The Calabi invariant for some groups of homeomorphisms

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Abstract

We show that the Calabi homomorphism extends to some groups of homeomorphisms on exact symplectic manifolds.

The proof is based on the uniqueness of the generating Hamiltonian (proved by Viterbo) of continuous Hamiltonian isotopies (introduced by Müller and Oh).

1 Introduction

1.1 The Calabi homomorphism

Let (M, ω) be a symplectic manifold, supposed to be *exact*, that is $\omega = d\lambda$ for some 1-form λ called *Liouville form*. Equivalently, this also means that there exists a vector field X such that the Lie derivative satisfies: $\mathcal{L}_X \omega = \omega$. The vector field X is called the *Liouville vector field* and is related to the 1-form λ by the relation $\iota_X \omega = \lambda$. For instance, cotangent bundles are exact symplectic manifolds.

Thanks to the work of Banyaga [1, 2], the algebraic structure of the group $\mathrm{Ham}_c(M, \omega)$ of smooth compactly supported Hamiltonian diffeomorphisms of (M, ω) is quite well understood: there exists a group homomorphism, defined by Calabi [3]

$$\mathrm{Cal} : \mathrm{Ham}_c(M, \omega) \to \mathbb{R},$$

whose kernel $\ker(\mathrm{Cal})$ is a simple group.

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The Calabi homomorphism is defined as follows. Let $\phi \in \mathrm{Ham}_c(M,\omega)$ and let H be a compactly supported Hamiltonian function generating ϕ, i.e., a smooth function $[0,1]\times M \to \mathbb{R}$ such that:

- ϕ is the time one map of the flow $(\phi_H^t)_{t\in[0,1]}$ of the only time dependent vector field X_H satisfying at any time $t \in [0,1]$,

$$\iota_{X_H(t,\cdot)}\omega = dH(t,\cdot),$$

- there exists a compact set in M that contains all the supports of the functions $H_t = H(t,\cdot)$, for $t \in [0,1]$.

Then, by definition,

$$\mathrm{Cal}(\phi) = \int_0^1 \int_M H(t,x)\omega^d dt, \tag{1}$$

where d is half the dimension of M. This expression does not depend on the choice of the generating function H, and gives a group homomorphism.

1.2 Question and results

We consider the following question.

Question 1.2.1. *To which groups of homeomorphisms does the Calabi homomorphism extend?*

Note that it is known (see e.g. [5]) that the Calabi homomorphism does not behave continuously with respect to the C^0-topology. For instance, one can consider the following example.

Example 1.2.2. Let $\phi \in \mathrm{Ham}_c(\mathbb{R}^2, rdr\wedge d\theta)$, and consider the sequence (ϕ_n) in $\mathrm{Ham}_c(\mathbb{R}^2, rdr\wedge d\theta)$ given by

$$\phi_n(r,\theta) = \frac{1}{n}\phi^{n^4}(nr,\theta).$$

This sequence converges in the C^0-sense to Id, but one can easily check that its Calabi invariant remains constant.

Let us consider the group G of all homeomorphisms ϕ such that (on some interval where it is well defined) the isotopy $t \mapsto [\mu_t, \phi]$ is a C^0-Hamiltonian isotopy (in the sense of [11], see Section 2.1 for the precise definition of G). Here, μ_t denotes the flow generated by the Liouville vector field X, and $[\mu_t,\phi] = \mu_t\circ\phi\circ\mu_t^{-1}\circ\phi^{-1}$. We will prove the following extension result for the Calabi invariant.

Theorem 1.2.3. *The Calabi homomorphism extends to a group homomorphism $G \to \mathbb{R}$.*

Let us now consider the special case where (M, ω) is the standard symplectic vector space $(\mathbb{R}^{2d}, \omega_0 = \sum_{i=1}^{d} dp_i \wedge dq_i)$, where we denote by $(q_1, \dots, q_d, p_1, \dots, p_d)$ the coordinates in $\mathbb{R}^{2d}$. In this case, we can construct two interesting subgroups G_1 and G_2 of G. Let us introduce the first one.

