

On the geometry of the space of smooth fibrations

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Abstract

We study geometrical aspects of the space of smooth fibrations between two given manifolds M and B , from the point of view of Fréchet geometry. As a first result, we show that any connected component of this space is the base space of a Fréchet-smooth principal bundle with the identity component of the group of diffeomorphisms of M as total space. Second, we prove that the space of fibrations is also itself the total space of a smooth Fréchet principal bundle with structure group the group of diffeomorphisms of the base B .

1 Introduction and results

The aim of this paper is to study some geometrical properties of the space $\text{Fib}(M, B)$ of all smooth fibrations $\pi : M \rightarrow B$, with M and B smooth finite-dimensional manifolds. By "fibration" we always mean a locally trivial fibre bundle. According to Ehresmann Theorem [1], $\text{Fib}(M, B)$ is nothing but the space of all smooth surjective submersions from M to B . This space is known to be an open subset of the Fréchet manifold of all smooth maps $C^\infty(M, B)$, provided M is closed (see e.g. [2], p. 85). Throughout the paper, M (and B) will always be assumed to be closed and we will study $\text{Fib}(M, B)$ in the framework of Fréchet differential geometry (see [4] or [2] for comprehensive introductions).

This work is actually part of a larger project dealing with the study of the space of Lagrangian fibrations of symplectic manifolds, in order

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to derive applications to the theory of Hamiltonian completely integrable systems (this is the topic of subsequent papers [9, 3]). The present article consists of a preliminary study of the general (non-Lagrangian) case.

Suppose $\text{Fib}(M, B)$ is non empty and let $\pi_0 \in \text{Fib}(M, B)$ be a fibration. A π_0 -vertical diffeomorphism ϕ is a diffeomorphism of M which lifts the identity of B , i.e., which satisfies

$$\pi_0 \circ \phi = \pi_0.$$

We denote by $\text{Diff}(M)$ the group of diffeomorphisms of M , by $\text{Diff}_0(M)$ its identity component and by $\mathcal{G}_{\pi_0}$ the subgroup of all π_0 -vertical diffeomorphisms. Following [8, 7] where this group has been studied extensively, we will refer to it as the *gauge group*. In this paper, we prove two independent theorems on the geometry of $\text{Fib}(M, B)$. The first one relates the diffeomorphism group of M and $\text{Fib}(M, B)$.

Theorem 1. *Let $\pi_0 \in \text{Fib}(M, B)$.*

1. *The action of $\mathcal{G}_{\pi_0}$ (resp. $\mathcal{G}_{\pi_0} \cap \text{Diff}_0(M)$) on $\text{Diff}(M)$ (resp. $\text{Diff}_0(M)$) by right composition gives $\text{Diff}(M)$ (resp. $\text{Diff}_0(M)$) the structure of a Fréchet principal bundle.*
2. *The connected component of π_0 in $\text{Fib}(M, B)$ is naturally Fréchet diffeomorphic to the quotient space $\text{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \text{Diff}_0(M))$.*

The first and second parts of this theorem are proved respectively in Sections 3 and 4.

REMARK. The action of the gauge group $\mathcal{G}_{\pi_0}$ by left composition leads also to a principal bundle structure, naturally obtained from the previous one by conjugation by the inversion of diffeomorphisms.

REMARK. The diffeomorphism that we construct to prove the second part of Theorem 1 is induced by the map

$$\text{Diff}_0(M) \rightarrow \text{Fib}(M, B), \quad \phi \mapsto \pi_0 \circ \phi^{-1}.$$

Note that in general, the analogous map defined on $\text{Diff}(M)$ is not surjective onto $\text{Fib}(M, B)$, so that there is no similar result without the connected components assumptions. Indeed, for some manifolds M, B one can find two fibrations $\pi_1, \pi_2 : M \rightarrow B$ with non-diffeomorphic fibres, which therefore can not satisfy $\pi_1 = \pi_2 \circ \phi$ for any $\phi \in \text{Diff}(M)$. For instance, let $M = \mathbb{S}_3 \times \mathbb{S}_2$, $B = \mathbb{S}_2$, π_1 be the canonical projection onto its second factor and π_2 be the composition of the Hopf fibration

$\mathbb{S}_3 \rightarrow \mathbb{S}_2$ with the projection onto the first factor. The fibre of π_1 is then $\mathbb{S}_3$ while the fibre of π_2 is $\mathbb{S}_1 \times \mathbb{S}_2$.

One immediate consequence of Theorem 1 is that $\text{Diff}_0(M)$ acts transitively on each connected component of $\text{Fib}(M, B)$. This was already proved by Michor in [6] using the Nash-Moser implicit function Theorem. On the contrary, our proof is based on explicit constructions.

A corollary of this transitivity property is the following lemma, for which we can also give a very simple and direct proof:

Lemma. *Two fibrations π_0, π_1 lying in the same connected component of $\text{Fib}(M, B)$ have diffeomorphic fibres.*

Proof. Since $\text{Fib}(M, B)$ is a smooth Fréchet manifold, one can find a smooth loop Π in $\text{Fib}(M, B)$ going through π_0 and π_1 . This loop then defines a map

$$\hat{\Pi} : \mathbb{S}^1 \times M \rightarrow \mathbb{S}^1 \times B, \quad (s, x) \mapsto (s, \Pi(s)(x))$$

which is a submersion. Indeed, its differential is everywhere upper-triangular with submersive diagonal blocks. Then, according to Ehresmann Theorem [1], $\hat{\Pi}$ is a fibration and in particular its fibres are all diffeomorphic to each other. In particular the fibres of π_0 are diffeomorphic to those of π_1 . $\square$

The second theorem concerns the action of $\text{Diff}(B)$ on $\text{Fib}(M, B)$ by left composition. It was inspired by Michor's article [5] about the principal bundle structure of the space of embeddings.

Theorem 2. *Let $\pi_0 \in \text{Fib}(M, B)$. Suppose that π_0 admits a global section. Then, the action of $\text{Diff}_0(B)$ by left composition gives the connected component of π_0 in $\text{Fib}(M, B)$ the structure of a Fréchet $\text{Diff}_0(B)$ -principal bundle.*

We will prove this theorem in Section 5.

REMARK. Intuitively, two fibrations in the same orbit under the action of $\text{Diff}_0(B)$ define the same foliation of M , so that the quotient space can be viewed as the space of bundle-like foliations.

REMARK. Theorem 2 still holds when one replaces $\text{Diff}_0(B)$ by $\text{Diff}(B)$ and the connected component of π_0 by the subset of $\text{Fib}(M, B)$ consisting in all fibrations admitting a global section. This subset is a union of connected components as follows easily from Theorem 1.

