Symplectic action selectors and applications

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Abstract

These are the notes of a series of three lectures given in the university of Padova, 12-16 February 2018, during the winter school "Recent advances in Hamiltonian dynamics and symplectic topology". The goal of these lectures was to present a family of invariants called "action selectors" or "spectral invariants". They are symplectic invariants attached to Hamiltonian systems which have a lot of dynamical applications. We will see an overview of (some of my favourite) applications. The last lecture will show how they can be used to study C^0-symplectic geometry.

Contents

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1 Basic definitions and settings

1.1 Symplectic manifolds

In all these notes, (M, ω) will be a symplectic manifold which is either closed or convex at infinity, a technical assumption which includes cotangent bundles of closed manifolds, $\mathbb{R}^{2n}$. We will also often assume that $\langle \omega, \pi_2(M) \rangle = 0$, to simplify the presentation (though much of what will be presented can be extended to more general symplectic manifolds). Here are the basic examples we have in mind.

Examples 1.1. (a) $\mathbb{R}^{2n} = \{(q_1, \dots, q_n, p_1, \dots, p_n)\}$ endowed with $\omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$;

(b) T^*N, endowed with $\omega = d\lambda$ where $\lambda = p\, dq$ Liouville form, for every closed smooth manifold;

(c) $\mathbb{T}^{2n}$ endowed with $\omega_0 = dp_i \wedge dq_i$;

(d) closed surfaces Σ_g of genus $g \geqslant 1$, endowed with any volume form, and their products.

◀

Given two symplectic manifolds (M, ω), (M', ω'), a symplectic diffeomorphism is a map $\phi : M \to M'$ such that $\phi^*\omega' = \omega$. In particular, symplectic diffeomorphisms preserve the volume form $\omega^{\wedge \frac{1}{2}\dim M}$. Locally, all symplectic manifolds of a given dimension $2n$ are isomorphic to an open set in $(\mathbb{R}^{2n}, \omega_0)$ (this is the famous Darboux theorem, see e.g. [27]).

1.2 Hamiltonians

A Hamiltonian is a time-dependent smooth function:

$$H : [0,1] \times M \to \mathbb{R}.$$

We will often use the notation $H_t(x) := H(t, x)$. A Hamiltonian function H generates a Hamiltonian isotopy ϕ_H^t by integrating the (time-dependent) vector field X_{H_t} defined by

$$\omega(X_{H_t}, \cdot) = -dH_t$$

(gradient of H_t with respect to ω).

In Darboux coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ (i.e. for the symplectic form $\omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$), we have $X_H = \left(\frac{\partial H}{\partial p_1}, \dots \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, \dots, -\frac{\partial H}{\partial q_n}\right)$ (omitting the t's), so that ϕ_H^t is obtained by integrating the famous Hamilton equations:

$$\begin{cases} \dot{q}_i(t) = \frac{\partial H_t}{\partial p_i}(q(t), p(t)), \\ \dot{p}_i(t) = -\frac{\partial H_t}{\partial q_i}(q(t), p(t)). \end{cases}$$

For instance, the Hamiltonian $H(q, p) = \frac{1}{2}p^2 + V(q)$ on $\mathbb{R}^2$ leads to the Hamilton equations $\dot{q} = p$ and $\dot{p} = -\nabla V$, hence to the well known Newton equation $\ddot{q} = -\nabla V$.

Fact. *Every Hamiltonian diffeomorphism, i.e. every diffeomorphism of the form ϕ_H^t for some H and some t, is symplectic: ${\phi_H^t}^*\omega = \omega$.*

Proof. Exercise using Cartan's magic formula. □

1.3 Lagrangian submanifolds

A submanifold L is said Lagrangian if L satisfies

$$\omega|_L = 0 \quad \text{and} \quad \dim L = \tfrac{1}{2}\dim M.$$

These submanifolds are often considered as **the** important objects in a symplectic manifold. Here are some fundamental examples:

Examples 1.2. (a) $\mathbb{R}^n \times \{0\}^n$ in the standard symplectic space $\mathbb{R}^{2n}$ is Lagrangian;

(b) the zero section $O_N \subset T^*N$ is Lagrangian;

(c) more generally, the graph of a closed 1-form in T^*N is Lagrangian;
(d) a cotangent fiber T_q^*N in T^*N is Lagrangian;
(e) more generally, given a submanifold $V \subset N$, the conormal

$$\nu^*V = \{(q,p) \in T^*N \mid q \in V, T_qV \subset \ker p\}.$$

(f) a standard torus $\mathbb{T}^n \times \{pt\}^n$ in $\mathbb{T}^{2n}$ is Lagrangian;
(g) in a surface, a Lagrangian submanifold is nothing but an embedded curve;
(h) given a diffeomorphism $\phi : M \to M$, ϕ is symplectic iff its graph $\mathrm{gr}(\phi) \subset (\mathrm{M} \times \mathrm{M}, \omega \oplus (-\omega))$ is Lagrangian.

◀

If the symplectic form $\omega = d\lambda$ (for example in cotangent bundles), a Lagrangian submanifold is said *exact* if the 1-form $\lambda|_L$, which is closed since L is Lagrangian, is exact. The examples a,b,c,d,e above are exact Lagrangians.

1.4 The Hamiltonian action functional

From now on we assume that our Hamiltonians are 1-periodic in time, i.e., that they are smooth functions $H : \mathbb{S}^1 \times M \to \mathbb{R}$. Every Hamiltonian diffeomorphism can be generated by such a Hamiltonian.

We denote by $\mathcal{L}M$ the space of contractible smooth maps $\mathbb{S}^1 \to M$. The *Hamiltonian action functional* is defined as the map

$$\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}, \quad x \mapsto \int_{\mathbb{S}^1} H_t(x(t))\,dt - \int_{\mathbb{D}^2} u^*\omega, \tag{1}$$

where u is a capping disk of x, i.e. a smooth map $u : \mathbb{D}^2 \to M$, such that $u|_{\mathbb{S}^1} = x$. The condition $\langle \omega, \pi_2(M) \rangle = 0$ implies that the term $\int_{\mathbb{D}^2} u^*\omega$ does not depend on the choice of capping u.

In a cotangent bundle, $\omega = d(pdq)$, the action functional reduces to the classical functional:

$$\mathcal{A}_H(q,p) = \int_0^1 \left(H_t(q(t),p(t)) - p(t)\dot{q}(t)\right)\,dt.$$

The action functional is a central object in Hamiltonian dynamics because of the following fact.

Lemma 1.3. *The critical points of $\mathcal{A}_H$ are exactly the contractible 1-periodic orbits of the Hamiltonian isotopy of H. Thus, they are in bijection with a subset of set of the fixed points of ϕ_H^1.*

Proof. (borrowed from [4]) Let $x \in \mathcal{L}M$, $Y \in T_x\mathcal{L}M \simeq \Gamma(x^*TM)$, and u be a capping disk of x. We extend x to a path $s \mapsto \tilde{x}(s)$ in $\mathcal{L}M$, defined on a small interval $(-\varepsilon, \varepsilon)$, such that $\tilde{x}(0) = x$ and $\tilde{x}'(0) = Y$. We will use the notation $\tilde{x}(s,t) := \tilde{x}(s)(t)$. We also extend $\tilde{x}$ to the disk and get a map $\tilde{u} : (-\varepsilon, \varepsilon) \times \mathbb{D}^2 \to M$ obtained for each $s \in (-\varepsilon, \varepsilon)$ by gluing u to the cylinder $[0,s] \times [0,1] \to M$, $(\sigma, t) \mapsto \tilde{x}(\sigma, t)$. We then have:

$$\int_{\mathbb{D}^2} \tilde{u}^*\omega = \int_{\mathbb{D}^2} u^*\omega + \int_{[0,s]\times[0,1]} \tilde{x}^*\omega = \int_{\mathbb{D}^2} u^*\omega + \int_0^s \int_0^1 \omega(\tfrac{\partial \tilde{x}}{\partial s}, \tfrac{\partial \tilde{x}}{\partial t})\,dt\,d\sigma.$$

