Subleading asymptotics of link spectral invariants and homeomorphism groups of surfaces

Dan Cristofaro-Gardiner, Vincent Humilière, Cheuk Yu Mak
Sobhan Seyfaddini and Ivan Smith

June 20, 2022

Abstract

In previous work, we defined "link spectral invariants" for any compact surface and used these to study the algebraic structure of the group of area-preserving homeomorphisms; in particular, we showed that the kernel of Fathi's mass-flow homomorphism is never simple. A key idea for this was a kind of Weyl law, showing that asymptotically the link spectral invariants recover the classical Calabi invariant.

In the present work, we use the subleading asymptotics in this Weyl law to learn more about the algebraic structure of these homeomorphism groups in the genus zero case. In particular, when the surface has boundary, we show that the kernel of the Calabi homomorphism on the group of hameomorphisms is not simple, answering an old question of Oh and Müller; this contrasts the smooth case, where the kernel of Calabi is simple. We similarly show that the group of hameomorphisms of the two-sphere is not simple. Related considerations allow us to extend the Calabi homomorphism to the full group of compactly supported area-preserving homeomorphisms, answering a longstanding question of Fathi. In fact, we produce infinitely many distinct extensions.

Central to the applications is that we show that the subleading asymptotics for smooth, possibly time-dependent, Hamiltonians are always $O(1)$, and for certain autonomous maps recover the Ruelle invariant. The construction of a hameomorphism with "infinite Ruelle invariant" then shows that a normal subgroup with $O(1)$ subleading asymptotics is proper.

Contents

1 Introduction

Area-preserving homeomorphisms of surfaces

Let (M, ω) be a compact manifold possibly with boundary, equipped with a volume-form, and consider the group $\mathrm{Homeo}_c(M, \omega)$ of volume-preserving homeomorphisms that are the identity near the boundary, in the component of the identity.

When the dimension of M is at least three, there is a clear picture due to Fathi regarding the algebraic structure of this group: there is a mass-flow homomorphism, and its kernel is a simple group. In contrast, in dimension two the situation is much less understood despite the fact that many decades have passed since Fathi's work.[1]

We recently showed [6] that when $\dim(M) = 2$, the kernel of mass-flow is never simple. In fact, it contains as a proper normal subgroup the group $\mathrm{Hameo}(M, \omega)$ of *hameomorphisms*, whose definition we review in Definition 2.2. When M has boundary, we also showed that the classical Calabi homomorphism, which we review in Definition 2.1 and which measures the average rotation of the map, extends to Hameo from the group $\mathrm{Ham}_c(M, \omega)$ of Hamiltonian diffeomorphisms that are the identity near the boundary. It is then natural to ask the following.

[1] We refer the reader to [6] for background, including a definition of the mass-flow homomorphism. In the cases that we will be mainly concerned with here, that is the disc and the sphere, the kernel of mass-flow is just the group of area and orientation preserving homeomorphisms.

Question 1.1. *When M is closed, is* $\mathrm{Hameo}(M, \omega)$ *simple? When M has non-empty boundary, is the kernel of Calabi on* $\mathrm{Hameo}(M, \omega)$ *simple?*

This is an old question. For example, a variant appears in [27, Problem (4)]. Let us briefly explain why one might hope for a positive answer. Hameomorphisms are homeomorphisms with well-defined Hamiltonians, and it is natural to wonder whether the algebraic structure of the group of hameomorphisms could be like that of the group $\mathrm{Ham}_c(M, \omega)$; moreover, Banyaga showed [1] that Ham_c is simple when M is closed and the kernel of Calabi is simple when M has boundary.

Our first result shows that the structure of Hameo is more complicated than this.

Theorem 1.2. *The following groups are not simple:*

1. *The kernel of Calabi on* $\mathrm{Hameo}(\mathbb{D}^2, \omega)$.
2. *The group* $\mathrm{Hameo}(\mathbb{S}^2, \omega)$.

A two-term Weyl law

Theorem 1.2 is proved by studying the asymptotics of the "link spectral invariants" defined in our previous work [6]. In [6] we defined quasimorphisms

$$\mu_k : \mathrm{Diff}(\mathbb{S}^2, \omega) \to \mathbb{R}, \quad f_k : \mathrm{Homeo}_c(\mathbb{D}^2, \omega) \to \mathbb{R}$$

and we showed that these satisfy the important asymptotic formulae

$$\lim_{k \to \infty} f_k(g) = \mathrm{Cal}(g) \tag{1}$$

on $\mathrm{Diff}_c(\mathbb{D}^2, \omega)$, and

$$\lim_{k \to \infty} \mu_k(g) = 0.$$

We called this the "Calabi property". Here, Cal denotes the aforementioned Calabi homomorphism and Diff_c denotes the group of diffeomorphisms that are the identity near the boundary and that preserve ω, which we note for the reader coincides with the group Ham_c in the above cases. We refer the reader to our review in Section 2 for more details about the μ_k and f_k.

The above formulas are kinds of Weyl laws[2], and it is natural to ask what can be said about the subleading asymptotics. With many seemingly similar kinds of Weyl laws, this tends to be a hard question. For example, the above Calabi property was inspired by an analogous Weyl law for the related "ECH spectral invariants" defined in [19], see [9]. For these spectral invariants, all that is known is a bound on the growth rate of the subleading asymptotics [11] that is likely far from optimal, with the conjectural bound being $O(1)$ [20].

[2]For specialists, we note that the convergence to zero for the μ_k is what one would hope for in a Weyl law, since these invariants are defined via mean normalization of Hamiltonians.

In contrast, it turns out that we are able to say quite a lot about the subleading asymptotics of the μ_k. To state our result, let Ru denote the Ruelle invariant from [34] (see also [17, 18]), which we review in Section 2.1.3. We now state a result that is central to our proof of Theorem 1.2 and which is also of independent interest.

Theorem 1.3. *If $\psi \in \mathrm{Diff}(\mathbb{S}^2, \omega)$ (resp. $\psi \in \mathrm{Diff}_c(\mathbb{D}^2, \omega) \cap \ker(\mathrm{Cal})$), then the sequence $\{k\, \mu_k(\psi)\}_{k \in \mathbb{N}}$ (resp. $\{k\, f_k(\psi)\}_{k \in \mathbb{N}}$) is uniformly bounded. In fact, if $\psi = \phi_H^1$, where $H : \mathbb{D}^2 \to \mathbb{R}$ is an autonomous and compactly supported Hamiltonian on the disc with finitely many critical values, then*

$$\lim_k k\mu_k(\psi) = \lim_k k(f_k(\psi) - \mathrm{Cal}(\psi)) = \mathrm{Cal}(\psi) - \frac{1}{2}\mathrm{Ru}(\psi). \tag{2}$$

A similar result concerning the subleading asymptotics of the μ_k in the case of autonomous Hamiltonians on the sphere with finitely many critical values also holds, but for brevity (and because the Ruelle invariant is not defined over the sphere without further choices), we do not state it.

