. Such definitions are often made possible by symplectic rigidity results. As an example, the famous Gromov–Eliashberg Theorem (the group of symplectic diffeomorphisms is C^0–closed in the full group of diffeomorphisms) allows to define a symplectic homeomorphism as a homeomorphism which is a C^0–limit of symplectic diffeomorphisms.

One important motivation for C^0–symplectic topology is to try to define continuous Hamiltonian dynamics. As an example, this was the purpose of the definition by Oh and Müller [23] of the notion of a "continuous Hamiltonian isotopy" (we will contract this terminology to the shorter "hameotopy"), whose definition we now recall. Equip M with a distance d induced by

any Riemannian metric. We define the C^0–distance between two homeomorphisms ϕ, ψ by $d_{C^0}(\phi, \psi) := \max_x d(\phi(x), \psi(x))$. For two paths of homeomorphisms ϕ^t, ψ^t ($t \in [0,1]$) define $d_{C^0}(\phi^t, \psi^t) := \max_{t,x} d(\phi^t(x), \psi^t(x))$. In Remark 14, we briefly discuss an important property of this metric.

A path of homeomorphisms h^t is a *hameotopy* if there exists a sequence of smooth Hamiltonian functions $\{H_k\}$ such that

- $d_{C^0}(\phi^t_{H_k}, h^t) \to 0$,
- the Hamiltonian functions H_k converge uniformly to a continuous function $H : \mathbb{S}^1 \times M \to \mathbb{R}$.

Analogously to the smooth case, the function H is said to "generate" the isotopy h^t. A continuous function H generates at most one hameotopy [23]. The set of all time-independent functions H generating a hameotopy will be denoted by C^0_{Ham}. As noticed in [23], every $C^{1,1}$ function belongs to C^0_{Ham}. One important result of the theory is the uniqueness of the generating continuous Hamiltonian:

Theorem 1 (Viterbo [31], Buhovsky–Seyfaddini [3])**.** *Let $\{H_k\}$, $\{H'_k\}$ be two sequences of normalized smooth Hamiltonians on a closed manifold M. Suppose that their flows C^0–converge to the same continuous isotopy and that $H_k - H'_k$ converges uniformly to some continuous function H. Then H vanishes identically. In other words, given a hameotopy, the generating continuous Hamiltonian is unique.*

The first major theorem of this article is a result analogous to the above with the C^0–distance replaced by the spectral pseudo-distance γ. Since, in this generality, γ is not defined on the Hamiltonian diffeomorphisms group itself but only on its universal cover, we need to replace isotopies by their lift to the universal cover. We will denote by $\{\tilde{\phi}^t_H\}$ (or just $\tilde{\phi}^t_H$) the unique lift of the isotopy $\{\phi^t_H\}$ to $\widetilde{\mathrm{Ham}}(M, \omega)$ whose starting point, $\tilde{\phi}^0_H$, is the identity element. Said differently, for fixed $t \in \mathbb{R}$, $\tilde{\phi}^t_H$ is the element of $\widetilde{\mathrm{Ham}}(M, \omega)$ represented by the path $[0,1] \to \mathrm{Ham}(M, \omega)$, $s \mapsto \phi^{st}_H$.

Theorem 2. *Let (M, ω) denote a rational symplectic manifold, let U be a non-empty open subset of M, I be a non-empty open interval in $\mathbb{R}$ and $\{H_k\}$, $\{H'_k\}$ be two sequences of smooth Hamiltonians such that*

(i) *For any $t \in I$, $\gamma(\tilde{\phi}^t_{H_k}, \tilde{\phi}^t_{H'_k})$ converges to zero,*
(ii) *H_k and H'_k converge uniformly on $I \times U$ respectively to continuous functions H and H'.*

Then, $H - H'$ depends only on the time variable on $I \times U$.

Note that, since the spectral pseudo-distance is bounded from above by Hofer's distance, this theorem also holds with γ replaced by $\|\cdot\|$. Note also that if $U = M$ and if the sequences consist of normalized Hamiltonians, then $H = H'$.

Finally, let us emphasize the fact that if spectral invariants descend from $\widetilde{\mathrm{Ham}}(M,\omega)$ to $\mathrm{Ham}(M,\omega)$, then γ also descends (as a genuine norm) and Theorem 2 holds if we replace (i) by the much weaker assumption:

(i') For any $t \in I$, $\gamma(\phi^t_{H_k}, \phi^t_{H'_k})$ converges to zero.

For example, this is true if we assume the additional (rather strong) assumption that (M,ω) is weakly exact (that is, $\omega(\pi_2(M)) = 0$). Another example comes from the third author's [28]. Assume that (M,ω) is negatively monotone and that there exists a non-empty open set V such that for all k, H_k and H'_k lie in $C^\infty_c(\mathbb{S}^1 \times (M \backslash V))$, then Theorem 2 holds under (i') and (ii).

Applications. Theorem 1, which proves that hameotopies have unique normalized generating Hamiltonians, is one of the most foundational results in C^0 Hamiltonian dynamics; see [3, 23, 31] for some of the consequences of this theorem. In Section 4, we will show that Theorem 2 allows us to recover Theorem 1; to the best of our knowledge, this is the first proof of Theorem 1 via Floer-theoretic methods.

In addition to the above, Theorem 2 has other interesting consequences as well. In [14] (see [13] for a better presentation though in French), the first author suggested another attempt of defining continuous Hamiltonian dynamics. The idea is to introduce the abstract completion of the group of Hamiltonian diffeomorphisms with respect to the spectral metric. The paper is written in $\mathbb{R}^{2n}$ but everything there can be done on general, symplectically aspherical, closed manifolds (where, as mentioned above, γ descends to a non-degenerate norm on $\mathrm{Ham}(M,\omega)$). On the level of Hamiltonian functions, one can introduce a distance between two Hamiltonians by

$$\gamma_u(H,K) = \sup_{t\in[0,1]} \gamma(\phi^t_H, \phi^t_K),$$

and call a "generalized Hamiltonian" any element in the completion of the set of smooth Hamiltonians with respect to the distance γ_u. The canonical map $H \mapsto \phi^t_H$ naturally extends to the completions, and we can speak of the "flow" generated by a generalized Hamiltonian. These completions have applications to the study of Hamilton–Jacobi equations ([14, 13]). They are also needed for Viterbo's symplectic homogenization theory [32].

The main problem encountered with these completions is that their elements are a priori very abstract objects, that is, equivalence classes of Cauchy sequences for some abstract distance. However, some elements can be represented by honest continuous functions: Indeed, the inequality $\gamma_u \leqslant \|\cdot\|_{C^0}$ induces a map ι from $C^0_c(\mathbb{S}^1 \times M)$ to the set of generalized Hamiltonians. It follows that continuous Hamiltonians have a flow in the γ–completion of the Hamiltonian group. Like in the case of hameotopies, it is natural to wonder whether the generating continuous Hamiltonian is unique. Theorem 2 answers this question positively. It says in particular that the map ι is injective. In other words, the continuous function representing a given generalized Hamiltonian is unique.