We can now compute the differential of the action functional:

$$\begin{aligned} d\mathcal{A}_H(x)(Y) &= \frac{d}{ds}\mathcal{A}_H(\tilde{x})|_{s=0} \\ &= -\frac{d}{ds}\left(\int_{\mathbb{D}^2} \tilde{u}^*\omega\right)|_{s=0} + \int_0^1 \frac{d}{ds}(H_t(\tilde{x}(s,t))|_{s=0}\,dt. \\ &= -\int_0^1 \omega\left(\frac{\partial \tilde{x}}{\partial s}|_{s=0}, \frac{\partial \tilde{x}}{\partial t}|_{s=0}\right)\,dt + \int_0^1 \omega_{x(t)}(Y(t), X_H(t,x(t))\,dt. \end{aligned}$$

Thus,

$$d\mathcal{A}_H(x)(Y) = \int_0^1 \omega_{x(t)}(\dot{x} - X_H(t,x(t)), Y(t))\,dt. \tag{2}$$

This shows that x is a critical point of $\mathcal{A}_H$ if and only if $\dot{x} = X_H(t,x)$. □

Given a pair of Lagrangian submanifolds (L_0, L_1), it is also interesting to consider the *relative Hamiltonian action functional* defined as follows[1]. Fix a path $\gamma : [0,1] \to M$ satisfying $\gamma(0) \in L_0$ and $\gamma(1) \in L_1$ and let $\Omega(L_0, L_1; \gamma)$ denote the space of cuves $x : [0,1] \to M$ such that $x(0) \in L_0$, $x(1) \in L_1$ and which are homotopic to γ within such curves. We define

$$\mathcal{A}_H^{L_0,L_1} : \Omega(L_0, L_1; \gamma) \to \mathbb{R}\,, \qquad x \mapsto \int_0^1 H(t, x(t))\, dt - \int u^*\omega, \tag{3}$$

where $u : [0,1] \times [0,1] \to M$ is a homotopy between γ and x such that $u([0,1] \times \{0\}) \subset L_0$ and $u([0,1] \times \{1\}) \subset L_1$.

The second term $\int u^*\omega$ in Equation (3) is not always well-defined, i.e. may depend on the choice of u. When this term does not depend on u, we say that the pair of Lagrangians is *weakly-exact*. Here are some examples of weakly exact pairs.

Examples 1.4. (a) If ω is exact, a pair of exact Lagrangian submanifolds is weakly exact with respect to any choice of γ.

(b) In the torus $(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, dp \wedge dq)$, denote $L = \mathbb{S}^1 \times \{0\}$ and $L' = \{0\} \times \mathbb{S}^1$. Then, (L, L), (L', L') and (L, L') are weakly exact pairs with respect to any curve γ.

(c) If (L_0, L_1) is weakly exact with respect to γ in (M, ω) and (L'_0, L'_1) is weakly exact with respect to γ' in (M', ω'), then $(L_0 \times L'_0, L_1 \times L'_1)$ is weakly exact with respect to (γ, γ') in $(M \times M', \omega \oplus \omega')$.

◀

We also have a relative version of Lemma 1.3:

Lemma 1.5. *The critical points of $\mathcal{A}_H^{L_0,L_1}$ are exactly the orbits of the Hamitonian isotopy $(\phi_H^t)_{t\in[0,1]}$ that connect L_0 and L_1 and are homotopic to γ. Thus, they are in bijection with a subset of the intersection $\phi_H^1(L_0) \cap L_1$.*

Lemmas 1.3 and 1.5 show that it will be useful for the study of Hamiltonian dynamics to develop tools to study critical points of functionals.

2 Action selectors

2.1 Toy-model: critical value selectors

Let f be a continuous function on a compact topological space X. The *sublevel sets* of f are the subsets

$$f^{\leqslant t} = \{x \in X \mid f(x) \leqslant t\},$$

for $t \in \mathbb{R}$.

Definition 2.1 (Critical value selectors)**.** *For all homology class $\alpha \in H_*(X)\backslash\{0\}$ and all continuous function $f : M \to \mathbb{R}$, we define*

$$\rho(\alpha, f) = \inf_{[\sigma]=\alpha} \max_{|\sigma|} f,$$

where the infimum is over all cycles σ representing α and $|\sigma|$ stands for the support of the cycle.

The above definition can be reformulated in a more algebraic way:

$$\rho(\alpha, f) = \inf\{s \in \mathbb{R} \mid \alpha \in \mathrm{Im}(\iota_s)\}, \tag{4}$$

where ι_s is the natural map $H_*(f^{\leqslant s}) \to H_*(X)$.

Example 2.2. (a) Assume X is path-connected and that $\alpha = [\mathrm{pt}]$ is the class of a point in X, then $\rho([\mathrm{pt}], f) = \min f$.

(b) Assume X is an orientable closed manifold and $\alpha = [X]$ is the fondamental class of X. Then the support of every cycle representing $[X]$, is X itself. Hence, $\rho([X], f) = \max f$.

[1]Here the word "relative" means "relative to a pair of Lagrangian submanifolds".

(c) If $X = \mathbb{T}^2$ is the 2-torus, the homology of X can be generated by four classes: $[pt]$, $[X]$, the class $[\alpha]$ of a meridian curve and the class $[\beta]$ of a longitude curve. Figure 1 represents the four corresponding values of the selector when f is the height function for an embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$.

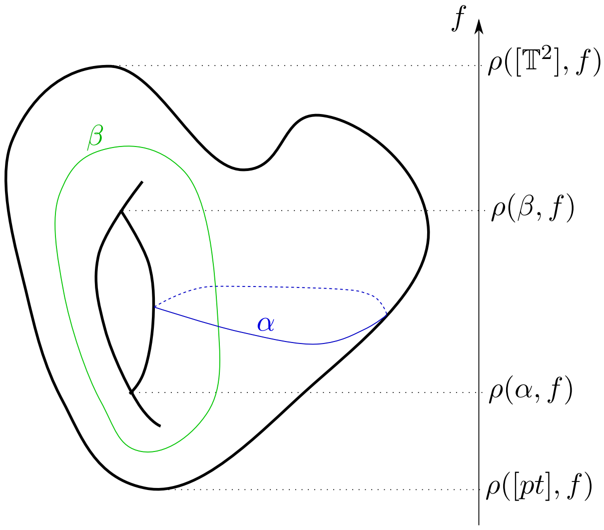

Figure 1: Illustration of Example 2.2.(c)

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Remark 2.3. It follows from standard Morse theory that if f is a smooth function on a smooth manifold, $\rho(\alpha, f)$ is a critical value of f. Indeed, the homology of the sublevel sets changes as we cross the value $\rho(\alpha, f)$.

