

COISOTROPIC RIGIDITY AND C^0 -SYMPLECTIC GEOMETRY

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ABSTRACT. We prove that symplectic homeomorphisms, in the sense of the celebrated Gromov–Eliashberg Theorem, preserve coisotropic submanifolds and their characteristic foliations. This result generalizes the Gromov–Eliashberg Theorem and demonstrates that previous rigidity results (on Lagrangians by Laudenbach–Sikorav, and on characteristics of hypersurfaces by Opshtein) are manifestations of a single rigidity phenomenon. To prove the above, we establish a C^0 -dynamical property of coisotropic submanifolds which generalizes a foundational theorem in C^0 -Hamiltonian dynamics: Uniqueness of generators for continuous analogs of Hamiltonian flows.

1. INTRODUCTION AND MAIN RESULTS

A submanifold C of a symplectic manifold (M, ω) is called coisotropic if for all $p \in C$, $(T_p C)^\omega \subset T_p C$ where $(T_p C)^\omega$ denotes the symplectic orthogonal of $T_p C$. For instance, hypersurfaces and Lagrangians are coisotropic. A coisotropic submanifold carries a natural foliation $\mathcal{F}$ which integrates the distribution $(TC)^\omega$; $\mathcal{F}$ is called the characteristic foliation of C . Coisotropic submanifolds and their characteristic foliations have been studied extensively in symplectic topology. The various rigidity properties that they exhibit have been of particular interest. For example, in [7] Ginzburg initiated a program for studying rigidity of coisotropic intersections. In this paper, we prove that coisotropic submanifolds, along with their characteristic foliations, are C^0 -rigid in the spirit of the Gromov–Eliashberg Theorem.

This celebrated theorem states that a diffeomorphism which is a C^0 -limit of symplectomorphisms is symplectic. Motivated by this, symplectic homeomorphisms are defined as C^0 -limits of symplectomorphisms (see Definition 9 and Remark 10). Area preserving homeomorphisms, and their products, are examples of symplectic homeomorphisms. Here is our main result.

Theorem 1. *Let C be a smooth coisotropic submanifold of a symplectic manifold (M, ω) . Let U be an open subset of M and $\theta: U \rightarrow V$ be a symplectic homeomorphism. If $\theta(C \cap U)$ is smooth, then it is coisotropic. Furthermore, θ maps the characteristic foliation of $C \cap U$ to that of $\theta(C \cap U)$.*

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An important feature of the above theorem is its locality: C is not assumed to be necessarily closed and θ is not necessarily globally defined. Here is an immediate, but surprising, consequence of Theorem 1.

Corollary 2. *If the image of a coisotropic submanifold via a symplectic homeomorphism is smooth, then so is the image of its characteristic foliation.*

Theorem 1 uncovers a link between two previous rigidity results and demonstrates that they are in fact extreme cases of a single rigidity phenomenon.

One extreme case, where C is a hypersurface, was established by Opshtein [23]. Clearly, in this case, the interesting part is the assertion on rigidity of characteristics, as the first assertion is trivially true.

Lagrangians constitute the other extreme case. When C is Lagrangian, its characteristic foliation consists of one leaf, C itself. In this case the theorem reads: *If θ is a symplectic homeomorphism and $\theta(C)$ is smooth, then $\theta(C)$ is Lagrangian.* In [11], Laudenbach–Sikorav proved a similar result: *Let L be a closed manifold and ι_k denote a sequence of Lagrangian embeddings $L \rightarrow (M, \omega)$ which C^0 -converges to an embedding ι . If $\iota(L)$ is smooth, then (under some technical assumptions) $\iota(L)$ is Lagrangian.* On one hand, their result only requires convergence of embeddings while Theorem 1 requires convergence of symplectomorphisms. On the other hand, Theorem 1 is local: It does not require the Lagrangian nor the symplectic manifold to be closed.

The above discussion raises the following question.

QUESTION. What can one say about C^0 -limits of coisotropic embeddings and their characteristic foliations?

We would like to point out that Theorem 1 is a coisotropic generalization of the Gromov–Eliashberg Theorem. Indeed, it implies that if the graph of a symplectic homeomorphism is smooth, then it is Lagrangian.

As we shall see, the proof of Theorem 1 relies on dynamical properties of coisotropic submanifolds. In particular, we use C^0 -Hamiltonian dynamics as defined by Müller and Oh [22]. To the best of our knowledge, this is one of the first extrinsic applications of this recent, yet promising, theory.

Following [22], we call a path of homeomorphisms ϕ^t a hameotopy if there exists a sequence of smooth Hamiltonian functions H_k such that the isotopies $\phi_{H_k}^t$ C^0 -converge to ϕ and the Hamiltonians H_k C^0 -converge to a continuous function H (see Definition 11). Then, H is said to generate the hameotopy ϕ^t , and to emphasize this we write ϕ_H^t ; the set of such generators will be denoted C_{Ham}^0 . A foundational result of C^0 -Hamiltonian dynamics is the uniqueness of generators Theorem (see [26, 3]) which states that the trivial hameotopy, $\phi^t = \text{Id}$, can only be generated by those functions in C_{Ham}^0 which solely depend on time (see also Corollary 4 below).

Let $H \in C^\infty([0, 1] \times M)$. Recall the following two dynamical properties of a coisotropic submanifold C : Assume that C is closed as a subset,

1. $H|_C$ is a function of time if and only if ϕ_H (preserves C and) flows along the characteristic foliation of C . By flowing along characteristics we mean that for any point $p \in C$ and any time $t \geq 0$, $\phi_H^t(p) \in \mathcal{F}(p)$, where $\mathcal{F}(p)$ stands for the characteristic leaf through p .
2. For each $p \in C$, $H|_{\mathcal{F}(p)}$ is a function of time if and only if the flow ϕ_H preserves C .

We will show that the above two properties hold for continuous Hamiltonians. The C^0 -analog of the first property, stated below, plays an important role in the proof of Theorem 1.

Theorem 3. *Denote by C a connected coisotropic submanifold of a symplectic manifold (M, ω) which is closed as a subset¹ of M . Let $H \in C_{\text{Ham}}^0$ with induced hamotopy ϕ_H . The restriction of H to C is a function of time if and only if ϕ_H preserves C and flows along the leaves of its characteristic foliation.*

This result answers a question raised by Buhovsky and Opshtein who asked if the above holds in the particular case where C is a smooth hypersurface. It also drastically generalizes the aforementioned uniqueness of generators Theorem. Indeed, if C is taken to be M , then the characteristic foliation consists of the points of M and the theorem follows immediately:

Corollary 4. *$H \in C_{\text{Ham}}^0$ is a function of time if and only if $\phi_H^t = \text{Id}$.*

After the first draft of this article was written, we were asked by Opshtein if the second of the aforementioned properties holds for C^0 Hamiltonians. Our next result provides an affirmative answer to Opshtein's question.

Theorem 5. *Denote by C a connected coisotropic submanifold of a symplectic manifold (M, ω) which is closed as a subset of M . Let $H \in C_{\text{Ham}}^0$ with induced hamotopy ϕ_H . The restriction of H to each leaf of the characteristic foliation of C is a function of time if and only if the flow ϕ_H preserves C .*

When C is a Lagrangian, Theorems 3 and 5 coincide and both state that: *The restriction of H to L is a function of time if and only if $\phi_H^t(L) = L$ for all t .* In an interesting manifestation of Weinstein's creed, "Everything is a Lagrangian submanifold!", the general case of Theorems 1, 3 and 5 will be essentially deduced from the a priori particular case of Lagrangians.

The results of this paper establish C^0 -rigidity of coisotropic submanifolds together with their characteristic foliations. It would be interesting to see if isotropic or symplectic submanifolds exhibit similar rigidity properties: If a smooth submanifold is the image of an isotropic (respectively symplectic) submanifold under a symplectic homeomorphism, is it isotropic (respectively symplectic)? Note that if in these questions one considers, instead of symplectic homeomorphisms, C^0 -limits of isotropic (respectively

¹ It is our convention that submanifolds have no boundary. Note that a submanifold is closed as a subset if and only if it is properly embedded.

symplectic) embeddings then Gromov's results on the h -principle provide negative answers in general. In short, isotropic and symplectic embeddings are not C^0 -rigid. (See [9, Section 3.4.2], or [5, Theorems 12.1.1 and 12.4.1].)

Defining C^0 -coisotropic submanifolds. As we will see in Section 6, an interesting feature of Theorem 1 is that it allows us to define C^0 -coisotropic submanifolds along with their C^0 -characteristic foliations. Roughly speaking, a C^0 -coisotropic will be defined to be a C^0 -submanifold of a symplectic manifold which is locally symplectic homeomorphic to a smooth coisotropic. The well-definedness of this notion is a consequence of Theorem 1. Furthermore, from the same theorem we conclude that a C^0 -coisotropic submanifold admits a unique C^0 -foliation which will be referred to as its C^0 -characteristic foliation.

As a consequence, we obtain a definition for C^0 -Lagrangian submanifolds as C^0 -coisotropic submanifolds of dimension n . Graphs of symplectic homeomorphisms and graphs of C^0 1-forms, closed in the sense of distributions, constitute examples of C^0 -Lagrangians. For further details, we refer the interested reader to Section 6.

Main tools: Lagrangian spectral invariants. In order to prove the main results, we use the theory of Lagrangian spectral invariants. One consequence of this theory is the existence of the spectral distance γ on the space of Lagrangians Hamiltonian isotopic to the 0-section in cotangent bundles introduced by Viterbo in [25].²

More precisely, we establish inequalities comparing γ to a capacity recently defined by Lisi–Rieser [15]. This capacity, which we denote by c_{LR} , is a relative (to a fixed Lagrangian) version of the Hofer–Zehnder capacity. We will now define c_{LR} . Fix a Lagrangian L . Recall that a Hamiltonian chord of a Hamiltonian H , of length T , is a path $x: [0, T] \rightarrow M$ such that $x(0), x(T) \in L$ and for all $t \in [0, T]$, $\dot{x}(t) = X_H^t(x(t))$. A Hamiltonian is said to be L -slow if all of its Hamiltonian chords of length at most 1 are constant. We denote by $\mathcal{H}(U)$ the set of admissible Hamiltonians, that is, smooth time-independent functions with compact support included in U , which are non-negative and reach their maximum at a point of L . For an open set U which intersects L , the relative capacity of U with respect to L is defined as

$$c_{\text{LR}}(U; L) = \sup\{\max f \mid f \in \mathcal{H}(U) \text{ } L\text{-slow}\}.$$

For instance, if B is the open ball of radius r in $\mathbb{R}^{2n}$ and $\mathcal{L}_0 = \mathbb{R}^n \times \{0\}$, then $c_{\text{LR}}(B; \mathcal{L}_0) = \frac{\pi r^2}{2}$; see [15].

In what follows, we denote by L_0 the 0-section of T^*L . The first energy-capacity inequality used in this paper is the following:

Lemma 6. *Let L be a smooth closed manifold and U_- and U_+ be open subsets of T^*L , so that $U_{\pm} \cap L_0 \neq \emptyset$. If a compactly supported Hamiltonian*

²One of the main features of γ is that it is bounded from above by Hofer's distance on Lagrangians. In particular, Lemmas 6 and 7 also hold with γ replaced by Hofer's distance.