Note that since Theorem 2 holds for any open set U, the uniqueness of the continuous generator is actually local. Therefore, the result can be applied to generalized Hamiltonians that can be represented by not everywhere continuous functions. Examples of such elements where provided in [14].

Theorem 2 also has consequences in terms of C^0–rigidity of the Poisson bracket. Recall that the Poisson bracket of two differentiable functions F, G on M, with Hamiltonian vector field X_F, X_G is given by

$$\{F, G\} = \omega(X_F, X_G).$$

A function F is called a first integral of $G \in C^0_{\mathrm{Ham}}$ if F is constant along the flow of G. When F and G are smooth, F is a first integral of G if and only if $\{F, G\} = 0$.

As one can see, the Poisson bracket is defined only in terms of the differentials of the involved functions. Nevertheless, it satisfies some rigidity with respect to the C^0–topology. This property was first discovered by Cardin and Viterbo [4]. Their theorem has opened an active domain of research and has been improved in several directions by many authors (see e.g. [1, 2, 6, 7, 15, 33] for some of the strongest results).

Here, we improve the result of Cardin and Viterbo in a new direction.

Theorem 3. *Let F_k and G_k be two sequences of smooth functions on a closed, rational symplectic manifold M such that:*

- *the sequence F_k converges uniformly to some continuous function F,*
- *the sequence G_k converges uniformly to some function $G \in C^0_{\mathrm{Ham}}$,*
- *the sequence of Poisson brackets $\{F_k, G_k\}$ converges uniformly to 0.*

Then, F is a first integral of G.

In particular, the theorem holds when F is C^0 and G is $C^{1,1}$. The result of Cardin and Viterbo was the same theorem but with both F and G of class $C^{1,1}$. In our case where F is only C^0, the proof is made more difficult by the fact that F does not have any flow in general.

After the first version of this paper was written, we were informed by Buhovsky that it is possible to prove Theorem 3 using the energy-capacity inequality. Furthermore, Buhovsky's method allows him to remove the rationality assumption in the statement of the theorem.

Theorem 3 allows us to relate two notions of Poisson commutativity for continuous Hamiltonians. First recall the definition proposed by Cardin and Viterbo [4]: Two continuous functions C^0–*commute* if they are uniform limits of functions whose Poisson bracket uniformly converges to 0. Another definition of commutativity for functions in C^0_{Ham} would simply be that their flows commute. Theorem 3 has the following immediate corollary.

Corollary 4. *If two functions in C^0_{Ham} commute in the sense of Cardin and Viterbo, then the hameotopies they generate commute.*

Key technique involved in the proof of the main result. In order to prove Theorem 2, we first establish a new variant of the energy-capacity inequality for closed monotone symplectic manifolds.

We denote by c_{HZ} the following version of the Hofer–Zehnder capacity: For an open set U,

$$c_{\mathrm{HZ}}(U) = \sup\{\max f \mid f \in C_c^\infty(U) \text{ slow and non-negative}\}.$$

Recall that H is called *slow* if its Hamiltonian flow $\{\phi_H^t\}_t$ has no non-trivial orbits of period at most 1. As an example, it is well known that for a symplectic ball B of radius r, $c_{\mathrm{HZ}}(B) = \pi r^2$.

Since these types of capacities are defined in a very different fashion than the action selector c – as well as other natural invariants like displacement energy –, comparison between them (energy-capacity-like inequalities) leads to interesting consequences (see e.g [8, Theorem 1] for such relations and further applications). The key result toward our proof of Theorem 2 is the following set of energy-capacity-like inequalities.

Theorem 5. *Let (M, ω) denote a monotone symplectic manifold. Suppose that U is an open subset of M and H is a smooth Hamiltonian such that $\forall (t, x) \in [0, 1] \times U$ we have $H(t, x) = C$. Then, at least one of the following two possibilities holds:*

(1) $\gamma(\tilde{\phi}_H^1) = c(H) + c(\bar{H}) \geqslant c_{\mathrm{HZ}}(U)$,
(2) $|c(H) - C| \leqslant c_{\mathrm{HZ}}(U)$ *and* $|c(\bar{H}) + C| \leqslant c_{\mathrm{HZ}}(U)$.

We will prove Theorem 5 in Section 3. It is evident from our proof that if (M, ω) is positively monotone and the second of the above possibilities holds, then the numbers $(c(H) - C)$ and $(c(\bar{H}) + C)$ are always non-negative.

The above result, combined with the fact that c and γ are both bounded by the Hofer norm (see Section 2), has an immediate corollary.

Corollary 6. *Let (M, ω) denote a monotone symplectic manifold, U an open subset of M, and H a normalized and smooth Hamiltonian such that for any $(t, x) \in [0, 1] \times U$ we have $H(t, x) = C \geqslant 2c_{\mathrm{HZ}}(U)$. Then $||\tilde{\phi}_H^1|| \geqslant c_{\mathrm{HZ}}(U)$.*

Our proof of Theorem 5 relies on the discreteness of $\omega(\pi_2(M))$ and hence it does not extend to irrational manifolds. However, when (M, ω) is rational, but not monotone, we can prove a weaker version of Theorem 5 which is sufficient for the applications considered in this article.

Theorem 7. *Let (M, ω) denote a rational symplectic manifold. Suppose that U is an open subset of M and H is a smooth Hamiltonian such that $\forall (t, x) \in [0, 1] \times U$ we have $H(t, x) = C$. Then, at least one of the following two possibilities holds:*

(1) $\gamma(\tilde{\phi}_H^1) = c(H) + c(\bar{H}) \geqslant c_{\mathrm{HZ}}(U)$,
(2) there exist k and $\bar{k} \in \mathbb{Z}$, depending on H, such that:
$0 \leqslant c(H) - C - k\Omega \leqslant c_{\mathrm{HZ}}(U)$ *and* $0 \leqslant c(\bar{H}) + C + \bar{k}\Omega \leqslant c_{\mathrm{HZ}}(U)$.

Moreover, if $c_{\mathrm{HZ}}(U) < \frac{1}{2}\Omega$ then we may choose $k = \bar{k}$.

Theorem 7 will be proven in Section 3; note that it is trivially true if $c_{\mathrm{HZ}}(U) \geqslant \Omega$.

Currently [16], we are in the process of proving energy-capacity-type inequalities, in the spirit of those appearing in this section, for Lagrangian spectral invariants as defined by Viterbo [30] for cotangent bundles or by Leclercq [19] for weakly exact Lagrangians in compact manifolds. Such inequalities could be potentially very helpful in obtaining new rigidity results for Lagrangian submanifolds.