Furthermore, if all the critical points with critical value $\rho(\alpha, f)$ are non-degenerate (in particular if f is Morse), then the critical value $\rho(\alpha, f)$ is attained at a critical point whose Morse index is the degree of α. ◀

The main feature of this selector is its continuity:

Proposition 2.4. *For all homology class $\alpha \in H_*(X) \setminus \{0\}$, the map $\rho(\alpha, \cdot) : C^0(X, \mathbb{R}) \to \mathbb{R}$ is Lipschitzian. More precisely,*

$$\forall f, g \in C^0(X, \mathbb{R}), \quad \min(f - g) \leqslant \rho(\alpha, f) - \rho(\alpha, g) \leqslant \max(f - g).$$

Proof. Let $\delta = -\min(f - g)$. The inclusions $f^{\leqslant s} \subset g^{\leqslant s+\delta} \subset X$ induces the commutative diagram

$$\begin{tikzcd} & H_*(X) \\ H_*(f^{\leqslant s}) \arrow[ur] \arrow[r] & H_*(g^{s+\delta}), \arrow[u] \end{tikzcd}$$

from which we deduce that if a class $\alpha \in H_*(X) \setminus \{0\}$ is in the image of $H_*(g^{s+\delta})$, then it is also in the image of $H_*(f^{\leqslant s})$. Using (4), we obtain $\rho(\alpha, g) \leqslant \rho(\alpha, f) + \delta$. This shows the first inequality. The second one is obtained similarly. □

Before coming back to symplectic geometry, let us give an interesting application of these critical value selectors.

Proposition 2.5. *[Lusternik-Schnirelman theory] Every smooth function on a closed manifold M admits at least as many critical points as the* cup length *of M which is the number:*

$$\mathrm{cl}(M) := \max\{k + 1 \,|\, \exists\, a_1, \dots, a_k \in H_*(M),\ \forall i, \deg(a_i) \neq \dim(M) \text{ and } a_1 \cdot \ldots \cdot a_k \neq 0\},$$

where $\cdot$ denotes the intersection product in homology[2].

For instance, the cup length of a sphere is 2, the cup length of a surface of genus at least one is 3, the cup length of a torus of dimension n is $n+1$. We can think of the cup length as a measure of the topological complexity of the manifold.

Proof. (borrowed from [41]) Let $k = \mathrm{cl}(M) - 1$ and let $a_1, \dots, a_k \in H_*(M)$ be classes of degree $\neq \dim(M)$, such that $a_1 \cdot \ldots \cdot a_k \neq 0$. We have almost by definition the following inequalities

$$\rho(a_1 \cdot \ldots \cdot a_k, f) \leqslant \ldots \leqslant \rho(a_1 \cdot a_2, f) \leqslant \rho(a_1, f) \leqslant \rho([M], f).$$

Arguing by contradiction, assume that a function $f : M \to \mathbb{R}$ has less critical points than $\mathrm{cl}(M)$. Then one of the above inequalities is an equality. Reformulating, there exist classes α and β, with $\deg \beta \neq \dim M$, such that

$$\rho(\alpha \cdot \beta, f) = \rho(\alpha, f).$$

Denote by ρ this common value and let K be the (compact) subset of M of all the critical points of f having ρ as critical value. For every open subset U of K, there exists $\varepsilon > 0$ such that f has no critical point in $f^{\leqslant \rho+\varepsilon} \setminus (f^{\leqslant \rho-\varepsilon} \cup U)$. By following the gradient lines of f (for an auxiliary Riemanian metric), and shrinking U if necessary, we can build a deformation of $f^{\leqslant \rho+\varepsilon} \setminus U$ into $f^{\leqslant \rho-\varepsilon}$. Thus, if a homology class admits a representant supported in $f^{\leqslant \rho+\varepsilon} \setminus U$, it also admits a representant supported in $f^{\leqslant \rho-\varepsilon}$.

By defnition, the class α has a representant supported in $f^{\leqslant \rho+\varepsilon}$. If the class β had a representant supported in the complement of U, then $\alpha \cdot \beta$ would have a representant in $f^{\leqslant \rho+\varepsilon} \setminus U$. Using the above deformation, $\alpha \cdot \beta$ would have a representant in $f^{\leqslant \rho-\varepsilon}$. But this is not the case since $\rho(\alpha \cdot \beta, f) = \rho$. This shows that the support of every representant of β meets U.

As a consequence, K cannot be a finite set of points, and we get a contradiction since we assumed that f admits less critical points than the finite set $\mathrm{cl}(M)$. ☐

Remark 2.6. The above proof actually shows that if $\rho := \rho(\alpha \cdot \beta, f) = \rho(\alpha, f)$, then every open neighborhood U of the set of critical points of f at level ρ is *homologically non trivial*, i.e. satisfies

$$H_i(U) \neq 0, \text{ for some } i \neq 0.$$

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2.2 Symplectic action selectors

It is possible to extend the previous construction to the action functional using the very powerful machinery of Floer homology. Very roughly, for a given (generically chosen) Hamiltonian H, the Floer homology groups $HF_*(H)$ are the homology groups of a chain complex whose underlying vector space is freely generated by the contractible 1-peridic orbits of H (= critical points of $\mathcal{A}_H$). This homology turns out to be canonically isomorphic to the singular homology of the ambient symplectic manifold M. For a detailed presentation of Floer theory, see [4].

For a (generically chosen) Hamiltonian H, the action selectors are defined in a similarly way as in Definition 2.1, by the formula:

$$c(\alpha, H) = \inf_{[\sigma]=\alpha} \max_{|\sigma|} \mathcal{A}_H,$$

where the infimum is taken over all Floer cycles σ representing α. Here $\alpha \in H_*(M) \setminus \{0\}$ and we say that a Floer cycle represents α if it represents the class in $HF(H)$ which corresponds to α under the canonical isomorphism $H_*(M) \simeq HF_*(H)$. A Floer cycle is a formal linear combination of 1-periodic orbits of H. By "support" of the cycle we simply mean the set of the orbits appearing in this combination.

This definition is only given for a generic choice of Hamiltonian, but action selectors can then be extended by continuity to all Hamiltonians.

[2] The cup length can be equivalently defined in terms of the cup product in cohomology. This explains the terminology

Theorem 2.7 (Viterbo [40], Schwarz [35], Frauenfelder-Schlenk [14], ...)**.** *Assume $\pi_2(M) = 0$. To all homology class $\alpha \in H_*(M)\setminus\{0\}$, and all compactly supported Hamiltonian H, one can associate a real number $c(\alpha, H)$, such that:*

1. *For all Hamiltonian H, $c(\alpha, H)$ belongs to* spec(H) *the set of critical values of $\mathcal{A}_H$.*
2. *For all Hamiltonians H, K,*
$$\int_0^1 \min_{x\in M}(H(t,x) - K(t,x))\,dt \leqslant c(\alpha, H) - c(\alpha, K) \leqslant \int_0^1 \max_{x\in M}(H(t,x) - K(t,x))\,dt \tag{5}$$
In particular, $c(\alpha, \cdot)$ is monotone and continuous.
3. *For every symplectic diffeomorphism $\psi : M \to M$, we have $c(\psi^*\alpha, H\circ\psi) = c(\alpha, H)$.*
4. *If H is a sufficiently C^2-small autonomous Hamiltonian, then*
$$c(\alpha, H) = \rho(\alpha, H).$$
5. *(Triangle inequality) For all Hamiltonians H, K, if we denote by $H\#K$ the Hamiltonian generating $\phi_H^t \circ \phi_K^t$,*
$$c(\alpha\cdot\beta, H\#K) = c(\alpha, H) + c(\beta, K),$$
6. *The number $\gamma(\alpha, \beta; H) := c(\beta, H) - c(\alpha, H)$ depends only on the time-one map ϕ_H^1.*

Remark 2.8. The spectrum spec(H) is a totally discontinuous subset of $\mathbb{R}$. In particular, a continuous function taking values in spec(H) has to be constant. This will be very useful for applications. ◀

We can get similar results for the relative action functional, using the corresponding version of Floer homology, called Lagrangian Floer homology.