REMARK. What happens when π_0 does not admit any global section remains an open problem.

Strangely enough, when M is symplectic and we consider the smaller space of Lagrangian fibrations, we have been able to prove the existence of the principal bundle structure (under the action of the whole $\text{Diff}(B)$) without any assumption on the existence of a global section [3]. The methods used therein involve the Nash-Moser Theorem but unfortunately do not apply here.

Acknowledgements

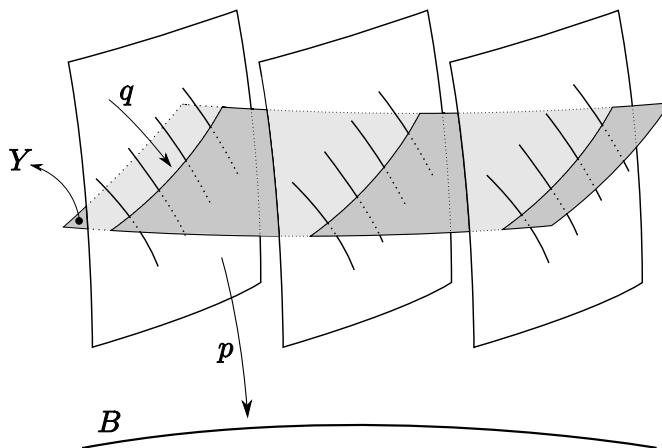
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2 Preparation Lemmas

In this section, we prove two independent lemmas that will be useful in the proof of both Theorems 1 and 2.

Lemma 3. *Let $p : X \rightarrow B$ be a fibre bundle and $Y \subset X$ a subbundle, i.e., a submanifold of X such that the restriction of p to it defines a fibre bundle over B . Then, there exists a tubular neighbourhood U of Y , whose associated projection $q : U \rightarrow Y$ is p -vertical, i.e.,*

$$p \circ q = p.$$



REMARK. Notice that in the case where $X \rightarrow B$ is a vector bundle, one can construct this tubular neighbourhood with the help of “local additions”, as defined in [5]. On the other hand, in the case where the

subbundle Y is actually the graph of a section of $X \rightarrow B$, this result was proven in [4, p. 95], with a similar proof.

Proof. The usual construction of tubular neighbourhoods is the following. We first consider the normal bundle N_Y to Y , which is a vector bundle over Y . Then, with the help of a Riemannian metric on X , we can construct a diffeomorphism ϕ from a neighbourhood of the 0-section in N_Y to a neighbourhood of Y in X . Namely, one identifies any element of N_Y with a tangent vector on X which is orthogonal to Y , and the exponential map provides a point in X . Then, the projection q corresponds through the diffeomorphism ϕ to the projection $N_Y \rightarrow Y$. Unfortunately, constructed in this way, q has no reason to be p -vertical.

Instead of fixing a metric on X and using the corresponding exponential map to construct ϕ , we rather take a smooth family of metrics $(g_b)_{b \in B}$ on the fibres $X_b = p^{-1}(b)$. Such a family can be obtained for example by restricting a given metric g on X to each fibre X_b . Then, at each point $x \in Y$, we denote $b = p(x)$ and we identify $N_Y(x)$ with the vector space W_x orthogonal to $T_x(Y \cap X_b)$ in $T_x X_b$, with respect to the metric g_b on X_b , simply by

$$\begin{aligned} W_x &\rightarrow N_Y(x) = T_x X / T_x Y \\ V &\rightarrow [V]. \end{aligned}$$

This is injective since the intersection of W_x with $T_x Y$ is trivial. It is also surjective because of dimension matching. Indeed, the dimension of $N_Y(x)$ equals $\dim X - \dim Y$. On the other hand, the dimension of W_x is just $\dim X_b - \dim Y_b$, where $Y_b = Y \cap X_b$. But this is equal to $(\dim X - \dim B) - (\dim Y - \dim B)$, hence to $\dim N_Y(x)$. This defines therefore an isomorphism between W_x and $N_Y(x)$.

The construction of the diffeomorphism ϕ is as follows. For any element in $N_Y(x)$ at some point $x \in Y$, we take the corresponding vector in W_x . Then, we use the exponential map associated to the metric g_b , where $b = p(x)$, and obtain a point which lies by construction in the same fibre X_b as x does. The smoothness of this map is clear. The fact that it is a diffeomorphism from a neighbourhood of the 0-section in N_Y to a neighbourhood of Y in X , follows from the property that the linearisation at x of the exponential map is simply the identity on $T_x X_b$. Finally, the fact that q is p -vertical follows directly from the construction of ϕ . $\square$

Lemma 4. *Let $\mathcal{F}$ be a Fréchet manifold together with an action of a Fréchet Lie group $\mathcal{G}$. Suppose that there exists a Fréchet space $\mathcal{E}$, such that for any $f \in \mathcal{F}$, there exist a $\mathcal{G}$ -invariant neighbourhood $\mathcal{W}_f$ of f*

in $\mathcal{F}$, an open set $\mathcal{V}_f$ in $\mathcal{E}$ and a Fréchet diffeomorphism

$$\Phi_f : \mathcal{W}_f \rightarrow \mathcal{G} \times \mathcal{V}_f,$$

which is equivariant under the action of $\mathcal{G}$, i.e., for all $g \in \mathcal{G}$, $\varphi \in \mathcal{W}_f$,

$$\Phi_f(g \cdot \varphi) = (g\Phi_f^1(\varphi), \Phi_f^2(\varphi)),$$

where Φ_f^1, Φ_f^2 denote the respective components of Φ_f .

Then, $\mathcal{F}$ has the structure of a Fréchet principal $\mathcal{G}$ -bundle.

Proof. Let $\mathcal{Q}$ denote the quotient space of $\mathcal{F}$ under the action of $\mathcal{G}$, endowed with the quotient topology. Let us check that $\mathcal{Q}$ is a Fréchet manifold.

For any $f \in \mathcal{F}$, the set $\mathcal{V}_f$ is then homeomorphic to some open set $\mathcal{U}_f$ in $\mathcal{Q}$ and the family $(\mathcal{U}_f)_{f \in \mathcal{F}}$ covers $\mathcal{Q}$. Let us check that $\mathcal{Q}$ is Hausdorff. Let q, q' be distinct elements of $\mathcal{Q}$. Let $\mathcal{U}$ denote one element of the family $(\mathcal{U}_f)_{f \in \mathcal{F}}$ containing q . If q' is in $\mathcal{U}$, since the Fréchet manifold $\mathcal{M}$ is hausdorff, there are two disjoint open subsets in $\mathcal{U}$ (and thus in $\mathcal{Q}$), containing respectively q and q' . If q' is not in $\mathcal{U}$, then we can also find two disjoint open sets containing q and q' by taking any open neighbourhood of q in $\mathcal{U}$ whose closure is included in $\mathcal{U}$ and the complement of its closure in $\mathcal{Q}$. This is possible since a Fréchet topology is metrizable.