Extension to non-closed manifolds. In this article, we have written our results for closed manifolds only, but each of them can be adapted to non-closed manifolds as soon as spectral invariants are properly defined and satisfy the standard properties (see Proposition 8 below). Of course, in this case, we only consider compactly supported Hamiltonians. (Note that in non-compact manifolds the requirement for a Hamiltonian to have compact support is a natural – and commonly used – normalization condition.)

Frauenfelder and Schlenk [9] defined the spectral invariant c on any weakly exact convex at infinity symplectic manifold. This has been extended to more general convex at infinity symplectic manifolds by Lanzat [18]. In the special case of $\mathbb{R}^{2n}$ spectral invariants can be defined using generating functions instead of Floer homology following Viterbo [30]. Our results also extend to this setting.

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2. A review of spectral invariants

In this section we briefly review the theory of spectral invariants on closed symplectic manifolds. For further details we refer the interested reader to [20, 22, 26].

Denote by $\Omega_0(M)$ the space of contractible loops in M and let $\Gamma := \frac{\pi_2(M)}{\ker(c_1)\cap\ker([\omega])}$. It is the group of deck transformations of the Novikov covering

of $\Omega_0(M)$, which is defined by the following expression:

$$\tilde{\Omega}_0(M) = \frac{\{[z,u] : z \in \Omega_0(M), u : D^2 \to M, u|_{\partial D^2} = z\}}{[z,u] = [z',u'] \text{ if } z = z' \text{ and } \bar{u}\#u' = 0 \text{ in } \Gamma},$$

where $\bar{u}\#u'$ denotes the sphere obtained by gluing u and u' along their common boundary with the orientation on u reversed. The disc u, appearing in the above definition, is referred to as the capping disc of z. Recall that the action functional of a Hamiltonian H is a map from $\tilde{\Omega}_0(M)$ to $\mathbb{R}$ defined by

$$\mathcal{A}_H([z,u]) = \int_{\mathbb{S}^1} H(t, z(t))dt \ - \int_{D^2} u^*\omega.$$

It is well known that the set of critical points of $\mathcal{A}_H$, denoted by $\mathrm{Crit}(\mathcal{A}_H)$, consists of equivalence classes of pairs, $[z,u] \in \tilde{\Omega}_0(M)$, such that z is a 1–periodic orbit of the Hamiltonian flow ϕ_H^t. The set of critical values of $\mathcal{A}_H$ is called the action spectrum of H and is denoted by $\mathrm{Spec}(H)$; it has Lebesgue measure zero. When H is non-degenerate, the set $\mathrm{Crit}(\mathcal{A}_H)$ can be indexed by the well known Conley–Zehnder index, $\mu_{\mathrm{CZ}} : \mathrm{Crit}(\mathcal{A}_H) \to \mathbb{Z}$, for every $\Sigma \in \Gamma$, the Conley–Zehnder index satisfies

$$\mu_{\mathrm{CZ}}([z, u\#\Sigma]) = \mu_{\mathrm{CZ}}([z,u]) - 2c_1(\Sigma). \tag{1}$$

Several conventions are used for defining this index. We fix our convention in the following fashion: suppose that g is a C^2–small Morse function. For every critical point p of g, we require that

$$i_{\mathrm{Morse}}(p) = \mu_{\mathrm{CZ}}([p, u_p]),$$

where $i_{\mathrm{Morse}}(p)$ is the Morse index of p and u_p is a trivial capping disc. Notice that the set of equivalence classes of pairs $[p,u]$ consists of equivalence classes $[p,\Sigma]$, with Σ in $\pi_2(M)$ (and $[p,\Sigma] = [p,\Sigma']$ if $\omega(\Sigma) = \omega(\Sigma')$) and that with our convention $\mu_{\mathrm{CZ}}([p,\Sigma]) = i_{\mathrm{Morse}}(p) - 2c_1(\Sigma)$. This observation will be useful in the proofs of Theorems 5 and 7.

Spectral invariants, or action selectors, are defined via Hamiltonian Floer theory. The procedure consists of filtering Floer homology by the values of the action functional and then associating to quantum cohomology classes (seen as Floer homology classes via the so-called PSS homomorphism [25]) the minimal action level at which they appear in the filtration. As mentioned in the introduction, the specific spectral invariant used in this article, denoted by $c(H)$ for $H \in C^\infty(\mathbb{S}^1 \times M)$, is the one associated to the neutral element $1 \in QH^*(M)$. We will now list, without proof, the basic properties of this spectral invariant. Recall that the composition of two Hamiltonian flows, $\phi_H^t \circ \phi_G^t$, and the inverse of a flow, $(\phi_H^t)^{-1}$, are Hamiltonian flows generated by $H\#G(t,x) = H(t,x) + G(t,(\phi_H^t)^{-1}(x))$ and $\bar{H}(t,x) = -H(t,\phi_H^t(x))$, respectively.

Proposition 8. *([21, 22, 26, 29])*
The spectral invariant $c : C^\infty(\mathbb{S}^1 \times M) \to \mathbb{R}$ has the following properties:

(1) (Shift) If $r : \mathbb{S}^1 \to \mathbb{R}$ is smooth then $c(H + r) = c(H) + \int_{\mathbb{S}^1} r(t)dt$.

(2) (Triangle Inequality) $c(H\#G) \leqslant c(H) + c(G)$.
(3) (Continuity) $|c(H) - c(G)| \leqslant \int_{\mathbb{S}^1} \max_{x\in M} |H_t - G_t| dt$.
(4) (Spectrality) If (M, ω) *is rational, then there exists* $[z, u] \in \mathrm{Crit}(\mathcal{A}_H)$ *such that* $c(H) = \mathcal{A}_H([z, u])$*, i.e.* $c(H) \in \mathrm{Spec}(H)$*. Furthermore, if* H *is non-degenerate then* $\mu_{\mathrm{CZ}}([z, u]) = 2n$.
(5) (Homotopy Invariance) Suppose that H *and* G *are normalized and generate the same element of* $\widetilde{Ham}(M)$*. Then,* $c(H) = c(G)$.

The spectral pseudo-norm γ is defined on $\widetilde{\mathrm{Ham}}(M, \omega)$ by the expression

$$\gamma(\phi_H^t) = c(H) + c(\bar{H}).$$

It induces a pseudo-distance (also denoted γ) defined by

$$\gamma(\phi_H^t, \phi_K^t) = \gamma((\phi_K^t)^{-1} \circ \phi_H^t) = c(\bar{K}\#H) + c(\bar{H}\#K).$$

Note that c and γ are both bounded by the Hofer distance $\|\cdot\|$. This easily follows from a slightly different version of Property *(3)*:

$$\int_{\mathbb{S}^1} \min_{x\in M}(H_t - G_t)dt \leqslant c(H) - c(G) \leqslant \int_{\mathbb{S}^1} \max_{x\in M}(H_t - G_t)dt.$$

It is well-known [26] that if $\omega|_{\pi_2(M)} = 0$, then γ descends to a genuine distance on $\mathrm{Ham}(M, \omega)$.