H satisfies $H|_{U_{\pm}} = \pm C_{\pm}$ with $C_{\pm} \in \mathbb{R}$ so that $C_{\pm} > c_{\text{LR}}(U_{\pm}; L_0)$, then $\gamma(\phi_H^1(L_0), L_0) \geq \min\{c_{\text{LR}}(U_-; L_0), c_{\text{LR}}(U_+; L_0)\}$.

This is the Lagrangian analog of the energy-capacity inequality proven for the Hamiltonian spectral distance in [10, Corollary 12]. Then, as in [10], we will derive a similar inequality for Hamiltonians (not necessarily constant but) with controlled oscillations on $U_{\pm}$, see Corollary 14.

Lemma 6 can also be established on compact manifolds for weakly exact Lagrangians via Leclercq [12] and for monotone Lagrangians via Leclercq–Zapolsky [13].

The second energy-capacity inequality is due to Lisi–Rieser [15]. This is a relative version of the standard energy-capacity inequality, see for example Viterbo [25].

Lemma 7. *Let L be a smooth closed manifold. Suppose that U is an open subset of T^*L , with $L_0 \cap U \neq \emptyset$. Assume that L' is a Lagrangian Hamiltonian isotopic to L_0 such that $L' \cap U = \emptyset$. Then $\gamma(L', L_0) \geq c_{\text{LR}}(U; L_0)$.*

A special case of this specific inequality appears in Barraud–Cornea [1] and Charette [4]. A similar inequality is worked out in Borman–McLean [2].

Finally, we will need an inequality which provides an upper bound for the spectral distance. Let g denote a Riemannian metric on a closed manifold L and denote by $T_r^*L = \{(q, p) \in T^*L \mid \|p\|_g \leq r\}$ the cotangent ball bundle of radius r . Suppose that $\phi_H^t(L_0) \subset T_r^*L$ for all $t \in [0, 1]$. Viterbo has conjectured [27] that there exists a constant $C > 0$, depending on g , such that $\gamma(\phi_H^1(L_0), L_0) \leq Cr$. This conjecture has many important ramifications; see [18, 27]. Lemma 8 below is a special case of Viterbo’s conjecture; a more precise version of the lemma appears in [21, Theorem 9.7].

Lemma 8. *Let L be a smooth closed manifold, $\mathcal{V}$ a proper open subset of L , and $V = \pi^{-1}(\mathcal{V}) \subset T^*L$, where $\pi: T^*L \rightarrow L$ is the standard projection. There exists $C > 0$, depending on the set $\mathcal{V}$, such that: For all $r > 0$, if H is a smooth, compactly supported Hamiltonian on T^*L such that $H|_V = 0$, and $\phi_H^t(L_0) \subset T_r^*L$ for all $t \in [0, 1]$ then $\gamma(\phi_H^1(L_0), L_0) \leq Cr$.*

Organization of the paper. In Section 2, we review the preliminaries on C^0 -Hamiltonian dynamics and Lagrangian spectral invariants. In Section 3, we prove energy-capacity inequalities (Lemmas 6 and 7) as well as the upper bound on the spectral distance (Lemma 8). In Section 4, we use these inequalities in order to prove localized versions of Theorem 3 in the special case of Lagrangians. In Section 5, we prove Theorems 1, 3, and 5 using the results of Section 4. In Section 6, we define C^0 -coisotropic submanifolds and their characteristic foliations. In the same section, we provide examples of such C^0 -objects.

In Appendix A, we provide relatively simple, and hopefully enlightening, proofs of Theorems 1 and 3 in the special case of closed Lagrangians in

cotangent bundles. We hope that this appendix will give the reader an idea of the proofs of the main results while avoiding the technicalities of Sections 4 and 5.

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2. PRELIMINARIES

2.1. Symplectic and Hamiltonian homeomorphisms. In this section we give precise definitions for symplectic and Hamiltonian homeomorphisms and recall a few basic properties of the theory of continuous Hamiltonian dynamics developed by Müller and Oh [22].

Given two manifolds M_1, M_2 , a compact subset $K \subset M_1$, a Riemannian distance d on M_2 , and two maps $f, g: M_1 \rightarrow M_2$, we denote

$$d_K(f, g) = \sup_{x \in K} d(f(x), g(x)).$$

We say that a sequence of maps $f_i: M_1 \rightarrow M_2$ C^0 -converges to some map $f: M_1 \rightarrow M_2$, if for every compact subset $K \subset M_1$, the sequence $d_K(f_i, f)$ converges to 0. This notion does not depend on the choice of the Riemannian metric.

Definition 9. *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A continuous map $\theta: U \rightarrow M_2$, where $U \subset M_1$ is open, is called symplectic if it is the C^0 -limit of a sequence of symplectic diffeomorphisms $\theta_i: U \rightarrow \theta_i(U)$.*

Let $U_1 \subset M_1$ and $U_2 \subset M_2$ be open subsets. If a homeomorphism $\theta: U_1 \rightarrow U_2$ and its inverse θ^{-1} are both symplectic maps, we call θ a symplectic homeomorphism.

Clearly, if θ is a symplectic homeomorphism, so is θ^{-1} . By the Gromov–Eliashberg Theorem a symplectic homeomorphism which is smooth is a symplectic diffeomorphism.

REMARK 10. More generally, one can define a symplectic homeomorphism to be a homeomorphism which is locally a C^0 -limit of symplectic diffeomorphisms. For simplicity, we do not use this more general definition, however

it is evident from the proof of Theorem 1 that it does hold for such homeomorphisms as well.

We now turn to the definition of Hamiltonian homeomorphisms (called hameomorphisms) introduced by Müller and Oh [22].

Definition 11. *Let (M, ω) be a symplectic manifold and $I \subset \mathbb{R}$ an interval. An isotopy $(\phi^t)_{t \in I}$ is called a hameotopy if there exist a compact subset $K \subset M$ and a sequence of smooth Hamiltonians H_i supported in K such that:*

- (1) *The sequence of flows $\phi_{H_i}^t$ C^0 -converges to ϕ^t uniformly in t on every compact subset of I ,*
- (2) *the sequence $H_i(t, \cdot)$ C^0 -converges to a continuous function $H(t, \cdot)$ uniformly in t on every compact subset of I .*

We say that H generates ϕ^t , denote $\phi^t = \phi_H^t$, and call H a continuous Hamiltonian. We denote by $C_{\text{Ham}}^0(M, \omega)$ (or just C_{Ham}^0) the set of all continuous functions $H: [0, 1] \times M \rightarrow \mathbb{R}$ which generate hameotopies parametrized by $[0, 1]$. A homeomorphism is called a hameomorphism if it is the time-1 map of some hameotopy parametrized by $[0, 1]$.

A continuous function $H \in C_{\text{Ham}}^0$ generates a unique hameotopy [22]. Conversely, Viterbo [26] and Buhovsky–Seyfaddini [3] proved that a hameotopy has a unique (up to addition of a function of time) continuous generator.

One can easily check that generators of hameotopies satisfy the same composition formulas as their smooth counterparts. Namely, if ϕ_H^t is a hameotopy, then $(\phi_H^t)^{-1}$ is a hameotopy generated by $-H(t, \phi_H^t(x))$; given another hameotopy ϕ_K^t , the isotopy $\phi_H^t \phi_K^t$ is also a hameotopy, generated by $H(t, x) + K(t, (\phi_H^t)^{-1}(x))$.

Moreover, we will repeatedly use the following simple fact: If $H \in C_{\text{Ham}}^0(V)$ for some open set V in a symplectic manifold (M, ω) and if $\theta: U \rightarrow V$ is a symplectic homeomorphism, then $H \circ \theta$ belongs to $C_{\text{Ham}}^0(U)$ and generates the hameotopy $\theta^{-1} \phi_H^t \theta$. This, in particular, holds for smooth $H: [0, 1] \times M \rightarrow \mathbb{R}$ supported in V .

2.2. Lagrangian spectral invariants. In [25], Viterbo defined Lagrangian spectral invariants on $\mathbb{R}^{2n}$ and cotangent bundles via generating functions. Then Oh [19] defined similar invariants via Lagrangian Floer homology in cotangent bundles which have been proven to coincide with Viterbo's invariants by Milinković [16]. They have been adapted to the compact case by Leclercq [12] for weakly exact Lagrangians and Leclercq–Zapolsky [13] for monotone Lagrangians. However, for the type of problems which we consider here (C^0 -convergence of Lagrangians), we can restrict ourselves to Weinstein neighborhoods and thus work only in cotangent bundles. We briefly outline below the construction of these invariants via Lagrangian Floer homology and collect their main properties in this situation. We refer to Monzner–Vichery–Zapolsky [18] which gives a very nice exposition of the theory.

Let L be a smooth compact manifold, L_0 denote the 0-section in T^*L , and λ the Liouville 1-form. To a compactly supported smooth time-dependent

Hamiltonian $H \in C_c^\infty([0, 1] \times T^*L)$ is associated the action functional

$$\mathcal{A}_H: \Omega(T^*L) \rightarrow \mathbb{R}, \quad \gamma \mapsto \int_0^1 H_t(\gamma(t)) dt - \int \gamma^* \lambda$$

where $\Omega(T^*L) = \{\gamma: [0, 1] \rightarrow T^*L \mid \gamma(0) \in L_0, \gamma(1) \in L_0\}$. The critical points of $\mathcal{A}_H$ are the chords of the Hamiltonian vector field X_H which start and end on L_0 . The spectrum of $\mathcal{A}_H$, denoted by $\text{Spec}(\mathcal{A}_H)$, consists of the critical values of $\mathcal{A}_H$. It is a nowhere dense subset of $\mathbb{R}$ which only depends on the time-1 map ϕ_H^1 , hence we sometimes denote it by $\text{Spec}(\phi_H^1)$.

Following Floer's construction, for a generic choice of Hamiltonian function, $\text{crit}(\mathcal{A}_H)$ is finite and one can form a chain complex $(CF_*(H), \partial_{H,J})$ whose generators are the critical chords and whose differential counts the elements of the 0-dimensional component of moduli spaces of Floer trajectories (i.e pseudo-holomorphic curves perturbed by H) which run between the critical chords (with boundary conditions on L_0). The differential relies on the additional data of a generic enough pseudo-complex structure, J .

This complex is filtered by the values of the action, that is, for $a \in \mathbb{R}$ a regular value of $\mathcal{A}_H$, one can consider only chords of action less than a . Such chords generate a subcomplex of the total complex $CF_*^a(H)$ (because the action decreases along Floer trajectories). We denote by i_*^a the inclusion $CF_*^a(H) \rightarrow CF_*(H)$. By considering homotopies between pairs (H, J) and (H', J') , one can canonically identify the homology induced by the respective Floer complexes $H_*(CF(H), \partial_{H,J})$ and $H_*(CF(H'), \partial_{H',J'})$ and by considering C^2 -small enough Hamiltonian functions, one can see that the resulting object $HF_*(L_0)$ is canonically isomorphic to the singular homology of L .

Thus, one can consider spectral invariants associated to any non-zero homology class α of L , defined as the smallest action level which detects α :

$$\ell(\alpha; H) = \inf\{a \in \mathbb{R} \mid \alpha \in \text{im}(H_*(i^a))\}$$

In what follows we will only be interested in the spectral invariants associated to the class of a point and the fundamental class which will be respectively denoted by $\ell_-(H) = \ell([\text{pt}]; H)$ and $\ell_+(H) = \ell([L]; H)$.

These invariants were proven to be continuous with respect to the C^0 -norm on Hamiltonian functions so that they are defined for any (not necessarily generic) Hamiltonian. Moreover, they only depend on the time-1 map ϕ_H^1 induced by the flow of H ; hence they are well-defined on $\text{Ham}^c(T^*L, d\lambda)$.