Remark 1.4. In the statement of the above theorem, we are implicitly invoking the fact that we can regard any $\psi \in \mathrm{Diff}_c(\mathbb{D}^2, \omega)$ as a map of the two-sphere by embedding $\mathbb{D}^2$ as a hemisphere and extending by the identity, for our conventions see Section 2.2.4, when we write $\mu_k(\psi)$ in (2); we will continue to do this throughout this paper. The invariants μ_k and f_k can be thought of as invariants of (possibly time-dependent) Hamiltonians as well, by setting $\mu_k(H) := \mu_k(\phi_H^1)$ and $f_k(H) := f_k(\phi_H^1)$. This viewpoint is helpful and adopted in [6], as well as Section 3 here.

In view of Theorem 1.3 it is natural to ask if (2) holds more generally. For the aforementioned ECH spectral invariants, essentially the same question was asked, under a genericity assumption on the contact form [20]. In the ECH case, simple examples exist, for example the boundary of the round sphere, with no well-defined subleading asymptotic limit at all; in this sense, then, the genericity assumption can not be dropped. In our case, however, we know of no such analog, and indeed Theorem 1.3 asserts that in the simplest cases, the subleading asymptotics in fact always recover Ruelle. We therefore pose as a question the following.

Question 1.5. *Is it the case that for any $\psi \in \mathrm{Diff}_c(\mathbb{D}^2, \omega)$,*

$$\lim_k k\mu_k(\psi) = \lim_k k(f_k(\psi) - \mathrm{Cal}(\psi)) = \mathrm{Cal}(\psi) - \frac{1}{2}\mathrm{Ru}(\psi)?$$

We emphasize that, in contrast to the ECH case, we are not requiring any genericity in ψ in the above question. We return to this question in Section 7, where we explore some heuristic considerations.

Remark 1.6. There is a cousin of ECH, called PFH, defined for area-preserving diffeomorphisms of surfaces. One can define homogenized spectral invariants with PFH, see [7], and it seems likely that these agree with the μ_k; this would follow from [4], for example, if it was known that the homogenized PFH spectral invariants are quasimorphisms. If this agreement of spectral invariants is proved, then it would imply the analogous two-term Weyl law as in Theorem 1.3 for these PFH invariants. For “one-term” Weyl laws for PFH, computing the leading asymptotics, see [10, 12].

Remark 1.7. Our interest in the present work is with homogenized spectral invariants because for applications to the algebraic structure of $\mathrm{Homeo}_c(M, \omega)$, these seem preferable.

Infinitely many extensions of Calabi and the Simplicity Conjecture revisited

Consideration of the asymptotics of the μ_k also leads to the resolution of an old question about the aforementioned Calabi homomorphism.

Question 1.8 ([14])**.** *Does* Cal *extend from* $\mathrm{Diff}_c(\mathbb{D}^2, \omega)$ *to* $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$*?*

Question 1.8 has a long history which is closely connected to the question of whether or not the group $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$ is simple; see for example [18, Sec. 2.2]. It is known that no C^0-continuous extension can exist, because the kernel of Cal is C^0-dense. It was also recently shown that this group is in fact not simple [8], resolving the longstanding “Simplicity Conjecture”. However, the question of whether an extension as a homomorphism exists has remained open.

One might guess that no such extension exists. For example, many groups of homeomorphisms satisfy an automatic continuity property, see for example [23], and as was stated above, it is known that a continuous extension can not exist; see also Remark 5.1 below. On the contrary, however, we have the following result.

Theorem 1.9. *The Calabi homomorphism admits infinitely many extensions to the group* $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$*.*

It follows from Theorem 1.9 that the group $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$ is not simple. This gives another proof of the aforementioned “Simplicity Conjecture”. It should be emphasized that our proof uses the nontrivial construction of the f_k from [6], so is not self-contained; on the other hand, it does give a new proof, deducing nonsimplicity purely algebraically from the existence of a geometrically constructed homomorphism out of $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$. This kind of argument for proving non-simplicity is much more in line with how non-simplicity is proved for related groups, see the summary in [8, Sec. 1.1.1], so it is natural to hope for a proof like this. Moreover, this perspective has value in finding new normal subgroups: to keep the introduction focused, we defer the precise statement regarding these subgroups to Section 5.2 below.

Remark 1.10. That one can partially extend the Calabi homomorphism, namely to a class of radial maps, via a limit construction involving (unhomogenized) link spectral invariants is stated in [29, Remark 2].

Simplicity

Let $G := \mathrm{Homeo}_c(\Sigma, \omega)$, where Σ is some compact surface. Given Theorem 1.2, it is natural to ask if *some* simple non-trivial normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$ exists. After all, there certainly exist groups (e.g. $\mathbb{Z}$) with no simple normal subgroups at all.

Theorem 1.11. *The commutator subgroup $[G, G]$ is simple.*

The proof of Theorem 1.11 is completely independent of our other results, and does not use link spectral invariants at all. In fact, we should note that from a certain point of view, Theorem 1.11 is not too surprising. Indeed, the commutator subgroup of $[G, G]$ is normal in G, so standard arguments as in [14], see in particular the exposition in [8, Prop. 2.2]), show that $[G, G]$ is perfect; and, for many transformation groups, perfectness and simplicity are equivalent. So, it might be the case that Theorem 1.11 is known to experts. However, we are unable to find any proof in the literature and so we include one here.

It would be very interesting to find a geometric characterization of $[G, G]$. In the diffeomorphism case, Banyaga has shown [1] that $[G, G]$ is the kernel of Cal.

Themes of the proofs and outline of the paper

A crucial fact for many of our arguments is the following estimate from [6] on the defect of the f_k. (We refer the reader to 2.2.2 for preliminaries about quasimorphisms.)

Lemma 1.12 ([6])**.** *The f_k and μ_k are quasimorphisms of defect $\frac{2}{k}$.*

This is a key property that powers many of our arguments and one goal of our paper is to illustrate the usefulness of this fact. The basic idea is that this defect property allows us to detect interesting normal subgroups and construct interesting homomorphisms; on the other hand, our two-term Weyl law from above allows us to recover the Calabi and Ruelle invariants, which are among the most studied invariants of area-preserving disc maps, from the f_k, for a wide class of diffeomorphisms.