Finally, we end this section with the following lemma which will be used in the proof of Theorem 5.

Lemma 9. *Suppose that* H *is a not necessarily non-degenerate Hamiltonian on a symplectic manifold* (M, ω)*. Let* $A = \{\gamma : \exists u \text{ s.t. } c(H) = \mathcal{A}_H([\gamma, u])\}$*. If all the orbits* $\gamma \in A$ *are non-degenerate, then there exists a capped orbit* $[\gamma, u]$ *such that* $c(H) = \mathcal{A}_H([\gamma, u])$ *and* $\mu_{CZ}([\gamma, u]) = 2n$.

Proof. For each $\gamma \in A$, let U_γ denote a neighborhood of γ which contains no other periodic orbits of H; such neighborhoods exist because the orbits contained in A are all isolated. Pick a sequence of non-degenerate Hamiltonians, $\{H_i\}_i$, C^3–approximating H such that for every i and every $\gamma \in A$, $H_i|_{U_\gamma} = H|_{U_\gamma}$. For each i, let $[\gamma_i, u_i]$ denote a capped orbit of H_i such that $c(H_i) = \mathcal{A}_{H_i}([\gamma_i, u_i])$ and $\mu_{CZ}([\gamma_i, u_i]) = 2n$. Such $[\gamma_i, u_i]$ exists by *spectrality* of the invariant c and non-degeneracy of H_i.

By the Arzela–Ascoli theorem, a subsequence of the orbits γ_i, which we will denote by γ_i as well, C^1–converges to an orbit γ' of H. Since the orbits γ_i C^1–converge to γ', one can construct a capping disc u' for γ' such that $\omega(u_i)$ converges to $\omega(u')$. It follows that $c(H) = \mathcal{A}_H([\gamma', u'])$ and thus $\gamma' \in A$.

Now, γ' is isolated and H_i coincides with H on $U_{\gamma'}$. Hence, $\gamma_i = \gamma'$ for large i and thus $\omega(u_i) = \omega(u')$ for large i. It then follows that the capped orbits $[\gamma', u_i]$, for sufficiently large i, satisfy the conclusion of our lemma. □

3. Proofs of the energy-capacity-type inequalities

The main goal of this section is to prove Theorems 5 and 7. We will also state and prove two additional results which will be used in the proof of Theorem 2. Our arguments will use the following notion:

Definition 10. *(See* [29, Definition 4.3]*) Let $f : M \to \mathbb{R}$ be an autonomous Hamiltonian. A critical point p of f is said to be flat if the linearized flow $(\phi_f^t)_* : T_pM \to T_pM$ has no non-constant periodic orbits of period at most 1. The function f is called flat if all of its critical points are flat.*

The importance of the above notion stems from the fact that if p is a non-degenerate and flat critical point of f, then the Morse index of p coincides with the Conley–Zehnder index of $[p, u_p]$. In Theorem 4.5 of [29], Usher proves that a slow and autonomous Hamiltonian on a closed manifold can be C^0–approximated, up to any precision, by Hamiltonians which are slow, flat, and Morse. In our proof of Theorem 5 we will need the following variant of Usher's theorem.

Theorem 11. *(Usher* [29, Theorem 4.5]*) Let $H : M \to \mathbb{R}$ denote a slow Hamiltonian whose support is contained in U. For any $\delta > 0$ there exists a slow Hamiltonian $K : M \to \mathbb{R}$ such that $\|K - H\|_{C^0} < \delta$, the support of K is contained in U, and all critical points of K that are contained in the interior of its support are non-degenerate and flat.*

We will not prove the above theorem as it can easily be extracted from the proof of Theorem 4.5 in [29]. Theorems 5 and 7 are similar in nature and their proofs have significant overlaps. Hence, we will provide a single argument proving both theorems at once.

Proofs of Theorems 5 and 7. Observe that by the shift property of spectral invariants we may assume, without loss of generality, that $C = 0$.

For any $\delta > 0$ pick a time independent Hamiltonian $f \in C_c^\infty(U)$ such that f is slow, $0 \leqslant f$, and $c_{\mathrm{HZ}}(U) - \delta \leqslant \max(f)$. Since f is slow we have $c(f) = \max(f)$ and $c(-f) = 0$; for a proof of this fact see Proposition 4.1 of [29]. Note that the conventions used in [29] are different from ours. By Theorem 11, we may assume that the critical points of f that are contained in the interior of the support of f are non-degenerate and flat. Consider the Hamiltonian $H_s = H + sf$. Its 1–periodic orbits consist of 1–periodic orbits of the flow of H together with the critical points of f. Hence,

$$\mathrm{Spec}(H_s) = \mathrm{Spec}(H) \cup \{sf(p) - \omega(\Sigma) : p \in \mathrm{Crit}(f),\ \Sigma \in \pi_2(M)\},$$

where $\mathrm{Crit}(f)$ denotes the set of critical points of f. Similarly, define $\bar{H}_s = \bar{H} + sf$. We have:

$$\mathrm{Spec}(\bar{H}_s) = \mathrm{Spec}(\bar{H}) \cup \{sf(p) - \omega(\Sigma) : p \in \mathrm{Crit}(f),\ \Sigma \in \pi_2(M)\}.$$

By the spectrality property we know that $c(H_s) \in \mathrm{Spec}(H_s)$ and $c(\bar{H}_s) \in \mathrm{Spec}(\bar{H}_s)$. However, suppose that one of the following two situations holds:

$$c(H_s) \in \mathrm{Spec}(H) \text{ for all } s \in [0,1] \tag{2}$$

$$\text{or} \quad c(\bar{H}_s) \in \mathrm{Spec}(\bar{H}) \text{ for all } s \in [0,1] \tag{3}$$

If (2) holds, then it follows, from the continuity property of spectral invariants, that $c(H) = c(H_1) = c(H+f)$. Using the triangle inequality we obtain $c(f) \leqslant c(\bar{H}) + c(H+f)$. Combining these with the fact that $c(f) = \max(f)$, we get

$$c_{\mathrm{HZ}}(U) - \delta \leqslant \max(f) \leqslant c(\bar{H}) + c(H).$$

We arrive at the same conclusion if (3) holds.