Theorem 2.9. *Let (L_0, L_1) be a weakly-exact pair of Lagrangians, with respect to an intersection point, which are either compact or "well-behaved" at infinity. Assume that $L_0\cap L_1$ have a connected and clean intersection[3]. Then, to all homology class $\alpha \in H_*(L_0\cap L_1)\setminus\{0\}$, and all compactly supported Hamiltonian H, one can associate a real number $\ell(\alpha, L_0, L_1; H)$, such that:*

1. *For all Hamiltonian H, $\ell(\alpha, L_0, L_1; H)$ belongs to* spec$(L_0, L_1; H)$ *the set of critical values of $\mathcal{A}_H^{L_0,L_1}$.*
2. *For all Hamiltonians H, K,*
$$\int_0^1 \min_{L_i}(H_t - K_t)\,dt \leqslant \ell(\alpha, L_0, L_1; H) - \ell(\alpha, L_0, L_1; K) \leqslant \int_0^1 \max_{L_i}(H_t - K_t)\,dt, \tag{6}$$
for $i = 0, 1$. In particular, $\ell(\alpha, L_0, L_1; \cdot)$ is monotone and continuous.
3. *For every symplectic diffeomorphism ψ, we have*
$$\ell(\psi^*\alpha, \psi^{-1}(L_0), \psi^{-1}(L_1); H\circ\psi) = \ell(\alpha, L_0, L_1; H).$$
4. *For every smooth function f on N, denoting $\pi : T^*N \to N$ the projection,*
$$\ell(\alpha, O_N, O_N; \pi_* f) = \rho(\alpha, f).$$
5. *(Triangle inequality)*
$$\ell(\alpha\cdot\beta, L_0, L_2; H\#K) = \ell(\alpha, L_0, L_1; H) + \ell(\beta, L_1, L_2; K),$$
6. *At least in the case of closed connected $L := L_0 = L_1$, we have $\gamma(\alpha, \beta; L, L; H) := \ell(\beta, L, L; H) - \ell(\alpha, L, L; H)$ depends only on the image Lagrangian $\phi_H^1(L)$.*

We will be particularly interested in the cases where L_0 is the zero section $O_N \subset T^*N$ and L_1 is either the zero section O_N or a cotangent fiber T_q^*N.

The references for the above theorem are somewhat spread in the litterature. This statement gathers results from [40], [29], [25], [44], [28], [34], [21], ...

[3]Two submanifolds L_0, L_1 are said to have clean intersection if $L_0\cap L_1$ is a submanifold and if at any $p \in L_0\cap L_1$, we have $T_p(L_0\cap L_1) = T_pL_0 \cap T_pL_1$.

3 A selection of applications of action selectors

3.1 Fixed points of Hamiltonian diffeomorphisms

Floer theory was originally invented to study the following conjecture:

Conjecture 3.1 (Arnold [3]). *On a closed symplectic manifold, a Hamiltonian diffeomorphism admits at least as many fixed points as a function should have critical points.*

In view of Proposition 2.5, a weak version of the conjecture is: *Every Hamiltonian diffeomorphism admits at least* $\mathrm{cl}(M)$ *fixed points.* This was established independently by Floer and Hofer on aspherical manifolds.

Theorem 3.2 (Floer [13], Hofer [16]). *Assume* (M, ω) *is a closed aspherical symplectic manifold. Then, every Hamiltonian diffeomorphism admits at least* $\mathrm{cl}(M)$ *fixed points.*

Proof. (borrowed from Howard [19]) Assume it does not hold. Then, the spectrum of the Hamiltonian H generating ϕ has less than $\mathrm{cl}(M)$ elements. Thus, there exist $\alpha, \beta \in H_*(M) \setminus \{0\}$ with $\deg(\beta) < \dim(M)$, such that $c(\alpha \cdot \beta; H) = c(\alpha; H)$.

Let U be any open neighborhood of the fixed-point set of ϕ. Denote by $f : M \to \mathbb{R}$ a smooth function such that $f = 0$ on $\bar{U}$ and $f < 0$ on $M \setminus \bar{U}$. Then, for any $a \in H_*(M) \setminus \{0\}$, and $\varepsilon > 0$ small enough, we have

$$c(a, H \# \varepsilon f) = c(a, H).$$

Indeed, it is easy to check that for all $s \in [0, 1]$, $\mathrm{spec}(H \# \varepsilon s f) = \mathrm{spec}(H)$. Since $\mathrm{spec}(H)$ is totally discontinuous, and since the action selectors are continuous, they have to remain constant along the deformation.

The triangle inequality then gives:

$$\begin{aligned} c(\alpha \cdot \beta, H \# \varepsilon f) &\leqslant c(\alpha, H) + c(\beta, \varepsilon f) \\ c(\alpha \cdot \beta, H) &\leqslant c(\alpha, H) + \rho(\beta, \varepsilon f). \end{aligned}$$

Since $c(\alpha \cdot \beta, H) = c(\alpha, H)$, we deduce that

$$0 \leqslant c(\beta, \varepsilon f).$$

On the other hand, $c(\beta, \varepsilon f) \leqslant 0$ since $f \leqslant 0$, hence $0 \leqslant c(\beta, \varepsilon f)$, which can be reformulated as:

$$c(\beta \cdot [M], \varepsilon f) = c([M], \varepsilon f).$$

We have seen that it implies that $\bar{U}$, which is the set of critical points of f at level 0, is homologically non-trivial. This shows that the set of fixed points of ϕ is itself homologically non-trivial, hence not finite, and gives a contradiction. $\square$

Remark 3.3. We actually proved that if the total number of action selectors of ϕ is smaller than $\mathrm{cl}(M)$, then the set of fixed points of ϕ is homologically non-trivial. ◀

3.2 Symplectic capacities

A symplectic capacity is a way to measure the "symplectic size" of an open subset of a symplectic manifold. For simplicity, we assume in this section that $(M, \omega) = (\mathbb{R}^{2n} \simeq \mathbb{C}^n, \omega_0)$.

Definition 3.4 (Ekeland-Hofer [12]). *A symplectic capacity is a map* $c : \{$ *open subsets of* $\mathbb{R}^{2n} \} \to [0, +\infty]$ *satisfying the following properties:*

(i) If $U \subset V \subset \mathbb{R}^{2n}$*, then* $c(U) \leqslant c(V)$*,*

(ii) For all $\lambda \in (0, +\infty)$ *and* $U \in \mathbb{R}^{2n}$*, we have* $c(\lambda U) = \lambda c(U)$*,*

(iii) For all symplectic diffeomorphism ϕ *and* $U \in \mathbb{R}^{2n}$*, we have* $c(\phi(U)) = c(U)$*,*

(iv) For all $r > 0$*, we have* $c(B^{2n}(r)) = \pi r^2 = c(Z(r))$*, where* $Z(r) = B^2(r) \times C^{2n-2} \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$*.*

It is absolutely not evident that a symplectic capacity exists. Before giving an example of symplectic capacity, let us note that the following celebrated theorem follows readily from the existence of a symplectic capacity.

Theorem 3.5 (Gromov non-squeezing [15])**.** *For $R > r$, there is no symplectic diffeomorphism which maps $B^{2n}(R)$ into $Z(r)$.*

In constrast, note that for all R, r, there exists a volume preserving diffeomorphism which maps $B^{2n}(R)$ into $Z(r)$. Thus, this theorem shows that symplectic maps behave differently (are more rigid) than volume preserving maps.

