

Hofer's distance on diameters and the Maslov index

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Abstract

We prove that Hofer's distance between two diameters of the open 2-disk admits an upper bound in terms of the Maslov index of their intersection points.

1 Introduction and result

This note is inspired by Khanevsky's article [2]. We will adopt his settings throughout the paper. Let us recall them briefly. Let $D \subset \mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$ be the open unit 2-disk, endowed with the symplectic structure $\omega = \frac{1}{\pi} dx \wedge dy$ (so that the total area of D is 1). We call *standard diameter* and denote by L_0 the intersection of D with the x -axis $\mathbb{R} \times \{0\}$. Given a time-dependent Hamiltonian H on D , we denote by X_H its Hamiltonian vector field defined by $\omega(X_H, \cdot) = dH$, and by ϕ_H^t the Hamiltonian isotopy generated by H . All Hamiltonian functions will be assumed compactly supported. By definition, a *diameter* is a curve which is isotopic to the standard diameter via a compactly supported Hamiltonian isotopy. We denote by $\mathcal{E}$ the set of all diameters.

The set $\mathcal{E}$ can be endowed with the so-called *Hofer distance* which is defined as follows:

$$d(L_1, L_2) = \inf \left\{ \int_0^1 \left(\max_{x \in D} H_t - \min_{x \in D} H_t \right) dt \right\},$$

where the infimum is taken over all smooth and compactly supported time dependent Hamiltonians H_t such that $\phi_H^1(L_1) = L_2$. The distance d may be defined on more general symplectic manifolds. Its

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non-degeneracy is a deep result (see [1]). The value of $\int_0^1 (\max H_t - \min H_t) dt$ is called the *Hofer energy* of the isotopy generated by H_t .

An interesting fact pointed out in Khanevsky's paper is that Hofer's distance descends to a metric on the *reduced diameter space* which is defined as $\hat{\mathcal{E}} = \mathcal{E} / \sim$, where $L \sim L'$ iff $L = \phi(L')$ with $\phi \in \mathcal{S}$, $\mathcal{S}$ being the group of Hamiltonian diffeomorphisms which globally fix the standard diameter L_0 . Several other invariants of $\mathcal{E}$ descend to $\hat{\mathcal{E}}$. Khanevsky considers two of them: the number of intersection points between diameters and (much more involved) an invariant r_A constructed from the Entov-Polterovich quasimorphism and the Calabi morphism. Roughly speaking, he proves that r_A is dominated by d (this implies that $\hat{\mathcal{E}}$ is unbounded) and that d is dominated by the number of intersection points. In the present note we are interested in a third invariant which is the Maslov index.

The Maslov index of a transverse intersection point of two given Lagrangians is an integer which is invariant under Hamiltonian transformations. Unless we choose some convention, this index is defined up to an additive constant but the difference of index between two points (the "index gap") is well defined. It can be constructed in a natural way by an abstract construction [3]. However, in our case, it can be constructed in a naive way as follows.

First, applying a Hamiltonian diffeomorphism, we may suppose that one of our two diameters is the standard one. Then, consider a diameter L intersecting L_0 transversely. Note that in our context, every diameter coincides with the standard diameter in the complement of a compact set. Therefore, we say that L and L_0 are transverse if we can write $L \cap L_0 = (-1, -1 + \varepsilon] \cup \{x_i\}_{i \in \{1, \dots, N\}} \cup [1 - \varepsilon', 1)$, where $\varepsilon, \varepsilon' > 0$ and the points $x_1, \dots, x_N$ are transverse intersections (in the usual sense). We suppose that the points $x_1, \dots, x_N$ are ordered by their position on L (not L_0). We choose a convention for the index of x_1 , namely we set $\mu(x_1) = 0$. Then, we construct the Maslov index μ of the other intersection points inductively as follows.

We denote by D^+ the upper half-disk and D^- the lower half-disk. For $i \in \{1, \dots, N\}$, we set $\mu(x_{i+1}) = \mu(x_i) + \delta$, where

$$\delta = \begin{cases} 1, & \text{if } L|_{[x_i, x_{i+1}]} \subset D^- \text{ and } x_i < x_{i+1} \text{ on } L_0 \\ -1, & \text{if } L|_{[x_i, x_{i+1}]} \subset D^- \text{ and } x_i > x_{i+1} \text{ on } L_0 \\ -1, & \text{if } L|_{[x_i, x_{i+1}]} \subset D^+ \text{ and } x_i < x_{i+1} \text{ on } L_0 \\ 1, & \text{if } L|_{[x_i, x_{i+1}]} \subset D^+ \text{ and } x_i > x_{i+1} \text{ on } L_0 \end{cases}$$

REMARK 1.1.

1. Intuitively, $\mu(x_i)$ measures how much L "twists" in the positive direction before reaching x_i .

2. The index μ descends to the reduced diameter space $\hat{\mathcal{E}}$.
3. Note that by definition, the index gap between two intersection points that are consecutive on L is 1 or -1 . The abstract definition of the Maslov index implies that we can invert the roles of L and L_0 and therefore that the index gap between two intersection points that are consecutive on L_0 is also either 1 or -1 .

To avoid any confusion, we will sometimes denote $\mu(x, L)$ instead of $\mu(x)$. We then set

$$\mu_{\max}(L) = \max_{i=1, \dots, N} \mu(x_i, L) \quad \text{and} \quad \mu_{\min}(L) = \min_{i=1, \dots, N} \mu(x_i, L).$$

The maximal index gap $\mu_{\max}(L) - \mu_{\min}(L)$ descends to the reduced diameter space $\hat{\mathcal{E}}$. It is therefore natural to try to compare it with the other invariants considered by Khanevsky. The following inequality is a consequence of Remark 1.1.4.

$$2(\mu_{\max}(L) - \mu_{\min}(L)) \leq \sharp(L \cap L_0) + 1.$$

Our main result is then the following theorem.