We put this together as follows. In our previous work, we studied twist maps with "infinite Calabi" invariant, defined via the leading asymptotics of the f_k, to show that Hameo is proper. Here, we study twist maps with "infinite Ruelle invariant," defined via the asymptotics of the kf_k, to show nonsimplicity on Hameo. More precisely, we define a subgroup of elements with $O(1)$ subleading asymptotics and we show that this contains all smooth Hamiltonian diffeomorphisms, but we show that it is proper by constructing a hameomorphism with "infinite Ruelle invariant;" see Proposition 4.1.

We further comment on the contrast between "infinite Ruelle" and "infinite Calabi" in Remark 4.4.

Summary of our knowledge of the normal subgroup structure

It seems to us useful to summarize in one place what is known about the normal subgroup structure for the groups that concern us here, and what remains to be understood.

We start with the case of diffeomorphisms, established by Banyaga, for the sake of comparison. Let G^∞ denote the relevant group. As mentioned above, in the case of $\mathbb{S}^2$, we have

$$[G^\infty, G^\infty] = \mathrm{Diff}(\mathbb{S}^2, \omega),$$

and in the case of $\mathbb{D}^2$ we have

$$[G^\infty, G^\infty] = \ker(\mathrm{Cal}) \subsetneq \mathrm{Diff}_c(\mathbb{D}^2, \omega).$$

Moreover, $[G^\infty, G^\infty]$ is simple; and, in the disc case, we have

$$G^\infty/[G^\infty, G^\infty] \simeq \mathbb{R}. \tag{3}$$

The case of homeomorphisms seems quite different: a striking phenomena, which seems genuinely new, is a plethora of normal subgroups arising from different geometric considerations.

To elaborate, we described above the subgroup Hameo, which one can think of as those homeomorphisms that can be said to have Hamiltonians. There is another normal subgroup FHomeo, containing Hameo, whose precise definition we skip for brevity: one can think of it as the largest normal subgroup for which Hofer's geometry can be defined. Buhovsky has recently shown [2] that FHomeo and Hameo do not coincide. As mentioned above, we showed in [8, 6], resolving in particular the Simplicity Conjecture, that FHomeo is proper. We can therefore summarize the situation regarding these groups, prior to this work, as follows. Let G denote the group of area and orientation preserving homeomorphisms of $\mathbb{S}^2$ or the group of compactly supported area-preserving homeomorphisms of $\mathbb{D}^2$.

For $\mathbb{S}^2$, we have

$$[G, G] \subset \mathrm{Hameo}(\mathbb{S}^2, \omega) \subsetneq \mathrm{FHomeo}(\mathbb{S}^2, \omega) \subsetneq G$$

For $\mathbb{D}^2$ we have

$$[G, G] \subset \ker(\mathrm{Cal}) \subsetneq \mathrm{Hameo}(\mathbb{D}^2, \omega) \subset \mathrm{FHomeo}(\mathbb{D}^2, \omega) \subsetneq G,$$

where here Cal denotes the extension of the Calabi homomorphism mentioned above that we established in [6]; one expects the inclusion of Hameo into FHomeo to be proper by the arguments in [2].

Our work here shows that the left most inclusions are proper, by constructing an explicit normal subgroup, and that $[G, G]$ is simple. The normal subgroup we construct to show properness does contain $[G, G]$, but we do not know if this inclusion is proper.

To set the context for describing more normal subgroups, it is natural to wonder if (3) has any counterpart for homeomorphisms. We do not currently understand the quotient $G/[G, G]$; in fact none of the quotients G/H where H is any of the above normal subgroups, can be identified. On the other hand, in this paper we find some "quasimorphism subgroups" that can be assumed to contain any of the above H whose quotients are isomorphic to $\mathbb{R}$, see our Section 5.

There are yet more normal subgroups that are not our focus in the present work but certainly of interest. First of all, one can construct normal subgroups via “fragmentation norms”, see [21]; it is not currently known how these relate to the normal subgroups above. One can also find normal subgroups between FHomeo and G by pulling back from the quotient subgroups corresponding to growth rates of infinite twist maps, see [29].

Organization of the paper

The outline of the paper is now as follows. After reviewing the preliminaries, we start with the computation in the smooth case, proving Theorem 1.3; this is the content of Section 3. We then move to the case of hameomorphisms in Section 4: the outcome of the computation from the previous section gives an explicit formula for the subleading asymptotics in the smooth case, and this motivates our definition for a hameomorphism with unbounded subleading asymptotics, see Section 4, which is the key step in proving Theorem 1.2. Section 5 uses related ideas to extend the Calabi invariant: the idea is that, just as the subleading asymptotics are a suitable replacement for Ruelle, the leading asymptotics allow for infinitely many extensions of Calabi. In Section 6, we prove the simplicity result Theorem 1.11. Finally, in the last section, we take up the question of whether the subleading asymptotics of the μ_k recover Ruelle for arbitrary Hamiltonians. This is not relevant to the algebraic considerations here, but is a very interesting question in its own right; we give a heuristic examination.

Acknowledgements

DCG thanks Lei Chen, Igor Frenkel, Dan Pomerleano, Christian Rosendal, and Rich Schwartz for helpful and interesting discussions and/or correspondence. SS thanks Yannick Bonthonneau for helpful conversations. IS is grateful to Peter Varju for helpful discussions.

DCG also thanks the National Science Foundation for their support under Awards #1711976 and #2105471. VH is partially supported by the Agence Nationale de la Recherche grants ANR-21-CE40-0002 and ANR-CE40-0014. CM is supported by the Simons Collaboration on Homological Mirror Symmetry. SS is partially supported by the ERC Starting Grant number 851701.

2 Preliminaries

We begin by reviewing the relevant background material and elaborating on some definitions mentioned in the introduction.

2.1 The groups, the Calabi homomorphism and the Ruelle invariant

2.1.1 Basic notions

Let S be either the standard 2-sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$ or the standard closed 2-disc $\mathbb{D}^2$ in $\mathbb{R}^2$. We assume that S is endowed with an area form ω. In the case of the disc, unless otherwise stated, all our maps will be assumed compactly supported, i.e. functions on $\mathbb{D}^2$ are assumed to vanish in some neighborhood of the boundary of $\mathbb{D}^2$ and homeomorphisms of $\mathbb{D}^2$ are assumed to coincide with the identity in some neighborhood of the boundary of $\mathbb{D}^2$.