Theorem 3.6 (Hofer-Zehnder capacity [18])**.** *Call* admissible *any autonomous Hamiltonian H that has no periodic orbit of period $\leqslant 1$ other than its critical points. Then, the map c defined by*

$$c(U) = \sup\{\max(H) \mid H \in C_c^\infty(U) \text{ admissible}\},$$

for all open subset $U \subset \mathbb{R}^{2n}$, is a symplectic capacity.

Proof. We need to justify the four points of Definition 3.4. (i) is fairly obvious. (ii) and (iii) follow from the easy fact that for all $\lambda > 0$ and all symplectic diffeomorphism ϕ, and all autonomous Hamiltonian H the following three assertions are equivalent:

- H is admissible,
- $\lambda^2 H(\frac{\cdot}{\lambda})$ is admissible[4],
- $H \circ \phi$ is admissible[5].

To prove that $c(B^{2n}(r)) \geqslant \pi r^2$, we consider Hamiltonians of the form $H = f(\pi\|x\|^2)$, with $\|\cdot\|$ the standard euclidean norm on $\mathbb{R}^{2n}$, and $f : [0, \infty) \to [0, \infty)$. An easy computation shows that the 1-periodic orbits correspond either to the point 0 or to the points $x \neq 0$ such that $f'(\pi\|x\|^2) \in \mathbb{Z}$. Moreover if $f'(\pi\|x\|^2) = 0$, then x is a critical point of H. Therefore, such a Hamiltonian is admissible if and only if f' does not take any other integral value than 0. One can find functions f supported in $[0, \pi r^2]$ whose derivative f' take values in $(-1, 0]$ and whose maximum is arbitrary close to πr^2. These functions provide admissible Hamiltonians supported in $B^{2n}(r)$, whose maximum is arbitrary close to πr^2, hence shows $c(B^{2n}(r)) \geqslant \pi r^2$.

There remains to prove that $c(Z(r)) \leqslant \pi r^2$, which is the hard part. We need the following lemma

Lemma 3.7 (Energy-capacity inequality, Hofer [17])**.** *Let H be an admissible Hamiltonian supported in open subset U, and let K be another Hamiltonian satisfying $\phi_K^1(U) \cap U = \emptyset$. Then,*

$$\max H \leqslant \int_0^1 (\max K_t - \min K_t)\, dt.$$

Proof. (borrowed from [37]) Let us first note that for an admissible Hamiltonian, and for all $s \in [0, 1]$, the Hamiltonian sH has no other 1-periodic orbits than the critical points of H and that the action of such periodic orbits is nothing but the corresponding critical value of sH. Thus we can write:

$$\tfrac{1}{s}\mathrm{spec}(sH) = \mathrm{Critval}(H), \text{ for all } s \in [0, 1].$$

By Sard theorem, $\mathrm{Critval}(H)$ is totally discontinuous, thus the continuous function $s \mapsto \frac{1}{s}c([M], sH)$, which takes values in $\mathrm{Critval}(H)$, has to be constant. By property 4 of Theorem 2.7, $\frac{1}{s}c([M], sH) = \max H$ for $s > 0$ small enough. At $s = 1$, we thus obtain:

$$c([M], H) = \max(H). \tag{7}$$

For $s \in [0, 1]$, we now consider the Hamiltonian F^s generating the isotopy

$$\begin{cases} \phi_H^{2st}, & t \in [0, \frac{1}{2}] \\ \phi_K^{2t-1} \circ \phi_H^s, & t \in [\frac{1}{2}, 1]. \end{cases}$$

[4]The flow of $\lambda^2 H(\frac{\cdot}{\lambda})$ is conjugate to that of H by the λ-dilation.
[5]The flow of $H \circ \phi$ is conjugate to that of H by ϕ.

In particular, $\phi^1_{F^s} = \phi^1_K \circ \phi^s_H$.

The condition on ϕ^1_K implies that the fixed points of ϕ^1_K are in the complement of U. Since H is supported in U, we deduce that $\phi^1_{F^s}$ has the same fixed points as ϕ^1_K. The 1-periodic orbits of F^s are obtained by concatenating an orbit of K with a constant path. Thus, they have the same action as that of K. We conclude that for all s, $\mathrm{spec}(F^s) = \mathrm{spec}(K)$.

As before, since $s \mapsto c([M], F^s)$ is continuous and takes its values in the totally discontinuous subset $\mathrm{spec}(K)$, we deduce that $c([M], F^s)$ remains constant. In particular,

$$c([M], F_1) = c([M], K).$$

Now, using the properties of action selectors,

$$\int_0^1 \min K_t \, dt \leqslant c([M], F_1) - c([M], H) \quad \text{and} \quad c([M], F_1) = c([M], K) \leqslant \int_0^1 \max K_t \, dt.$$

Thus $c([M], H) \leqslant \int_0^1 (\max K_t - \min K_t) \, dt$. Using (7), this finishes the proof of the energy-capacity inequality. □

We can now use Lemma 3.8 to prove the inequality $c(Z(r)) \leqslant \pi r^2$. First note that for all $\varepsilon > 0$, a 2 dimensional disk of area πr^2 can be mapped by an area preserving map of $\mathbb{R}^2$ into the rectangle $[0,1] \times [0, \pi r^2 + \varepsilon]$. Hence $Z(r)$ can be mapped by a symplectic map into the subset $V = [0,1] \times [0, \pi r^2 + \varepsilon] \times \mathbb{R}^{2n-2}$.

Denote $(q_1, p_1, \ldots, q_n, p_n)$ the standard coordinates in $\mathbb{R}^{2n}$ Let H be a admissible Hamiltonian supported in V, and let K be a Hamiltonian which coincides with $(\pi r^2 + \varepsilon) q_1$ on U. The vector field X_K coincides with $(\pi r^2 + \varepsilon) \frac{\partial}{\partial y_1}$, hence $\phi^1_K(\mathrm{supp}(H)) \cap \mathrm{supp}(H) = \emptyset$. Using an approriate cut-off function, such a Hamiltonian can be found satisfying $\int_0^1 (\max K_t - \min K_t) \, dt \leqslant \pi r^2 + 2\varepsilon$. By the energy-capacity inequality, we deduce that $\max(H) \leqslant \pi r^2 + 2\varepsilon$, for all admissible $H \in C^\infty_c(V)$. This shows $c(Z(r)) \leqslant c(V) \leqslant \pi r^2 + 2\varepsilon$ for all ε, hence $c(Z(r)) \leqslant \pi r^2$. □

3.3 A relative version of the energy-capacity inequality

The energy-capacity inequality, Lemma 3.7, tells us in particular that if a Hamiltonian diffeomorphism ϕ^1_K displaces a ball B of radius r, i.e. $\phi^1_K(B) \cap B = \emptyset$, then $\int_0^1 (\max K_t - \min K_t) \, dt \geqslant \pi r^2$. This is a surprising fact: the C^0 norm of the Hamiltonian has some control on the dynamics, which is defined only with the help of the Hamiltonian vector field, hence of the differential of the Hamiltonian.

A relative version of this statement was recently established by Lisi and Rieser (a proof can be found in [20] or in [6]), and is even more surprising.

Lemma 3.8 (Relative energy-capacity inequality)**.** *Let L be a weakly exact closed Lagrangian, $B = i(B(0,r))$ a symplectic ball, s.t. $i(\mathbb{R}^n \cap B(0,r)) \subset L$ and K Hamiltonian s.t. $\phi^1_K(B) \cap L = \emptyset$, Then,*

$$\int_{\mathbb{S}^1} (\max_L K_t - \min_L K_t) dt \geqslant \tfrac{1}{2} \pi r^2.$$

The proof presented in [20] goes along the same lines as the one presented in Section 3.2, using Lagrangian action selectors and their properties (Theorem 2.9) instead of the Hamiltonian action selectors.