Definition 1.2.4. *We denote by G_1 the identity component of the group of compactly supported bilipschitz symplectic homeomorphisms.*

Remark 1.2.5. Since Lipschitz maps are almost everywhere differentiable, the pull-back of a differential form by a bilipschitz map is well-defined as a differential form with L^∞ coefficients. Therefore, as in the smooth case, a bilipschitz homeomorphism ϕ of M is symplectic if $\phi^*\omega = \omega$.

Note that a bilipschitz homeomorphism which is the C^0-limit of smooth symplectomorphisms is symplectic in this sense. This follows from the Gromov-Eliashberg rigidity theorem (see e.g. [9], p59)

Let us now introduce our second group.

Definition 1.2.6. *A C^1 compactly supported function $S : \mathbb{R}^{2d} \to \mathbb{R}$ is called an* admissible generating function *if in any point $(x, \eta) \in \mathbb{R}^{2n}$ the maps*

$$x_i \mapsto x_i + \frac{\partial S}{\partial \eta_i}(x, \eta) \text{ and } \eta_i \mapsto \eta_i + \frac{\partial S}{\partial x_i}(x, \eta), \text{ for } i \in \{1, \dots, n\},$$

are increasing homeomorphisms of $\mathbb{R}$.

A homeomorphism f of $\mathbb{R}^{2d}$ is called an admissible homeomorphism *if there exists an admissible generating function S such that for any $x, y, \eta, \xi \in \mathbb{R}^{2d}$,*

$$f(x, y) = (\xi, \eta) \quad \Longleftrightarrow \quad \begin{cases} \xi = x + \frac{\partial S}{\partial \eta}(x, \eta) \\ y = \eta + \frac{\partial S}{\partial x}(x, \eta) \end{cases} . \tag{2}$$

We denote by G_2 the group generated by admissible homeomorphisms.

Remark 1.2.7. The relation (2) means that under the symplectic identification

$$j : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}, (x, y; \xi, \eta) \mapsto (x, \eta; y - \eta, \xi - x),$$

one has $j(\mathrm{graph}(f)) = \mathrm{graph}(dS)$.

REMARK 1.2.8. It is well known that compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2d}$ which are in some C^1-neighbourhood of the identity are exactly the admissible homeomorphisms that are diffeomorphisms. It follows that the Hamiltonian group is the group generated by admissible diffeomorphism. Therefore, the group G_2 is a non-smooth generalization of the Hamiltonian group.

REMARK 1.2.9. For any admissible generating function S, there exists an admissible homeomorphism f associated to S by (2).

To prove this, first note that for any $x, \eta \in \mathbb{R}^n$, the maps $\eta \mapsto \eta + \frac{\partial S}{\partial x}(x,\eta)$ and $x \mapsto x + \frac{\partial S}{\partial \eta}(x,\eta)$ are homeomorphisms of $\mathbb{R}^n$.

Indeed, one sees easily that $\eta \mapsto \eta + \frac{\partial S}{\partial x}(x,\eta)$ is continuous and injective. Since it is compactly supported, it is also proper and hence is an embedding. Finally, this implies that it is onto, because otherwise its image would contain non-contractible spheres $\mathbb{S}_{n-1}$. The same argument holds for $x \mapsto x + \frac{\partial S}{\partial \eta}(x,\eta)$.

Then, the map f is given by the formula

$$f(x,y) = (\alpha(x, \beta(x,\cdot)^{-1}(y)), \beta(x,\cdot)^{-1}(y)),$$

where $\alpha(x,\eta) = x + \frac{\partial S}{\partial \eta}(x,\eta)$ and $\beta(x,\eta) = \eta + \frac{\partial S}{\partial x}(x,\eta)$. Proving that f is indeed an homeomorphism will be left to the reader.

Our result is then the following.

Theorem 1.2.10. *The following inclusions hold*

$$\mathrm{Ham}(\mathbb{R}^{2d}, \omega_0) \subset G_1 \subset G_2 \subset G.$$

REMARK 1.2.11. In the special case of the (2-dimensional) open disk, the fact that the Calabi homomorphism extends to G_1 was already proved by Haissinsky [6][1]. His methods are completely different.

Let us also mention that Gambaudo and Ghys have proved that two diffeomorphisms of the disk that are conjugated by an area preserving homeomorphism have the same Calabi invariant [5].