We will next show that if the first possibility, in either of Theorems 5 and 7, does not hold, then the second one must hold. Therefore, **for the rest of the proof**, we will suppose that $c_{\mathrm{HZ}}(U) > c(\bar{H}) + c(H)$. This implies that there exist δ and f as in the first paragraph of this proof such that (2) and (3) do not hold. Let $s_0 = \inf\{s \in [0,1] : c(H_s) \notin \mathrm{Spec}(H)\}$. Note that this means $c(H) = c(H_{s_0})$. Pick a sequence of numbers $s_i \in \{s \in [0,1] : c(H_s) \notin \mathrm{Spec}(H)\}$ such that $s_i \to s_0$. There exist critical points p_i of f contained in the interior of the support of f such that $c(H_{s_i}) = s_i f(p_i) - \omega(\Sigma_i)$, where $\Sigma_i \in \pi_2(M)$. By passing to a subsequence, we may assume that $p_i \to p$, where p is a critical point of f; note that p is not necessarily contained in the interior of the support of f. Now, the sequence $\omega(\Sigma_i)$ must converge because both $c(H_{s_i})$ and $s_i f(p_i)$ converge. Since $\omega(\pi_2(M))$ is discrete we conclude that $\omega(\Sigma_i) = \omega(\Sigma)$ for large i. It then follows that $c(H_{s_i}) = s_i f(p_i) - \omega(\Sigma)$ for large i, and

$$c(H) = c(H_{s_0}) = s_0 f(p) - \omega(\Sigma). \tag{4}$$

Similarly, let $r_0 = \inf\{r \in [0,1] : c(\bar{H}_s) \notin \mathrm{Spec}(\bar{H}_s)\}$. Repeating the same argument as above we find a capped orbit $[q, \Sigma']$ such that

$$c(\bar{H}) = c(\bar{H}_{r_0}) = r_0 f(q) - \omega(\Sigma'). \tag{5}$$

We will prove Theorems 5 and 7 by carefully analyzing the numbers $\omega(\Sigma)$ and $\omega(\Sigma')$.

Proof of Theorem 7: Since (M, ω) is rational, there exist integers k_1 and k_2 such that $\omega(\Sigma) = k_1\Omega$ and $\omega(\Sigma') = k_2\Omega$. From Equations (4) and (5) we get that $c(H) = s_0 f(p) - k_1\Omega$ and $c(\bar{H}) = r_0 f(q) - k_2\Omega$ so that point (2) of Theorem 7 holds with $k = -k_1$ and $\bar{k} = k_2$ since $0 \leqslant s_0 f(p),\ r_0 f(q) \leqslant c_{\mathrm{HZ}}(U)$.

Moreover, we have the following chain of inequalities:

$$0 \leqslant c(H) + c(\bar{H}) = s_0 f(p) + r_0 f(q) + (k - \bar{k})\Omega \leqslant c_{\mathrm{HZ}}(U)$$

which implies, if $c_{\mathrm{HZ}}(U) < \frac{1}{2}\Omega$, that

$$-\Omega < -2c_{\mathrm{HZ}}(U) \leqslant -s_0 f(p) - r_0 f(q) \leqslant (k - \bar{k})\Omega \leqslant c_{\mathrm{HZ}}(U) < \frac{1}{2}\Omega$$

which can be satisfied only if $k = \bar{k}$.

Proof of Theorem 5: We now assume (M,ω) to be monotone. We can apply Lemma 9 to H_{s_i} and assume that $\mu_{\mathrm{CZ}}([p_i,\Sigma_i]) = 2n$. On the other hand, H_{s_i} coincides with $s_i f$ on a neighborhood of p_i and thus

$$\mu_{\mathrm{CZ}}([p_i,\Sigma_i]) = i_{\mathrm{Morse}}(p_i) - 2c_1(\Sigma_i),$$

where $i_{\mathrm{Morse}}(p_i)$ is the Morse index of p_i with respect to f. Here, we have used the assumption that the critical points p_i of f are flat and non-degenerate, and hence $i_{\mathrm{Morse}}(p_i) = \mu_{\mathrm{CZ}}([p_i,u_{p_i}])$. Because $i_{\mathrm{Morse}}(p_i) \leqslant 2n$ we conclude that $c_1(\Sigma_i) \leqslant 0$. Recall that for large i, $\omega(\Sigma_i) = \omega(\Sigma)$, and thus, by monotonicity, $c_1(\Sigma_i) = c_1(\Sigma)$. Therefore,

$$c_1(\Sigma) \leqslant 0.$$

Similarly, we have

$$c_1(\Sigma') \leqslant 0.$$

Recall that $\omega = \lambda c_1$ on $\pi_2(M)$. First, suppose that $\lambda > 0$. Because $c_1(\Sigma), c_1(\Sigma') \leqslant 0$ we get that

$$c(H) = s_0 f(p) - \lambda c_1(\Sigma) \geqslant s_0 f(p) \text{ and,}$$
$$c(\bar{H}) = r_0 f(q) - \lambda c_1(\Sigma') \geqslant r_0 f(q).$$

Combining these inequalities with the assumption that $c(H)+c(\bar{H}) < c_{\mathrm{HZ}}(U)$ we conclude that

$$0 \leqslant c(H) \leqslant c_{\mathrm{HZ}}(U) \ \text{ and } \ 0 \leqslant c(\bar{H}) \leqslant c_{\mathrm{HZ}}(U).$$

This proves Theorem 5 for positively monotone symplectic manifolds. Next, suppose that $\lambda \leqslant 0$ and repeat the same argument as in the previous paragraph to get that $c(H) \leqslant s_0 f(p)$ and $c(\bar{H}) \leqslant r_0 f(q)$. Thus, $c(H),\ c(\bar{H}) \leqslant c_{\mathrm{HZ}}(U)$. Combining this with the fact that $c(H) + c(\bar{H}) \geqslant 0$ we obtain

$$|c(H)| \leqslant c_{\mathrm{HZ}}(U) \ \text{ and } \ |c(\bar{H})| \leqslant c_{\mathrm{HZ}}(U)$$

which concludes the proof of Theorem 5. □

We now focus on the rational case (proofs in the particular case of monotone manifolds are quite similar only slightly easier). Theorem 7 has the following straightforward corollary.

Corollary 12. *Let U_- and U_+ denote non-empty open subsets of (M,ω), and C_- and C_+ real numbers such that $\frac{1}{4}\Omega > C_\pm > c_{\mathrm{HZ}}(U_\pm)$. If a Hamiltonian H satisfies $H|_{U_\pm} = \pm C_\pm$, then $\gamma(\tilde{\phi}_H^t)$ is greater than or equal to at least one of $c_{\mathrm{HZ}}(U_-)$ and $c_{\mathrm{HZ}}(U_+)$.*

Proof. Let H be as above. Apply Theorem 7 to H on both U_- and U_+ and get two integers k, l such that

$$\gamma(\tilde{\phi}_H^t) \geqslant c_{\mathrm{HZ}}(U_+) \quad \text{or} \quad \begin{cases} C_+ + k\Omega \leqslant c(H) \leqslant C_+ + k\Omega + c_{\mathrm{HZ}}(U_+) \\ -C_+ - k\Omega \leqslant c(\bar{H}) \leqslant -C_+ - k\Omega + c_{\mathrm{HZ}}(U_+) \end{cases}$$

$$\gamma(\tilde{\phi}_H^t) \geqslant c_{\mathrm{HZ}}(U_-) \quad \text{or} \quad \begin{cases} -C_- + l\Omega \leqslant c(H) \leqslant -C_- + l\Omega + c_{\mathrm{HZ}}(U_-) \\ C_- - l\Omega \leqslant c(\bar{H}) \leqslant C_- - l\Omega + c_{\mathrm{HZ}}(U_-) \end{cases}$$