This inequality has applications to C^0-symplectic geometry that will be presented in the last lecture.

3.4 Graph selectors and variational solutions to evolution Hamilton-Jacobi equations

Let L be a closed exact Lagrangian in a cotangent bundle T^*N and set for all $q \in N$,

$$u_L(q) = \ell([pt], L, T^*_q N; 0).$$

In the case where $L = \phi_H^1(0_N)$ is Hamiltonian isotopic to the zero section, the fonction u_L can also be defined by:

$$u_L(q) = \ell([pt], 0_N, T_q^*N; H).$$

This function u_L is called a graph selector because of the following result.

Theorem 3.9 (Chaperon [11], Oh [29], Amorim-Oh-Santos [1])**.** *The function u_L is Lipschitz and there exists a dense open subset of N on which du_L exists and belongs to L.*

The graph selector plays a major role in the proof of the higher dimensional Birkhoff theorem by Marie-Claude Arnaud. A Tonelli Hamiltonian is a autonomous Hamiltonian which is fiberwise stricly convex and grows superlinearly at infinity.

Theorem 3.10 (Arnaud [2], see also [5], [1])**.** *Every closed exact Lagrangian which is invariant by a Tonelli Hamiltonian is the graph of an exact 1-form.*

Very roughly, the idea of the proof is to use weak-KAM/Aubry-Mather theory to show that $L = \mathrm{graph}(du_L)$.

The graph selector can also be used to build certain weak solutions to evolution Hamilton-Jacobi equations, that is, to the following problem:

$$\begin{cases} \partial_t u + H_t(q, \partial_q u(q)) = 0 \\ u(0, x) = u_0(x), \end{cases} \tag{8}$$

where $H : S^1 \times T^*N \to \mathbb{R}$ is a given Hamiltonian (that we rather see as a function $\mathbb{R} \times T^*N \to \mathbb{R}$, $u_0 : N \to \mathbb{R}$ is the initial function, and the unknown u is a function $\mathbb{R} \times N \to \mathbb{R}$.

This problem can be reformulated in a geometric way, as follows. Let Λ be the isotropic submanifold of $T^*\mathbb{R} \times T^*N$ defined by

$$\Lambda = \{(0,\, -H_0(q, du_0(q))\,,\, q\,, du_0(q)) \,|\, q \in N\},$$

and let $\Sigma = \hat{H}^{-1}(0)$, where $\hat{H}(t, \tau, q, p) = \tau + H_t(q, p)$. One can easily check that a function u is a solution (8) if and only if $\Lambda \subset \mathrm{graph}(du) \subset \Sigma$.

The Lagrangian submanifold $L = \bigcup_{s\in\mathbb{R}} \phi_{\hat{H}}^s(\Lambda)$ satisfies $\Lambda \subset L \subset \Sigma$, hence the next result.

Theorem 3.11. *[39, 32] With the above notations, the graph selector u_L solves* (8) *on a dense open set. We call it the* variational solution *of the evolution Hamilton-Jacobi equation.*

In the case of a Tonelli Hamiltonian, it was proved by Joukovskaia in her PhD thesis that the variational solution coincides with the viscosity solution, but this is not true for general Hamiltonians. Recent results on the relations between these two types of solutions have been obtained by Wei [43] and Roos [33].

3.5 Symplectic homogenization: a generalization of Mather's α-function

If H is a Tonelli Hamiltonian, with associated Lagrangian $\mathcal{L}$, one can define the Lagrangian action of a curve q on the base N:

$$\mathbb{L}(q) = \int_0^1 \mathcal{L}(t, q(t), \dot{q}(t))\, dt.$$

If $(q(t), \dot{q}(t))$ is a solution of the corresponding Euler-Lagrange equation and is associated to a solution $(q(t), p(t))$ of the Hamilton equations via the Legendre transform, then we have:

$$\mathbb{L}(q) = \mathcal{A}_H(q, p).$$

The Mather α-function at zero is defined as:

$$\alpha_H(0) := -\lim_{k\to\infty} \frac{1}{k} \inf\{\mathbb{L} \,|\, q : [0, k] \to N\}.$$

Theorem 3.12 (Monzner-Vichery-Zapolsky [28], building on Viterbo [42])**.** *For every Tonelli Hamiltonian H,*

$$\alpha_H(0) = \lim_{k\to\infty} \frac{1}{k} \ell([N], O_N, O_N, H^{\sharp k}).$$

The invariant $\ell([N], O_N, O_N, H)$ is a priori defined only for compactly supported H. However, it can be extended to every Hamiltonian with complete flow (e.g. Tonelli) by applying $\ell([N], O_N, O_N, \cdot)$ to a cut off of H sufficiently far away from the zero section. The fact that the right hand side converges is a direct consequence of the triangle inequality, which implies that the sequence $\ell([N], O_N, O_N, H^{\sharp k})$ is subadditive.

Vichery extended another aspect of Mather's theory: the existence of invariant measures. For $c \in H^1(M)$, we denote $\alpha_H(c) := \lim_{k\to\infty} \frac{1}{k}\ell([N], \mathrm{graph}(\eta), \mathrm{graph}(\eta), H^{\sharp k})$, where η is any 1-form representing c. The value of $\alpha_H(c)$ corresponds to the value of Mather α-function at c in the case of a Tonelli Hamiltonian.

Theorem 3.13 (Vichery [38]). *For every η in the Clarke subdifferential $\partial\,\alpha_H(p) \subset H_1(N)$, there exists an invariant measure μ s.t*

$$\mathcal{A}_H(\mu) = \alpha_H(p) \quad \textit{and} \quad \rho(\mu) = \eta,$$

where ρ denotes the rotation vector of the measure.

The following very interesting problem is widely open, as far as I know.

Problem. *Which other aspects of Mather's theory can be extended to general (non-convex) Hamiltonians in a cotangent bundle?*

4 Introduction to C^0-symplectic geometry

We now give a short introduction to the very recent and rapidly developping field of C^0-symplectic geometry. This can not be considered as a survey of the field since many of its aspects are ignored (in particular all the works related to function theory on symplectic manifolds or to continuous Hamiltonian dynamics are absent from these notes). Action selectors are a central tool in this field.

4.1 The Gromov-Eliashberg theorem and symplectic homeomorphisms

The birth of C^0-symplectic geometry coincides with the following celebrated theorem.

Theorem 4.1 (Gromov-Eliashberg, $\simeq$ 1983). *If a diffeomorphism is a C^0-limit of symplectic diffeomorphisms, then it is a symplectic diffeomorphism.*

Idea of the proof . Knowing the existence of a symplectic capacity (this existence is the hard part of the proof), one proves that a limit of symplectic maps has to preserve the capacity (rather easy) and that if a smooth map preserves the capacity then it is (anti-)symplectic (not too difficult). See [18] for details. ☐

This theorem asserts that symplectic diffeomorphisms are C^0-rigid. This is very surprising! Indeed, being a symplectic diffeomorphism is a condition on the differential of the diffeomorphism, which should not a priori behave well under C^0-limits. This is a manifestation of symplectic rigidity. The Gromov-Eliashberg theorem opens the field of *C^0-symplectic topology*, which may be defined as the study of the behavior of symplectic objects under C^0-limits, or more broadly, as the search of which aspects of symplectic geometry can be generalized to continuous (not smooth) settings. As a first step in this direction, this theorem allows to define:

Definition 4.2. *A homeomorphism ϕ between two symplectic manifolds is called a symplectic homeomorphism if it is (locally) the C^0-limit of a sequence of symplectic diffeomorphisms.*

By the Gromov–Eliashberg theorem, a symplectic homeomorphism which is smooth is a symplectic diffeomorphism. Here are the some examples. Many more examples can be constructed, for instance by taking compositions, products...