Now, let $q = \Phi_f^2(f)$ and $q' = \Phi_{f'}^2(f')$ be two distinct elements in $\mathcal{Q}$, with two neighbourhoods $\mathcal{U}_f, \mathcal{U}_{f'}$ such that $\mathcal{U}_f \cap \mathcal{U}_{f'} \neq \emptyset$, and such that there exist two homeomorphisms $\phi : \mathcal{U}_f \rightarrow \mathcal{V}_f$, $\phi' : \mathcal{U}_{f'} \rightarrow \mathcal{V}_{f'}$. Then, for any fixed $g \in \mathcal{G}$, the transition map can be written

$$\phi' \circ \phi^{-1}|_{\phi(\mathcal{U}_f \cap \mathcal{U}_{f'})} = \Phi_{f'}^2 \circ (\Phi_f)^{-1}|_{\{g\} \times \phi(\mathcal{U}_f \cap \mathcal{U}_{f'})}$$

and hence is smooth. Therefore, the family of sets $(\mathcal{U}_f)_{f \in \mathcal{F}}$ is a smooth Fréchet atlas for $\mathcal{Q}$.

Finally, we see that the family of maps $(\Phi_f)_{f \in \mathcal{F}}$ are smooth local equivariant trivializations whose second coordinate correspond to the natural projection $\mathcal{F} \rightarrow \mathcal{Q}$. We thus have a principal bundle structure. $\square$

3 The principal bundle structure of $\text{Diff}(M)$

In this section, we prove that given a fixed fibration $\pi_0 : M \rightarrow B$ the following right action of the gauge group

$$\begin{aligned} \mathcal{G}_{\pi_0} \times \text{Diff}(M) &\longrightarrow \text{Diff}(M) \\ (\psi, \phi) &\longmapsto \phi \circ \psi \end{aligned}$$

gives $\text{Diff}(M)$ the structure of a Fréchet principal bundle with structure group $\mathcal{G}_{\pi_0}$, as claimed in the first point of Theorem 1. We leave to the reader to check that the proof works if one replaces $\text{Diff}(M)$ by $\text{Diff}_0(M)$ and $\mathcal{G}_{\pi_0}$ by $\mathcal{G}_{\pi_0} \cap \text{Diff}(M)$.

The proof is divided in three steps:

- In Section 3.1 we show that the orbits of the $\mathcal{G}_{\pi_0}$ -action are Fréchet submanifolds of $\text{Diff}(M)$ and in particular that $\mathcal{G}_{\pi_0}$ is indeed a Fréchet Lie group.
- Then, in Section 3.2 we construct a $\mathcal{G}_{\pi_0}$ -invariant neighbourhood $\mathcal{U}$ of $\text{Id} \in \text{Diff}(M)$ together with a Fréchet submanifold $\mathcal{S} \subset \mathcal{U}$, transverse to the $\mathcal{G}_{\pi_0}$ -orbits.
- Finally, Section 3.3 provides the construction of the local charts of the Fréchet principal bundle

$$\text{Diff}(M) \supset \mathcal{U} \longrightarrow \mathcal{S} \times \mathcal{G}_{\pi_0}.$$

Thanks to the transitive action of $\text{Diff}(M)$ onto itself by left composition, we then obtain a chart near any $\phi \in \text{Diff}(M)$.

Throughout the proof, we will use intensively the standard identification of smooth maps on M with smooth sections of $M \times M$, namely

$$\begin{aligned} C^\infty(M, M) &\xrightarrow{\cong} \Gamma(M, M \times M) \\ \phi &\longmapsto \hat{\phi} = (\text{Id}, \phi). \end{aligned}$$

Here and always except when stated explicitly, $M \times M$ is viewed as a trivial bundle over the first factor. Notice also that through this identification, diffeomorphisms of M correspond to an open subset of $\Gamma(M, M \times M)$, which we denote by $\bar{\text{Diff}}(M)$.

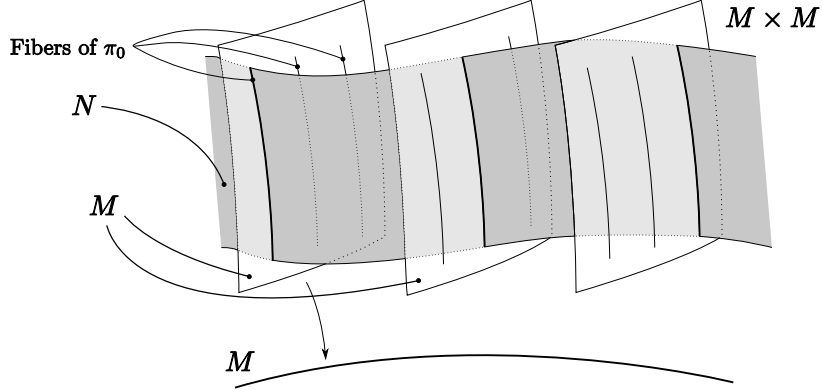
3.1 The structure group $\mathcal{G}_{\pi_0}$

We construct a subbundle $N \subset M \times M$ over M which will provide later a suitable principal bundle chart for $\text{Diff}(M)$ around Id . We define the subset N by

$$N = \{(x, y) \in M \times M \mid \pi_0(x) = \pi_0(y)\}.$$

This is nothing but the pullback bundle of $M \xrightarrow{\pi_0} B$ by the (same) smooth map $\pi_0 : M \rightarrow B$, hence a subbundle of $M \times M$ over M . This subbundle N provides a nice parametrisation of the $\mathcal{G}_{\pi_0}$ -orbits because of the following equivalence.