Theorem 1.2. *For any diameter L transverse to L_0 , with at least two transverse intersection points with L_0 ,*

$$d(L, L_0) \leq \mu_{\max}(L) - \mu_{\min}(L) - \frac{1}{2}.$$

REMARK 1.3. The maximal index gap $\mu_{\max}(L) - \mu_{\min}(L)$ vanishes if and only if there is only one transverse intersection point. In that case, $d(L, L_0) \leq \frac{1}{2}$.

REMARK 1.4. This implies $d(L, L_0) \leq \frac{1}{2} \cdot \sharp(L \cap L_0)$. Therefore our result implies a linear estimate as in [2] but with $\frac{1}{2}$ as multiplicative constant, while Khanevsky's constant is $\frac{1}{8}$. Nevertheless, in many cases, our upper bound is much better. Moreover, after I completed this note, Michael Khanevsky explained to me an argument based on his paper's settings (in particular "diameter trees") showing that it might be possible to improve the multiplicative constant 1 in Theorem 1.2 to the constant $\frac{1}{4}$.

REMARK 1.5. As far as I know, the analogous problem in higher dimension is completely open.

Let us give an idea of the proof of Theorem 1.2. Suppose that $\mu_{\min}(L) < 0$. We first concentrate on intersection points of minimal index. Given such a point, it is possible to construct a very simple Hamiltonian isotopy with several properties including the fact that it removes the intersection point (see Section 3). We then show that

such simple transformations can be performed in an appropriate order so that all intersection points of minimal index are removed and so that the total energy needed for these transformation is less than 1. We get a new diameter L' satisfying $\mu_{\min}(L') \leq \mu_{\min}(L) + 1$ satisfying $d(L', L) = 1$. We then proceed by induction until $\mu_{\min} = 0$, and work similarly for points of positive index (see Section 4).

REMARK 1.6. A consequence of our proof is that for any diameter L with $\mu_{\min}(L) = 0$, there exists a non-negative Hamiltonian function H such that $d(\phi_H^1(L_0), L) \leq 1$.

Aknowlegments

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2 Two lemmas on the Maslov index

Let L be a fixed diameter. We first recall the abstract definition of the Maslov index gap between two transverse intersection points, as introduced by Viterbo in [3]. Let x, y be two transverse intersection points. Denote by $\gamma : [0, 1] \rightarrow L$ a curve on L with $\gamma(0) = x$ and $\gamma(1) = y$. The derivative $\dot{\gamma}$ determines a path $[\dot{\gamma}]$ in the projective space $\mathbb{R}P^1 = \mathbb{R}/\mathbb{Z}$. This path can be turned into a loop after concatenation with a path $\tau : [0, 1] \rightarrow \mathbb{R}P^1$, such that $\tau(0) = [\dot{\gamma}](1)$, $\tau(1) = [\dot{\gamma}](0)$ and such that for any s , $\tau(s)$ is transverse to L_0 . Let $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{R}$ be the lift of this loop satisfying $\tilde{\gamma}(0) = 0$. The index gap $\mu(y, L) - \mu(x, L)$ is then the Maslov index of the loop, i.e, the integer $\tilde{\gamma}(1)$.

We will need the two following lemmas.

Lemma 2.1. *Let x, y be two transverse intersection points between L and L_0 . Let $(h^t)_{t \in [0, 1]}$ be a compactly supported Hamiltonian isotopy of the disk. Suppose that for any $t \in [0, 1]$, the points x and y remain intersection points of L_0 and $h^t(L)$. Suppose moreover that the tangent line to $h^t(L)$ at x (resp. y) remains constant for $t \in [0, 1]$. Then the index gap remains constant:*

$$\forall t \in [0, 1], \mu(y, h^t(L)) - \mu(x, h^t(L)) = \mu(x, L) - \mu(y, L).$$

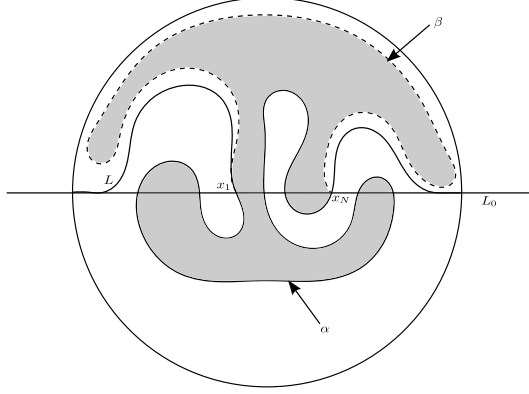


Figure 1: Proof of Lemma 2.2

Proof. — Let γ be a curve joining x to y on L , as above. Let α_t be a continuous family of curves joining x to $h^t(x)$ on $h^t(L)$ and β_t a continuous family of curves joining $h^t(y)$ to y on $h^t(L)$. Then consider the family of curves γ_t obtained as a smooth reparametrization of the concatenation of three paths:

$$\gamma_t = \alpha_t * (h^t \circ \gamma) * \beta_t.$$

Our assumptions imply that the family of loops $[\dot{\gamma}_t] * \tau$, where τ is as above, is a homotopy with fixed end points in $\mathbb{R}P^1$. Thus, $\tilde{\gamma}_t(1) = \tilde{\gamma}(1)$ for any t , and the lemma follows. $\square$

As in the introduction, we write $L \cap L_0 = (-1, -1+\varepsilon] \cup \{x_i\}_{i \in \{1, \dots, N\}} \cup [1-\varepsilon', 1)$, where $\varepsilon, \varepsilon' > 0$ and the points $x_1, \dots, x_N$ are transverse intersections ordered by their position on L .