As mentioned in the introduction, our main characters will be

$$G = \mathrm{Homeo}_c(S, \omega)$$

the group of (compactly supported) area preserving homeomorphisms of S and its smooth counter-part

$$G^\infty = \mathrm{Diff}_c(S, \omega)$$

the group of area preserving diffeomorphisms of S. When $S = \mathbb{S}^2$ we will drop the subscript 'c' from the notation. The group G is known to be the C^0-closure of G^∞. A smooth Hamiltonian $H = (H_t)_{t\in[0,1]} : [0, 1] \times S \to \mathbb{R}$ generates an isotopy $(\phi^t_H)_{t\in[0,1]}$ obtained by integrating the time dependent vector field X_{H_t} defined by $\omega(X_{H_t}, \cdot) = dH_t$. It is known that G^∞ coincides with the Hamiltonian group $\mathrm{Ham}_c(S, \omega)$, i.e. that any $\psi \in G^\infty$ is of the form $\psi = \phi^1_H$ for some Hamiltonian H.

In the disc case, $G^\infty = \mathrm{Ham}_c(\mathbb{D}^2, \omega)$ admits a non-trivial group homomorphism $\mathrm{Cal} : G^\infty \to \mathbb{R}$, called the Calabi homomorphism, which we now recall.

Definition 2.1. Let $\psi \in \mathrm{Diff}_c(\mathbb{D}^2, \omega)$. Since $G^\infty = \mathrm{Ham}_c(\mathbb{D}^2, \omega)$, the diffeomorphism ψ is the time-one map of a Hamiltonian H, i.e. $\psi = \phi^1_H$. The quantity

$$\mathrm{Cal}(\psi) = \int_0^1 \int_{\mathbb{D}^2} H\, \omega\, dt$$

turns out to be independent of the choice of Hamiltonian H and is called the Calabi invariant of ψ. This defines a map $\mathrm{Cal} : G^\infty \to \mathbb{R}$ which is a group homomorphism [3] (see also [25]).

As mentioned in the introduction, we can think of the Calabi homomorphism as measuring the "average rotation" of the map, see [15, 17].

2.1.2 Some normal subgroups of G

We are interested in this work in a particular normal subgroup of G. The key definition is as follows. As above, we denote by S a surface which is either $\mathbb{S}^2$ or $\mathbb{D}^2$.

Definition 2.2 (Oh-Müller [27])**.** A homeomorphism $\psi \in G$ is called a hameomorphism (or sometimes a strong Hamiltonian homeomorphism) if there exist a compact subset $K \subset S$, a sequence of Hamiltonians $H_i : [0,1] \times S \to \mathbb{R}$, $i \in \mathbb{N}$, supported in K and an isotopy $(\psi^t)_{t\in[0,1]}$ with $\psi^0 = \mathrm{Id}$ and $\psi^1 = \psi$, such that

(i) $\phi^t_{H_i}$ converges to ψ^t in the C^0 topology and uniformly in $t \in [0,1]$,

(ii) H_i is a Cauchy sequence with respect to the Hofer norm $\|\cdot\|$.

The set of all hameomorphisms is denoted $\mathrm{Hameo}(S,\omega)$.

Remark 2.3. Several variants of the above definition may be found in the literature. In particular, one sometimes replaces the convergence with respect to the Hofer norm $\|\cdot\|$ with uniform convergence. However, it was proved by Müller [26] that this change in the definition gives rise to the same group of hameomorphisms.

In [6], we used the following weaker variant. We called ψ a hameomorphism if there exist a compact set K and a sequence of Hamiltonians H_i, supported in K, such that the time-1 maps $\phi^1_{H_i}$ converge to ψ and H_i is Cauchy with respect to the Hofer norm. This weaker notion gives rise to another normal subgroup of G, which we will denote Hameo' in this remark. We clearly have the inclusion $\mathrm{Hameo} \subset \mathrm{Hameo}'$, but we do not know whether equality holds.

Our reason to change from one notion to another is to have stronger statements. Indeed, in [6] (see also the discussion in Theorem 2.4 below), we extended the Calabi homomorphism to Hameo' which is a priori a stronger result than just extending to Hameo. Here, we find a normal subgroup of G which is strictly smaller than Hameo (resp. ker(Cal) in Hameo); this is a priori a stronger statement than finding a subgroup in Hameo' (resp. ker(Cal) in Hameo').

We can use the Calabi homomorphism from above to get some additional subgroups, as the following shows.

Theorem 2.4 ([6], Theorem 1.4)**.** *The Calabi homomorphism on G^∞ extends canonically to a group homomorphism* $\mathrm{Hameo}(\mathbb{D}^2,\omega) \to \mathbb{R}$*. Moreover, for any $\psi \in$* $\mathrm{Hameo}(\mathbb{D}^2,\omega)$ *and any sequence H_i as in Definition 2.2 the extension of the Calabi homomorphism satisfies*

$$\mathrm{Cal}(\psi) = \lim_{i\to\infty} \mathrm{Cal}(\phi^1_{H_i}).$$

This gives another normal subgroup of G in the case of the disc, namely the kernel of $\mathrm{Cal} : \mathrm{Hameo}(\mathbb{D}^2,\omega) \to \mathbb{R}$.

2.1.3 The Ruelle invariant

We now recall the construction of the Ruelle quasi-morphism, following [17]. Recall that $G^\infty := \mathrm{Diff}_c(\mathbb{D}^2,\omega)$ denotes the group of compactly-supported area-preserving diffeomorphisms of the 2-disc. We fix a trivialization

$$T\mathbb{D}^2 \cong \mathbb{D}^2 \times \mathbb{R}^2 \tag{4}$$

(which is unique up to homotopy). The group G^∞ is contractible, so if $g \in G^\infty$ we may pick an isotopy $\{g_t\}$ from Id to g, again unique up to homotopy. For a point $z \in \mathbb{D}^2$, let

$$v_t(z) \in \mathbb{R}^2 \backslash \{0\}$$

denote the first column of $dg_t(z) \in SL(2, \mathbb{R})$ expressed in the trivialisation (4), and

$$\mathrm{Ang}_g(z) \in \mathbb{R}$$

the variation in the angle of $v_t(z)$, measured with respect to a fixed direction (say the x-axis) and integrated over $0 \leqslant t \leqslant 1$. The uniqueness of the choice of $\{g_t\}$ up to homotopy shows this does not depend on the choice of isotopy from g to Id. The function $z \mapsto \mathrm{Ang}_g(z)$ is smooth and so integrable. Setting

$$r(g) := \int_{\mathbb{D}^2} \mathrm{Ang}_g(z)\, \omega$$

we obtain the *Ruelle invariant*

$$\mathrm{Ru}(g) := \lim_{p \to \infty} r(g^p)/p.$$

This is a non-trivial homogeneous quasi-morphism on G^∞ (and on the kernel of the Calabi homomorphism).