[1]Area preserving quasiconformal maps of the plane are bilipschitz. Therefore, Haissinsky's result is precisely the fact that the Calabi homomorphism extends to G_1.

1.3 Motivation

Our motivation for this work comes from two distinct problems. The first one is the following:

Question 1.3.1 (Fathi [4]). *Is the group* $\mathrm{Homeo}_c(\mathbb{D}_2, area)$ *of compactly supported area preserving homeomorphisms of the disk a simple group ?*

Several non-trivial normal subgroups of $\mathrm{Homeo}_c(\mathbb{D}_2, area)$ have been defined by Ghys [2], Müller-Oh [12] and recently by Le Roux [8]. But so far, no one has been able to prove that any of them is a proper subgroup.

Our study is inspired by the work of Müller and Oh. They introduced on any symplectic manifold (M, ω) a group denoted $\mathrm{Hameo}(M, \omega)$, whose elements are homeomorphisms called *hameomorphisms* (as the contraction of "Hamiltonian homeomorphisms"). This group contains all compactly supported Hamiltonian diffeomorphisms and, in the case of the disk, forms a normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}_2, area)$. A. Fathi noticed that if one could extend the Calabi homomorphism to the group of hameomorphisms, then it would be necessarily a proper subgroup, and $\mathrm{Homeo}_c(\mathbb{D}_2, area)$ would not be simple.

In the present paper, we propose a different approach: instead of constructing a group which is known to be normal but on which it is unknown whether the Calabi homomorphism extends, we construct a group (namely G) to which the Calabi invariant extends but for which it is unknown whether it is normal.

Another motivation is a very natural general problem: how can one generalize Hamiltonian dynamics in a non-smooth context? or (less optimistic) which properties of Hamiltonian maps can be extended? The present paper concentrates on a particular aspect: the Calabi homomorphism.

Our interest in the groups G_1 and G_2 comes from the fact that they give large families of examples of elements of G, but also from the fact that they are quite natural generalizations of the Hamiltonian group, which could be considered to study the extension of other aspects of Hamiltonian dynamics. As an example, all the rigidity results obtained on Hamiltonian diffeomorphisms using generating functions techniques may also hold for the elements of G_2 (and thus of G_1).

Several other possible groups generalizing the Hamiltonian group have already been considered in literature. The group $\mathrm{Hameo}(M, \omega)$ mentioned above is one of them, another has been studied by the author in [7]. But this direction of research is still to be developed.

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2 The group G and the Calabi invariant

2.1 The group G

To define the group G we first need the following notion.

Definition 2.1.1 (Müller-Oh [12]). *A* C^0-Hamiltonian isotopy *is a path* $(\phi^t)_{t\in[0,\delta]}$ *of homeomorphisms of* M *for which there exist a compact set* K *and a sequence of smooth Hamiltonian functions* H_n *on* M *with support in* K*, such that*

- (H_n) *converges to some continuous function* $H : [0,\delta]\times M \to \mathbb{R}$ *in the* C^0*-sense,*
- $(\phi^t_{H_n})$ *converges to* ϕ^t *in the* C^0*-sense, uniformly in* $t \in [0,\delta]$.

The function H *is called a* C^0-Hamiltonian function *generating* (ϕ^t).

Remark 2.1.2. The elements of C^0-Hamiltonian isotopies are *symplectic homeomorphisms*, i.e., homeomorphisms which are the C^0 limit of a sequence of symplectic diffeomorphisms supported in a common compact set.

It is not difficult to check that if (ϕ^t) and (ψ^t) are two C^0-Hamiltonian isotopies generated by F and G, then $((\phi^t)^{-1})$ and $(\phi^t \circ \psi^t)$ are C^0-Hamiltonian isotopies generated by $-F(t,(\phi^t)^{-1}(x))$ and $F(t,x) + G(t,\phi^t(x))$, and that if f is any symplectic homeomorphism, $(f^{-1}\circ\phi^t\circ f)$ is a C^0-Hamiltonian isotopy generated by $F(t,f(x))$. This means that the computations are the same as in the smooth case.

The main result concerning C^0-Hamiltonian isotopies is:

Theorem 2.1.3 (Viterbo [13]). *A given* C^0*-Hamiltonian isotopy is generated by a unique* C^0*-Hamiltonian function.*

This theorem is the only non-trivial result needed in this paper. Its proof needs at some point a (hard!) rigidity result in symplectic topology due to Gromov.