Lemma 5. *Let $\phi \in \text{Diff}(M)$ be a diffeomorphism. Then the corresponding $\hat{\phi} \in \Gamma(M, M \times M)$ lies in $\Gamma(M, N)$ if and only if $\phi \in \mathcal{G}_{\pi_0}$.*



Proof. If $\phi \in \mathcal{G}_{\pi_0}$, then for each x , $\pi_0 \circ \phi(x)$ equals $\pi_0(x)$ since ϕ is vertical with respect to π_0 . But this precisely means that $(x, \phi(x))$ lies in N for all $x \in M$, i.e., $\hat{\phi} \in \Gamma(M, N)$. Conversely, if $(x, \phi(x)) \in N$ for all $x \in M$, this means by definition that $\pi_0(x) = \pi_0 \circ \phi(x)$ for all x , hence $\phi \in \mathcal{G}_{\pi_0}$. $\square$

We will use this characterisation of $\mathcal{G}_{\pi_0}$ in the proof of Lemma 9 below. But it also follows from this lemma that $\mathcal{G}_{\pi_0}$ is identified through the correspondance $\phi \mapsto \hat{\phi}$ with the intersection of the open subset $\widehat{\text{Diff}}(M) \subset \Gamma(M, M \times M)$ and $\Gamma(M, N)$. On the other hand, it is well-known [2, Exp 4.2.2] that the set of sections of a subbundle is a Fréchet submanifold of the set of sections of the bundle. Therefore $\mathcal{G}_{\pi_0}$ is a Fréchet submanifold of $\text{Diff}(M)$. Moreover $\mathcal{G}_{\pi_0}$ is also a subgroup of $\text{Diff}(M)$, which is a Fréchet Lie group. This way, we recover the following result of [7].

Lemma 6. *The gauge group $\mathcal{G}_{\pi_0}$ is a Fréchet Lie group.*

REMARK. Lemma 6 was already proved in [7, 8], where it is also shown that $\mathcal{G}_{\pi_0}$ is the space of sections of a Fréchet fibre bundle over B , whose standard fibres are identified with the group of diffeomorphisms of the standard fibre of π_0 .

Notice that the corresponding Lie algebra is $\Gamma(M, V_{\pi_0}) \subset \mathfrak{X}(M)$, where $V_{\pi_0} \subset TM$ is the π_0 -vertical tangent bundle of M , i.e., $V_{\pi_0}(x) = \ker D\pi_0(x)$.

Notice also that the orbit of any $\phi_0 \in \text{Diff}(M)$ is also a Fréchet submanifold of $\text{Diff}(M)$. Indeed, this orbit is simply the image of $\mathcal{G}_{\pi_0}$ by the left composition map $L_{\phi_0} : \text{Diff}(M) \rightarrow \text{Diff}(M)$ which sends any ϕ to $\phi_0 \circ \phi$. This map is smooth [2, Exp. 4.4.5] and its inverse $L_{\phi_0}^{-1}$ as well. It is therefore a Fréchet diffeomorphism of $\text{Diff}(M)$ and the result follows.

3.2 The local section $\mathcal{S}$

We now need a tubular neighbourhood of N in $M \times M$ with special properties, reflecting the fact that fibres of N over two different points $x, x' \in M$ satisfying $\pi_0(x) = \pi_0(x')$ are identified, since both are naturally identified with $\pi_0^{-1}(\pi_0(x))$.

Lemma 7. *There exists a tubular neighbourhood $U \subset M \times M$ of N , which is invariant under the action of $\mathcal{G}_{\pi_0}$ on the second factor and whose projection $P : U \rightarrow N$ has the form*

$$P(x, y) = (x, P_2(x, y))$$

with $P_2 : U \rightarrow M$ satisfying $P_2(\psi(x), y) = P_2(x, y)$ for any $\psi \in \mathcal{G}_{\pi_0}$.

Proof. In order to get the required $\mathcal{G}_{\pi_0}$ -invariance property of our tubular neighbourhood inside $M \times M$, we will first make a construction in $B \times M$ and then lift it to $M \times M$.

Let us consider the trivial bundle $B \times M$ over B . Similarly as above, one defines

$$\tilde{N} = \{(b, y) \in B \times M \mid b = \pi_0(y)\},$$

which is a subbundle of $B \times M$ over B . Its fibre over b is simply $\{b\} \times \pi_0^{-1}(b)$. Now, according to Lemma 3, we can construct a tubular neighbourhood $\tilde{U} \subset B \times M$ of $\tilde{N}$ such that the fibres of its associated projection $\tilde{P} : \tilde{U} \rightarrow \tilde{N}$ are included in the fibres of $B \times M$, i.e., $\tilde{P}$ has the form $\tilde{P}(b, y) = (b, \tilde{P}_2(b, y))$.

On the other hand, one can assume that $\tilde{U}$ is invariant under the action of $\mathcal{G}_{\pi_0}$ on the second factor of $B \times M$. Indeed, for any neighbourhood $V \subset B \times B$ of the diagonal, the set $\rho^{-1}(V)$, where $\rho : B \times M \rightarrow M$ is defined by $\rho(b, x) = (b, \pi_0(x))$, is a neighbourhood of $\tilde{N}$ in $B \times M$. Then, we can take V so small that $\rho^{-1}(V)$ is contained in $\tilde{U}$. To see this, fix a metric on $B \times M$ and consider the distance δ between $\tilde{N}$ and the boundary of $\tilde{U}$. It is non-vanishing by compactness of M and B . Then one can take V with a diameter smaller than δ , implying $\rho^{-1}(V) \subset \tilde{U}$. In other word, up to taking $\tilde{U}$ smaller, we can assume it has the form $\tilde{U} = \rho^{-1}(V)$, which is by construction invariant under the action of $\mathcal{G}_{\pi_0}$ on the second factor of $B \times M$.

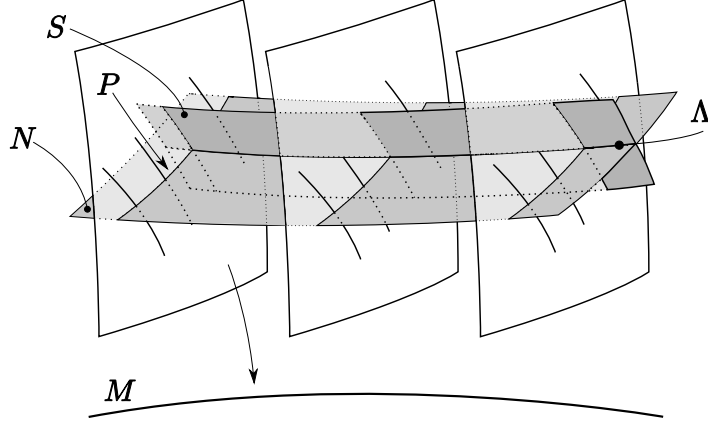
Now, if we define $\hat{\pi}_0 : M \times M \rightarrow B \times M$ by $\hat{\pi}_0(x, y) = (\pi_0(x), y)$, it follows that $\hat{\pi}_0^{-1}(\tilde{N})$ is precisely N and that $U = \hat{\pi}_0^{-1}(\tilde{U})$ is a neighbourhood of N . Then we check easily that the map

$$P : (x, y) \longmapsto (x, \tilde{P}_2 \circ \hat{\pi}_0(x, y))$$

is indeed a projection from U onto N . Moreover, by construction for any $\psi \in \mathcal{G}_{\pi_0}$ and any point $x \in M$, one has $\pi_0(x) = \pi_0(\psi(x))$. This implies that

$$\tilde{P}_2 \circ \hat{\pi}_0(\psi(x), y) = \tilde{P}_2 \circ \hat{\pi}_0(x, y)$$

and therefore $P_2(\psi(x), y) = P_2(x, y)$. On the other hand, it is straightforward to check that U is invariant under the action of $\mathcal{G}_{\pi_0}$ on the second factor of $B \times M$, because $\tilde{U}$ is so. $\square$