Lemma 2.2. *We suppose that both arcs $L|_{[0, x_1]}$ and $L|_{[x_N, 1]}$ are included in D^+ . Then, $\mu(x_N) - \mu(x_1) = 1$.*

Proof. — Let ψ be a Hamiltonian diffeomorphism such that $L = \psi(L_0)$. Let γ be a smooth path on D which is the concatenation of two paths: a path α that parametrizes the arc $L|_{[x_1, x_N]}$ and a path β from x_N to x_1 disjoint from L_0 and L (see Figure 1). The path γ is a simple curve which is the boundary of an embedded disk. Therefore, the Maslov index of $[\dot{\gamma}]$ equals 2. Since the contribution of β to the Maslov index is obviously 1, it follows that the contribution of α is also 1. Thus, $\mu(x_N) - \mu(x_1) = 1$. $\square$

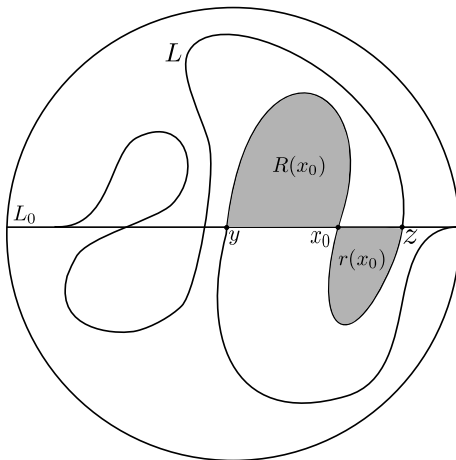


Figure 2: Large and small regions of a minimal index point

3 Removing points of minimal index

Let L be a diameter transverse to L_0 . In this section we describe simple Hamiltonian transformations that allow to remove an intersection point with extremal index. We describe it only for points of minimal index. Points of maximal index can be treated in a similar way.

We start with the following remark on the shape of L near a point of minimal index. Let x_0 be a point with minimal Maslov index. Consider the two transverse intersection points of L and L_0 which are consecutive to x_0 on L . Here, we make the assumption that those two points exist, i.e., that $x_0 \neq x_1$ and $x_0 \neq x_N$. The minimality of $\mu(x)$ imposes that one of them (call it y) is on the left of x_0 on L_0 and the other (call it z) is on its right. It also forces the arc $\alpha = L|_{[y, x_0]}$ to lie in D^+ and the arc $\beta = L|_{[x_0, z]}$ to lie in D^- (see figure 2).

We assume that one of the two closed regions delimited respectively by α and L_0 on one side, β and L_0 on the other side has a smaller area. This region will be called the *small region* of x_0 and denoted by $r(x_0)$. Its area will be called the *weight* of x_0 and denoted by $w(x_0)$. The largest region will be called the *large region* of x_0 and denoted $R(x_0)$. By *half-disk* we will mean any intersection of a smooth closed disk with one of the two standard half disks D^+ and D^- (figure 2).

Lemma 3.1 (Removing a minimal index point, figure 3). *Let $\varepsilon > 0$, U a neighbourhood of $r(x_0) \cup R(x_0)$, and A any half disk included in the interior of $R(x_0)$, of area less than $w(x_0)$ and such that any connected component of $A \cap L$ is an arc with at least one extremity on L_0 . Then there exists a Hamiltonian diffeomorphism ϕ supported in U*

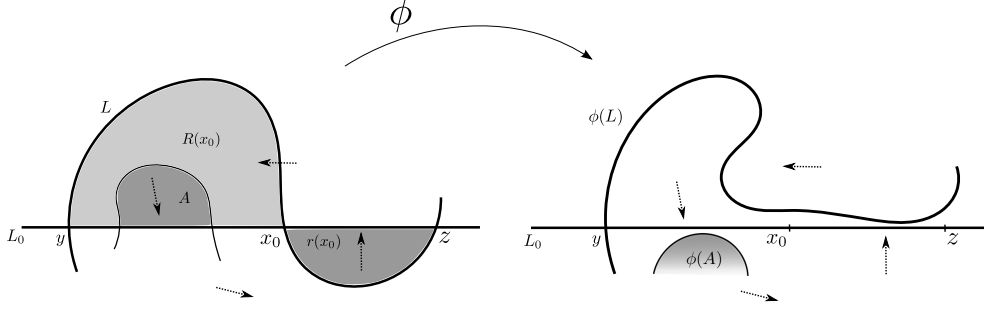


Figure 3: Illustration of Lemma 3.1 (the dotted arrows indicate roughly how ϕ acts)

with $\|\phi\| \leq w(x_0) + \varepsilon$, such that

- ϕ removes at least the intersection points x_0 and z and does not create new intersection points, i.e., $\phi(L) \cap L_0 \subset L \cap L_0 - \{x_0, z\}$,
- ϕ does not change the index gap between any two remaining intersection points,
- ϕ maps the small region $r(x_0)$ into the opposite half disk. In particular it removes all intersection points contained in $r(x_0)$.
- ϕ maps A into the opposite half disk.

We will use several times the following fact and therefore state it in a separate lemma (see Figure 4).

Lemma 3.2. *Let $R = [a, b] \times [c, d]$, $R' = [a', b'] \times [c', d']$ be two disjoint closed rectangles in $\mathbb{R}^2$ with same area α . We suppose moreover that $a' < b' < a < b$. Let γ (resp. γ') be a curve that does not meet $R \cup R'$ and joins a point of $[a, b] \times \{c\}$ (resp. $[a, b] \times \{d\}$) to a point of $[a', b'] \times \{c'\}$ (resp. $[a', b'] \times \{d'\}$). We also suppose that both curves are disjoint. Let $\varepsilon > 0$ and let V be an open neighbourhood of $R \cup R' \cup \gamma \cup \gamma'$. Then there exists a smooth autonomous Hamiltonian function H satisfying*

- $H = 0$ on the unbounded component of $\mathbb{R}^2 - V$ and $H = \alpha + 2\varepsilon$ on the bounded component of $\mathbb{R}^2 - V$.
- $\min H = 0$ and $\max H = \alpha + 2\varepsilon$
- On each rectangle R and R' , H is an affine function of x . More precisely, $H(x, y) = (c - d + \varepsilon)(x - b) + \varepsilon$ on R and $H(x, y) = (d' - c' + \varepsilon)(x - a') + \varepsilon$ on R' .
- $\phi_H^1(R) = R'$ and $\phi_H^1(R') = R$.