Definition 2.1.4. *We denote by G the set of all compactly supported symplectic homeomorphisms ϕ for which there exists some $\delta > 0$ small enough, such that the isotopy $([\mu_t, \phi])_{t\in[0,\delta]}$ is a C^0-Hamiltonian isotopy.*

Remark 2.1.5. As in the introduction, $\mu_t(x)$ denotes the flow (when it is defined) of the Liouville vector field X, at time t and point $x \in M$. Note that it satisfies $\mu_t^*\omega = e^t\omega$.

Let ϕ be a compactly supported homeomorphism of M. Then there exists a real number $\delta > 0$, such that for any $t \in [0, \delta]$, μ^t and $(\mu^t)^{-1}$ are well defined on the support of ϕ. Thus, the conjugation $\mu_t \circ \phi \circ \mu_t^{-1}$ is well defined on $\mu_t(\mathrm{Supp}(\phi))$. In the complement of this set, it is the identity where it is defined. Therefore, we can extend it to a well defined homeomorphism still denoted $\mu_t \circ \phi \circ \mu_t^{-1}$ just by setting it to equal the identity where it is not defined.

Clearly, G contains $Ham_c(M, \omega)$.

Proposition 2.1.6. *The set G is a group. Moreover, if the first cohomology group $H^1(M, \mathbb{R})$ vanishes, G does not depend on the choice of the Liouville vector field.*

Proof . — Let $\phi, \psi \in G$. For δ small enough $([\mu_t, \phi])_{t\in[0,\delta]}$ and $([\mu_t, \psi])_{t\in[0,\delta]}$ are C^0-Hamiltonian isotopies. Then, note that

$$[\mu_t, \phi \circ \psi] = [\mu_t, \phi] \circ (\phi \circ [\mu_t, \psi] \circ \phi^{-1}),$$

and

$$[\mu_t, \phi^{-1}] = \phi^{-1} \circ [\mu_t, \phi]^{-1} \circ \phi.$$

We conclude with Remark 2.1.2 that G is a group.

Suppose now that $H^1(M, \mathbb{R}) = 0$, and that μ'_t is the flow of another Liouville vector field. Then, $\eta_t = \mu'_t \circ \mu_t^{-1}$ is a smooth symplectic isotopy which is Hamiltonian since $H^1(M, \mathbb{R}) = 0$. The Hamiltonian generating (η_t) is not compactly supported but Remark 2.1.2 still applies to the identity

$$[\mu'_t, \phi] = \eta_t \circ [\mu_t, \phi] \circ (\phi \circ \eta_t^{-1} \circ \phi^{-1}), \tag{3}$$

showing that G would be the same if it was defined with another Liouville vector field. $\square$

2.2 Examples: fibered rotation in $\mathbb{R}^2$

In this section, we give a sufficient condition for a fibered rotation of $\mathbb{R}^2$ to be in G.

By definition a *fibered rotation* is an homeomorphism ϕ of $\mathbb{R}^2$ described in polar coordinates (r, θ) by the formula

$$\phi(r, \theta) = (r, \theta + \rho(r)),$$

for some continuous *angular* function $\rho : (0, +\infty) \to \mathbb{R}$ with bounded support. It is easily checked that any fibered rotation lies in the identity component of the group of compactly supported area preserving homeomorphism of $\mathbb{R}^2$.

We consider μ_t the Liouville flow given by $\mu_t(r, \theta) = (e^{t/2}r, \theta)$. Its commutator with a fibered rotation is given by

$$[\mu_t, \phi](r, \theta) = (r, \theta - \rho(r) + \rho(e^{-t/2}r)).$$

If ϕ is moreover a diffeomorphism, the generating Hamiltonian of the isotopy $t \mapsto [\mu_t, \phi]$ is

$$H(t, r, \theta) = \frac{1}{2} r\rho(e^{-t/2}r) - \frac{1}{2}\int_0^r \rho(e^{-t/2}s)\, ds.$$

Now suppose that ρ is a continuous and integrable angular function, such that $r\rho(r)$ converges to 0 when r tends to 0. Suppose also that ρ_k is a sequence of smooth angular functions with bounded support and satisfying $\rho_k(r) = \rho(1/k)$ for $r \leqslant 1/k$ and $|\rho_k(r) - \rho(r)| \leqslant 1/k$ for $r > 1/k$. Then, the associated sequence of fibered rotations (ϕ_k) converges in the C^0-sense to ϕ, and the sequence of Hamiltonians (H_k) generating the isotopies $t \mapsto [\mu_t, \phi_k]$ also C^0-converges.