Examples 4.3. (a) On a surface (Σ, ω) endowed with an area form, every homeomorphism which preserves the area and the orientation is a symplectic homeomorphism [36, 30];

(b) Let α be a continuous 1-form which is closed in the sense of distributions[6], then the map

$$\phi : T^*M \to T^*M, \quad (q,p) \mapsto (q, p + \alpha(q))$$

is a symplectic homeomorphism ([20], Prop 26);

(c) If H is a C^1 Hamiltonian such that the (C^0) vector field X_H is uniquely integrable with flow ϕ_H^t. Then every such map ϕ_H^t is a symplectic homeomorphism.

◀

Given this definition, the central problem will be the following.

Main Problem. *Do these symplectic homeomorphisms behave similarly as the smooth symplectic maps (C^0-rigidity) or do they present some sort of exotic behavior (flexibility) ?*

4.2 Symplectic homeomorphisms and submanifolds

In this section, we survey one aspect of the rigidity/flexiblity properties of C^0 symplectic geometry: the behavior of symplectic homeomorphisms with respect to submanifolds. Recall that a submanifold V is said:

- Symplectic if $\omega|_V$ is non-degenerate (local model $\mathbb{C}^d \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$)
- Isotropic if $\omega|_V = 0$ (iff $TV \subset TV^{\perp\omega}$, local model $\mathbb{R}^d \subset \mathbb{C}^n$)
- Coisotropic if $TV^{\perp\omega} \subset TV$ (local model $\mathbb{C}^{n-k} \times \mathbb{R}^k \subset \mathbb{C}^n$)
- Lagrangian if $TV^{\perp\omega} = TV$ (local model $\mathbb{R}^n \subset \mathbb{C}^n$)

Moreover, if V is coisotropic, $TV^{\perp\omega}$ is an integrable distribution on V. The corresponding foliation is called the *characteristic foliation* of V.

Theorem 4.4 (H.-Leclercq-Seyfaddini[20]). *Let V be a smooth coisotropic submanifold and ψ a symplectic homeomorphism. Assume $\psi(V)$ is a* smooth *submanifold. Then $\psi(V)$ is coisotropic and ψ maps the characteristic foliation of V to that of $\psi(V)$.*

Remark 4.5.

(a) There was partial earlier results by Laudenbach-Sikorav [23] (in the Lagrangian case) and Opshtein [31](in the case of hypersurfaces).

(b) In particular, a symplectic homeomorphisms maps Lagrangians to Lagrangians.

(c) This theorem recovers the Gromov-Eliashberg theorem, using the fact that a map is symplectic iff its graph in $M \times \overline{M}$ is Lagrangian.

(d) An interesting feature is that it is a local statement. The submanifold is not assumed closed.

(e) More results on what happens transversally to the characteristic foliation (i.e. on the *reduction*) have been obtained [10, 21], but will not reported here.

◀

However, symplectic homeomorphisms do not always behave like their smooth analogs:

Theorem 4.6 (Buhovsky-Opshtein [10]). *There exists a compactly supported symplectic homeomorphism of $\mathbb{R}^6 \simeq \mathbb{C}^3$, which preserves the symplectic plane $P = \mathbb{C} \times \{0\} \times \{0\}$, and whose restriction to P is smooth and acts on the unit disk as $z \mapsto \frac{1}{2}z$. In particular its restriction to P is smooth but not symplectic.*

It is believed that the set of technics introduced to prove this theorem (called "quantitative h-principle technics" by their authors), can be applied to find a symplectic homeomorphism which sends a smooth isotropic submanifold to a smooth symplectic submanifold and conversely. Therefore the type of rigidity that appears in Theorem 4.4 is specific to coisotropic submanifolds.

[6] A 1-form α is closed in the sense of distribution if and only if its integral over every smooth contractible curve vanishes and if and only if it is locally the differential of a C^1 function.

We will now present a short proof of theorem 4.4 in the case of a closed weakly exact Lagrangian submanifold. This will rely on the relative energy-capacity inequality (Lemma 3.8), hence on action selectors. The proof for a general coisotropic submanifold presented in [20] actually consists in reducing to this special case.

Proof of theorem 4.4 in the case of a closed weakly-exact Lagrangian submanifold. Let ψ be a symplectic homeomorphism. By definition, there exists a sequence of symplectic maps ψ_n which C^0 converges to ψ. Let L be a closed weakly-exact Lagrangian submanifold such that $\psi(L)$ is also a smooth submanifold. We want to show that $L' = \psi(L)$ is Lagrangian, i.e. $T_yL'^{\perp\omega} \subset T_yL'$ at all point $y = \psi(x)$, $x \in L$.

Let $v \in T_yL'^{\perp\omega}$. Then, there exists a function H which is constant on L' and such that $X_H(y) = v$. Indeed, such a function can be constructed by performing a cutoff of the function $\omega(\cdot, v)$ in local coordinates where L' is a linear subspace.

Now assume that $v \notin T_yL'$. Then, for $s > 0$ small enough, $\phi^s_H(y) \notin L'$, hence $\psi^{-1}\phi^s_H\psi(x) \notin L$. It follows that there exists a small ball B centered at x, such that, $\psi^{-1}\phi^s_H\psi(B) \cap L = \emptyset$. Thus, for n large enough:

$$\phi^1_{sH\circ\psi_n}(B) \cap L = \psi_n^{-1}\phi^s_H\psi_n(B) \cap L = \emptyset$$

The relative energy-capacity of Lemma 3.8 then yields

$$\max_L sH \circ \psi_n - \min_L sH \circ \psi_n \geqslant \tfrac{1}{2}\pi r^2,$$

and taking limit, we get $\max_L H \circ \psi - \min_L H \circ \psi \geqslant \frac{1}{2s}\pi r^2$. This is in contradiction with the fact that H is constant on L'. Thus we deduce $v \in T_yL'$ and we proved $TL'^{\perp\omega} \subset TL'$ $\square$

4.3 Fixed points of Hamiltonian homeomorphisms

In this last part, we discuss the fate of the Arnold conjecture when leaving the world of diffeomorphisms to that of homeomorphisms. We first need to define Hamiltonian homeomorphisms. We adopt the following definition in the present lecture.

Definition 4.7. *A Hamiltonian homeomorphism is a homeomorphism which is a C^0-limit of Hamiltonian diffeomorphisms.*

Examples 4.8. (a) If $f : N \to \mathbb{R}$ is a C^1-function, then $(q, p) \mapsto (q, p + df(q))$ is a Hamiltonian homeomorphism of T^*N.

(b) More generally, if H is a C^1-function on M, whose (C^0) Hamiltonian vector field X_H is uniquely integrable, then the time t map of ϕ^t_H is a Hamiltonian homeomorphism.

(c) On a surface, Hamiltonian homeomorphisms are the homeomorphisms which preserve area, are isotopic to Id and have vanishing rotation vector.

◀

This last example shows that on a surface a smooth Hamiltonian homeomorphism is a Hamiltonian diffeomorphism. It is absolutely not known whether this holds in higher dimension. It is conjectured, and known in many cases [22, 7], that if a smooth Hamiltonian homeomorphism belongs to the identity component of the group of symplectic diffeomorphisms, then it is Hamiltonian. This conjecture is known as the C^0-flux conjecture.

We can now state our question, motivated by Theorem 3.2.