The next step is to construct a submanifold $\mathcal{S} \subset \text{Diff}(M)$ which is transverse to the $\mathcal{G}_{\pi_0}$ -orbits. We consider the subbundle $N \subset M \times M$ defined at the beginning of the section, and the associated tubular neighbourhood $U \xrightarrow{P} N$ of Lemma 7. Then we denote by $\Delta \subset M \times M$ the diagonal and by ι_Δ the associated inclusion. Clearly, Δ lies actually in N . Then we define $S = P^{-1}(\Delta)$. It is a submanifold of $M \times M$, but in fact will see that it is even a subbundle of $M \times M$ transverse to N .

To prove that it is a subbundle, first notice that S is nothing but the total space of the induced bundle $\iota_\Delta^* P$ over Δ . Then, since the restriction to Δ of the first projection $P_M : M \times M$ is a diffeomorphism $\Delta \rightarrow M$, one gets by composition a bundle $S \rightarrow M$. Finally, since $P_M \circ P = P_M$, the projection of this last bundle is nothing but the restriction of P_M to S , so that it is indeed a subbundle of $M \times M$. Since TS contains the vertical direction of the bundle P over N , S is transverse to N .

Consequently, the set of sections of S or more precisely

$$\mathcal{S} = \left\{ \phi \in \text{Diff}(M) \mid \hat{\phi} \in \Gamma(M, S) \right\},$$

is a Fréchet submanifold of $\text{Diff}(M)$. The following characterisation will be useful.

Lemma 8. *A diffeomorphism ϕ lies in $\mathcal{S}$ if and only if $P \circ \hat{\phi} = \text{Id}$.*

Proof. First, the left composition of a section $\hat{\phi} \in \Gamma(M, M \times M)$ by P is still a section. Therefore, $P \circ \hat{\phi}$ equals Id if and only if its image $P \circ \hat{\phi}(M)$ coincides with the image of Id , namely Δ . But this happens precisely when the image of $\hat{\phi}$ lies in S , by definition of S . $\square$

3.3 The principal bundle charts

We now have all the technical tools for defining the principal bundle charts on $\text{Diff}(M)$. We first define a chart near Id . We consider the open set U from Lemma 7 and define the set

$$\mathcal{U} = \left\{ \phi \in \text{Diff}(M) \mid \text{im}(\hat{\phi}) \subset U \right\}$$

which is an open neighbourhood of Id in $\text{Diff}(M)$. It is also $\mathcal{G}_{\pi_0}$ -invariant because of the corresponding property for U .

On the other hand, in the definition of the principal bundle chart below, we will need that the composition $P \circ \hat{\phi}$ lies in $\widehat{\text{Diff}}(M)$, or equivalently that the second factor $P_2 \circ \hat{\phi}$ is a diffeomorphism. We thus have to restrict to the smaller set

$$\tilde{\mathcal{U}} = \left\{ \phi \in \mathcal{U} \mid P_2 \circ \hat{\phi} \in \text{Diff}(M) \right\},$$

which is open since the left composition by P_2 is a Fréchet smooth map. The set $\tilde{\mathcal{U}}$ turns out to be also $\mathcal{G}_{\pi_0}$ -invariant. Indeed, for any $\phi \in \tilde{\mathcal{U}}$ and any $\psi \in \mathcal{G}_{\pi_0}$, we compute $P_2 \circ \widehat{\phi \circ \psi}$. This gives

$$P_2 \circ (\text{Id}, \phi \circ \psi) = P_2 \circ (\psi, \phi \circ \psi)$$

because of the second property of P given in Lemma 7. But this is equal to $P_2 \circ \hat{\phi} \circ \psi$ which is a diffeomorphism of M since both $P_2 \circ \hat{\phi}$ and ψ are so.

Lemma 9. *The map*

$$\begin{aligned} \Phi : \tilde{\mathcal{U}} &\longrightarrow \mathcal{S} \times \mathcal{G}_{\pi_0} \\ \phi &\longmapsto (\phi_{\mathcal{S}}, \psi), \end{aligned}$$

where $\hat{\psi} = P \circ \hat{\phi}$ and $\phi_{\mathcal{S}} = \phi \circ \psi^{-1}$, is well-defined and is a Fréchet smooth diffeomorphism, whose inverse is

$$\Phi^{-1} : (\phi_{\mathcal{S}}, \psi) \longmapsto \phi_{\mathcal{S}} \circ \psi.$$

Proof. First we check that this definition makes sense. Since the image of P lies in N , it follows that $\hat{\psi}$ is a section of N and, thanks to Lemma 5, that $\psi \in \mathcal{G}_{\pi_0}$. It remains to check that $\phi_{\mathcal{S}}$ lies indeed in $\mathcal{S}$. According to Lemma 8, we only need to check that $P \circ \hat{\phi}_{\mathcal{S}} = \text{Id}$, or equivalently that $P_2 \circ \hat{\phi}_{\mathcal{S}} = \text{Id}$. The composition $P_2 \circ \hat{\phi}_{\mathcal{S}}$ equals $P_2 \circ (\text{Id}, \phi \circ \psi^{-1})$. Now, if we use the property of P_2 given in Lemma 7, we obtain $P_2 \circ (\psi^{-1}, \phi \circ \psi^{-1})$ and thus $P_2 \circ \hat{\phi} \circ \psi^{-1}$. But this is exactly $\psi \circ \psi^{-1}$ and thus Id . The map Φ is therefore well defined. It is also smooth since it is made of compositions and inversion of diffeomorphisms, which are both Fréchet smooth [2, Exp. 4.4.5 and 4.4.6].