Their main properties are collected in the following theorem, which corresponds to [18, Theorem 2.20], except for (7) which we prove below. Note that, except for (6) and (7), these properties already appear in Viterbo [25].

Theorem 12. *Let L be a smooth closed connected manifold. Let L_0 denote the 0-section of T^*L . There exist two maps $\ell_\pm: \text{Ham}^c(T^*L, d\lambda) \rightarrow \mathbb{R}$ with the following properties:*

- (1) *For any $\phi \in \text{Ham}^c(T^*L, d\lambda)$, $\ell_\pm(\phi)$ lie in $\text{Spec}(\phi)$.*
- (2) *$\ell_- \leq \ell_+$.*

- (3) For any two Hamiltonian functions H and K ,
 $\int_0^1 \min(H_t - K_t) dt \leq \ell_-(\phi_H^1) - \ell_-(\phi_K^1) \leq \int_0^1 \max(H_t - K_t) dt$, and
 $\int_0^1 \min(H_t - K_t) dt \leq \ell_+(\phi_H^1) - \ell_+(\phi_K^1) \leq \int_0^1 \max(H_t - K_t) dt$.
- (4) For any ϕ and $\phi' \in \text{Ham}^c(T^*L, d\lambda)$, $\ell_+(\phi\phi') \leq \ell_+(\phi) + \ell_+(\phi')$.
- (5) For any $\phi \in \text{Ham}^c(T^*L, d\lambda)$, $\ell_\pm(\phi) = -\ell_\mp(\phi^{-1})$.
- (6) If $H|_{L_0} \leq c$ (respectively $H|_{L_0} \geq c$ or $H|_{L_0} = c$), then $\ell_\pm(\phi_H^1) \leq c$ (respectively $\ell_\pm(\phi_H^1) \geq c$ or $\ell_\pm(\phi_H^1) = c$).
- (7) If f is a L_0 -slow admissible Hamiltonian, then $\ell_+(\phi_f^1) = \max(f|_{L_0})$ and $\ell_-(\phi_f^1) = 0$.
- (8) For any ϕ and $\phi' \in \text{Ham}^c(T^*L, d\lambda)$ such that $\phi(L_0) = \phi'(L_0)$, $\ell_+(\phi) - \ell_-(\phi) = \ell_+(\phi') - \ell_-(\phi')$.

Proof of item (7). Since f is L_0 -slow, $\text{Spec}(\phi_f^1)$ consists of critical values of f corresponding to critical points lying in L_0 . Now for all $s \in [0, 1]$, since f is autonomous, $\text{Spec}(\phi_f^s) = \text{Spec}(\phi_{sf}^1) = s \cdot \text{Spec}(\phi_f^1)$. Since in cotangent bundles spectral invariants lie in the spectrum regardless of degeneracy of f , by continuity of $\ell_\pm$ there exist $p_\pm \in \text{crit}(f) \cap L_0$ such that $\ell_\pm(\phi_f^s) = s \cdot f(p_\pm)$. Now we claim that for small times s , $\ell_+(\phi_{sf}^1)$ (respectively $\ell_-(\phi_{sf}^1)$) is the maximum (respectively the minimum) of sf so that $f(p_+) = \max(f)$ (respectively $f(p_-) = \min(f)$) which concludes the proof.

It remains to prove the claim. Since f is autonomous, the path of Lagrangians it generates is a geodesic with respect to Hofer's distance for small times, see Milinković [17, Theorem 8]. That is, for $s \in (0, 1)$ small enough

$$(1) \quad \max(sf) - \min(sf) = \text{length}(\{L_t\}_{t \in [0, s]}) = d_{\text{Hof}}(L_0, L_s)$$

with $L_t = \phi_f^t(L_0)$. (As f reaches its extrema on L_0 , the extrema of sf over T^*L and L_0 coincide and we remove them from the notation.) On the other hand, by choosing s small enough we can ensure that $L_s = \Gamma_{dS}$, the graph of the differential of some smooth function $S: L \rightarrow \mathbb{R}$. Now in general $d_{\text{Hof}}(L_0, \Gamma_{dS}) \leq \text{osc}(S)$ and by [17, Proposition 1.(6)]: $\text{osc}(S) = \ell_+(\phi_{\tilde{S}}^1) - \ell_-(\phi_{\tilde{S}}^1)$ where $\tilde{S}$ lifts S to T^*L by pullback via the natural projection and cutoff far from the Lagrangian isotopy. Since $\phi_{\tilde{S}}^1(L_0) = \Gamma_{dS}$, by Property (8), we can replace $\phi_{\tilde{S}}^1$ by ϕ_{sf}^1 thus (1) leads to the inequality $\max(sf) - \min(sf) \leq \ell_+(\phi_{sf}^1) - \ell_-(\phi_{sf}^1)$. Finally, since $\min(sf) \leq \ell_-(\phi_{sf}^1) \leq \ell_+(\phi_{sf}^1) \leq \max(sf)$, we deduce that $\ell_-(\phi_{sf}^1) = \min(sf)$ and $\ell_+(\phi_{sf}^1) = \max(sf)$. $\square$

In view of Property (8), Viterbo (followed by Oh) derived an invariant γ of Lagrangians Hamiltonian isotopic to the 0-section, defined as follows.

Definition 13. For a Hamiltonian diffeomorphism $\phi \in \text{Ham}^c(T^*L, d\lambda)$ we set $\gamma(\phi) = \ell_+(\phi) - \ell_-(\phi)$ and for a Lagrangian L Hamiltonian isotopic to the 0-section, we set $\gamma(L, L_0) = \gamma(\phi)$ for any $\phi \in \text{Ham}^c(T^*L, d\lambda)$ such that $\phi(L_0) = L$.

From the properties of spectral invariants, it is immediate that for all ϕ and $\psi \in \text{Ham}^c(T^*L, d\lambda)$, $0 \leq \gamma(\phi\psi) \leq \gamma(\phi) + \gamma(\psi)$, and that $\gamma(\phi) = \gamma(\phi^{-1})$. Moreover, $\gamma(L, L_0) = 0$ implies $L = L_0$ as proven in [25].

One of the main properties of γ which will be used in what follows is the fact that

$$(2) \quad \gamma(\phi_H^1(L_0), L_0) = \gamma(\phi_H^1) \leq \max_{t \in [0,1]} (\text{osc}(H_t|_{L_0}))$$

where $\text{osc}(H_t|_{L_0}) = \max_{L_0}(H_t) - \min_{L_0}(H_t)$. This inequality can be directly derived from Property (6) of Theorem 12. Note that this yields

$$\forall t_0 \in \mathbb{R}, \quad \gamma(\phi_H^{t_0}(L_0), L_0) \leq t_0 \cdot \max_{t \in [0, t_0]} (\text{osc}(H_t|_{L_0})).$$

3. ENERGY-CAPACITY INEQUALITIES

3.1. The energy-capacity inequality for Hamiltonians constant on open sets. We prove Lemma 6 by mimicking the proof of the corresponding inequality in [10] (here for Lagrangians, in the easier world of aspherical objects). Then we prove a corollary which will be used in the proof of the main result.

Proof of Lemma 6. First, assume that $\ell_-(\phi_H^1) \leq 0$. Then for any admissible L_0 -slow function with support in U_+ , $f \in \mathcal{H}(U_+)$, we define the 1-parameter family of Hamiltonians $H_s(t, x) = H(t, x) - sf(x)$ with $s \in [0, 1]$. Since H is constant on U_+ , H_s generates $\phi_{H_s}^1 = \phi_H^1 \phi_f^{-s}$. By triangle inequality and duality (i.e Properties (4) and (5)) of spectral invariants, we get

$$(3) \quad \ell_+(\phi_f^s) \leq \ell_+(\phi_{H_s}^{-1}) + \ell_+(\phi_H^1) = \ell_+(\phi_H^1) - \ell_-(\phi_{H_s}^1).$$

Then notice that

$$\text{Spec}(\mathcal{A}_{H_s}) = \text{Spec}(\mathcal{A}_H) \cup \{C_+ - sf(p) \mid p \in \text{crit}(f) \cap U_+\}.$$

Since for all $p \in \text{crit}(f) \cap U_+$, $sf(p) \leq \max(f) \leq c_{\text{LR}}(U_+; L_0) < C_+$, there exists $\varepsilon > 0$, such that $\text{Spec}_\varepsilon(\mathcal{A}_{H_s})$, defined as $\text{Spec}(\mathcal{A}_{H_s}) \cap (-\infty, \varepsilon)$, does not depend on s and coincides with $\text{Spec}_\varepsilon(\mathcal{A}_H)$ which is totally discontinuous. Since the map $s \mapsto \ell_-(\phi_{H_s}^1)$ is continuous and maps 0 to $\text{Spec}_\varepsilon(\mathcal{A}_H)$, it has to be constant so that $\ell_-(\phi_{H_1}^1) = \ell_-(\phi_H^1)$ and, from (3), Property (7) of spectral invariants immediately leads to

$$\max(f) = \ell_+(\phi_f^1) \leq \ell_+(\phi_H^1) - \ell_-(\phi_H^1) = \gamma(\phi_H^1(L_0), L_0).$$

Since this holds for any L_0 -slow function in $\mathcal{H}(U_+)$, we get $\gamma(\phi_H^1(L_0), L_0) \geq c_{\text{LR}}(U_+; L_0)$.

Now, assume that $\ell_-(\phi_H^1) \geq 0$ and consider $\bar{H}(t, x) = -H(t, \phi_H^t(x))$. By assumption, ϕ_H^t is the identity on U_- and $\bar{H}|_{U_-} = -H|_{U_-} = C_-$. Since $\bar{H}$ generates $\phi_{\bar{H}}^1 = (\phi_H^1)^{-1}$,

$$\ell_-(\phi_{\bar{H}}^1) = -\ell_+(\phi_H^1) \leq -\ell_-(\phi_H^1) \leq 0.$$

Then the first case gives that $\gamma(\phi_H^1) \geq c_{\text{LR}}(U_-; L_0)$. Since $\gamma(\phi_H^1) = \gamma(\phi_H^1) = \gamma(\phi_H^1(L_0), L_0)$, we get that when $\ell_-(\phi_H^1) \geq 0$, $\gamma(\phi_H^1) \geq c_{\text{LR}}(U_-; L_0)$.

So $\gamma(\phi_H^1) \geq \min\{c_{\text{LR}}(U_-; L_0), c_{\text{LR}}(U_+; L_0)\}$ regardless the sign of $\ell_-(\phi_H^1)$ which concludes the proof. $\square$

From this (and as in [10]) we infer the same result but for Hamiltonian functions which are allowed to have controlled oscillations on $U_\pm$.

Corollary 14. *Let L be a smooth closed manifold and $U_\pm$ be open subsets of T^*L such that $U_\pm \cap L_0 \neq \emptyset$. Let H be a Hamiltonian so that for all $t \in [0, 1]$*

- (1) $H_t|_{U_+} > c_{\text{LR}}(U_+; L_0)$, and $H_t|_{U_-} < -c_{\text{LR}}(U_-; L_0)$,
- (2) $\text{osc}(H_t|_{U_\pm}) < \varepsilon$, for some $\varepsilon > 0$.

Then, $\gamma(\phi_H^1(L_0), L_0) \geq \min\{c_{\text{LR}}(U_-; L_0), c_{\text{LR}}(U_+; L_0)\} - 2\varepsilon$.