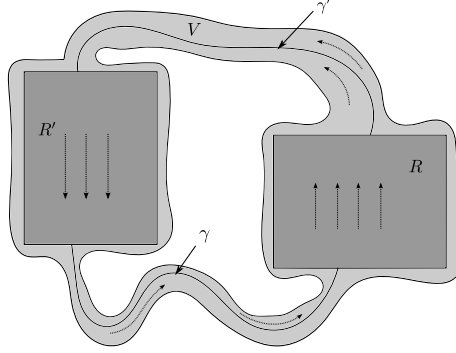


Figure 4: Illustration of Lemma 3.2.

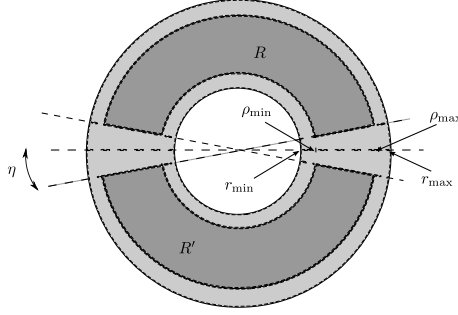


Figure 5: Image of Figure 4 by some area preserving diffeomorphism.

Note that the second condition on H implies that the Hofer energy of the time one map ϕ_H^1 is less or equal to $\alpha + 2\varepsilon$. Note also that the third condition means that the flow lines restricted to both rectangles are vertical straight lines.

Proof. Under some area preserving map, the situation is conjugated to that where V is a standard annulus of the form $\{(x, y) \in \mathbb{R}^2 \mid r_{\min}^2 < x^2 + y^2 < r_{\max}^2\}$. The transformation can also be chosen so that the rectangles become angular sectors of a sub-annulus. In polar coordinates (ρ, θ) , we may suppose that R is now

$$\{(\rho, \theta) \in (0, +\infty) \times (-\pi, \pi] \mid \rho_{\min} \leq \rho \leq \rho_{\max} \text{ and } \eta \leq \theta \leq \pi - \eta\}$$

and that R' is

$$\{(\rho, \theta) \in (0, +\infty) \times (-\pi, \pi] \mid \rho_{\min} \leq \rho \leq \rho_{\max} \text{ and } -\pi + \eta \leq \theta \leq -\eta\},$$

for some real numbers $\rho_{\min}$, $\rho_{\max}$ and η such that $r_{\min} < \rho_{\min} < \rho_{\max} < r_{\max}$, and $0 < \eta$. The straight vertical lines in the rectangles become pieces of circles centered at the origin. See figure 5.

Therefore, we only need to find H in this situation, which is easy: take H in the form $H(\rho, \theta) = h(\rho)$, where h is some non-increasing function which satisfies the following properties.

- if $\rho \leq r_{\min}$ then $h(\rho) = \alpha + 2\varepsilon$,
- if $\rho_{\min} \geq \rho \geq \rho_{\max}$ then $h(\rho) = \frac{1}{2}(\rho_{\max}^2 - \rho^2) + \varepsilon$,
- if $\rho \geq r_{\max}$ then $h(\rho) = 0$.

The associated flow acts as a linear rotation on the sub-annulus delimited by $\rho_{\min}$ and $\rho_{\max}$. The time one map makes half a turn and hence sends R onto R' and R' onto R as wished. $\square$

Proof of Lemma 3.1. — First, recall that at any moment of the proof we can apply a small Hamiltonian perturbation and that all objects may thus be supposed to be in generic position. Let $\varepsilon > 0$ sufficiently small and U an open neighbourhood of $r(x_0) \cup R(x_0)$. The first step of the proof is to note that after conjugation by some area preserving map of the plane, we may suppose that D^- and D^+ are rectangles, that $r(x_0)$ is included in a rectangle R of area less than $w(x_0) + \varepsilon$ and that this rectangle R is included in $U \cap D^-$. We may also suppose that A lies inside a rectangle $R' \subset D^+$ included in $R(x_0)$ and with the same area as R , with $R \cap R' = \emptyset$. The arcs which are the connected components of $L \cap R'$ either have both extremities on L_0 or may be assumed to be pieces of straight vertical lines with one extremity on L_0 .

In this situation, let γ' be a smooth curve included in $D^+ \cap U$ joining (as in Lemma 3.2) the top sides of R and R' , and γ be a smooth curve in $D^- \cap U$ joining the bottom sides of R and R' . Let V be a neighborhood of $R \cup R' \cup \gamma \cup \gamma'$ included in U and sufficiently narrow to avoid meeting L_0 except in the neighbourhood of R' and R . Now, we may apply Lemma 3.2 which provides us with a Hamiltonian function H . Then set $\phi = \phi_H^\tau$ where $\tau < 1$ will be appropriately chosen in the sequel.

Since ϕ_H^1 sends R onto R' , $r(x_0)$ is sent entirely into D^+ by ϕ provided $1 > \tau > \inf\{t \mid \phi_H^t(r(x_0)) \subset D^+\}$. In particular the intersection points x_0 and z disappear. In the same way A is sent into D^- . We have to make sure that no new intersection point appears. These new points would necessarily come from the arcs of L meeting V . If V is sufficiently narrow, such an arc must meet R , R' or $\gamma \cup \gamma'$. First remark that such an arc cannot meet R except if it is included in $r(x_0)$ (R is supposed sufficiently narrow). But then, the whole arc is sent to D^+ and no new intersection point appears. Then, note that the assumption made on the intersection $R' \cap L$ implies that along the