Question 4.9 (Arnold conjecture for homeomorphisms)**.** *Is it true that for every Hamiltonian homeomorphism of a closed symplectic manifold M, the number of fixed points is always at least the cup length of M.*

It turns out that the answer is positive on surfaces.

Theorem 4.10 (Matsumoto [26], Le Calvez [24])**.** *For every Hamiltonian homeomorphism on a closed connected surface of genus g, the number of fixed points is at least 2 for $g = 0$ and at least 3 for $g \geqslant 1$.*

However, in higher dimension, things work completely differently, and the Arnold conjecture is not true anymore.

Theorem 4.11 (Buhovsky-H.-Seyfaddini [9])**.** *On every connected compact manifold M of dimension at least 4, there exists a Hamiltonian homeomorphism which admits a unique fixed point.*

Remark 4.12.

(a) As a limit of Hamiltonian diffeomorphisms, which all admit a fixed point, compactness implies that a Hamiltonian homeomorphism must always admit at least 1 fixed point.

(b) One can find such a Hamiltonian homeomorphism smooth in the complement of the fixed point.

(c) One can find such a Hamiltonian homeomorphism in any normal subgroup of the group of symplectic homeomorphisms which contains the group of Hamiltonian diffeomorphisms. Heuristically, any reasonable definition of Hamiltonian homeomorphism should lead to such a normal subgroup, thus this remark shows that whatever reasonnable definition we choose for Hamiltonian homeomorphisms, the Arnold conjecture will not hold in dim $\geqslant 4$ for these maps.

◀

Idea of the proof . (borrowed from the introduction of [9]) The construction of a homeomorphism f, as prescribed in Theorem 4.11, is done in two main steps. The first step, which is the most difficult of the two, can be summarized in the following statement.

Lemma 4.13. *On every closed connected symplectic manifold of dimension at least* 4*, there exists a Hamitonian homeomorphism ψ and a continuously embedded tree T such that:*

(i) $\psi(T) = T$,

(ii) All the fixed points of ψ are contained in T.

The proof of this lemma forms the main technical part of the proof. This is also where the dimension at least four is needed. An important ingredient used in the construction of the invariant tree T is the Buhovsky-Opshtein quantitative h-principle technics [10]. Roughly, the idea is to start from an arbitrary C^2-small Morse function H. Its time-1 map ϕ_H^1 admits only finitely many fixed points which correspond to the critical points of H. Given two such critical points x_1, y_1, we construct a closed curve γ_1, connecting x_1 to y_1, and a Hamiltonian homeomorphism ψ_1 which is C^0-close to Id, such that $\psi_1 \circ \phi_H^1$ has the same fixed points as ϕ_H^1 and the curve γ_1 is invariant by $\psi_1 \circ \phi_H^1$. We perform this construction sufficiently many times to construct curves $\gamma_1, \gamma_2, \dots, \gamma_N$ whose union constitute the required tree T. The homeomorphism ψ is then obtained as a composition: $\psi = \psi_N \circ \dots \circ \psi_1 \circ \phi_H^1$.

The end of the construction consists of "collapsing" the invariant tree T to a single point which will be the fixed point of the homeomorphism f. To proceed, we fix a point $p \in M$ and we construct a sequence $\phi_i \in \mathrm{Symp}(M, \omega)$ such that ϕ_i converges uniformly to a map $\phi : M \to M$ with the following two properties:

1. $\phi(T) = \{p\}$,
2. ϕ is a proper symplectic diffeomorphism from $M \setminus T$ to $M \setminus \{p\}$.

The Hamiltonian homeomorphism f is then obtained as follows: $f(p) = p$ and

$$\forall x \in M \setminus \{p\},\ f(x) = \phi \circ \psi \circ \phi^{-1}(x).$$

It is not difficult to see that p is the unique fixed point of f. Indeed, on $M \setminus \{p\}$, the map f is conjugate to $\psi : M \setminus T \to M \setminus T$ which is fixed point free by construction. By picking the above sequence of symplectomorphisms ϕ_i carefully, it is possible to ensure that the sequence of conjugations $\phi_i \psi \phi_i^{-1}$ converges uniformly to f. It follows that f can be written as the uniform limit of a sequence of Hamiltonian diffeomorphisms. □

In the previous construction, we clearly see how we collapse all the fixed points into a single points by a limiting process. In some way the unique fixed point that we get has a multiplicity. This leads us to the following question.

Question 4.14. *Is it possible to assign multiplicities to fixed points so that the Arnold conjecture holds for homeomorphisms?*

At this point we do not have a precise answer to this question, but some progress have been made using action selectors.

Recall that under the condition $\langle \omega, \pi_2(M) \rangle = 0$, the quantity

$$\gamma(\alpha, \beta, H) = c(\beta, H) - c(\alpha, H)$$

does not depend on H but only on ϕ_H^1. Therefore, we adopt the notation $\gamma(\alpha, \beta, \phi_H^1)$ from now on.

Definition 4.15. *We call essential spectrum of a Hamiltonian diffeomorphism ϕ the subset of $\mathbb{R}$:*

$$\mathrm{EssSpec}(\phi) = \{\gamma(\alpha, [pt], \phi) \mid \alpha \in H_*(M) \setminus \{0\}\}.$$

A very recent result asserts that this essential spectrum may be defined for homeomorphisms as well.

Theorem 4.16 (Buhovsky-H.-Seyfaddini [8]). *Assume $\langle \omega, \pi_2(M) \rangle = 0$. Then the map $\gamma(\alpha, \beta, \cdot)$ is continuous for the C^0-distance on the group of Hamiltonian diffeomorphism and extend by continuity to Hamiltonian homeomorphisms. Hence* $\mathrm{EssSpec}(\phi)$ *is well defined for a Hamiltonian homeomorphism ϕ.*

Note that for a Hamiltonian diffeomorphism ϕ, $\gamma(\alpha, \beta, \phi)$ is the difference of two critical values of the action functional. Thus, a relation $\gamma(\alpha, \beta, \phi) \neq 0$ witnesses the presence of two distinct fixed points. Therefore, we have

$$\#\mathrm{Fix}(\phi) \geqslant \#\mathrm{EssSpec}(\phi). \tag{9}$$

However, for homeomorphisms, the action functional is not defined and the number $\gamma(\alpha, \beta, \phi)$ is not a difference of critical values. Thus we can not deduce from $\gamma(\alpha, \beta, \phi) \neq 0$ the presence of two distinct fixed points. In particular, we a priori do not have (9) for Hamiltonian homeomorphisms. A relation $\gamma(\alpha, \beta, \phi) \neq 0$ in the presence of a unique fixed point may be interpreted as the fact that the fixed point has multiplicity at least two.

We noticed (Remark 3.3) that for a Hamiltonian diffeomorphism, if the essential spectrum has strictly less elements than $\mathrm{cl}(M)$, then the set of fixed points is homologically non-trivial. It turns out that the same statement can be proved for homeomorphisms.

Theorem 4.17 (Buhovsky-H.-Seyfaddini [8]). *Assume $\langle \omega, \pi_2(M) \rangle = 0$. Then for every Hamiltonian homeomorphism ϕ, either* $\#\mathrm{EssSpec}(\phi) \geqslant \mathrm{cl}(M)$ *or* $\mathrm{Fix}(\phi)$ *is homologically non-trivial hence infinite.*

This shows in particular that if a Hamltonian homeomorphism admits fewer fixed points than prescribed by the Arnold conjecture, then its essential spectrum has at least $\mathrm{cl}(M)$ elements. If we interpret the number of elements in the essential spectrum as a count of multiplicity of fixed points, this provides a positive answer to Question 4.14.

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