As a consequence, *any fibered rotation associated to an integrable angular function* ρ *such that* $r\rho(r) \xrightarrow{r \to 0} 0$, *belongs to* G.

REMARK 2.2.1. This gives examples of elements that are in G but not in G_2: if ρ is not finite (nearby 0), the fibered rotation ϕ cannot be in G_2. Indeed, the angle between a vector and its image by an admissible homeomorphism is bounded by π. Therefore, this angle has to be finite for elements of G_2.

2.3 Extension of the Calabi homomorphism

In this section, we prove that the Calabi homomorphism extends to G. Let us first give a new formula for the Calabi invariant, for which we need to choose a Liouville form instead of choosing an isotopy.

Lemma 2.3.1. *Let $\phi \in \mathrm{Ham}_c(M,\omega)$ and let $H_{\lambda,\phi}$ be the generating Hamiltonian function of the smooth Hamiltonian isotopy $([\mu_t,\phi])$. Then,*

$$\mathrm{Cal}(\phi) = \frac{1}{d+1}\int_M H_{\lambda,\phi}(0,x)\omega^d.$$

Proof. First note that if ϕ is the time one map of some Hamiltonian function H, and if we suppose $\mu_\delta \circ \phi \circ \mu_\delta^{-1}$ to be well defined, then it can be generated by the Hamiltonian function $e^\delta H \circ \mu_\delta^{-1}$. After an easy change of variables in Equation (1), one gets

$$\mathrm{Cal}(\mu_\delta \circ \phi \circ \mu_\delta^{-1}) = e^{(d+1)\delta}\mathrm{Cal}(\phi),$$

where d is half the dimension of M. Thus,

$$\mathrm{Cal}([\mu_\delta,\phi]) = (e^{(d+1)\delta} - 1)\mathrm{Cal}(\phi).$$

Hence, applying formula (1) to $H_{\lambda,\phi}$,

$$\mathrm{Cal}(\phi) = \frac{1}{e^{(d+1)\delta}-1}\int_0^\delta \int_M H_{\lambda,\phi}(t,x)\omega^d dt.$$

Now, letting δ converge to 0, we get the desired formula. □

Once this formula obtained, extending the Calabi homomorphism to G is very easy, even though it relies on the "hard symplectic topology" uniqueness Theorem 2.1.3.

Proof. let $\phi \in G$ and let H be the **unique** C^0-Hamiltonian function generating $([\mu_t,\phi])_{t\in[0,\delta]}$ for some small δ. We set:

$$\widetilde{\mathrm{Cal}}(\phi) = \frac{1}{d+1}\int_M H(0,x)\omega^n.$$

By Lemma 2.3.1, $\widetilde{\mathrm{Cal}}$ coincide with Cal on $Ham_c(M,\omega)$. Moreover using Remark 2.1.2 and the formulas in the proof of Proposition 2.1.6, one checks easily that $\widetilde{\mathrm{Cal}} : G \to \mathbb{R}$ is a group homomorphism. □

Remark 2.3.2. If $H^1(M,\mathbb{R}) = 0$, then $\widetilde{\mathrm{Cal}}$ does not depend on the choice of the Liouville vector field. This is an immediate consequence of Equation (3).

3 Proof of the inclusions

3.1 Proof of the inclusion $G_1 \subset G_2$

The argument is very classical in the smooth case, we just rewrite it in the bilipschitz case.