Now, the image of the claimed inverse Φ^{-1} is included in $\tilde{U}$. Indeed, since $\mathcal{S}$ is included in U , then $\mathcal{S}$ is included in $\mathcal{U}$. It is moreover in $\tilde{U}$ thanks to Lemma 8. Finally, $\tilde{U}$ is $\mathcal{G}_{\pi_0}$ -invariant, hence the image of Φ^{-1} is included in $\tilde{U}$.

Then, it is straightforward to see that the claimed inverse Φ^{-1} is a left inverse of Φ . Let us check that it is also a right inverse. For any $\phi_{\mathcal{S}} \in \mathcal{S}$ and any $\psi \in \mathcal{G}_{\pi_0}$ we have to compute $(\phi'_{\mathcal{S}}, \psi') = \Phi(\phi_{\mathcal{S}} \circ \psi)$. First, we have $\psi' = P_2 \circ \widehat{\phi_{\mathcal{S}} \circ \psi}$, i.e.,

$$\psi' = P_2 \circ (\text{Id}, \phi_{\mathcal{S}} \circ \psi).$$

But the property of P_2 given in Lemma 7 shows that this equals $P_2 \circ \hat{\phi}_{\mathcal{S}} \circ \psi$. On the other hand, since $\hat{\phi}_{\mathcal{S}}$ lies in $\mathcal{S}$, one has $P_2 \circ \hat{\phi}_{\mathcal{S}} = \text{Id}$, hence $\psi' = \psi$. Therefore, $\Phi(\phi_{\mathcal{S}} \circ \psi)$ is equal to $(\phi'_{\mathcal{S}}, \psi)$ with $\phi'_{\mathcal{S}}$ given by $\phi_{\mathcal{S}} \circ \psi \circ \psi^{-1}$ which thus coincides with $\phi_{\mathcal{S}}$. We have thus proved that the map $(\phi_{\mathcal{S}}, \psi) \mapsto \phi_{\mathcal{S}} \circ \psi$ is indeed the (double-sided) inverse of Φ . $\square$

Therefore, near $\text{Id} \in \text{Diff}(M)$, we have constructed a diffeomorphism Φ from a $\mathcal{G}_{\pi_0}$ -invariant neighbourhood $\tilde{U}$ of Id to $\mathcal{S} \times \mathcal{G}_{\pi_0}$. It is actually $\mathcal{G}_{\pi_0}$ -equivariant, as one can easily check on the inverse Φ^{-1} .

Then, near any $\phi \in \text{Diff}(M)$, we can construct a similar diffeomorphism

$$\Phi_{\phi_0} : \tilde{U}_{\phi_0} \longrightarrow \mathcal{S}_{\phi_0} \times \mathcal{G}_{\pi_0},$$

where $\tilde{U}_{\phi_0}$ and $\mathcal{S}_{\phi_0}$ are obtained respectively from $\tilde{U}$ and $\mathcal{S}$ by left composition with ϕ_0 . The map Φ_{ϕ_0} is simply defined by

$$\Phi_{\phi_0}(\phi) = (\phi_0 \circ \Phi^{\mathcal{S}}(\phi_0^{-1} \circ \phi), \Phi^{\mathcal{G}_{\pi_0}}(\phi_0^{-1} \circ \phi)),$$

where $\Phi^{\mathcal{S}}$ and $\Phi^{\mathcal{G}_{\pi_0}}$ are respectively the $\mathcal{S}$ and $\mathcal{G}_{\pi_0}$ component of Φ .

Finally, it follows from Lemma 4 that the maps Φ_{ϕ_0} form a principal bundle atlas for $\text{Diff}(M)$ with the gauge group $\mathcal{G}_{\pi_0}$ as structure group.

4 The quotient space $\text{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \text{Diff}_0(M))$

In this section, we prove the second part of Theorem 1.

Proof. First recall that, as already mentioned in the introduction, $\text{Fib}(M, B)$ is an open subset of $C^\infty(M, B)$. Let $\pi_0 \in \text{Fib}(M, B)$. The smooth map

$$\Psi : \text{Diff}_0(M) \rightarrow \text{Fib}(M, B), \quad \phi \mapsto \pi_0 \circ \phi^{-1}$$

induces a map

$$\tilde{\Psi} : \text{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \text{Diff}_0(M)) \rightarrow \text{Fib}(M, B).$$

In order to prove that $\tilde{\Psi}$ is a diffeomorphism onto the component of π_0 , we will prove that it is injective, smooth, that it admits a local smooth inverse near any point and finally that its image is the connected component of π_0 .

Proving the injectivity is easy: for any two diffeomorphisms $\phi_1, \phi_2 \in \text{Diff}_0(M)$ satisfying $\pi_0 \circ \phi_1^{-1} = \pi_0 \circ \phi_2^{-1}$, then $\phi_1^{-1} \circ \phi_2$ obviously belongs to $\mathcal{G}_{\pi_0}$ and thus ϕ_1 and ϕ_2 represent the same element in $\text{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \text{Diff}_0(M))$.

The fact that $\tilde{\Psi}$ is smooth follows from the first part of Theorem 1. Indeed, for any φ in the quotient $\text{Diff}_0(M)/\mathcal{G}_{\pi_0}$, the first part of Theorem 1 implies that there exists a smooth section σ of the bundle $\text{Diff}_0(M) \rightarrow \text{Diff}_0(M)/\mathcal{G}_{\pi_0}$ defined on a neighbourhood of φ . On this neighbourhood, $\tilde{\Psi}$ can be written $\tilde{\Psi} = \Psi \circ \sigma$ and therefore is smooth.

To construct a local inverse to $\tilde{\Psi}$, we will make use of the following lemma which relates the spaces of sections of two different fibrations of the same manifold.

Lemma 10 (Roy [9]). *Let X be a smooth manifold and $p_1, p_2 : X \rightarrow M$ be two fibrations over M . Suppose there exists a common section $s_0 : M \rightarrow X$, i.e., a smooth map satisfying*

$$p_1 \circ s_0 = p_2 \circ s_0 = \text{Id}_M.$$

Then, there exists open subsets $V_1 \subset \Gamma(p_1)$ and $V_2 \subset \Gamma(p_2)$ containing s_0 and such that the map

$$V_1 \rightarrow V_2, \quad \alpha \mapsto \alpha \circ (p_2 \circ \alpha)^{-1}$$

is a diffeomorphism.

For the sake of completeness, we briefly recall its proof.