Gambaudo and Ghys [17, Proposition 2.9] give a formula for the Ruelle invariant in the special case of an autonomous Hamiltonian flow of a function $H \in C_c^\infty(\mathbb{D}^2)$ with finitely many critical values. Suppose $\xi \in \mathbb{R}$ is a regular value of H, so $H^{-1}(\xi)$ is a finite disjoint union of circles. Each such circle C bounds a disc in $\mathbb{D}^2$, and we associate the sign $+1$, respectively -1, to $C \subset H^{-1}(\xi)$ depending on whether H increases, respectively decreases, as one crosses from the exterior to the interior region.

Then

$$\mathrm{Ru}(H) := \mathrm{Ru}(\phi_H^1) = \int_{\mathbb{R}} n_H(\xi) d\xi \tag{5}$$

where the integer $n_H(\xi) \in \mathbb{Z}$ is the signed sum of values ± 1 over the connected components C of $H^{-1}(\xi)$.

Specialising further to the case of a smooth function $H \in C_c^\infty(\mathbb{D}^2)$ which is Morse with critical points p_i, this simplifies to ([18, Section 2.4]):

$$\mathrm{Ru}(H) = \sum_i (-1)^{\mathrm{ind}(p_i)} H(p_i) \tag{6}$$

where $\mathrm{ind}(p_i)$ is the Morse index of p_i.

2.2 Monotone links, spectral invariants and quasimorphisms

The material for this section was developed in [6]. We refer the reader to this paper for further details.

2.2.1 Monotone links and their spectral invariants

We call a Lagrangian link (or Lagrangian configuration) any subset of the form $\underline{L} = L_1 \cup \cdots \cup L_k$ where the L_i's are pairwise disjoint smooth simple closed curves in $\mathbb{S}^2$, see Figure 1. A Lagrangian link is called *monotone* if the connected components of its complement all have the same area $\frac{\text{area}(\mathbb{S}^2)}{k+1}$.

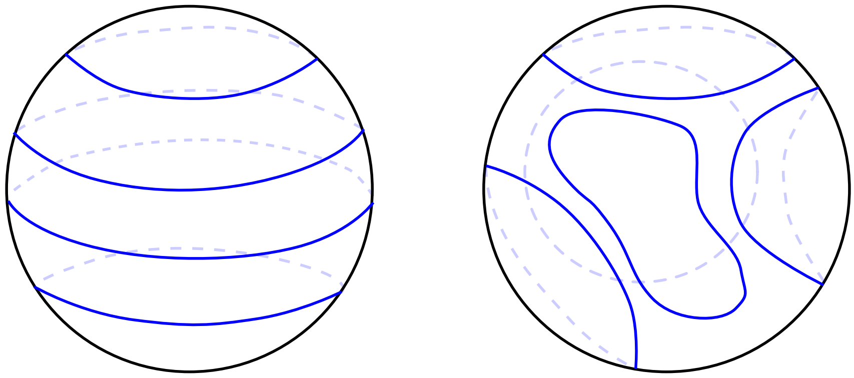

Figure 1: Two examples of Lagrangian links on $\mathbb{S}^2$ with respectively $k = 4$ and $k = 5$ components.

Remark 2.5. In [6], we introduced a more general notion of η-monotonicity, where η is a non-negative real parameter. We will not need this more general notion in the present paper. What we call monotonicity here corresponds to 0-monotonicity.

Let $\underline{L}$ be a monotone link with k components. We can take the product of the components to form the associated connected submanifold $\text{Sym}(\underline{L})$ inside the k-fold symmetric product $\text{Sym}^k(\mathbb{S}^2)$. The symplectic form ω on $\mathbb{S}^2$ induces a singular symplectic form on $\text{Sym}^k(\mathbb{S}^2)$ whose singular locus is away from $\text{Sym}(\underline{L})$ and makes $\text{Sym}(\underline{L})$ a Lagrangian submanifold. After smoothing the symplectic form near the singular locus, the Lagrangian Floer cohomology of $\text{Sym}(\underline{L})$ with itself is well-defined and non-zero [6, Lem. 6.10]. It enables us to define the link spectral invariants as follow.

Given a Hamiltonian function $H : [0,1] \times \mathbb{S}^2 \to \mathbb{R}$, we define $\text{Sym}(H) : [0,1] \times \text{Sym}^k(\mathbb{S}^2) \to \mathbb{R}$ to be $\text{Sym}(H)_t([x_1, \ldots, x_k]) := \sum_{i=1}^k H_t(x_i)$. The Lagrangian link spectral invariant $c_{\underline{L}}(H)$ is defined to be $\frac{1}{k} c_{\text{Sym}(\underline{L})}(\text{Sym}(H))$, where $c_{\text{Sym}(\underline{L})}(\text{Sym}(H))$ is the Lagrangian spectral invariant of $\text{Sym}(H)$ with respect to the Lagrangian submanifold $\text{Sym}(\underline{L})$ [6, Equation (54)]. We have shown in [6] that it is well-defined and independent from the choice of smoothing of the symplectic form as long as the smoothing is sufficiently local. For a Hamiltonian diffeomorphism $\psi \in \text{Diff}(\mathbb{S}^2, \omega)$ $(= \text{Ham}(\mathbb{S}^2, \omega))$ and a mean-normalized generating Hamiltonian H (i.e. $\int_{\mathbb{S}^2} H_t \omega dt = 0$ for all $t \in [0,1]$, and $\psi = \phi_H^1$), we have shown in [6] that $c_{\underline{L}}(\psi) := c_{\underline{L}}(H)$ is well-defined and independent of the choice of H.

Remark 2.6. In the specific case of a link $\underline{L}$ by parallel circles, similar invariants were previously constructed by Polterovich and Shelukhin [29] using orbifold Floer cohomology [16], [5] and the computational techniques in [22].