Proof. Fix $\eta > 0$. We choose disjoint open subsets $V_\pm$ such that $U_\pm \Subset V_\pm$ (with $\Subset$ denoting compact containment) and $\text{osc}_{V_\pm}(H) < \text{osc}_{U_\pm}(H) + \eta$. We also choose cut-off functions $\rho_\pm$ with support in $V_\pm$, such that $0 \leq \rho_\pm \leq 1$ and $\rho_\pm|_{U_\pm} = 1$. Then we define

$$h = H - \rho_+(H - C_+) - \rho_-(H + C_-)$$

with $C_+ = \inf(H|_{[0,1] \times U_+})$ and $C_- = -\sup(H|_{[0,1] \times U_-})$. By triangle inequality, we get $\gamma(\phi_H^1) \geq \gamma(\phi_h^1) - \gamma((\phi_H^1)^{-1}\phi_h^1)$ and we now bound the right-hand side terms.

First, notice that h satisfies the requirements of Lemma 6: $h|_{U_\pm} = \pm C_\pm$ with by assumption $C_+ > c_{\text{LR}}(U_+; L_0)$, and $C_- > c_{\text{LR}}(U_-; L_0)$. Thus we immediately get that $\gamma(\phi_h^1) \geq \min\{c_{\text{LR}}(U_-; L_0), c_{\text{LR}}(U_+; L_0)\}$. Now by Property (3) of spectral invariants,

$$\gamma((\phi_H^1)^{-1}\phi_h^1) \leq \text{osc}_{T^*L}((H - h) \circ (\phi_H)^{-1}) \leq \text{osc}_{V_- \cup V_+}(H - h)$$

so that

$$\begin{aligned} \gamma((\phi_H^1)^{-1}\phi_h^1) &\leq \text{osc}_{V_+}(\rho_+(H - C_+)) + \text{osc}_{V_-}(\rho_-(H + C_-)) \\ &\leq \text{osc}_{V_+}(H) + \text{osc}_{V_-}(H) \leq \text{osc}_{U_+}(H) + \text{osc}_{U_-}(H) + 2\eta \\ &\leq 2\varepsilon + 2\eta. \end{aligned}$$

Finally, $\gamma(\phi_H^1(L_0), L_0) \geq \min\{c_{\text{LR}}(U_-; L_0), c_{\text{LR}}(U_+; L_0)\} - 2\varepsilon - 2\eta$ for any $\eta > 0$. $\square$

3.2. The energy-capacity inequality for Lagrangians displaced from an open set. We give a proof of Lemma 7 from Lisi–Rieser [15], for the reader's convenience. The method of proof is now classical and goes back to Viterbo [25] (see also Usher's proof of the analogous result [24] for compact manifolds, itself heavily influenced by Frauenfelder–Ginzburg–Schlenk [6]). Recall that in cotangent bundles the spectral invariants only depend on the endpoints of Hamiltonian isotopies; this drastically simplifies the proof.

³In the next few lines we loosely denote $\max_{t \in [0,1]} \text{osc}(f_t|_U)$ by $\text{osc}_U(f)$ for readability.

Proof of Lemma 7. Assume that $\phi_H^1(L_0) \cap U = \emptyset$. Choose a L_0 -slow function f in $\mathcal{H}(U)$. For $s \in [0, 1]$, consider the Hamiltonian diffeomorphism $\phi_{sf}^1 \phi_H^1$. It is the end of the isotopy defined as the concatenation

$$\theta^t = \begin{cases} \phi_H^{2t}, & t \in [0, 1/2], \\ \phi_{sf}^{2t-1} \phi_H^1, & t \in [1/2, 1]. \end{cases}$$

A Hamiltonian chord of θ is a path $t \mapsto \gamma(t)$ such that $\gamma(0) \in L_0$, $\gamma(1) \in L_0$ and for all t , $\gamma(t) = \theta^t(\gamma(0))$. In particular, for such a chord, $\phi_{sf}^1 \phi_H^1(\gamma(0)) \in L_0$. However, since by assumption $\phi_H^1(L_0) \cap \text{supp}(sf) = \emptyset$, necessarily $\phi_H^1(\gamma(0))$ is not in the support of f and $\gamma(t) = \phi_H^{2t}(\gamma(0))$ for all $t \leq 1/2$ and remains constant $\gamma(t) = \phi_H^1(\gamma(0))$ for $t \geq 1/2$.

This means that for all s , the set of Hamiltonian chords remains constant and so does $\text{Spec}(\phi_{sf}^1 \phi_H^1)$. Since this set is nowhere dense and ℓ_+ is continuous (and takes its values in the action spectrum), the function $s \mapsto \ell_+(\phi_{sf}^1 \phi_H^1)$ is constant so that $\ell_+(\phi_H^1) = \ell_+(\phi_f^1 \phi_H^1)$. Thus,

$$\ell_+(\phi_f^1) \leq \ell_+(\phi_f^1 \phi_H^1) + \ell_+((\phi_H^1)^{-1}) = \ell_+(\phi_H^1) - \ell_-(\phi_H^1) = \gamma(\phi_H^1)$$

by Properties (4) and (5) of spectral invariants. Since this holds for any L_0 -slow function $f \in \mathcal{H}(U)$ and since for such a function $\max(f) = \ell_+(\phi_f^1)$ by Property (7) of spectral invariants, the result follows. $\square$

REMARK 15. Lemma 7 can be reformulated as follows:

Let L, L_0 , and U be as in Lemma 7. Suppose that H is a smooth Hamiltonian such that $L_0 \cap \phi_H^1(U) = \emptyset$. Then, $\gamma(\phi_H^1(L_0), L_0) \geq c_{\text{LR}}(U; L_0)$.

The equivalence of the above statement to Lemma 7 follows easily from the fact that $\gamma(\phi_H^1(L_0), L_0) = \gamma((\phi_H^1)^{-1}(L_0), L_0)$.

REMARK 16. Let L, L_0 , and U be as in Lemma 7. Suppose that H is a smooth Hamiltonian such that either $\phi_H^1(L_0) \cap U = \emptyset$ or $L_0 \cap \phi_H^1(U) = \emptyset$. Then, it follows from Lemma 7, the previous remark, and Inequality (2) that $\max_{t \in [0, 1]}(\text{osc}(H_t|_{L_0})) \geq c_{\text{LR}}(U; L_0)$.

3.3. An upper bound for the spectral distance. In this section we prove Lemma 8 which establishes Viterbo's conjecture in a special case. The fact that Viterbo's conjecture holds under the additional assumptions of Lemma 8 seems to be well known to experts; we provide a proof here for the sake of completeness.

Proof of Lemma 8. Note that since $\phi_H^t(L_0) \subset T_r^*L$ for all $t \in [0, 1]$, modifying H outside of T_r^*L leaves $\phi_H^1(L_0)$, and hence $\gamma(\phi_H^1(L_0), L_0)$, unchanged. Therefore, by cutting H off outside of T_r^*L and replacing r with $2r$ we may assume that H is supported inside $T_r^*L \setminus V$.

Pick $f: L \rightarrow \mathbb{R}$ to be a Morse function on L whose critical points are all contained in the open set $\mathcal{V}$. Because f has no critical points inside $L \setminus \mathcal{V}$, we may assume, by rescaling, that

$$(4) \quad \|df|_{L \setminus \mathcal{V}}\|_g \geq 1.$$

Let $\beta: T^*L \rightarrow \mathbb{R}$ denote a non-negative cutoff function such that $\beta = 1$ on T_R^*L , where R is picked so that $R \gg r$. Let $F = \beta\pi^*f: T^*L \rightarrow \mathbb{R}$. By picking R to be sufficiently large, we can ensure that, for $t \in [0, 2r]$ and (q, p) in a neighborhood of T_r^*L , the Hamiltonian flow of F is given by the formula

$$\phi_F^t(q, p) = (q, p + tdf(q)).$$

This, combined with (4), implies that $\phi_F^t(q, 0) \notin T_r^*L$ for $r < t \leq 2r$ and any point $q \in L \setminus \mathcal{V}$. Hence, we see that $\phi_F^t(L_0)$ is outside the support of H for $r < t \leq 2r$. Therefore, $\phi_H^1\phi_F^{2r}(L_0) = \phi_F^{2r}(L_0)$, and so

$$\gamma(\phi_H^1\phi_F^{2r}(L_0), L_0) = \gamma(\phi_F^{2r}(L_0), L_0).$$

Thus we get,

$$\begin{aligned} \gamma(\phi_H^1(L_0), L_0) &= \gamma(\phi_H^1) \leq \gamma(\phi_H^1\phi_F^{2r}) + \gamma((\phi_F^{2r})^{-1}) = \gamma(\phi_F^{2r}) + \gamma(\phi_F^{2r}) \\ &\leq 4r \operatorname{osc}(F) = 4r \operatorname{osc}(f) \end{aligned}$$

and the result follows with $C = 4 \operatorname{osc}(f)$. $\square$

4. LOCALIZED RESULTS FOR LAGRANGIANS

The main goal of this section is to establish a suitable localized version of Theorem 3 for Lagrangians; since we seek localized statements we do not assume that the Lagrangians in question are necessarily closed. Not surprisingly, in this new setting Theorem 3 does not hold as stated. The localized results of this section, which have more complicated statements and proofs, are more powerful and they constitute the main technical steps towards proving Theorems 1 and 3.

We prove the analog of the direct implication of Theorem 3 in Section 4.1. The analog of the converse implication is proven in Section 4.2. Since L is a Lagrangian, its characteristic foliation has a single leaf, L itself, and thus in this section we make no mention of characteristic foliations.

4.1. C^0 -Hamiltonians constant on a Lagrangian preserve it. In this subsection, we show that if the restriction of $H \in C_{\text{Ham}}^0$ to a Lagrangian L is a function of time, then the associated hameotopy ϕ_H preserves L , locally. More precisely,

Proposition 17. *Let $L \subset M$ denote a Lagrangian (not necessarily closed) and $H \in C_{\text{Ham}}^0$ with associated hameotopy ϕ_H . If $H|_L = c(t)$, is a function of time, then for any point $p \in L$ there exists $\varepsilon > 0$ such that $\phi_H^t(p) \in L$ for all $t \in [0, \varepsilon]$.*

Our proof of the above proposition will use the following simple lemma on the local structure of Lagrangians.

Lemma 18. *Let $L \subset M$ denote a Lagrangian. Around each point $p \in L$ there exists a neighborhood $L_p \subset L$ such that L_p is contained in a closed Lagrangian torus $T \subset M$.*

Proof. Let U denote a Darboux ball around p , equipped with the standard coordinates (x_i, y_i) , such that $L \cap U = \{(x_i, y_i) \mid -a < x_i < a, y_i = 0\}$. Let $L_p = \{(x_i, y_i) \mid -\frac{a}{2} \leq x_i \leq \frac{a}{2}, y_i = 0\}$. Then, for all k , the projection of L_p onto the $\langle x_k, y_k \rangle$ plane is the segment $[-\frac{a}{2}, \frac{a}{2}] \times \{0\}$ which can be completed to a smooth embedded loop, say T_k , in the $\langle x_k, y_k \rangle$ plane. Here, we can make sense of the projection onto a coordinate plane by identifying U with a standard ball in $\mathbb{R}^{2n}$. We set $T = T_1 \times \cdots \times T_n$. This is a Lagrangian torus inside U containing L_p . $\square$

Proof of Proposition 17. By replacing H by $H - c(t)$ we can suppose that $H|_L = 0$. Apply Lemma 18 to obtain L_p and T as described in the lemma and note that by replacing L with L_p we may make the following simplifying assumption: *there exists a Lagrangian torus T in M such that $L \subset T$.*

We will now prove the proposition under this simplifying assumption. Let $H_i: [0, 1] \times M \rightarrow \mathbb{R}$ denote a sequence of smooth Hamiltonians such that H_i converges uniformly to H and ϕ_{H_i} converges to ϕ_H in C^0 -topology. Take $W \Subset V$ to be open subsets of M such that

$$(5) \quad p \in W \cap L \quad \text{and} \quad \bar{V} \cap (T \setminus L) = \emptyset.$$

Recall that the symbol $\Subset$ denotes compact containment and $\bar{V}$ denotes the closure of V . The second condition in (5) allows us to pick a cutoff function $\beta: M \rightarrow \mathbb{R}$ such that $\beta|_V = 1$ and $\beta|_{T \setminus L} = 0$. By shrinking V , if needed, we may assume that β is supported in a Weinstein neighborhood of T .