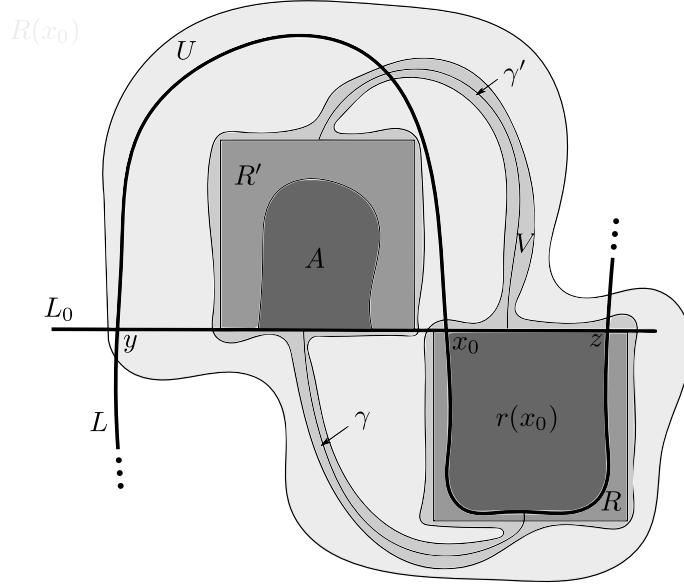


Figure 6: Proof of Lemma 3.1.

isotopy, the intersection points either disappear or do not move at all and no new point appears. Finally, the arcs which correspond to intersections between L and $\gamma \cup \gamma'$ do not create new intersection points either provided τ is sufficiently close to $\inf\{t \mid \phi_H^t(r(x_0)) \subset D^+\}$.

The intersection points that remain after applying ϕ are either those not contained in the support of ϕ or some of those included in R' . In both cases these points satisfy the assumptions of Lemma 2.1. Thus, for any pair of remaining points u, v , Lemma 2.1 gives: $\mu(v, \phi(L)) - \mu(u, \phi(L)) = \mu(v, L) - \mu(u, L)$.

Finally, according to Lemma 3.2, our diffeomorphism ϕ has energy less than approximately the area of the rectangle R . But this area is also approximately the area of $r(x_0)$, that is $w(x_0)$. $\square$

REMARK 3.3. Note that if we take the isotopy ϕ_H^t of the proof of Lemma 3.1 at some smaller time, we can construct for any half-disk $A \subset R(x_0)$ of area less than $w(x_0)$ a diffeomorphism similar to that in Lemma 3.1 that does not remove the intersection point x_0 , but decreases $r(x_0)$ by approximately the area of A . The energy needed is approximately the area of A . We will call this procedure a *partial removing* of x_0 associated to A .

REMARK 3.4. We consider again an isotopy like the one appearing in the proof of Lemma 3.1. Let B be a half disk included in D^- and

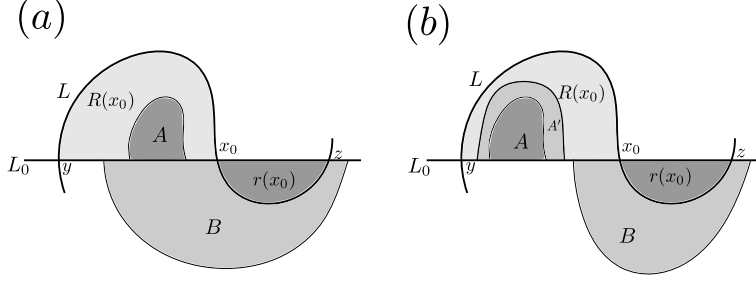


Figure 7: Illustration of Remark 3.4

containing $r(x_0)$. It will be useful to figure out how the area of B behaves along the isotopy ϕ_H^t in the following two cases.

First, if $A \cap L_0 \subset B \cap L_0$, then it is possible to choose the support of the isotopy so that it does not meet $\partial B \setminus L_0$. In this case, the area of B remains constant along the isotopy (see Figure 7 (a)).

Second, if A is entirely on the left of B (i.e., $\forall a \in A \cap L_0, \forall b \in B \cap L_0, a \leq b$). Then, for any other half disk A' containing A and on the left of B , we can choose the path γ (resp. γ') so that once it has entered B (resp. A') it does not exit B (resp. A') anymore (i.e., $\gamma \cap B$ is connected). This choice made, we see that along our (area preserving) isotopy, the respective areas of A , A' and B decrease at the same speed exactly (see Figure 7 (b)).

With the help of these remarks, we can prove the following refinement of Lemma 3.1, illustrated by Figure 8.

Lemma 3.5. *Let $\varepsilon > 0$ and U be an open neighbourhood of $r(x_0) \cup R(x_0)$. Let $A_1, \dots, A_p = A'_1 \dots A'_n, B_1, \dots, B_n$ be closed half disks such that*

- *the A_i 's are included in $R(x_0)$, are pairwise disjoint and are ordered by their position on L_0 from the right to the left,*
- $\sum_{i=1}^{p-1} \text{area}(A_i) < w(x_0)$,
- $A_p = A'_1 \subset \dots \subset A'_n \subset R(x_0) \subset D_+$,
- $r(x_0) \subset B_n \subset \dots \subset B_1 = B \subset D_-$,
- *for any index i , A'_i is on the left of B_i and intersects at one point exactly,*
- *for any indices i, j , either $A_i \subset A'_j$ or $A_i \cap A'_j = \emptyset$.*

Then, there exists a Hamiltonian diffeomorphism ϕ supported in U with $\|\phi\| \leq w(x_0) + \varepsilon$, such that