Thus, either we directly get $\gamma(\tilde{\phi}_H^t) \geqslant c_{\mathrm{HZ}}(U_\bullet)$ for $\bullet$ being either $+$ or $-$, or we have:

$$c(H) + c(\bar{H}) \geqslant -C_- - C_+ + (l-k)\Omega, \qquad \text{and}$$
$$c(H) + c(\bar{H}) \leqslant -C_- - C_+ + (l-k)\Omega + c_{\mathrm{HZ}}(U_-) + c_{\mathrm{HZ}}(U_+).$$

Since $0 \leqslant c(H)+c(\bar{H})$ the second inequality forces $l-k$ to be positive. Then from the first inequality we obtain:

$$\gamma(\tilde{\phi}_H^t) = c(H) + c(\bar{H}) \geqslant \frac{1}{2}\Omega \geqslant c_{\mathrm{HZ}}(U_-) + c_{\mathrm{HZ}}(U_+)$$

which concludes the proof. □

Now, using cut-off functions and this corollary, we can prove the following lemma which will be the main ingredient of the proof of Theorem 2.

Lemma 13. *Let F and G be Hamiltonians. Let $U_\pm$ be non-empty, disjoint, open subsets such that $c_{\mathrm{HZ}}(U_-) = c_{\mathrm{HZ}}(U_+)$ (we denote this common value by $c_{\mathrm{HZ}}(U)$) and*

(1) $c_{\mathrm{HZ}}(U) < \inf_{U_+}(F) - \sup_{U_+}(G) < \frac{1}{4}\Omega$, *and symmetrically* $-\frac{1}{4}\Omega < \sup_{U_-}(F) - \inf_{U_-}(G) < -c_{\mathrm{HZ}}(U)$,

(2) $\mathrm{osc}_{U_\pm}(F) + \mathrm{osc}_{U_\pm}(G) < \frac{1}{3}c_{\mathrm{HZ}}(U)$.

Then $\gamma(\tilde{\phi}_F^t, \tilde{\phi}_G^t) \geqslant \frac{1}{3}c_{\mathrm{HZ}}(U)$.

Proof. Fix $\varepsilon > 0$. We choose disjoint open subsets $V_\pm$ such that $\overline{U_\pm} \subset V_\pm$ and $\mathrm{osc}_{V_\pm}(F) < \mathrm{osc}_{U_\pm}(F) + \varepsilon$ and $\mathrm{osc}_{V_\pm}(G) < \mathrm{osc}_{U_\pm}(G) + \varepsilon$. We also choose cut-off functions $\rho_\pm$ with support in $V_\pm$, such that $0 \leqslant \rho_\pm \leqslant 1$ and $\rho_\pm|_{U_\pm} = 1$.

We define intermediate functions, f and g, by

$$f = F - \rho_+(F - a_+) - \rho_-(F - a_-) \quad \text{with } a_+ = \inf_{U_+}(F) \text{ and } a_- = \sup_{U_-}(F),$$
$$g = G - \rho_+(G - b_+) - \rho_-(G - b_-) \quad \text{with } b_+ = \sup_{U_+}(G) \text{ and } b_- = \inf_{U_-}(G).$$

By triangle inequality, we get

$$\gamma(\tilde{\phi}_F^t, \tilde{\phi}_G^t) \geqslant \gamma(\tilde{\phi}_f^t, \tilde{\phi}_g^t) - \gamma(\tilde{\phi}_f^t, \tilde{\phi}_F^t) - \gamma(\tilde{\phi}_G^t, \tilde{\phi}_g^t) \tag{6}$$

and we now bound the quantities appearing on the right-hand side.

Bounding $\gamma(\tilde{\phi}_f^t, \tilde{\phi}_g^t)$. Define $\varphi = \bar{g}\#f$, that is,

$$\varphi(t,x) = -g(t, \phi_g^t(x)) + f(t, \phi_g^t(x))$$

which generates $(\phi_g^t)^{-1} \circ \phi_f^1$. Notice that φ is constant on both open sets $U_\pm$: $\varphi|_{U_\pm} = a_\pm - b_\pm$ and that, by assumption,

$$\frac{1}{4}\Omega > C_+ = a_+ - b_+ = \inf_{U_+}(F) - \sup_{U_+}(G) > c_{\mathrm{HZ}}(U),$$
$$\frac{1}{4}\Omega > C_- = -(a_- - b_-) = -(\sup_{U_-}(F) - \inf_{U_-}(G)) > c_{\mathrm{HZ}}(U).$$

Thus, by applying Corollary 12 to φ we get: $\gamma(\tilde{\phi}^t_f, \tilde{\phi}^t_g) = \gamma(\tilde{\phi}^t_\varphi) \geqslant c_{\mathrm{HZ}}(U)$.

Bounding $\gamma(\tilde{\phi}^t_f, \tilde{\phi}^t_F)$ and $\gamma(\tilde{\phi}^t_G, \tilde{\phi}^t_g)$. By general property of γ, and definition of f

$$\begin{aligned}\gamma(\tilde{\phi}^t_f, \tilde{\phi}^t_F) &\leqslant \mathrm{osc}_M(F-f) = \mathrm{osc}_M(\rho_+(F-a_+) + \rho_-(F-a_-)) \\ &\leqslant \mathrm{osc}_{V_+}\big(\rho_+(F-a_+)\big) + \mathrm{osc}_{V_-}\big(\rho_-(F-a_-)\big) \\ &\leqslant \mathrm{osc}_{V_+}(F) + \mathrm{osc}_{V_-}(F) \leqslant \mathrm{osc}_{U_+}(F) + \mathrm{osc}_{U_-}(F) + 2\varepsilon\end{aligned}$$

For the same reasons, we also have $\gamma(\tilde{\phi}^t_g, \tilde{\phi}^t_G) \leqslant \mathrm{osc}_{U_+}(G) + \mathrm{osc}_{U_-}(G) + 2\varepsilon$ and (6) leads to

$$\begin{aligned}\gamma(\tilde{\phi}^t_F, \tilde{\phi}^t_G) &\geqslant c_{\mathrm{HZ}}(U) - (\mathrm{osc}_{U_+}(F) + \mathrm{osc}_{U_+}(G)) - (\mathrm{osc}_{U_-}(F) + \mathrm{osc}_{U_-}(G)) - 4\varepsilon \\ &\geqslant \frac{1}{3}c_{\mathrm{HZ}}(U) - 4\varepsilon\end{aligned}$$

for any $\varepsilon > 0$. This concludes the proof. □

4. Uniqueness of generators

In this section, we prove Theorems 2 and 1.