Let $G_i = \beta H_i$ and $G = \beta H$. Observe that G_i converges uniformly to G and $G|_T = 0$. We pick $\varepsilon > 0$ such that

$$(6) \quad \forall t \in [0, \varepsilon], \quad \phi_H^t(W) \subset V.$$

For i large enough $\phi_{H_i}^t(W) \subset V$ for all $t \in [0, \varepsilon]$. Since, $G_i|_V = H_i|_V$ we conclude that, for large i ,

$$(7) \quad \forall (t, x) \in [0, \varepsilon] \times W, \quad \phi_{G_i}^t = \phi_{H_i}^t.$$

For a contradiction, suppose that $\phi_H^{t_0}(p)$ is not contained in L for some $t_0 \in [0, \varepsilon]$. From (5) and (6) we conclude that $\phi_H^{t_0}(p) \notin T$. Hence, we can find a small ball $B \subset W$ around p which intersects T non-trivially and such that $\phi_H^{t_0}(B) \cap T = \emptyset$. Hence, for i large enough, we have $\phi_{H_i}^{t_0}(B) \cap T = \emptyset$. From (7) we get that $\phi_{G_i}^{t_0}(B) \cap T = \emptyset$.

We picked β such that the Hamiltonians G_i all have support in a Weinstein neighborhood of T . Therefore, we can pass to T^*T , apply Lemma 7 and its consequences as stated in Remarks 15 and 16 and conclude that

$$t_0 \cdot \max_{t \in [0, t_0]} (\text{osc}(G_i(t, \cdot)|_{T_0})) \geq c_{\text{LR}}(B; T_0),$$

where T_0 stands for the 0-section in T^*T . Here, we have used the fact that $\phi_{G_i}^{t_0}$ is the time-1 map of the flow of the Hamiltonian $t_0 G_i(t_0 t, x)$. Since G_i converges uniformly to G , the same inequality must hold for G but this contradicts the fact that $G|_T = 0$. $\square$

4.2. C^0 -Hamiltonians preserving a Lagrangian are constant on it.

In this subsection, we show that if $H \in C_{\text{Ham}}^0$ generates a hameotopy ϕ_H which (locally) preserves a Lagrangian L then, the restriction of H to L is (locally) a function of time. More precisely,

Proposition 19. *Let $L \subset M$ denote a Lagrangian, $\mathcal{U}$ an open subset of L , and $H \in C_{\text{Ham}}^0$ with associated hameotopy ϕ_H . Suppose that $\phi_H^t(\mathcal{U}) \subset L$ for all $t \in [0, 1]$ and let $\mathcal{V}$ denote the interior of $\cap_{t \in [0, 1]} \phi_H^t(\mathcal{U})$. Then, $H(t, \cdot)|_{\mathcal{V}}$ is a locally constant function for each $t \in [0, 1]$.*

We were recently informed by Y.-G. Oh that it is possible to extract the above proposition from [20, Theorem 4.9]; the techniques of [20] are different than ours.

Our proof of the proposition uses the following consequence of Corollary 14. This result can be viewed as a Lagrangian analog of the uniqueness of generators Theorem [10, Theorem 2]. The argument presented here is similar to the proof of the mentioned uniqueness theorem.

Proposition 20. *Let L be a smooth closed manifold and $\{H_k\}_k$ a sequence of smooth, uniformly compactly supported Hamiltonian functions on T^*L —that is, there exists a compact K such that $\cup_k \text{supp}(H_k) \subset K$ —, so that*

- (1) *for all $t \in [0, 1]$, $\gamma(\phi_{H_k}^t(L_0), L_0)$ converges to 0, and*
- (2) *$\{H_k\}_k$ uniformly converges to a continuous function H .*

Then, H restricted to L_0 is a function of time.

The proofs of this section repeatedly use the following simple fact: Let H denote a Hamiltonian, C^0 or smooth. The time- t flow of $\tilde{H}(t, x) = aH(t_0 + at, x)$ is given by the expression: $\phi_{\tilde{H}}^t = \phi_H^{t_0+at}(\phi_H^{t_0})^{-1}$.

Proof of Proposition 20. If $H|_{[0, 1] \times L_0}$ is not a function of time, there exist $t_0 \in [0, 1)$ and $x_+, x_- \in L_0$ such that $H_{t_0}(x_+) > H_{t_0}(x_-)$. Up to a shift (and cutoff far from $K \cup L_0$), we can assume that $H_{t_0}(x_+) = -H_{t_0}(x_-) = \Delta > 0$.

Now, let $\varepsilon = \Delta/4$ and notice that there exist $\delta \in (0, 1]$ and $r > 0$ such that $[t_0, t_0 + \delta] \subset [0, 1]$ and that there exist symplectically embedded balls, centered at $x_{\pm}$, $B_{\pm} = \iota_{\pm}(B_{\mathbb{C}^n}(0, r))$, with real part mapped to L_0 , which are disjoint and such that

$$\text{osc}_{[t_0, t_0 + \delta] \times B_{\pm}} H < \varepsilon.$$

By shrinking either δ or r , we can assume that the Lisi-Rieser capacity of the balls with respect to L_0 satisfy

$$c_{\text{LR}}(B_{\pm}; L_0) = \delta(\Delta - \varepsilon).$$

Then set $F_k(t, x) = \delta H_k(t_0 + \delta t, x)$ and define F accordingly. By construction, the function F satisfies

$$\forall (t, x) \in [0, 1] \times B_+, \quad F(t, x) > c_{\text{LR}}(B_{\pm}; L_0),$$

$$\forall (t, x) \in [0, 1] \times B_-, \quad F(t, x) < -c_{\text{LR}}(B_{\pm}; L_0)$$

and

$$\text{osc}_{[0,1] \times B_{\pm}}(F) < \delta\varepsilon.$$

Since the Hamiltonians F_k converge uniformly to F , they satisfy the same inequalities for k large enough. Thus, by Corollary 14,

$$\gamma(\phi_{F_k}^1(L_0), L_0) \geq c_{\text{LR}}(B_{\pm}; L_0) - 2\delta\varepsilon = \delta(\Delta - 3\varepsilon) = \frac{1}{4}\delta\Delta > 0.$$

Hence, $\gamma(\phi_{F_k}^1(L_0), L_0)$ is uniformly bounded away from 0. However, $\phi_{F_k}^1 = \phi_{H_k}^{t_0+\delta}(\phi_{H_k}^{t_0})^{-1}$ so that, by Properties (4) and (5) of spectral invariants,

$$\begin{aligned} \gamma(\phi_{F_k}^1(L_0), L_0) &= \gamma(\phi_{F_k}^1) \leq \gamma(\phi_{H_k}^{t_0+\delta}) + \gamma((\phi_{H_k}^{t_0})^{-1}) = \gamma(\phi_{H_k}^{t_0+\delta}) + \gamma(\phi_{H_k}^{t_0}) \\ &\leq \gamma(\phi_{H_k}^{t_0+\delta}(L_0), L_0) + \gamma(\phi_{H_k}^{t_0}(L_0), L_0) \end{aligned}$$

which goes to 0 when k goes to infinity because of Assumption (1) and we get a contradiction. $\square$

Proof of Proposition 19. Assume, for a contradiction, that the conclusion of the proposition fails to hold. We can therefore find $p \in \mathcal{V}$ and $t_0 \in [0, 1]$ such that $H(t_0, \cdot)$ is not constant on any neighborhood of p in $\mathcal{V}$. First, note that, up to time reparametrization, we may assume that $t_0 = 0$. Indeed, replace H with $\tilde{H}(t, x) = aH(t_0 + at, x)$, where $a = 1 - t_0$, and $\mathcal{U}$ with $\tilde{\mathcal{U}} = \phi_{\tilde{H}}^{t_0}(\mathcal{U})$. Then, $\mathcal{V}$ is contained in the interior of $\cap_{t \in [0, 1]} \phi_{\tilde{H}}^t(\tilde{\mathcal{U}})$ and $\tilde{H}(0, \cdot)$ is not constant on any neighborhood of p .

Apply Lemma 18 to obtain L_p and T as described in the lemma. By shrinking L_p , if needed, we may assume that $L_p \Subset \mathcal{V}$. Let $U \subset M$ denote a small open set around p which is contained in a Weinstein neighborhood of T and such that $L_p \cap U = L \cap U = T \cap U \Subset L_p$. Furthermore, towards the end of this proof we will need to apply Lemma 8, and so we pick U such that the projection of U to T along the cotangent fibers in the Weinstein neighborhood is a proper subset of T whose complement contains a ball.

Since $L_p \cap U \Subset L_p \Subset \mathcal{V}$, it follows that there exists a small $\varepsilon > 0$ such that $\phi_H^t(L_p \cap U), (\phi_H^t)^{-1}(L_p \cap U) \Subset L_p$ for all $t \in [0, \varepsilon]$. Replacing H with $\tilde{H}(t, x) = \varepsilon H(\varepsilon t, x)$, we may assume that

$$\phi_H^t(L_p \cap U), (\phi_H^t)^{-1}(L_p \cap U) \Subset L_p \text{ for all } t \in [0, 1].$$

Next, let B denote an open neighborhood of p which is compactly contained in U . Once again, as in the previous paragraph, by a reparametrization in time, where H is replaced with $\tilde{H}(t, x) = \varepsilon H(\varepsilon t, x)$ for a sufficiently small ε , we may assume that $\phi_H^t(B), (\phi_H^t)^{-1}(B) \Subset U$ for all $t \in [0, 1]$.

Pick $q \in B$ such that $H(0, p) \neq H(0, q)$ and take a symplectomorphism ψ supported in B such that ψ preserves T and $\psi(p) = q$. Consider the continuous Hamiltonian $G = (H \circ \psi - H) \circ \phi_H$. It is supported in $\cup_{t \in [0, 1]} (\phi_H^t)^{-1}(B) \Subset U$, and moreover, the flow of G is $(\phi_H^t)^{-1} \psi^{-1} \phi_H^t \psi$. We will now prove that this flow preserves T globally. Note that the flow is supported in U and pick $x \in T \cap U \subset L_p$. Since $\phi_H^t(T \cap U) \subset T$, and $\psi(x) \in T \cap U$, we see that $\phi_H^t \psi(x) \in T$. First, suppose that $\phi_H^t \psi(x) \notin B$. Then, $\phi_H^t \psi(x)$ is outside

the support of ψ^{-1} and so $(\phi_H^t)^{-1}\psi^{-1}\phi_H^t\psi(x) = \psi(x)$ which is in T . Next, suppose that $\phi_H^t\psi(x) \in B \cap T$. Then, $\psi^{-1}\phi_H^t\psi(x) \in B \cap T$, and so it suffices to check that $(\phi_H^t)^{-1}(B \cap T) \subset T$: this is because $B \cap T \subset U \cap T \subset U \cap L_p$ and $(\phi_H^t)^{-1}(L_p \cap U) \Subset L_p$ for all $t \in [0, 1]$. We have proven that the flow of G preserves T globally.