2.2.2 Quasimorphisms for diffeomorphism groups

Recall that a quasimorphism on a group Γ is a map $f : \Gamma \to \mathbb{R}$ for which there exists a constant $D > 0$ such that for any $a, b \in \Gamma$,

$$|f(ab) - f(a) - f(b)| \leqslant D$$

The constant D is called a defect of f. A quasimorphism f is said to be homogeneous if it satisfies $f(a^k) = kf(a)$ for any $k \in \mathbb{Z}$ and $a \in \Gamma$.

The spectral invariant $c_{\underline{L}}$ may be used to construct quasimorphisms on $G^\infty = \mathrm{Diff}(\mathbb{S}^2, \omega)$. This was proved in [6], inspired by an older (and famous) construction of Entov and Polterovich [13].

Let $\underline{L}$ be a monotone Lagrangian link with k components and $\phi \in \mathrm{Diff}(\mathbb{S}^2, \omega)$ and let us introduce the homogenized spectral invariant

$$\mu_k(H) = \lim_{n \to \infty} \frac{1}{n} c_{\underline{L}}(H^{\sharp n}),$$

for any Hamiltonian H on $\mathbb{S}^2$; these are the μ_k mentioned in the introduction. The above limit does not depend on the choice of the link $\underline{L}$; see Theorem 2.7 below. Here the notation $H^{\sharp n}$ means the n-times composition of H, where the composition of Hamiltonians is defined by $(H \sharp K)_t(x) = H_t(x) + K_t \circ (\phi_H^t)^{-1}(x)$. It is well known that $H \sharp K$ generates $\phi_H^t \circ \phi_K^t$, thus $H^{\sharp k}$ generates the isotopy $(\phi_H^t)^k$.

Note that μ_k has a shift property (see [6]), namely for any Hamiltonian H and any constant $c \in \mathbb{R}$, we have

$$\mu_k(H + c) = \mu_k(H) + c. \tag{7}$$

As above, we obtain invariants associated to elements of $\mathrm{Diff}(\mathbb{S}^2, \omega)$ (still denoted μ_k) by:

$$\mu_k(\varphi) = \mu_k(H), \tag{8}$$

for any mean-normalized Hamiltonian H such that $\phi_H^1 = \varphi$. This does not depend on the choice of H, see [6].

Theorem 2.7 ([6] Thm. 7.6, Thm. 7.7)**.** *For fixed k, the map $\mu_k : \mathrm{Diff}(\mathbb{S}^2, \omega) \to \mathbb{R}$ does not depend on the choice of Lagrangian link $\underline{L}$. Moreover the following properties hold*

1. *(Hofer continuity and monotonocity) For all Hamiltonians H, K,*

$$\int_0^1 \min_{x \in \mathbb{S}^2}(H_t(x) - K_t(x))dt \leqslant \mu_k(H) - \mu_k(K) \leqslant \int_0^1 \max_{x \in \mathbb{S}^2}(H_t(x) - K_t(x))dt.$$

2. *(Lagrangian control) Let H be a Hamiltonian and $\underline{L} = L_1 \cup \cdots \cup L_k$ be a Lagrangian link, such that for all $i = 1, \ldots, k$ the restriction of H to L_i is a function of t denoted c_i. Then,*

$$\mu_k(H) = \frac{1}{k} \sum_{i=1}^{k} \int_0^1 c_i(t)dt.$$

3. *(Quasimorphism) The map μ_k is a homogeneous quasimorphism of defect $\frac{2}{k}$.*

The first item implies that the quasimorphisms $\mu_k : \mathrm{Diff}(\mathbb{S}^2, \omega) \to \mathbb{R}$ are Lipschitz continuous with respect to Hofer distance d_H on $\mathrm{Diff}(\mathbb{S}^2, \omega)$ defined by

$$d_H(\varphi, \psi) := \inf_{\varphi=\phi_H^1, \psi=\phi_K^1} \|H - K\|, \tag{9}$$

where the norm is given by $\|H\| := \int_0^1 (\max_{\mathbb{S}^2} H_t - \min_{\mathbb{S}^2} H_t) dt$ (See e.g. [28] for an introduction to Hofer's distance). A consequence of item 2, proved in [6], is that the quasimorphisms μ_k are linearly independent. In the case $k = 1$, we recover the Entov-Polterovich quasimorphism [13].

Remark 2.8. As a consequence of the first and second items, for any Hamiltonian H, we have

$$\frac{1}{k} \sum_{i=1}^{k} \int_0^1 \min_{x \in L_i} H_t(x) dt \leqslant \mu_k(H) \leqslant \frac{1}{k} \sum_{i=1}^{k} \int_0^1 \max_{x \in L_i} H_t(x) dt$$

2.2.3 Quasimorphisms on the sphere

We now introduce quasimorphisms on the sphere. Denote

$$f_k := \mu_k - \mu_1.$$

By (8) and the shift property (7), we have

$$f_k(H) = f_k(\phi_H^1)$$

for all Hamiltonians H (not only for mean-normalized ones). The f_k give quasimorphisms on $\mathrm{Diff}(\mathbb{S}^2, \omega)$ which have similar properties to the μ_k. Our motivation for introducing them is their C^0-continuity, which is not satisfied by the μ_k. We collect in the next theorem their useful properties.

Theorem 2.9 ([6]). 1. *(C^0-continuity) For all $k \geqslant 1$, the quasimorphism f_k is continuous with respect to C^0 topology and extends continuously to* $\mathrm{Homeo}(\mathbb{S}^2, \omega)$

2. *(Support control) For all $k \geqslant 1$ and $\phi \in$ $\mathrm{Homeo}(\mathbb{S}^2, \omega)$ whose support is included in a disc of area $\leqslant \frac{1}{k+1}$. Then, $f_k(\phi) = 0$.*

Remark 2.10. In fact, for any positive integers k, k', the difference $\mu_{k'} - \mu_k$ extends continuously to a quasimorphism on $\mathrm{Homeo}(\mathbb{S}^2, \omega)$. Its defect is bounded above by the sum of the defects of μ_k and $\mu_{k'}$, i.e. by $\frac{2}{k} + \frac{2}{k'}$. In particular, f_k has defect $\frac{2}{k} + 2$.

2.2.4 Inducing quasimorphisms on the disc

Let $\iota : \mathbb{D}^2 \to \mathbb{S}^2$ be a smooth symplectic embedding which identifies the disc $\mathbb{D}^2$ with the northern (or southern) hemisphere. Then we have an inclusion $\mathrm{Homeo}_c(\mathbb{D}^2, \omega) \subset \mathrm{Homeo}(\mathbb{S}^2, \omega)$ and the maps f_k induce by restriction quasimorphisms on $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$.