Proof of Theorem 2. Assume that the conclusion of the theorem is false; i.e. $H - H'$ is a function of time and space variables on $I \times U$. Then there exist t_0 and, up to a shift of (say) H' by a constant, $x_+ \neq x_- \in U$ such that

$$\Delta = H(t_0, x_+) - H'(t_0, x_+) = H'(t_0, x_-) - H(t_0, x_-) > 0.$$

First, notice that there exist $\delta_0 \in]0, 1]$ and $r_0 > 0$ such that for any $\delta \leqslant \delta_0$, $J = [t_0, t_0 + \delta] \subset I$ and for any $r \leqslant r_0$, the balls $B_\pm = B_r(x_\pm)$ are disjoint, included in U, and

$$\left(\frac{5}{4}\frac{1}{M_+}, \frac{4}{3}\frac{1}{M_+}\right) \cap \left(\frac{5}{4}\frac{1}{M_-}, \frac{4}{3}\frac{1}{M_-}\right) \neq \emptyset \tag{7}$$

with $M_+ = \sup_{J\times B_+}(H) - \inf_{J\times B_+}(H')$ and $M_- = \sup_{J\times B_-}(H') - \inf_{J\times B_-}(H)$. (Even though J, $B_\pm$, and $M_\pm$ depend on δ and/or r, we omit them from the notation for readability.) Indeed, let $\eta = \frac{\Delta}{32}$ and choose δ_0 and r_0 small enough such that

$$\begin{cases} \sup_{J\times B_+}(H) \leqslant H(t_0, x_+) + \eta & \text{and} \quad \inf_{J\times B_+}(H') \geqslant H'(t_0, x_+) - \eta \\ \inf_{J\times B_-}(H) \geqslant H(t_0, x_-) - \eta & \text{and} \quad \sup_{J\times B_-}(H') \leqslant H'(t_0, x_-) + \eta \end{cases}$$

Then $\Delta \leqslant M_\pm \leqslant \Delta + 2\eta$, so that $|M_+ - M_-| \leqslant 2\eta = \frac{\Delta}{16}$ which in turn ensures that (7) holds. Next, notice that we can also assume δ_0 and r_0 are small enough so that, for any $\delta \leqslant \delta_0$ and $r \leqslant r_0$

$$\inf_{J\times B_+}(H) - \sup_{J\times B_+}(H') > \frac{4}{5}M_+ \text{ and } \inf_{J\times B_-}(H') - \sup_{J\times B_-}(H) > \frac{4}{5}M_- \tag{8}$$

(since these inequalities obviously hold for $\delta = 0$ and $r = 0$ and H and H' are continuous). We choose such a δ.

Recall that $c_{\mathrm{HZ}}(B_+) = c_{\mathrm{HZ}}(B_-) = \pi r^2$ (which we denote $c_{\mathrm{HZ}}(B)$) so that we can choose r small enough such that $c_{\mathrm{HZ}}(B) < \delta\frac{3}{4}\Delta$. This in particular implies that $c_{\mathrm{HZ}}(B) < \delta\frac{3}{4}M_\pm$. Finally, we choose r small enough so that

$$c_{\mathrm{HZ}}(B) < \frac{3}{16}\Omega\ . \tag{9}$$

Now that r and δ are fixed, we choose σ such that

$$\frac{\delta\sigma}{c_{\mathrm{HZ}}(B)} \in \left(\frac{5}{4}\frac{1}{M_+}, \frac{4}{3}\frac{1}{M_+}\right) \cap \left(\frac{5}{4}\frac{1}{M_-}, \frac{4}{3}\frac{1}{M_-}\right).$$

Notice that, by definition, $\sigma < \frac{4}{3}\frac{1}{M_+}\frac{c_{\mathrm{HZ}}(B)}{\delta} \leqslant 1$. This implies that for all $t \in [0,1]$, $t_0 + \sigma\delta t \in J$ and we define L_k, L'_k, L and L' by: $L^\star_\bullet(t,x) = \sigma\delta H^\star_\bullet(t_0 + \sigma\delta t, x)$ (with $\bullet$ being either nothing or an integer and $\star$ being either nothing or $'$).

In view of the constants we chose, we get that

$$\begin{aligned}\inf_{[0,1]\times B_+}(L) &\geqslant \delta\sigma \inf_{J\times B_+}(H) \geqslant \delta\sigma\left(\frac{4}{5}M_+ + \sup_{J\times B_+}(H')\right) \qquad \text{by (8)}\\ &\geqslant \delta\sigma\frac{4}{5}M_+ + \sup_{[0,1]\times B_+}(L')\end{aligned}$$

so that, by definition of σ,

$$\inf_{[0,1]\times B_+}(L) - \sup_{[0,1]\times B_+}(L') \geqslant \delta\sigma\frac{4}{5}M_+ > c_{\mathrm{HZ}}(B) \tag{10}$$

We also get:

$$\begin{aligned}\sup_{[0,1]\times B_+}(L) &\leqslant \delta\sigma \sup_{J\times B_+}(H) = \delta\sigma\left(M_+ + \inf_{J\times B_+}(H')\right)\\ &\leqslant \delta\sigma M_+ + \inf_{[0,1]\times B_+}(L')\end{aligned}$$

so that

$$\sup_{[0,1]\times B_+}(L) - \inf_{[0,1]\times B_+}(L') < \frac{4}{3}c_{\mathrm{HZ}}(B)$$

which (together with (10)) leads to

$$\mathrm{osc}_{[0,1]\times B_+}(L) + \mathrm{osc}_{[0,1]\times B_+}(L') < \frac{1}{3}c_{\mathrm{HZ}}(B)\ .$$

Since the quantity $\mathrm{osc}_{[0,1]\times B_+}(L) + \mathrm{osc}_{[0,1]\times B_+}(L')$ is non-negative, the bound on the capacity of B in terms of the rationality constant Ω, (9), finally ensures that:

$$\inf_{[0,1]\times B_+}(L) - \sup_{[0,1]\times B_+}(L') \leqslant \sup_{[0,1]\times B_+}(L) - \inf_{[0,1]\times B_+}(L') < \frac{4}{3}c_{\mathrm{HZ}}(B) < \frac{1}{4}\Omega\ .$$

By collecting all the above results, we get

$$c_{\mathrm{HZ}}(B) < \inf_{[0,1]\times B_+}(L) - \sup_{[0,1]\times B_+}(L') < \frac{1}{4}\Omega\ , \quad \text{and}$$

$$\mathrm{osc}_{[0,1]\times B_+}(L) + \mathrm{osc}_{[0,1]\times B_+}(L') < \frac{1}{3}c_{\mathrm{HZ}}(B)$$

and since L_k and L'_k converge uniformly on U to L and L' respectively, they also satisfy all these inequalities as soon as k is large enough.

Now, by considering the situation on B_-, we obtain the (symmetric) properties required in order to apply Lemma 13 which allows us to conclude that $\gamma(\tilde{\phi}^t_{L_k}, \tilde{\phi}^t_{L'_k})$ is bounded from below by $\frac{1}{3}c_{\mathrm{HZ}}(B)$ for k big enough.