First remark that the space of bilipschitz compactly supported maps of $\mathbb{R}^{2d}$ carries the structure of a topological group induced by the distance defined as follows. Let f, g be two such maps. We endow $\mathbb{R}^{2d}$ with the standard euclidean norm $\|\cdot\|$ and we denote $d_{C^0}(f,g) = \sup_{x \in \mathbb{R}^{2d}} \|f(x) - g(x))\|$ and

$$\mathrm{dil}(f,g) = \sup_{x \neq y \in \mathbb{R}^{2d}} \frac{|\|f(x) - g(x)\| - \|f(y) - g(y)\||}{\|x - y\|}.$$

Then, the bilipschitz distance between f and g is given by

$$D(f,g) = d_{C^0}(f,g) + d_{C^0}(f^{-1}, g^{-1}) + \mathrm{dil}(f,g) + \mathrm{dil}(f^{-1}, g^{-1}).$$

Let $f \subset G_1$ be close to the identity. Then, if we denote $q : \mathbb{R}^{2d} \to \mathbb{R}^d$, $(x,y) \mapsto x$ and $p : \mathbb{R}^{2d} \to \mathbb{R}^d$, $(x,y) \mapsto y$ the canonical projections, the maps

$$y \mapsto p \circ f(x,y) \quad \text{and} \quad \xi \mapsto q \circ f^{-1}(\xi, \eta)$$

are Lipschitz-close to the identity and thus (by standard arguments) are bilipschitz homeomorphisms of $\mathbb{R}^{2d}$, close to the identity, in any given points x,η. Now let $\alpha(\cdot, \eta)$ be the inverse of $q \circ f^{-1}(\cdot, \eta)$ and $\beta(x, \cdot)$ be the inverse of $p \circ f(x, \cdot)$ which are again close to the identity in the Lipschitz sense. It is not difficult to check that the maps α and β are Lipschitz.

Since f is symplectic, the 1-form $f^*(ydx) - ydx$ is closed hence exact. Therefore, the Lipschitz 1-form $\alpha(x,\eta)dx + \beta(x,\eta)d\eta = (ydx - \eta d\xi) + d(\eta\xi)$ is the differential of a $C^{1,1}$ function σ.

Now, since f is compactly supported, $\alpha(x,\eta) = x$ and $\beta(x,\eta) = \eta$ out of a compact set, thus, up to a constant shift, $\sigma(\eta, \xi) = \langle x, \eta \rangle$ out of a compact set. Then, it is easily checked that the function $S : \mathbb{R}^{2d} \to \mathbb{R}$, $(x,\eta) \mapsto \sigma(x,\eta) - \langle x, \eta \rangle$ satisfies the relation (2).

The map S is an admissible generating function. To prove this, just remark that in any points x, η, the maps $\alpha(\cdot,\eta)$ and $\beta(x,\cdot)$ are Lipschitz close to the identity. Therefore, the maps for any $i \in \{1, \dots, n\}$

$$x_i \mapsto q_i \circ \alpha(x,\eta) = x_i + \frac{\partial S}{\partial \eta_i}(x,\eta),$$

$$\eta_i \mapsto p_i \circ \beta(x,\eta) = \eta_i + \frac{\partial S}{\partial x_i}(x,\eta),$$

where $q_i : (x, \eta) \mapsto x_i$ and $p_i : (x, \eta) \mapsto \eta_i$ are the canonical projections onto $\mathbb{R}$, are also Lipschitz close to the identity and hence are increasing homeomorphisms of $\mathbb{R}$.

We have proved that any element in a neighbourhood of the identity in G_1 is admissible, and moreover that it is associated to a small $C^{1,1}$ function. Conversely, with the help of Remark 1.2.9, the same kind of argument (again standard in the smooth case) shows that any $C^{1,1}$-small function is admissible and gives rise to a homeomorphism which is in G_1 and close to the identity. Now note that if S is a $C^{1,1}$-small function, then for any $t \in [0,1]$, tS is again a $C^{1,1}$-small admissible generating function. It follows that any element of G_1 can be linked to the identity by a continuous path of elements of G_1.

This proves that G_1 is a locally arcwise connected topological group and thus that its connected components are arcwise connected. As a consequence, any element f in G_1 can be linked to the identity by a continuous path. Cutting this path into sufficiently small pieces, f can be written as a composition of elements in a neighbourhood of the identity. It follows that any element in G_1 is a product of admissible homeomorphisms. $\square$

3.2 Proof of the inclusion $G_2 \subset G$

The inclusion $G_2 \subset G$ will follow from the following proposition. We denote by $\Psi(S)$ the admissible homeomorphism associated to an admissible generating function S.