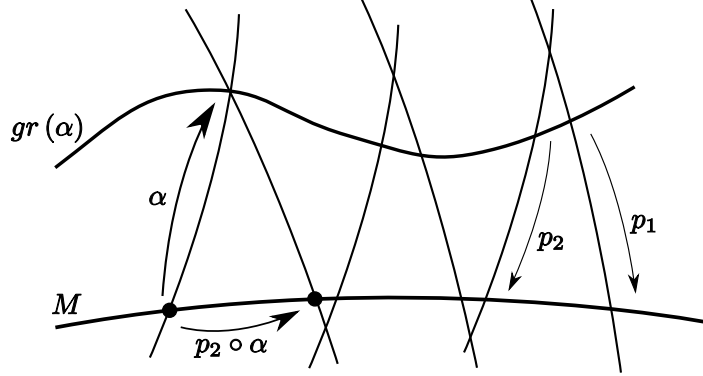


Figure 1: On this picture, the graph of s_0 is identified with M

Proof. First of all, the map $A : \Gamma(p_1) \rightarrow C^\infty(M, M)$, $\alpha \mapsto p_2 \circ \alpha$ is smooth and in particular continuous so that the pre-image V_1 of the open subset $\text{Diff}(M) \subset C^\infty(M, M)$ is an open subset of $\Gamma(p_1)$. Moreover, $s_0 \in V_1$ since $A(s_0) = \text{Id}$. The map $\alpha \mapsto \alpha \circ (p_2 \circ \alpha)^{-1}$ is well-defined on V_1 and smooth, because built of compositions and inversion of diffeomorphisms. On the other hand, the image of α is indeed a section of p_2 , since $p_2 \circ \alpha \circ (p_2 \circ \alpha)^{-1} = \text{Id}$.

Reversing the roles of p_1 and p_2 one constructs similarly a map $\beta \mapsto \beta \circ (p_1 \circ \beta)^{-1}$, defined on some open set $V_2 \subset \Gamma(p_2)$ containing s_0 . It is then easily checked that, up to taking smaller V_1 and V_2 , this map and the previous one are inverse to each other. $\square$

Let us now use Lemma 10 to construct a local inverse to $\tilde{\Psi}$ around π_0 . Denote by $p_M : M \times B \rightarrow M$ and $p_B : M \times B \rightarrow B$ the natural projections on the first and the second factor respectively, and $s_0 : M \rightarrow M \times B$, $x \mapsto (x, \pi_0(x))$. It turns out that the graph of s_0 is a subbundle of p_B precisely because $\pi_0 : M \rightarrow B$ is a submersion. Therefore, according to Lemma 3, there exists a tubular neighbourhood X of the graph of s_0 whose projection q is p_B -vertical.

We now define the two necessary fibrations for applying Lemma 10. The first one $p_1 : X \rightarrow M$ is simply the restriction of p_M to X . The second one $p_2 : X \rightarrow M$ is given by $p_2 = p_1 \circ q$. Now, we see easily that s_0 , which is obviously a section of p_1 , is also a section of p_2 . This is because the projection q acts as the identity on the graph of s_0 . Therefore, we can apply Lemma 10 and deduce that for any section $s = (\text{Id}, \pi)$ of p_1 close to s_0 , there exists a diffeomorphism $f \in \text{Diff}(M)$ (depending smoothly on s and hence on π), such that $s \circ f$ is a section of p_2 . Note by the way that since $\text{Fib}(M, B)$ is open in $C^\infty(M, B)$, if s is sufficiently close to s_0 , then π is a fibration. On the other hand, by construction any section of p_2 has the form (g, π_0) , with $g : M \rightarrow M$.

Indeed, for any $x \in M$, the fibre $p_2^{-1}(x)$ is contained in $M \times \{b\}$, with $b = \pi_0(x)$. Thus, one has

$$(f, \pi \circ f) = s \circ f = (g, \pi_0).$$

This proves in particular the crucial property $\pi \circ f = \pi_0$.

We denote $[f]$ the image of a diffeomorphism $f \in \text{Diff}_0(M)$ in the quotient $\text{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \text{Diff}_0(M))$. The map $\chi : \pi \mapsto [f]$ is our candidate for being the local smooth inverse of $\tilde{\Psi}$ around π_0 . Indeed, for any π near π_0 , one has

$$\tilde{\Psi}(\chi(\pi)) = \tilde{\Psi}([f]) = \Psi(f) = \pi_0 \circ f^{-1} = \pi.$$

Conversely, for any $[f]$ close to $[\text{Id}]$, $\chi(\tilde{\Psi}([f]))$ is given by

$$\chi(\Psi(f)) = \chi(\pi_0 \circ f^{-1}) = [g],$$

where $g \in \text{Diff}(M)$ satisfies $\pi_0 \circ f^{-1} \circ g = \pi_0$. But this precisely means that $f^{-1} \circ g \in \mathcal{G}_{\pi_0}$, hence $[g] = [f]$. Now, around any element $\tilde{\Psi}(\phi) = \pi_0 \circ \phi^{-1}$ in the image of $\tilde{\Psi}$, we construct the local inverse of $\tilde{\Psi}$ by $\pi \mapsto \phi \circ [\chi(\pi \circ \phi)]$. Here, the left composition of ϕ with a class $[f]$ is defined to be $[\phi \circ f]$, which makes sense because $\phi \circ (f \circ \mathcal{G}_{\pi_0}) = (\phi \circ f) \circ \mathcal{G}_{\pi_0}$. We leave to the reader to check that this is indeed a local inverse of $\tilde{\Psi}$.

Let us now finish our proof. It remains to show that the image of $\tilde{\Psi}$ is exactly the connected component of π_0 in $\text{Fib}(M, B)$. First, the image of $\tilde{\Psi}$ is connected since $\text{Diff}_0(M)$ is. Then, let π_1 be an element in the connected component of π_0 in $\text{Fib}(M, B)$, so that there exists a path $(\pi_t)_{t \in [0,1]}$ joining them. The compactness of $[0, 1]$ allows us to find $0 = t_0 < \dots < t_k = 1$ such that for any $i = 1, \dots, k$, π_{t_i} belongs to the domain of the local inverse as constructed previously around $\pi_{t_{i-1}}$ (just replace π_0 with $\pi_{t_{i-1}}$). Therefore, for any $i = 1, \dots, k$, there exists an element ϕ_i in $\text{Diff}_0(M)$ such that

$$\pi_{t_i} = \pi_{t_{i-1}} \circ \phi_i.$$

Thus $\pi_1 = \pi_0 \circ (\phi_k \circ \dots \circ \phi_1)$ which implies that π_1 lies in the image of $\tilde{\Psi}$. $\square$