- $\phi(L) \cap L_0 \subset L \cap L_0 - \{x_0, z\}$,

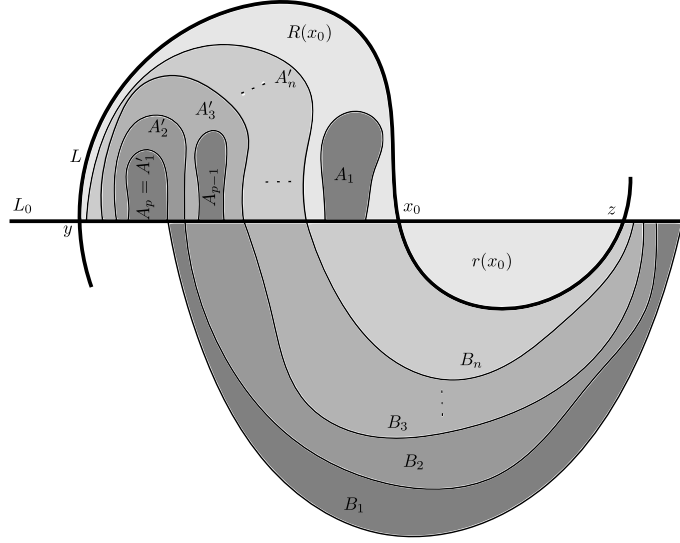


Figure 8: Settings of Lemma 3.5

- ϕ does not change the index gap between two remaining intersection points,
- ϕ maps the small region $r(x_0)$ into the opposite half disk,
- ϕ maps each A_i , $1 \leq i \leq p-1$ into the opposite half disk,
- for any index $j \in \{1, \dots, n\}$, the map ϕ decreases the areas of A'_j and B_j by exactly the same amount, except in the special case $j = 1$ and $w_0 > \sum_{i=1}^p \text{area}(A_i)$, for which $A'_1 = A_p$ is entirely mapped to the opposite half disk.

Proof . — We first apply successively partial removings of x_0 associated to each A_i with $1 \leq i \leq p-1$ (see Remark 3.3). We get diffeomorphisms $\phi_1, \dots, \phi_{p-1}$. We can choose them so that the support of each ϕ_i meets neither the A_j 's for $j > i$, nor the A'_j 's that do not contain A_i . Moreover, according to Remark 3.4, this can be done so that for any index j , if A'_j contains A_i , then the areas of A'_j and B_j are reduced by the same amount exactly, and if A'_j does not contain A_i , then A'_j and B_j remain globally unchanged.

After the action of the composition $\phi_{p-1} \circ \dots \circ \phi_1$, the half-disks $A_1, \dots, A_{p-1}$ have been moved to D_- , the small region $r(x_0)$ has now area $c = w(x_0) - \sum_{i=1}^{p-1} \text{area}(A_i)$ and moreover $A_p = A'_1$ and B_1 remain unchanged.

Then, two cases may occur. If $c > \text{area}(A_p)$, then we apply Lemma 3.1 for a half-disk A containing A_p and of area approximately c . If

$c < \text{area}(A_p)$, then we apply Lemma 3.1 for a half-disk A included in A_p and of area approximately c . In the first case, this gives a diffeomorphism ϕ_p which maps A_p to D_- . In the second case, this gives a diffeomorphism ϕ_p which, according to Remark 3.4, can be chosen so that both areas of $A_p = A'_1$ and B_1 are decreased by c . Finally we see that the diffeomorphism $\phi = \phi_p \circ \cdots \circ \phi_1$ suits our needs. $\square$

4 Proof of the theorem

We will prove Theorem 1.2 by induction on the maximal index gap $\mu_{\max}(L) - \mu_{\min}(L)$. Before to start, we apply an arbitrarily small perturbation on L so that we may assume that $\mu(x_1) = \mu(x_N) = 0$. This can be achieved by creating new transverse intersection points near the boundary to be in the situation of Lemma 2.2 and then add one more intersection point to get $\mu(x_1) = \mu(x_N)$.

First, suppose that $\mu_{\max}(L) - \mu_{\min}(L) = 1$. In this case, the intersection points of L and L_0 are in a very simple configuration: the indices are successively $0, \pm 1, 0, \pm 1, \dots$ and there exists some area preserving map of the plane, that preserves the x -axis and sends L to the graph of the differential of a compactly supported function $S : \mathbb{R} \rightarrow \mathbb{R}$. In this case, the Hofer distance is less than $\max S - \min S$. This value can be interpreted as the difference between the areas of some domains delimited by L and L_0 respectively in D_+ and D_- . This leads to the estimate $d(L, L_0) \leq \frac{1}{2}$, which proves the theorem in the case $\mu_{\max}(L) - \mu_{\min}(L) = 1$.

Suppose now that $\mu_{\max}(L) - \mu_{\min}(L) > 1$. Then, at least one of the two integers $\mu_{\min}(L)$ and $\mu_{\max}(L)$ does not vanish. Suppose $\mu_{\min}(L) < 0$. We are going to construct a Hamiltonian diffeomorphism h , with energy $\|h\| \leq 1$, such that $\mu_{\min}(h(L)) = \mu_{\min}(L) + 1$. If $\mu_{\max}(L) > 0$, an analogous construction could also be performed to reduce $\mu_{\max}(L)$ by 1. In any case, we are able to reduce the maximal index gap $\mu_{\max} - \mu_{\min}$ by 1. It follows by induction that there exists a Hamiltonian diffeomorphism g that sends L to a diameter L' with $\mu_{\max}(L') - \mu_{\min}(L') = 1$ and $\|g\| \leq \mu_{\max}(L) - \mu_{\min}(L) - 1$. This will conclude the proof of Theorem 1.2 since

$$d(L, L_0) \leq d(L, L') + d(L', L_0) \leq \|g\| + \frac{1}{2}.$$

Let us now suppose $\mu_{\min}(L) < 0$ and construct a Hamiltonian diffeomorphism h with energy less than 1 and such that $\mu_{\min}(h(L)) = \mu_{\min}(L) + 1$.

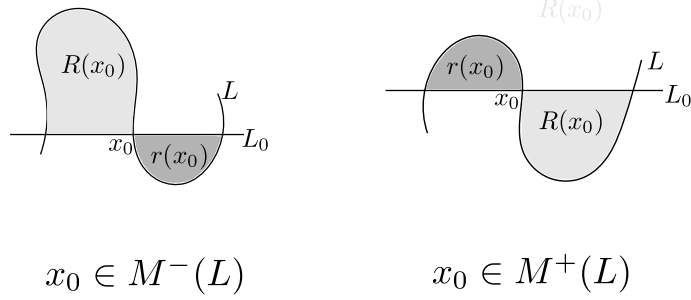


Figure 9: Intersection points belonging to M^- or M^+ .