Proposition 3.2.1. *Let $t \mapsto S^t$ be a C^1 path of admissible generating functions, which is the uniform limit for the C^1 distance of a sequence of smooth paths $t \mapsto S^t_n$ of smooth admissible generating functions, and whose derivative $t \mapsto \frac{dS^t}{dt}$ is the uniform limit for the C^0 distance of the derivatives $t \mapsto \frac{dS^t_n}{dt}$. Then, the path $t \mapsto \Psi(S_t)$ is a C^0-Hamiltonian isotopy.*

Remark 3.2.2. This proposition gives new examples of C^0-Hamiltonian isotopies. As an example, the argument shows that any continuous path in G_1 is a C^0-Hamiltonian isotopy.

To prove Proposition 3.2.1, we will need two (classical) lemmas.

Lemma 3.2.3. *For any integer $k \geqslant 0$, the map Ψ is a homeomorphism between the set of C^{k+1} admissible generating functions (endowed with the C^{k+1}-topology) and the set of C^k admissible (diffeo)homeomorphisms (endowed with the C^k-topology).*

Lemma 3.2.4. *Let $t \mapsto S_t$ be a smooth path of smooth admissible generating functions and denote by H the compactly supported Hamiltonian function that generates the Hamiltonian isotopy $t \mapsto \Psi(S_t)$. Then, for any time t and any point $x, y \in \mathbb{R}^{2d}$,*

$$H(t,x,y) = \frac{\partial S_t}{\partial t}(x, p \circ \Psi(S_t)(x,y)).$$

Proof . — This is nothing but the classical Hamilton-Jacobi equation (see e.g. [10], p.283). □

Proof of Proposition 3.2.1. — With the notation of Proposition 3.2.1, let H_n be the Hamiltonian function generating the Hamiltonian isotopy $\Psi(S_n^t)$. By Lemma 3.2.3, the isotopies $(\phi_{H_n}^t) = (\Psi(S_n^t))$ C^0-converge to $\Psi(S^t)$. Moreover, by Lemma 3.2.4, the Hamiltonian functions

$$H_n(t,x,y) = \frac{\partial S_n^t}{\partial t}(x, p \circ \Psi(S_n^t)(x,y))$$

also C^0-converge. This shows that $(\Psi(S_t))$ is a C^0-Hamiltonian isotopy. □

Proof of the inclusion $G_2 \subset G$. — Let f be an admissible homeomorphism associated to an admissible generating function S. Let us first prove that S can be approximated for C^1 topology by smooth admissible generating functions.

Let χ be a smooth non-negative function, defined on $\mathbb{R}^{2d}$, whose support is contained in a disk centered in 0 and with integral equal to 1. For any positive integer k, we set $\chi_k = k^{2d}\chi(\frac{\cdot}{k})$. Then, it is well known that the sequence of smooth functions (S_k) defined by

$$S_k(x,\eta) = \chi_k * S(x,\eta) = \int_{\mathbb{R}^{2d}} S(x-u, \eta - v)\chi_k(u,v)\, du\, dv,$$

C^1-converges to S as k goes to infinity. Moreover, there exists a compact set that contains the supports of every S_k.

Let us now prove that the S_k are admissible generating functions. Set

$$\alpha(x,\eta) = x + \frac{\partial S}{\partial \eta}(x,\eta) \text{ and } \beta(x,\eta) = \eta + \frac{\partial S}{\partial x}(x,\eta).$$

We want to prove that for any indices i, the maps $x_i \mapsto q_i \circ (\chi_k * \alpha(x,\eta))$ and $\eta_i \mapsto p_i \circ (\chi_k * \beta(x,\eta))$ are increasing homeomorphisms of $\mathbb{R}$.

They are clearly continuous. Since they are compactly supported, we only need to show that they are increasing. Let us prove it for $x_1 \mapsto q_1 \circ (\chi_k * \alpha(x, \eta))$. The proof is similar for the others.