Note that $G|_{T \cap U}$ is not a function of time only: $G(0, p) = H(0, q) - H(0, p) \neq 0$ and $G = 0$ near the boundary of U . Hence, we have obtained a C^0 -Hamiltonian G , supported in U , whose flow ϕ_G^t preserves T globally, but $G(0, \cdot)$ is not constant on T . Because $G \in C_{\text{Ham}}^0$ there exist smooth Hamiltonians G_i such that $\{G_i\}$ converges uniformly to G and $\{\phi_{G_i}^t\}$ converges to ϕ_G^t . Furthermore, we can ensure that all G_i 's are supported in U . This can be achieved by picking a corresponding sequence of smooth Hamiltonians H_i for H and defining $G_i = (H_i \circ \psi - H_i) \circ \phi_{H_i}$. For large i , G_i is supported in U .

Since U is contained in a Weinstein neighborhood of T , we can pass to T^*T and work with the Lagrangian spectral invariants of the 0-section T_0 associated to the Hamiltonians G_i . Recall that in the second paragraph of the proof we picked the set U so that Lemma 8 could be applied. For any fixed $r > 0$, because ϕ_G^t preserves T_0 , we have $\phi_{G_i}^t(T_0) \subset T_r^*T$ for sufficiently large i . The Hamiltonians G_i are all supported in U and hence using Lemma 8 we conclude that $\gamma(\phi_{G_i}^1(T_0), T_0) \leq Cr$, i.e. $\gamma(\phi_{G_i}^1(T_0), T_0) \rightarrow 0$. Of course, by the same reasoning we obtain that $\gamma(\phi_{G_i}^t(T_0), T_0) \rightarrow 0$ for all $t \in [0, 1]$. Then, Proposition 20 implies that $G|_T = c(t)$, which contradicts the fact that $G|_T$ is not a function of time only. $\square$

5. C^0 -RIGIDITY OF COISOTROPIC SUBMANIFOLDS AND THEIR CHARACTERISTIC FOLIATIONS

This section is devoted to the proofs of Theorems 1, 3 and 5. We begin by proving Theorem 3 and then deduce Theorems 1 and 5 from it.

Before going into the proof, recall (see [14, Proposition 13.7] and [8]) that coisotropic submanifolds admit *coisotropic charts*, that is, for every point $p \in C$, there is a pair (θ, U) where U is an open neighborhood of p and $\theta: U \rightarrow V \subset \mathbb{R}^{2n}$ is a symplectic diffeomorphism which maps p to 0 and C to the standard coisotropic linear subspace

$$\mathcal{C}_0 = \{(x_1, \dots, x_n, y_1, \dots, y_n) \mid (y_{n-k+1}, \dots, y_n) = (0, \dots, 0)\}.$$

Such a diffeomorphism sends the characteristic foliation of C to that of $\mathcal{C}_0$, whose leaf through a point $q = (a_1, \dots, a_n, b_1, \dots, b_{n-k}, 0, \dots, 0) \in \mathcal{C}_0$ is the affine subspace

$$\begin{aligned} \mathcal{F}_0(q) = \{ & (a_1, \dots, a_{n-k}, x_{n-k+1}, \dots, x_n, b_1, \dots, b_{n-k}, 0, \dots, 0) \\ & \mid (x_{n-k+1}, \dots, x_n) \in \mathbb{R}^k \}. \end{aligned}$$

The first step of the proof, is establishing the next lemma which is a version of the first implication of Theorem 3 that does not require the coisotropic submanifold to be a closed subset but holds only for small times.

Lemma 21. *Let (M, ω) be a symplectic manifold and C a coisotropic submanifold of M . Let $H \in C_{\text{Ham}}^0(M, \omega)$ with induced hameotopy ϕ_H . Assume that the restriction of H to C only depends on time. Then, for every $p \in C$, there exists $\varepsilon > 0$ such that for all $t \in [0, \varepsilon]$, $\phi_H^t(p)$ belongs to $\mathcal{F}(p)$, the characteristic leaf of C through p .*

Before going into the details of the proof of Lemma 21, we make the following observation. The lemma holds for coisotropic submanifolds of arbitrary codimension but its proof will follow from the particular case of Lagrangians. As mentioned in the introduction, this is not surprising in view of Weinstein's creed: "Everything is a Lagrangian submanifold!" [28].

Proof. Let $p \in C$ and let (U, θ) be a coisotropic chart as defined above. For $i \in \{1, \dots, n - k\}$ consider the Lagrangian linear subspaces

$$\Lambda_i = \{(x_1, \dots, x_n, y_1, \dots, y_n) \mid x_i = 0 \text{ and } \forall j \neq i, y_j = 0\},$$

and their pull backs $L_i = \theta^{-1}(\Lambda_i)$. Clearly, for all $i \in \{1, \dots, n - k\}$, $L_i \subset C \cap U$ and

$$\mathcal{F}(p) \cap U = \bigcap_{i=1}^{n-k} L_i.$$

Let H be as in the statement of Lemma 21. Then for any i , the restriction of H to L_i is a function of time since L_i is included in C . Thus by Proposition 17 there exists $\varepsilon_i > 0$ such that for all $t \in [0, \varepsilon_i]$, $\phi_H^t(p) \in L_i$. Taking $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_{n-k}\}$, we get

$$\forall t \in [0, \varepsilon], \phi_H^t(p) \in \bigcap_{i=1}^{n-k} L_i \subset \mathcal{F}(p).$$

□

We can now prove Theorem 3.

Proof of Theorem 3. Let $H \in C_{\text{Ham}}^0$ such that $H|_C$ is a function of time only and pick $p \in C$. For a contradiction, assume that for some $t > 0$, $\phi_H^t(p) \notin \mathcal{F}(p)$ and set $t_0 = \inf\{t > 0 \mid \phi_H^t(p) \notin \mathcal{F}(p)\}$. Note that since C is a closed subset, the point $\phi_H^{t_0}(p)$ belongs to C . Then, consider the Hamiltonian $K_t = -H_{-t+t_0}$, so that $\phi_K^t = \phi_H^{-t+t_0}(\phi_H^{t_0})^{-1}$. Its restriction to C is also a function of time. Lemma 21 applied to K at the point $\phi_H^{t_0}(p)$ implies that for some small $t > 0$, $\phi_H^{t_0-t}(p) \in \mathcal{F}(\phi_H^{t_0}(p))$. But by definition of t_0 , we also have $\phi_H^{t_0-t}(p) \in \mathcal{F}(p)$, hence $\phi_H^{t_0}(p) \in \mathcal{F}(p)$. Now apply Lemma 21 again to H_{t+t_0} at the point $\phi_H^{t_0}(p)$. We get that for some $\varepsilon' > 0$ and all $t \in [t_0, t_0 + \varepsilon']$, $\phi_H^t(p) \in \mathcal{F}(p)$ which contradicts the definition of t_0 . Thus, $\phi_H^t(p) \in \mathcal{F}(p)$ and the direct implication of Theorem 3 follows.

We now prove the converse. Assume that the flow of $H \in C_{\text{Ham}}^0$ preserves each leaf of the characteristic foliation. We are going to show first that the function H_0 is locally constant.

Let $p \in C$ and $\theta: U \rightarrow V$ be a coisotropic chart around p , with $\theta(p) = 0$. For $\sigma > 0$ small enough, the set $\cap_{t \in [0, \sigma]} (\phi_H^t)^{-1}(U)$ contains p in its interior. Denote by U' this interior for some fixed σ . Similarly, for $s \in (0, \sigma]$ small enough, $\cap_{t \in [0, s]} \phi_H^t(U')$ contains p in its interior. Let U'' be an open neighborhood of p contained in this interior, and with the property that $\theta(U'')$ is convex. Let q be any other point in U'' and Λ be a linear Lagrangian subspace included in C_0 , containing $\theta(q)$ and the standard leaf $\mathcal{F}_0(0)$. The subspace Λ can be written as the union of the leaves $\mathcal{F}_0(x)$ for all $x \in \Lambda$.

Now, consider the Lagrangian $L = \theta^{-1}(\Lambda \cap V)$. Let $\mathcal{U} = L \cap U'$ and $\mathcal{V} = L \cap U''$. By construction, $q \in \mathcal{V}$. By assumption $\phi_H^t(\mathcal{U}) \subset L$ for all $t \in [0, s]$. We may apply Proposition 19 to L and the continuous Hamiltonian $K_t(x) = sH_{st}(x)$ which generates the hameotopy ϕ_H^{st} . We get that for any $t \in [0, 1]$, K_t is locally constant on $\mathcal{V}$. Equivalently, for any $t \in [0, s]$, H_t is locally constant on $\mathcal{V}$. Now since $\theta(U'')$ is convex and Λ is linear, $\theta(U'') \cap \Lambda$ is connected. It follows that $\mathcal{V}$ is also connected and therefore $H_t(p) = H_t(q)$. To summarize, we proved that for t small enough, H_t is constant on $U'' \cap C$. In particular, H_0 is locally constant on C .

Since C is assumed to be connected, this means that H_0 is constant on C . The argument we followed for $t = 0$ applies for any other initial time. Thus, H_t must be constant on C for any t . $\square$

The proof of Theorem 1 relies on the first implication of Theorem 3 and the following characterization of coisotropic submanifolds and their characteristic foliations:

A submanifold is coisotropic if and only if the flow of every autonomous Hamiltonian constant on it preserves it. Moreover, the leaf through a point p is locally the union of the orbits of p under the flows of all such Hamiltonians.

The next lemma is based on this characterization.

Lemma 22. *Let C be a submanifold in a symplectic manifold (M, ω) . Assume that every point $p \in C$ admits an open neighborhood V such that any $H \in C_c^\infty(V)$, with $H|_C \equiv 0$, satisfies $\phi_H^t(p) \in C$ for every $t \in [0, +\infty)$. Then C is coisotropic.*

Moreover, for such a neighborhood V , there exists a smaller neighborhood $W \Subset V$ such that, the leaf $\mathcal{F}(p)$ of the characteristic foliation of C passing through p satisfies

$$W \cap \mathcal{F}(p) = W \cap \{\phi_H^t(p) \mid t \in [0, +\infty), H \in C_c^\infty(V), H|_C \equiv 0\}.$$

Proof. Let $p \in C$ and let V be an open subset as in the statement of the lemma. Assume that C coincides locally with $f_1^{-1}(0) \cap \dots \cap f_k^{-1}(0)$ for some smooth functions $f_1, \dots, f_k$ whose differentials are linearly independent at p . By multiplying by an appropriate cutoff function, we can assume that

these functions are defined everywhere on M , have compact support in V , and vanish on C .

The Hamiltonian vector fields at p of $f_1, \dots, f_k$ span $(T_p C)^\omega$, and by assumption belong to $T_p C$. Thus $(T_p C)^\omega \subset T_p C$ and C is coisotropic.