Let $M(L)$ be the (finite) set of transverse intersection points of L with L_0 whose index is $\mu_{\min}(L)$. Since $\mu_{\min}(L) < 0$, the points x_1 and x_N do not belong to $M(L)$. Let $M^-(L)$ (resp. $M^+(L)$) be the set of all $x \in M(L)$ whose small region is contained in the lower (resp. upper) half-disk (Figure 9). The points of $M(L)$ are ordered by their position on L_0 . We will denote by $\preceq$ this order relation and by $\prec$ the associated strict order. Namely, $x \prec y$ means that x is strictly on the left of y . In a similar way, a region delimited by L , L_0 and two intersection points y, z (i.e., a Whitney disk) will be said to be *on the left* of an intersection point x if y and z are on the left of x .

When we consider a minimum point in $(M^-(L), \preceq)$, we can refine Lemma 3.1 as follows.

Lemma 4.1. *Let x_0 be the minimum in $(M^-(L), \preceq)$. Then there exists a Hamiltonian diffeomorphism ϕ meeting the conditions of Lemma 3.1 and satisfying moreover the following:*

- (i) $M(\phi(L)) \subsetneq M(L)$ and $\forall q \in M^-(\phi(L)), q \succ x_0$.
- (ii) For any $q \in M(\phi(L))$, consider the region $\mathcal{R}$ of q that is included in D^- where q is seen as a point of $M(L)$. If $\mathcal{R}$ is on the right of $r(x_0)$, then $\mathcal{R}$ does not meet the support of ϕ .
- (iii) $M^+(\phi(L)) \subset M^+(L)$.

Let us postpone the proof of this lemma to the end of the section and achieve the proof of Theorem 1.2.

Applying Lemma 4.1 to the minimum point x_1 in $(M^-(L), \preceq)$, we get a Hamiltonian diffeomorphism ϕ_1 meeting several conditions. Then, we repeat inductively the operation on the successive minimum points $x_1, x_2, x_3, \dots$, constructing diffeomorphisms $\phi_1, \phi_2, \phi_3, \dots$ until M^- is empty. The process stops at some point since according to property (i) in Lemma 4.1 the (finite) cardinal of $M(L)$ decreases at

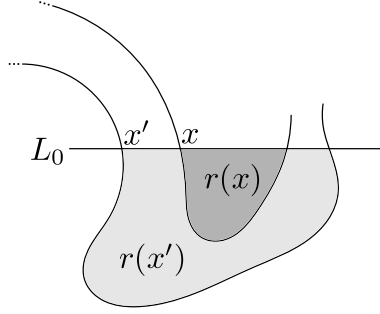


Figure 10: If $r(x) \subset r(x')$, then $x' \preceq x$.

each step. We have constructed a finite family of Hamiltonian diffeomorphisms $\phi_1, \dots, \phi_p$ such that if we write $\psi = \phi_p \circ \dots \circ \phi_1$ then $M^-(\psi(L)) = \emptyset$.

Moreover, the energy of each ϕ_i is approximately the area $w(x_i)$ of the corresponding small region $r(x_i)$. But from conditions (i) and (ii) in Lemma 4.1, the small regions of the points involved in the process are pairwise disjoint. Indeed, these conditions imply that each $r(x_i)$, which is a priori a region of x_i seen as a point in $M(\phi_{i-1} \dots \phi_1(L))$, is actually already a region of x_i seen as a point in $M(L)$ (the region is invariant under each successive diffeomorphism). But, two such regions are either disjoint or nested. This last possibility cannot happen for two small regions involved in our process: the largest of those two small regions would be associated to the $\preceq$ -smallest intersection point (Figure 10), thus we would apply Lemma 4.1 first to this larger small region during the process and this would remove the other intersection point. As a consequence, the energy of ψ is bounded above as follows:

$$\|\psi\| \leq \sum_{i=1}^p \|\phi_i\| \leq \sum_{i=1}^p (w(x_i) + \varepsilon) \leq \text{area}(D^-) + p\varepsilon = \frac{1}{2} + p\varepsilon.$$

Since we can do it for any small enough $\varepsilon > 0$, we get $\|\psi\| \leq \frac{1}{2}$.

Now we apply the same process to the points of $M^+(L)$: we exchange the roles of $M^+(L)$ and $M^-(L)$, which means in particular that we have to start from the right. We get a Hamiltonian diffeomorphism ψ' with Hofer energy less than $\frac{1}{2}$ and such that $M^+(\psi' \circ \psi(L)) = \emptyset$. Moreover, it follows from Lemma 4.1 (iii) that $M^-(\psi' \circ \psi(L)) \subset M^-(\psi(L)) = \emptyset$. Hence $M(\psi' \circ \psi(L)) = \emptyset$.

As a conclusion, we can move with energy at most 1 any diameter with given $\mu_{\min}$ to a diameter with minimal index $\mu_{\min} + 1$. As explained in the beginning of the section, this concludes the proof of Theorem 1.2. $\square$

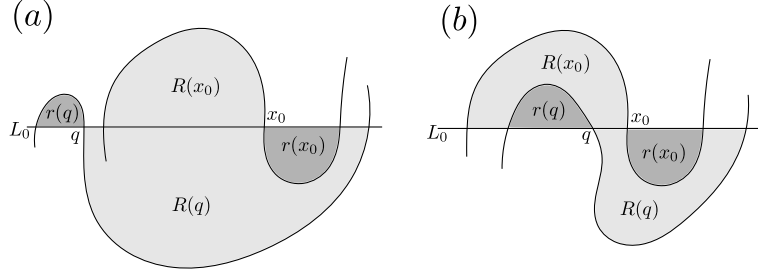


Figure 11: Two possibilities for $x_0 \in R(q)$: (a) $q \notin R(x_0)$, (b) $q \in R(x_0)$.

Proof of Lemma 4.1. —

(i) According to Lemma 3.1, ϕ removes some intersection points letting the remaining intersection points at the same place and with the same index. Thus, $M(\phi(L)) \subset M(L)$ is obvious.