Fix $\eta, x_2, \ldots, x_d$ and $x_1 < x_1'$ and denote $x = (x_1, x_2, \ldots, x_d)$ and $x' = (x_1', x_2, \ldots, x_d)$. We want to compare $q_1 \circ (\chi_k * \alpha(x, \eta))$ with $q_1 \circ (\chi_k * \alpha(x', \eta))$. By assumption, for all $(u, v) \in \mathbb{R}^{2d}$,

$$q_1 \circ \alpha(x - u, \eta - v) \ < \ q_1 \circ \alpha(x' - u, \eta - v),$$

thus the following integral is non-negative:

$$\int_{R^{2d}} \chi_k(u, v) \left[q_1 \circ \alpha(x' - u, \eta - v) - q_1 \circ \alpha(x - u, \eta - v) \right] \, du \, dv.$$

It is moreover positive because it is the integral of a non-negative continuous function which is non-identically zero. This integral is nothing but $q_1 \circ (\chi_k * \alpha(x, \eta)) - q_1 \circ (\chi_k * \alpha(x', \eta))$. Therefore the map $x_1 \mapsto q_1 \circ (\chi_k * \alpha(x, \eta))$ is an increasing homeomorphism of $\mathbb{R}$.

Now, remark that the conjugation $\mu_t \circ f \circ \mu_t^{-1}$ of f by the Liouville flow $\mu_t : x \mapsto e^{t/2} x$ is also admissible, since it is associated to the generating function $S_t(x, \eta) = e^t S(e^{-t/2} x, e^{-t/2} \eta)$. The path $t \mapsto S_t$ satisfies the hypothesis of Proposition 3.2.1, since it is the limit of the path $S_k^t(x, \eta) = e^t S_k(e^{-t/2} x, e^{-t/2} \eta)$. Therefore the isotopy $t \mapsto \mu_t \circ f \circ \mu_t^{-1}$ is a C^0-Hamiltonian isotopy starting at f, and $t \mapsto [\mu_t, f]$ is a C^0-Hamiltonian isotopy, starting at Id.

We have proved that any admissible homeomorphism is in G. As a consequence, $G_2 \subset G$. $\square$

References

[1] Augustin Banyaga. *The structure of classical diffeomorphism groups*, volume 400 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997.

[2] Abed Bounemoura. *Simplicité des groupes de transformations de surfaces*. Ensaios Matematicos, 2008.

[3] Eugenio Calabi. On the group of automorphisms of a symplectic manifold. In *Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969)*, pages 1–26. Princeton Univ. Press, Princeton, N.J., 1970.

[4] Albert Fathi. Structure of the group of homeomorphisms preserving a good measure on a compact manifold. *Ann. Sci. École Norm. Sup. (4)*, 13(1):45–93, 1980.

[5] Jean-Marc Gambaudo and Etienne Ghys. Enlacements assymptotiques. *Topology*, 36(6):1355–1379, 1997.

[6] Peter Haïssinsky. L'invariant de Calabi pour les homéomorphismes quasiconformes du disque. *C. R. Math. Acad. Sci. Paris*, 334(8):635–638, 2002.

[7] Vincent Humiliere. On some completions of the space of hamiltonian maps. *Bull. Soc. Math. France*, 136, fascicule 3 (2008), 373-404.

[8] Frédéric Le Roux. Simplicity of the group of area preserving homeomorphisms of the disc and fragmentation of symplectic diffeomorphisms, 2008. preprint.

[9] Helmut Hofer and Edouard Zehnder. *Symplectic invariants and Hamiltonian dynamics*. Birkhäuser Advanced Texts. Birkäuser Verlag, Basel, 1994.

[10] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[11] Yong-Geun Oh. The group of Hamiltonian homeomorphisms and continuous Hamiltonian flows, 2006. arXiv.org:math/0601200.

[12] Yong-Geun Oh and Stefan Muller. The group of Hamiltonian homeomorphisms and C^0 symplectic topology, 2004.

[13] Claude Viterbo. Erratum to: "On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows" [Int. Math. Res. Not. **2006**, Art. ID 34028, 9 pp.; mr2233715]. *Int. Math. Res. Not.*, pages Art. ID 38784, 4, 2006.

[14] Alan Weinstein. Symplectic manifolds and their Lagrangian submanifolds. *Advances in Math.*, 6:329–346 (1971), 1971.