Now, since the characteristic leaves are preserved by smooth Hamiltonians constant on C , we have the inclusion

$$\mathcal{F}(p) \supset \{\phi_H^t(p) \mid t \in [0, +\infty), H \in C_c^\infty(V), H|_C \equiv 0\}.$$

Conversely, consider the map

$$F: \mathbb{R}^k \rightarrow C, \quad (v_1, \dots, v_k) \mapsto \phi_{\sum_{i=1}^k v_i f_i}^1(p).$$

Since, $\sum_{i=1}^k v_i f_i$ is constant on C , its flow preserves the characteristics, hence F takes values in the characteristic leaf $\mathcal{F}(p)$ through p . The partial derivatives of F at 0 are $\partial_{v_i} F(0) = X_{f_i}(p)$ and in particular they are linearly independent and span $T_p \mathcal{F}(p) = (T_p C)^\omega$. The inverse function theorem then shows that F is a diffeomorphism from a neighborhood of 0 to a neighborhood of p in $\mathcal{F}(p)$. This shows

$$W \cap \mathcal{F}(p) \subset W \cap \{\phi_H^t(p) \mid t \in [0, +\infty), H \in C_c^\infty(V), H|_C \equiv 0\}$$

for some neighborhood of p in M and finishes the proof of Lemma 22. $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let C be a smooth coisotropic submanifold, and $\theta: U \rightarrow V$ be a symplectic homeomorphism. Assume $C' = \theta(U \cap C)$ is smooth. Let $p' \in C'$ and $p = \theta^{-1}(p')$. By passing to an appropriate Darboux chart around p , we may assume that $U \subset \mathbb{R}^{2n}$, $p = 0 \in \mathbb{R}^{2n}$, and $C = \mathcal{C}_0$. We are going to prove that any function $H \in C_c^\infty(V)$, with $H|_{C'} \equiv 0$, satisfies $\phi_H^t(p') \in C'$ for all $t \in [0, +\infty)$. According to Lemma 22, this will imply that C' is coisotropic.

Let H be such a function and consider the function $H \circ \theta$. It is compactly supported in U and can be extended by 0 outside U to a continuous compactly supported function $K: \mathbb{R}^{2n} \rightarrow \mathbb{R}$. Since H is smooth and θ is a symplectic homeomorphism, $K \in C_{\text{Ham}}^0(\mathbb{R}^{2n}, \omega_0)$. Since $K|_{\mathcal{C}_0} = 0$, Theorem 3 yields $\phi_K^t(0) \in \mathcal{F}_0(0)$ for any $t \geq 0$. Since K has support in U , we have

$$(8) \quad \forall t \geq 0, \phi_K^t(0) \in \mathcal{F}_0(0) \cap U \subset \mathcal{C}_0 \cap U.$$

Since $\phi_H^t = \theta \phi_K^t \theta^{-1}$, we deduce $\phi_H^t(p') \in C'$ as desired and hence that C' is coisotropic.

Denote $\mathcal{F}'$ the characteristic foliation of C' . From (8), we deduce that for any $H \in C_c^\infty(V)$, $H|_{C'} \equiv 0$,

$$\forall t \geq 0, \phi_H^t(p') \in \theta(\mathcal{F}_0(0) \cap U).$$

Now according to Lemma 22, there exists a neighborhood $W \subset V$ such that

$$W \cap \mathcal{F}'(p') = W \cap \{\phi_H^t(p') \mid t \in [0, +\infty), H \in C_c^\infty(V), H|_{C'} \equiv 0\}.$$

Thus,

$$W \cap \mathcal{F}'(p') \subset W \cap \theta(\mathcal{F}_0(0)).$$

We get the reverse inclusion by switching the roles of C and C' , and we see that θ sends locally $\mathcal{F}_0(0)$ onto $\mathcal{F}'(p')$. $\square$

Let us now turn to the proof of Theorem 5. The proof has three main ingredients: the Lagrangian case in Theorem 3 (i.e., Propositions 17 and 19), Theorem 1, and the fact that the graph of the characteristic foliation, given by

$$\Gamma(\mathcal{F}) = \{(x, x') \in M \times M \mid x \in C, x' \in \mathcal{F}(x)\},$$

is Lagrangian in the product $M \times M$ endowed with the symplectic form $\omega \oplus (-\omega)$, as long as it is a submanifold.

Proof of Theorem 5. Let $p \in C$ and (U, θ) be a coisotropic chart of C around p sending p to 0. The symplectic diffeomorphism $\Theta = \theta \times \theta$, defined on $U \times U$ maps $\Gamma(\mathcal{F})$ to the graph of the standard characteristic foliation

$$\begin{aligned} \Gamma(\mathcal{F}_0) = \{ & (x_1, \dots, x_n, y_1, \dots, y_n, x'_1, \dots, x'_n, y'_1, \dots, y'_n) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \mid \\ & \forall i \in \{n-k+1, \dots, n\}, y_i = y'_i = 0 \text{ and} \\ & \forall j \in \{1, \dots, n-k\}, x_i = x'_i, y_i = y'_i\}. \end{aligned}$$

Since $\Gamma(\mathcal{F}_0)$ is a Lagrangian submanifold of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, then $\Lambda = \Theta^{-1}(\Gamma(\mathcal{F}_0)) = (U \times U) \cap \Gamma(\mathcal{F})$ is a Lagrangian submanifold of $M \times M$.

Now note that if $H \in C^0_{\text{Ham}}$ then the function $K : [0, 1] \times M \times M \rightarrow \mathbb{R}$ given by $K_t(x, x') = H_t(x) - H_t(x')$ is a continuous Hamiltonian generating the hameotopy $\phi_H \times \phi_H$ (recall that the symplectic form on $M \times M$ is $\omega \oplus (-\omega)$).

Assume for a contradiction that H is a function of time on every leaf of the characteristic foliation $\mathcal{F}$ of C and that for some point $q \in C$ and some time $t < 1$, $\phi_H^t(q) \notin C$. Set $t_0 = \sup\{t \geq 0 \mid \phi_H^t(q) \in C\}$. Since C is a closed subset of M , $p = \phi_H^{t_0}(q) \in C$ and we can assume that the above construction yielding the construction of Λ is performed in the neighborhood of this point. Consider the "time-reparametrized" Hamiltonians $\tilde{H}, \tilde{K}$ given by $\tilde{H}(t, x) = (1 - t_0)H(t_0 + (1 - t_0)t, x)$ and $\tilde{K}(t, x, x') = \tilde{H}(t, x) - \tilde{H}(t, x')$. The fact that H is a function of time on any leaf implies that the restriction of $\tilde{K}$ to $\Gamma(\mathcal{F})$ is identically 0. In particular, it vanishes on the Lagrangian Λ and according to Proposition 17 there exists $\varepsilon > 0$ such that for all $t \in [0, \varepsilon]$, $\phi_{\tilde{K}}^t(p, p) = (\phi_{\tilde{H}}^t \times \phi_{\tilde{H}}^t)(p, p) \in \Lambda$. This implies that

$$\phi_H^{t_0 + \varepsilon(1-t_0)}(q) = \phi_H^\varepsilon(p) \in C,$$

which contradicts the maximality of t_0 .

Conversely, assume that the flow ϕ_H^t preserves C . By Theorem 1, ϕ_H^t sends leaves to leaves and in particular, it preserves the graph of the foliation. Therefore, for any point $p \in C$, we may apply Proposition 19 to the Lagrangian Λ and the continuous Hamiltonian K . We get that on a neighborhood of (p, p) , and for small times t , K_t is constant. Since $K_0(p, p) = 0$

we get that K_0 vanishes in a neighborhood of (p, p) . But this implies that H_0 is constant on a neighborhood of p in the leaf $\mathcal{F}(p)$. The argument can be performed for any $p \in C$ and at any initial time t instead of 0. It shows that H_t is locally constant, hence constant, on leaves. $\square$

6. DEFINING C^0 -COISOTROPIC SUBMANIFOLDS AND THEIR CHARACTERISTIC FOLIATIONS

In this section we will use Theorem 1 to define C^0 -coisotropic submanifolds and their characteristic foliations. Below, we assume that $\mathbb{R}^{2n}$ is equipped with the standard symplectic structure. Recall from the beginning of Section 5 that every coisotropic submanifold of codimension k is locally symplectomorphic to

$$\mathcal{C}_0 = \{(x_1, \dots, x_n, y_1, \dots, y_n) \mid (y_{n-k+1}, \dots, y_n) = (0, \dots, 0)\} \subset \mathbb{R}^{2n},$$

and that the leaf of its characteristic foliation, $\mathcal{F}_0$, passing through $p = (a_1, \dots, a_n, b_1, \dots, b_{n-k}, 0, \dots, 0)$ is given by

$$\mathcal{F}_0(p) = \{(a_1, \dots, a_{n-k}, x_{n-k+1}, \dots, x_n, b_1, \dots, b_{n-k}, 0, \dots, 0) \mid (x_{n-k+1}, \dots, x_n) \in \mathbb{R}^k\}.$$

Definition 23. A codimension- k C^0 -submanifold C of a symplectic manifold (M, ω) is C^0 -coisotropic if around each point $p \in C$ there exists a C^0 -coisotropic chart, that is, a pair (U, θ) with U an open neighborhood of p and $\theta: U \rightarrow V \subset \mathbb{R}^{2n}$ a symplectic homeomorphism, such that $\theta(C \cap U) = \mathcal{C}_0 \cap V$.

A codimension- n C^0 -coisotropic submanifold is called a C^0 -Lagrangian.

EXAMPLE. Graphs of symplectic homeomorphisms are C^0 -Lagrangians. Graphs of differentials of C^1 functions and, more generally, graphs of C^0 1-forms, closed in the sense of distributions, provide a family of non trivial examples; see Proposition 26 for a proof.

Conversely, we could ask whether every continuous 1-form whose graph is a C^0 -Lagrangian is closed in the sense of distributions. An affirmative answer in a particular case appears in Viterbo [26, Corollary 22].

As a consequence of Theorem 1, C^0 -coisotropic submanifolds carry (C^0-) characteristic foliations in the following sense.

Proposition 24. Any C^0 -coisotropic submanifold C admits a unique C^0 -foliation $\mathcal{F}$ which is mapped to $\mathcal{F}_0$ by any C^0 -coisotropic chart.

Proof. If such a foliation exists it has to coincide with $\theta^{-1}(\mathcal{F}_0)$ on the domain of any C^0 -coisotropic chart φ . The only thing to check is that for any two C^0 -coisotropic charts $\theta_1: U_1 \rightarrow V_1$ and $\theta_2: U_2 \rightarrow V_2$, the foliations $\theta_1^{-1}(\mathcal{F}_0)$ and $\theta_2^{-1}(\mathcal{F}_0)$ coincide on $U_1 \cap U_2$. But this follows immediately from Theorem 1 applied to $C = \mathcal{C}_0$ and $\theta = \theta_1 \theta_2^{-1}: \theta_2(U_1 \cap U_2) \rightarrow \theta_1(U_1 \cap U_2)$. $\square$

Theorem 1 states that a smooth C^0 -coisotropic submanifold is coisotropic and its natural C^0 -foliation coincides with its characteristic foliation.

EXAMPLE. If $C = \theta(C')$, with C' a smooth coisotropic submanifold and θ a symplectic homeomorphism, then $\mathcal{F} = \theta(\mathcal{F}')$ where $\mathcal{F}'$ is the characteristic foliation of C' .