Let $q \in M^-(\phi(L))$. First, $q \neq x_0$ because the intersection x_0 is removed by ϕ . If $q \in M^-(L)$ then $q \succ x_0$ by definition of x_0 . Let us show that ϕ may be chosen so that $q \in M^+(L)$ implies $q \succ x_0$. The two conditions $q \in M^+(L)$ and $q \in M^-(\phi(L))$ mean that either the area of $r(q)$ is expanded by ϕ or that of $R(q)$ is reduced by ϕ (maybe both). We will study these two cases separately. But we must first remark that the vector field generating ϕ is oriented downward for points of L_0 located on the left of (a neighbourhood of) x_0 and is oriented upward for points of L_0 located on the right of x_0 (see Figures 6 and 4).

Suppose now that $r(q)$ is expanded by ϕ . Then, since it is included in D^+ , it cannot be entirely on the left of x_0 because otherwise, ϕ would decrease it. Since q is on the right of $r(q)$, it follows that q is on the right of x_0 .

Suppose now that $R(q)$ is reduced by ϕ . Then a similar argument implies that $R(q)$ is not entirely on the left of x_0 and two cases could happen: either $R(q)$ is entirely on the right of x_0 , or x_0 belongs $R(q)$. The first case implies $q \succ x_0$ as wished. In the second case, there are again two possibilities (Figure 11): (a) $q \notin R(x_0)$, (b) $q \in R(x_0)$. We want to avoid these two cases. To avoid the case (a) we only have to choose the map ϕ to have a support sufficiently small so that $R(q)$ remains invariant. Avoiding the case (b) requires a bit more work and we will need to apply the refined Lemma 3.5 to appropriate regions A_i , A'_j and B_j , $1 \leq i \leq p$, $0 \leq j \leq n$.

Suppose that the set C of points $q \in R(x_0) \cap M^+(L)$ such that $r(x_0) \subset R(q)$ is not empty. We want to choose ϕ so that C does not meet $M^-(\phi(L))$. To do so, we classify the points in C into a partition

of four sets C_1, C_2, C_3 and C_4 , constructed by the following algorithm.

We initialize our sets as $C_1 = \emptyset, C_2 = \emptyset, C_3 = \emptyset$ and $C_4 = C$. We also initialize some auxiliary variables: a set $E = C$, a real variable $\alpha = 0$ and a point $q = \max(E)$. Then we iterate the following operations until $E = \emptyset$.

- (1) If $E \neq \emptyset$ and $\alpha + \omega(q) < w(x_0)$, then modify all the variables by the rules (the symbol " $:=$ " means "is replaced by"):

$$\begin{aligned} C_1 &:= C_1 \cup \{q\} \\ C_2 &:= C_2 \\ C_3 &:= C_3 \cup \{q' \in E \mid q' \in r(q), q' \neq q\} \\ C_4 &:= C_4 \setminus \{q' \in E \mid q' \in r(q)\} \\ \alpha &:= \alpha + w(q) \\ E &:= E \setminus \{q' \in E \mid q' \in r(q)\}. \end{aligned}$$

- (2) If $E \neq \emptyset$ and $\alpha + \omega(q) > w(x_0)$, then modify the variables by the rules:

$$\begin{aligned} C_1 &:= C_1 \\ C_2 &:= C_2 \cup \{q\} \\ C_3 &:= C_3 \\ C_4 &:= C_4 \setminus \{q\} \\ \alpha &:= \alpha \\ E &:= \{q' \in E \mid r(q') \subset r(q), q' \neq q\}. \end{aligned}$$

The result of the algorithm is illustrated by Figure 12. Once it has been performed, we decide to call $A_1, A_2, \dots$ the small regions of the points of C_1 , ordered from the right to the left. We denote by $A'_1, A'_2, \dots$ the small regions of the points of C_2 , ordered by inclusion from the smallest to the largest. We also denote by $B_1, B_2, \dots$ the large regions associated to $A'_1, A'_2, \dots$. This family of regions satisfies the settings of Lemma 3.5 which provides us with a diffeomorphism ϕ with good properties. Let us analyse how it acts on the set C . First, points in C_1 are removed. Then, points in C_2 remain but the area of their small region decreases by exactly the same amount as their large region. Points in C_3 have their small regions included in small regions of points in C_1 and thus are removed. Finally points in C_4 have their small and large regions disjoint from the support of ϕ (they are on the left of its support). In any cases, we see that no points of C belong to $M_-(\phi(L))$, which is exactly what we wished for.

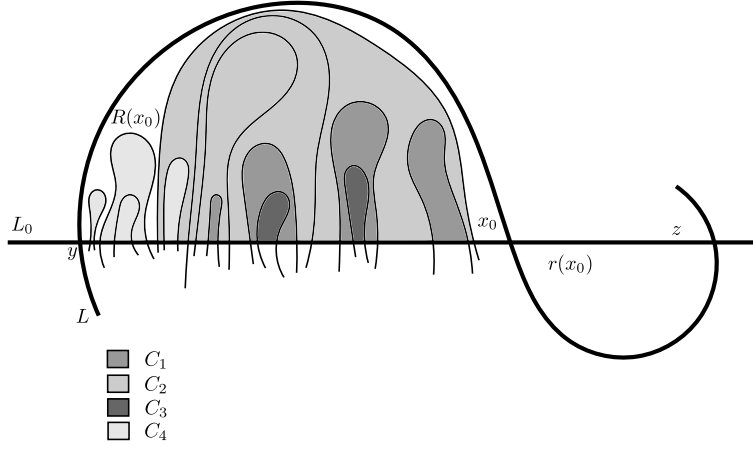


Figure 12: Small regions of points in C are represented. The small regions are colored according to their belonging to C_1 , C_2 , C_3 or C_4 at the end of the algorithm.

(ii) This condition is obviously satisfied provided the support of ϕ is chosen sufficiently narrow.

(iii) Let $x \in M^+(\phi(L))$. Suppose $r(x) \subset D^-$ where x is seen as an intersection point of L and L_0 (i.e. $x \in M^-(L)$). Then either the region $r(x)$ is increased by ϕ or $R(x)$ is decreased (it may be both). But this imposes $x \prec x_0$. Contradiction. $\square$

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