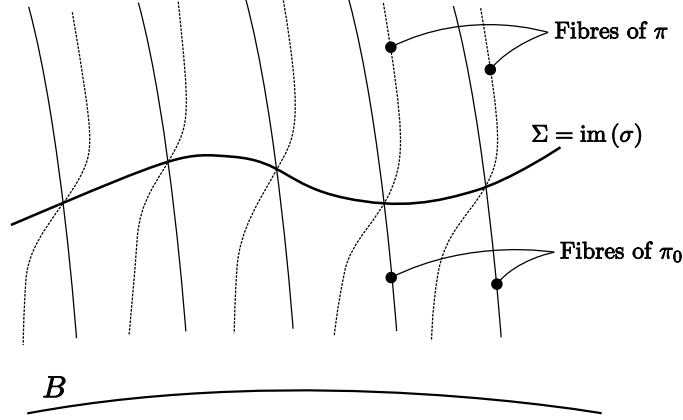
5 The principal bundle structure of $\text{Fib}(M, B)$

This section is devoted to the proof of Theorem 2.

Proof. — Let $\pi_0 \in \text{Fib}(M, B)$ and σ be a global section of π_0 , i.e., $\pi_0 \circ \sigma = \text{Id}_B$. We denote by $\text{Fib}_0(M, B)$ the connected component

of π_0 in $\text{Fib}(M, B)$ and consider the set $\mathcal{S}$ of all fibrations for which σ is a section:

$$\mathcal{S} = \{\pi \in \text{Fib}_0(M, B) \mid \pi \circ \sigma = \text{Id}_B\}.$$



As a first step, let us prove that $\mathcal{S}$ is a Fréchet submanifold of $\text{Fib}_0(M, B)$. To see this, first remark that since $\text{Fib}_0(M, B)$ is a homogeneous space (Theorem 1), we can work in the neighbourhood of π_0 without lack of generality. Then, remember that there exist an open set $\mathcal{U}$ in $\text{Fib}_0(M, B)$ containing π_0 and a Fréchet diffeomorphism $\Psi : \mathcal{U} \rightarrow \mathcal{V}$ onto a neighbourhood $\mathcal{V}$ of the zero section in the Fréchet linear space of sections $\Gamma(\pi_0^*TB)$ of the vector bundle π_0^*TB over M . Moreover, this chart Ψ can be chosen so that for all $x \in M$ and all $\pi \in \mathcal{U}$, $\pi(x) = \pi_0(x)$ if and only if $\Psi(\pi)(x) = 0$.

Now, if we denote by Σ the submanifold of M which is the image of the section σ , we see that for $\pi \in \mathcal{U}$,

$$\pi \in \mathcal{S} \iff \pi|_{\Sigma} = \pi_0|_{\Sigma} \iff \Psi(\pi)|_{\Sigma} = 0.$$

Let us consider the space $\Gamma_{\Sigma}(\pi_0^*TB)$ of sections of π_0^*TB which vanish on Σ and let us show that it is a closed subspace of $\Gamma(\pi_0^*TB)$ with a closed complement.

First, $\Gamma_{\Sigma}(\pi_0^*TB)$ is obviously closed, so that it only remains to construct a closed complement. To any vector field χ over B , we associate the section $\tilde{\chi}$ of π_0^*TB defined by

$$\tilde{\chi} = \chi \circ \pi_0.$$

Now, we consider $\mathcal{F} = \{\tilde{\chi} \mid \chi \in \Gamma(TB)\}$ the set of sections of π_0^*TB which lift vector fields on B . The set $\mathcal{F}$ is a closed linear subspace of

$\Gamma(\pi_0^*TB)$. Moreover, for any $s \in \Gamma(\pi_0^*TB)$, we can consider the vector field $s \circ \sigma$ over B . In the decomposition

$$s = (s - \widetilde{s \circ \sigma}) + \widetilde{s \circ \sigma},$$

the first term lies in $\Gamma_\Sigma(\pi_0^*TB)$ while the second is in $\mathcal{F}$, hence

$$\Gamma(\pi_0^*TB) = \Gamma_\Sigma(\pi_0^*TB) + \mathcal{F}.$$

Since moreover $\Gamma_\Sigma(\pi_0^*TB)$ and $\mathcal{F}$ have trivial intersection, the sum is direct and $\mathcal{F}$ is a closed complement of $\Gamma_\Sigma(\pi_0^*TB)$. This proves our claim that $\Gamma_\Sigma(\pi_0^*TB)$ is a closed subspace of $\Gamma(\pi_0^*TB)$, with closed complement.

Consequently, the image of $\mathcal{S} \cap \mathcal{U}$ by Ψ coincides with the intersection of $\mathcal{V}$ with a closed complemented linear subspace of $\Gamma(\pi_0^*TB)$. This shows that $\mathcal{S}$ is a Fréchet submanifold of $\text{Fib}_0(M, B)$.

Let us now consider $\mathcal{W} = \{\pi \in \text{Fib}_0(M, B) \mid \pi \circ \sigma \in \text{Diff}_0(B)\}$. Since the right composition by σ is a continuous map, $\mathcal{W}$ is an open subset of $\text{Fib}_0(M, B)$. It is moreover invariant under the action of $\text{Diff}_0(B)$ by composition on the left. We are going to show that there is an equivariant Fréchet diffeomorphism

$$\mathcal{W} \simeq \text{Diff}_0(B) \times \mathcal{S},$$

where the action of $\text{Diff}_0(B)$ on the right hand side is given by composition on the left on the first factor. Because of Lemma 4, this will achieve the proof of Theorem 2.

Let $\pi \in \mathcal{W}$ and set

$$\Phi(\pi) = (\pi \circ \sigma, (\pi \circ \sigma)^{-1} \circ \pi).$$

The first factor is obviously in $\text{Diff}_0(B)$ and the second in $\mathcal{W} \cap \mathcal{S}$. Since composition and inversion are smooth maps Φ is also a smooth map. Moreover, the fact that Φ is equivariant is easily checked: for any $\phi \in \text{Diff}_0(B)$, $\pi \in \mathcal{W}$, one has

$$\begin{aligned} \Phi(\phi \circ \pi) &= (\phi \circ \pi \circ \sigma, (\phi \circ \pi \circ \sigma)^{-1} \circ \phi \circ \pi) \\ &= (\phi \circ (\pi \circ \sigma), (\pi \circ \sigma)^{-1} \circ \pi). \end{aligned}$$

Finally, Φ has a smooth inverse given by the map $(\phi, \pi_S) \mapsto \phi \circ \pi_S$. $\square$

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