One may wonder if every topological hypersurface is C^0 -coisotropic. It is possible to show, via an application of Proposition 24, that the boundary of the standard cube in $\mathbb{R}^4$ does not possess a C^0 -characteristic foliation, and hence, it is not C^0 -coisotropic.

The following proposition tells us that Theorems 3 and 5 hold for C^0 -coisotropic submanifolds.

Proposition 25. *Denote by C a connected C^0 -coisotropic submanifold of a symplectic manifold (M, ω) which is closed as a subset of M . Let $H \in C_{\text{Ham}}^0$ with induced hamotopy ϕ_H .*

- (1) *The restriction of H to C is a function of time if and only if ϕ_H preserves C and flows along the leaves of its (C^0) -characteristic foliation.*
- (2) *The restriction of H to each leaf of the characteristic foliation of C is a function of time if and only if the flow ϕ_H preserves C .*

The above can be proven by adapting the proofs of Theorems 3 and 5 to C^0 -coisotropics. We will not provide a proof for Proposition 25 here, and we only mention that to adapt the proofs one would have to introduce C^0 -coisotropic charts and use the following simple fact: if θ is a symplectic homeomorphism and $H \in C_{\text{Ham}}^0$ then $H \circ \theta \in C_{\text{Ham}}^0$ and $\phi_{H \circ \theta}^t = \theta^{-1} \phi_H^t \theta$.

Finally, we provide a family of non trivial examples of C^0 -Lagrangians.

Proposition 26. *Let α be a C^0 1-form on a smooth manifold N which is closed in the sense of distributions. Then, its graph, $\text{graph}(\alpha) \subset T^*N$, is a C^0 -Lagrangian.*

Proof. Since the statement is local, it is sufficient to prove it when N is an open set in $\mathbb{R}^n$. Then α can be written as $\alpha = \sum_{i=1}^n p_i(x) dx_i$, where $x_1, \dots, x_n$ are the canonical coordinates in $\mathbb{R}^n$ and $p_1, \dots, p_n$ continuous functions on N . The fact that α is closed is equivalent to the equations

$$(9) \quad \forall i, j \in \{1, \dots, n\}, \quad \partial_j p_i = \partial_i p_j,$$

where $\partial_i p_j$ is the i -th partial derivative of p_j in the sense of distributions.

We use convolution to approximate α . To that end, take a compactly supported smooth function ρ such that $\rho \geq 0$, and $\int_N \rho(x) dx = 1$ and set $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho(\frac{x}{\varepsilon})$ for every $\varepsilon > 0$. For any continuous function f on N , the functions

$$f * \rho_\varepsilon(x) = \int_N f(y) \rho_\varepsilon(x - y) dy$$

are well-defined on any compact subset of N for ε small enough. Moreover, for any ε , $f * \rho_\varepsilon$ is smooth, converges locally uniformly to f as ε goes to 0,

its differential satisfies $d(f * \rho_\varepsilon) = (df) * \rho_\varepsilon$ and converges in the sense of distributions to df .

Let $U \subseteq N$ be an open subset of N . Then, for ε small enough,

$$\alpha_\varepsilon = \sum_{i=1}^n p_i * \rho_\varepsilon dx_i$$

is a well-defined 1-form on U . It satisfies Equations (9) and thus is closed. Moreover, it converges uniformly to α on U .

Now let ϕ_ε be the family of symplectic diffeomorphisms of T^*U defined by $\phi_\varepsilon(x, p) = (x, p + \alpha_\varepsilon(x))$. They converge uniformly on U to the symplectic homeomorphism $\phi: T^*U \rightarrow T^*U$, $(x, p) \mapsto (x, p + \alpha(x))$ and $\text{graph}(\alpha)$ restricted to T^*U is $\phi(U)$. This shows that $\text{graph}(\alpha)$ is locally the image of a smooth Lagrangian by a symplectic homeomorphism. $\square$

APPENDIX A. THE MAIN RESULTS FOR CLOSED LAGRANGIAN

In this section we provide relatively simple proofs for Theorems 1, 3, and 5 in an enlightening and important special case. We suppose that $M = T^*L$ equipped with its canonical symplectic structure for some closed smooth manifold L . Denote by θ a symplectic homeomorphism of T^*L . And let $L' = \theta(L_0)$, where L_0 denotes the 0-section of T^*L .

Below, we will prove Theorems 1, 3, and 5 in the special case where the coisotropic C is taken to be the zero section L_0 . In this case Theorem 1 states the following:

Theorem 27. *If L' is smooth, then it is Lagrangian.*

In the settings considered in this appendix, Theorems 3 and 5 coincide and state the following:

Theorem 28. *Let $H \in C_{\text{Ham}}^0$ with induced hamotopy ϕ_H . The restriction of H to L_0 is a function of time if and only if ϕ_H preserves L_0 .*

We believe that the above special cases provide the reader with the opportunity to get an idea of the proofs of our main results without having to go through the technical details of Sections 4 and 5.

We will first show that Theorem 27 follows from Theorem 28. In order to do so we will need the following dynamical characterizations of isotropic and coisotropic submanifolds, respectively.

Lemma 29. *Let I denote a (smooth) submanifold of a symplectic manifold (M, ω) . The following are equivalent:*

- *I is isotropic,*
- *For every smooth Hamiltonian H , if ϕ_H preserves I , then $H|_I$ is a function of time only.*

Lemma 30. *Let C denote a (smooth) submanifold of a symplectic manifold (M, ω) . The following are equivalent:*

- C is coisotropic,
- For every smooth Hamiltonian H , if $H|_C$ is a function of time only, then ϕ_H preserves C .

We leave the proofs of the above lemmas, which follow from symplectic linear algebra, to the reader. In the proof of Theorem 1, we use Lemma 22 which is a variation of the second of the above two lemmas.

Proof of Theorem 27. Each of Lemmas 29 and 30 gives a different proof. We provide both proofs here.

First proof: Suppose that H is any smooth Hamiltonian whose flow ϕ_H preserves L' . Then $H \circ \theta \in C_{\text{Ham}}^0$ and its flow, $\theta^{-1}\phi_H^t\theta$, preserves L_0 . It follows from Theorem 28 that the restriction of $H \circ \theta$ to L_0 is a function of time only. Therefore, $H|_{L'}$ depends on time only, and so using Lemma 29 we conclude that L' is isotropic.

Second proof: Suppose that H is any smooth Hamiltonian whose restriction to L' is a function of time only. Then $H \circ \theta \in C_{\text{Ham}}^0$ and its restriction to L_0 depends only on time. It follows from Theorem 28 that the flow of $H \circ \theta$, which is $\theta^{-1}\phi_H^t\theta$, preserves L_0 and so the flow of H preserves L' . Using Lemma 30 we conclude that L' is coisotropic. $\square$

Proof of Theorem 28. To prove the direct implication suppose that $H_t|_{L_0} = c(t)$, where $c(t)$ is a function of time only. For a contradiction assume that ϕ_H does not preserve L_0 , then for some t_0 we have $\phi_H^{t_0}(L_0) \not\subset L_0$, and after the time reparametrization $t \mapsto t_0 t$ we may assume that $t_0 = 1$, that is, $\phi_H^1(L_0) \not\subset L_0$.

Since $H \in C_{\text{Ham}}^0$ there exists a sequence of smooth Hamiltonians $H_i: [0, 1] \times M \rightarrow \mathbb{R}$ such that H_i converges uniformly to H and ϕ_{H_i} converges to ϕ_H in C^0 -topology.

Because $\phi_H^1(L_0) \not\subset L_0$, there exists a ball B such that $B \cap L_0 \neq \emptyset$ and $\phi_H^1(B) \cap L_0 = \emptyset$. It follows that $\phi_{H_i}^{-1}(L_0) \cap B = \emptyset$ for large i . And so,

$$\gamma(\phi_{H_i}^1(L_0), L_0) = \gamma(\phi_{H_i}^{-1}(L_0), L_0) \geq c_{\text{LR}}(B; L_0) > 0.$$

Inequality (2) from Section 2.2 implies that

$$\max_{t \in [0, 1]} (\text{osc}(H_i(t, \cdot)|_{L_0})) \geq c_{\text{LR}}(B; L_0),$$

contradicting the fact that $H|_{L_0}$ is a function of time. We conclude that ϕ_H preserves L_0 .

Next, to prove the converse implication suppose that ϕ_H preserves L_0 . We will show that $H(0, \cdot)|_{L_0}$ is constant. A time reparametrization argument, where $H(t, x)$ is replaced with $\tilde{H}(t, x) = (1 - s)H(s + (1 - s)t, x)$, would then show that $H(s, \cdot)|_{L_0}$ is constant for any choice of $s \in [0, 1]$. This in turn would imply that $H|_{L_0}$ is a function of time.

Let B denote an open ball intersecting L_0 and U a small open neighborhood of $\overline{B}$, such that $L_0 \setminus \pi(U)$ has a non-empty interior, where $\pi: T^*L \rightarrow L_0$ is the natural projection. (Picking U in this way enables us to apply Lemma

8.) Let ψ be any symplectomorphism supported in B and preserving L_0 . Next, we pick $\varepsilon > 0$ such that $\phi_H^t(B), (\phi_H^t)^{-1}(B) \Subset U$ for all $t \in [0, \varepsilon]$. By a reparametrization in time, where $H(t, x)$ is replaced with $\varepsilon H(\varepsilon t, x)$, we may assume that $\phi_H^t(B), (\phi_H^t)^{-1}(B) \Subset U$ for all $t \in [0, 1]$.

Consider the C^0 -Hamiltonian $G = (H \circ \psi - H) \circ \phi_H$. We will now show that $G|_{L_0} = 0$. The support of G is included in $\cup_{t \in [0, 1]} (\phi_H^t)^{-1}(B) \subset U$, and moreover, its flow is $(\phi_H^t)^{-1} \psi^{-1} \phi_H^t \psi$. Because ψ and ϕ_H preserve L_0 the flow of G also preserves L_0 .

Since $G \in C_{\text{Ham}}^0$ there exist smooth Hamiltonians G_i such that $\{G_i\}$ converges uniformly to G and $\{\phi_{G_i}\}$ converges to ϕ_G . Furthermore, we can require that all G_i 's are supported in U . This can be achieved by picking a corresponding sequence of smooth Hamiltonians H_i for H and defining $G_i = (H_i \circ \psi - H_i) \circ \phi_{H_i}$. For large i , G_i is supported in U .

Fix a small $r > 0$. Because $\phi_G^t(L_0) = L_0$ for any $t \in [0, 1]$, for sufficiently large i we have $\phi_{G_i}^t(L_0) \subset T_r^* L_0$. Furthermore, the Hamiltonians G_i are all supported in U and hence we can apply Lemma 8 and conclude that $\gamma(\phi_{G_i}^1(L_0), L_0) \leq Cr$, i.e. $\gamma(\phi_{G_i}^1(L_0), L_0) \rightarrow 0$. Of course, the same reasoning yields $\gamma(\phi_{G_i}^t(L_0), L_0) \rightarrow 0$ for all $t \in [0, 1]$. Then, Proposition 20 implies that $G|_{L_0} = c(t)$. Since it has support in U , we conclude that $G|_{L_0} = 0$.

In particular, $G|_{L_0} = 0$ at time 0. Now since the ball B can contain any chosen pair of points $x_1, x_2 \in L_0$, and ψ can be chosen so that $\psi(x_1) = x_2$, we conclude that the restriction $H(0, \cdot)|_{L_0}$ is constant. $\square$

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