Let H be a Hamiltonian which is compactly supported in the disc. Then the Lagrangian control property yields $\mu_1(H) = 0$ hence

$$f_k(\phi_H^1) = \mu_k(H) = \mu_k(\phi_H^1) + \int_0^1 \int_{\mathbb{S}^2} H\, \omega\, dt.$$

and in particular we obtain the following strengthening of the bound on the defect in Remark 2.10 (which we already stated in Lemma 1.12).

Lemma 2.11. *The f_k restricted to the disc are quasimorphisms with defect $2/k$.*

Using the Lagrangian control property and a Lagrangian link $\underline{L}$ consisting of horizontal circles $L_i = \{(x, y, z) \in \mathbb{S}^2 \mid z = -1 + 2\frac{i}{k+1}\}$, $i = 1, \ldots, k$, we can compute f_k explicitly for Hamiltonians that only depend on the variable z, namely:

$$f_k(\varphi_H^1) = \frac{1}{k} \sum_{i=1}^{k} H(-1 + 2\tfrac{i}{k+1}). \tag{10}$$

This formula will be used in some subsequent sections.

3 The subleading asymptotics and the Ruelle invariant

In this section, we first show that the spectral invariants $\{\mu_k\}$ have O(1) subleading asymptotics, and then compute those asymptotics exactly in the case of autonomous disc maps with finitely many critical values.

3.1 $O(1)$ subleading asymptotics

The proof that the spectral invariants $\{\mu_k\}$ have $O(1)$ subleading asymptotics in the smooth case is an almost immediate consequence of the key inequality

$$|\mu_k(\psi_0\psi_1) - \mu_k(\psi_0) - \mu_k(\psi_1)| \leqslant \frac{2}{k} \tag{11}$$

from Lemma 1.12.

Theorem 3.1. *For any $\psi \in \mathrm{Diff}(\mathbb{S}^2, \omega)$, the sequence $\{k\, \mu_k(\psi)\}_{k \in \mathbb{N}}$ is uniformly bounded.*

Proof. Let $G_{O(1)} := \{\psi \in \mathrm{Diff}(\mathbb{S}^2, \omega) | k\mu_k(\psi) = O(1)\}$. Equation (11) shows both that if $\psi_0, \psi_1 \in G_{O(1)}$, then so is the product $\psi_0\psi_1$, and also that $\psi \in G_{O(1)}$ if and only if $\psi^{-1} \in G_{O(1)}$. Therefore, $G_{O(1)}$ is a subgroup of $\mathrm{Diff}(\mathbb{S}^2, \omega)$. Since $\mu_k(\psi)$ is invariant under conjugating ψ by elements in $\mathrm{Diff}(\mathbb{S}^2, \omega)$, $G_{O(1)}$ is a normal subgroup.

Since $\mathrm{Diff}(\mathbb{S}^2, \omega)$ is simple, to show $G_{O(1)} = \mathrm{Diff}(\mathbb{S}^2, \omega)$ it therefore suffices to show that $G_{O(1)}$ contains a single non-identity element. Let H be the height function (projection to z co-ordinate) of $\mathbb{S}^2 \subset \mathbb{R}^3$. Let $\underline{L}_k$ be the k-component monotone link all of whose components are level sets of H. By the Lagrangian control property, we have $\mu_k(H) = 0$. Since H is mean-normalized, we have $\mu_k(\phi_H^1) = 0$, but ϕ_H^1 is not the identity element in $\mathrm{Diff}(\mathbb{S}^2, \omega)$. The result follows. $\square$

By restricting the $\{\mu_k\}$ to Hamiltonians on $\mathbb{S}^2$ supported (for instance) in a hemisphere, we immediately obtain:

Corollary 3.2. *Let $\mathbb{D}^2$ be a disc in $\mathbb{S}^2$ with area at most half of that of $\mathbb{S}^2$. For any $\psi \in \mathrm{Diff}_c(\mathbb{D}^2, \omega)$, the sequence $\{k \cdot (f_k(\psi) - \mathrm{Cal}(\psi))\}_{k \in \mathbb{N}}$ is uniformly bounded.*

Proof. It follows from $f_k(\psi) = \mu_k(\psi) - \mu_1(\psi)$, $\mu_1(\psi) = -\mathrm{Cal}(\psi)$ and Theorem 3.1. $\square$

3.2 Autonomous Hamiltonians

For general smooth Hamiltonian diffeomorphisms on the disc, we know from above that $k \cdot (f_k(\psi) - \mathrm{Cal}(\psi))$ is bounded as $k \to \infty$, but not that this sequence has a well-defined limit. For autonomous maps with finitely many critical *values*, the limit does exist, and is determined by the classical Ruelle invariant from Section 2.1.3: showing this is the aim of this section.

The main result is the following.

Theorem 3.3. *Let $H : (\mathbb{D}^2, \omega) \to \mathbb{R}$ be a compactly supported autonomous Hamiltonian with finitely many critical values. Then,*

$$\lim_{k \to \infty} (k\mu_k(H) - (k+1)\mathrm{Cal}(H)) = -\frac{1}{2}\mathrm{Ru}(H). \tag{12}$$

Theorem 1.3 directly follows from Theorem 3.1, Corollary 3.2 and Theorem 3.3.

Remark 3.4. The coefficient $k+1$ of $\mathrm{Cal}(H)$ is the reciprocal of the monotonicity constant of a k-component link $\underline{L}_k$ (see [6, Definition 1.12]).

The proof will use monotone Lagrangian links $\underline{L}_k$ 'most' of whose connected components are contained in level sets of H. In order to describe these links, we need a generalization of the Reeb graph of a Morse function on $\mathbb{S}^2$.

Let $H : (\mathbb{S}^2, \omega) \to \mathbb{R}$ be an autonomous Hamiltonian with finitely many critical values. We define an equivalence relation $\sim$ on $\mathbb{S}^2$ via $x \sim y$ if and only if there is $c \in \mathbb{R}$ such that $H(x) = H(y) = c$ and for any open interval $I \subset \mathbb{R}$ containing c, x and y are in the same connected component of $H^{-1}(I)$. Let $R : \mathbb{S}^2 \to G := \mathbb{S}^2/\sim$ be the associated quotient map. There is a uniquely defined function $H_G : G \to \mathbb{R}$ such that $H = H_G \circ R$.

Lemma 3.5. *The space G can be equipped with the structure of a finite graph. Moreover, it is a tree.*

Proof. We define the set of vertices of G to be the H_G-preimage of the set of critical values of H.