However, by the definition of L_k, for all $t \in [0,1]$, $\phi^t_{L_k} = \phi^{t_0+\delta\sigma t}_{H_k}(\phi^{t_0}_{H_k})^{-1}$ so that $\tilde{\phi}^t_{L_k} = \tilde{\phi}^{t_0+\delta\sigma t}_{H_k}(\tilde{\phi}^{t_0}_{H_k})^{-1}$. Similarly, we have $\tilde{\phi}^t_{L'_k} = \tilde{\phi}^{t_0+\delta\sigma t}_{H'_k}(\tilde{\phi}^{t_0}_{H'_k})^{-1}$ and thus assumption *(i)* ensures that $\gamma(\tilde{\phi}^t_{L_k}, \tilde{\phi}^t_{L'_k})$ does go to 0 when k goes to infinity and we get a contradiction. □

As promised in the introduction, we will explain how one can recover Theorem 1 from Theorem 2. But, before doing so, we make a short digression to discuss an important property of C^0–convergence.

Remark 14. An important feature of C^0–convergence, which will be used below, is that if a sequence of homeomorphisms, ϕ_i, C^0–converges to a *homeomorphism* ϕ, then the sequence of inverses, ϕ_i^{-1}, C^0–converges to ϕ^{-1}. We will sketch a proof of this fact below.

First, note that $d_{C^0}(\phi_i, \phi) \to 0$ implies that $d_{C^0}(\psi\phi_i, \psi\phi) \to 0$ for any uniformly continuous map ψ. Taking $\psi = \phi^{-1}$, we get that $d_{C^0}(\phi^{-1}\phi_i, Id) \to 0$. Next, observe that d_{C^0} is right-invariant and so we get that

$$d_{C^0}(\phi^{-1}\phi_i, Id) = d_{C^0}(\phi^{-1}, \phi_i^{-1}) \to 0.$$

The above proof would fail without the assumption that ϕ^{-1} exists. In fact, it is possible for a sequence of homeomorphisms to converge (with respect to the above version of d_{C^0}), to a map which is not a homeomorphism. In that case, the sequence of inverses diverges. Some authors use a version of d_{C^0} which avoids the above issue by making it impossible for a sequence of homeomorphisms to converge to a map which is not a homeomorphism. For example, this is achieved in [23] by defining

$$\bar{d}_{C^0}(\phi, \psi) = \max_x(d(\phi(x), \psi(x)) + d(\phi^{-1}(x), \psi^{-1}(x))).$$

As pointed out by Müller and Oh, the group of homeomorphisms equipped with $\bar{d}_{C^0}$ is a complete metric space.

Proof. First, note that Theorem 1 follows from the following simpler statement: If H_k is a sequence of normalized smooth Hamiltonians which uniformly converges to some continuous function H and if the flows $\phi^t_{H_k}$ converge uniformly to Id, then $H = 0$.

Then remark that if we knew that the spectral distance is continuous with respect to the C^0–topology then this statement would follow directly from Theorem 2 on rational symplectic manifolds. Unfortunately, this is only partially known and we need a trick to get around this difficulty. We are going to show that for any connected and sufficiently small open subset $U \subset M$, the function H only depends on the time variable t. Since H is normalized, this will prove the statement. We use the same trick as in [3, Theorem 11].

Let U be an open connected subset of M small enough to admit a symplectic embedding to a closed rational symplectic manifold $\iota : U \hookrightarrow W$. Let ψ be a Hamiltonian diffeomorphism generated by a Hamiltonian function compactly supported in U. Since $\phi^t_{H_k}$ C^0–converges to Id, the isotopy $\phi^{-t}_{H_k}\psi^{-1}\phi^t_{H_k}\psi$ is supported in U for k large enough. Moreover, by Remark 14, it converges to Id in the C^0 sense. We may pushforward this isotopy using the embedding ι and get a Hamiltonian isotopy of W supported $\iota(U)$. This isotopy also converges to Id in the C^0 sense. Thus, according to [27, Theorem 1], its spectral pseudo-norm converges to 0. In other words, $\gamma(\tilde{\psi}^{-1}\tilde{\phi}^t_{H_k}\tilde{\psi}, \tilde{\phi}^t_{H_k})$ converges to 0. We may now apply our Theorem 2 in W and get that $H(t, \psi(x)) - H(t, x)$ only depends on the time variable on $[0, 1] \times U$. Since this holds for any psi, this proves our claim that H depends only on the time variable on $[0, 1] \times U$. □

5. C^0-rigidity of the Poisson bracket

This section is devoted to the proof of Theorem 3

Proof. We use the notation of Theorem 3. The assumption $G \in C^0_{\mathrm{Ham}}$ means that there exists a sequence of smooth functions G'_k (a priori different from G_k), which converges uniformly to G and such that the flows $\phi^t_{G'_k}$ converge in the C^0 sense to a continuous isotopy also denoted ϕ^t_G. Let s be a real number. We want to prove that $F = F \circ \phi^s_G$. The sequence of functions $F'_k = F_k \circ \phi^s_{G'_k}$ converges uniformly to $F \circ \phi^s_G$. In view of Theorem 2, if we show that $\gamma(\tilde{\phi}^t_{F_k}, \tilde{\phi}^t_{F'_k})$ converges to 0, then $F = F \circ \phi^s_G$ follows.

Let us recall two identities. For any smooth functions H, K,

$$\phi^t_{H\circ\phi^s_K} = \phi^{-s}_K \circ \phi^t_H \circ \phi^s_K, \tag{11}$$

$$H \circ \phi^s_K - H = \int_0^s \{H, K\} \circ \phi^\sigma_K \, d\sigma. \tag{12}$$

The triangle inequality for γ and (11) give

$$\gamma(\tilde{\phi}^t_{F_k}, \tilde{\phi}^t_{F'_k}) \leqslant \gamma(\tilde{\phi}^t_{F_k\circ\phi^s_{G_k}}, \tilde{\phi}^t_{F_k}) + \gamma(\tilde{\phi}^{-s}_{G_k}\tilde{\phi}^t_{F_k}\tilde{\phi}^s_{G_k}, \tilde{\phi}^{-s}_{G'_k}\tilde{\phi}^t_{F_k}\tilde{\phi}^s_{G'_k}).$$

The Lipschitz properties of γ with respect to the C^0–norm of Hamiltonians, the bi-invariance of γ and (12) yield

$$\gamma(\tilde{\phi}^t_{F_k}, \tilde{\phi}^t_{F'_k}) \leqslant s\|\{F_k, G_k\}\|_{C^0} + 2\|G_k - G'_k\|_{C^0}.$$

Hence $\gamma(\tilde{\phi}^t_{F_k}, \tilde{\phi}^t_{F'_k})$ converges to 0 as wanted. □

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