Take an interval I containing c in its interior and containing no other critical values. Then the number of connected components k_c of $H^{-1}(I)$ depends only on c but not on I. If, in addition, I is a closed interval, we know that $H^{-1}(I)$ is also closed and hence compact. Let J be an open interval containing I such that c is the only critical value in J. The collection of connected components of $H^{-1}(J)$ forms an open cover of $H^{-1}(I)$. By the compactness of $H^{-1}(I)$, it is covered by finitely many connected components of $H^{-1}(J)$. Since the number of connected components of $H^{-1}(I)$ and $H^{-1}(J)$ are the same, a connected component of $H^{-1}(J)$ contains precisely one connected component of $H^{-1}(I)$, which implies that k_c is finite. By definition, k_c is the number of vertices in G with H_G-value equal to c. Since there are only finitely many critical values, G has finitely many vertices.

For any two consecutive critical values $c_0 < c_1$ of H, and an interval $J \subset (c_0, c_1)$ containing no critical values, the number of connected components k_{c_0,c_1} of $H^{-1}(J)$ depends only on c_0, c_1 but not on J. By compactness of $\mathbb{S}^2$ again, k_{c_0,c_1} is finite. We define the connected components of $H_G^{-1}((c_0, c_1))$ to index the (open) edges of G; each such open edge is diffeomorphic to an open interval. The choice of connected component of $H^{-1}(J)$ also picks out connected components of $H_G^{-1}(c_i)$, and hence vertices of G, and we define the closure of an open edge in G to be a closed interval which connects the corresponding two vertices associated to the critical values c_0 and c_1. It is now clear that G has a structure of a finite graph.

Finally, if G were not a tree, then we would be able to lift a non-trivial 1-cycle from G to $\mathbb{S}^2$, contradicting $H_1(\mathbb{S}^2; \mathbb{Z}) = 0$. □

Let t be the number of vertices of G and enumerate the vertices $v_1, \ldots, v_t$. Let μ be the Borel measure on G such that for every open set $U \subset G$, we define

$$\mu(U) := \int_{R^{-1}(U)} \omega.$$

For $i = 1, \ldots, t$, let

$$m_i := \mu(v_i) \in [0, 1].$$

Note that for any $x \in G \backslash \{v_1, \ldots, v_t\}$, we have $\mu(x) = 0$. Many of the components of our desired links $\underline{L}_k$ will be of the form $R^{-1}(x)$ for some $x \in G \backslash \{v_1, \ldots, v_t\}$. Near $R^{-1}(v_i)$, the following simple observation will be useful in constructing $\underline{L}_k$.

Lemma 3.6. *Let $k \in \mathbb{N}$. For any open neighborhood J_i of v_i, there exist $S_{k,i} := \lfloor (k+1)m_i \rfloor$ pairwise disjoint circles in $R^{-1}(J_i)$ each of which bounds a disk of area $\frac{1}{k+1}$ in $R^{-1}(J_i)$.*

Proof. When $(k+1)m_i < 1$, the lemma is regarded as vacuously true so we assume $(k+1)m_i \geqslant 1$.

First, we show that $R^{-1}(J_i)$ is connected. Let $y_0, y_1 \in R^{-1}(J_i)$ be such that $R(y_0) = R(y_1)$. From the definition of the equivalence relation $\sim$, y_0 and y_1 belong to the same connected component of $R^{-1}(J_i)$. In other words, for any $x \in J_i$, there is only one connected component of $R^{-1}(J_i)$ whose R-image contains x. Furthermore, the restriction of R to any connected component of $R^{-1}(J_i)$ not containing $R^{-1}(v_i)$ is an S^1-fibre bundle. Together, these facts clearly imply that $R^{-1}(J_i)$ is connected.

Since $R^{-1}(J_i)$ is a connected proper open subset of the sphere, it is diffeomorphic to a planar domain.

Moreover, we have

$$(k+1)\omega(R^{-1}(J_i)) > (k+1)m_i \geqslant S_{k,i}.$$

It is therefore clear that we can find $S_{k,i}$-many pairwise disjoint circles in $R^{-1}(J_i)$ such that each bounds a disk of area $\frac{1}{k+1}$ in $R^{-1}(J_i)$. $\square$

For $1 \leqslant i \leqslant t$, let $\mathrm{val}(v_i)$ denote the valency of v_i.

Lemma 3.7. *Let $B_i = \mathrm{val}(v_i) - 1$. For any $1 \leqslant i \leqslant t$, any open neighborhood J_i of v_i, and for all $k \in \mathbb{N}$ sufficiently large such that $\frac{3}{k+1} < \mu(e)$ for every open edge $e \in G$, there is a monotone link $\underline{L}_k = \bigcup_{j=1}^k L_{k,j} \subset \mathbb{S}^2$ with k components such that*[3]*:*

- *For $j \in T_{k,1} := \{1, \ldots, k - \sum_{u=1}^t (S_{k,u} + B_u)\}$, $L_{k,j}$ is contained in a level set of H;*
- *For $i = 1, \ldots, t$ and $j \in T_{k,2,i} := \{k - \sum_{u=1}^{i-1}(S_{k,u} + B_u) + 1, \ldots, k - \sum_{u=1}^{i-1}(S_{k,u} + B_u) + S_{k,i}\}$, we have $L_{k,j} \subset R^{-1}(J_i)$;*
- *For $i = 1, \ldots, t$ and $T_{k,3,i} := \{k - \sum_{u=1}^{i-1}(S_{k,u} + B_u) + S_{k,i} + 1, \ldots, k - \sum_{u=1}^{i}(S_{k,u} + B_u)\}$, the set*

  $$R\left(\bigcup_{j \in \{T_{k,2,i} \bigcup T_{k,3,i}\}} L_{k,j}\right)$$

  Hausdorff converges to $\{v_i\}$ as k goes to ∞;
- *For a fixed $i \in \{1, \ldots, t\}$ and any $j_0, j_1 \in T_{k,2,i} \bigcup T_{k,3,i}$, the circles L_{k,j_0} and L_{k,j_1} are contained in the same connected component of $\mathbb{S}^2 \setminus \cup_{j \in T_{k,1}} L_{k,j}$.*

Note that $T_{k,1}$, $T_{k,2,i}$ and $T_{k,3,i}$ for all i together form a partition of $\{1, \ldots, k\}$. Roughly speaking, the components of $\underline{L}_k$ associated to the first three bullet points above are:

- S^1-fibres lying above open edges of G;
- components coming from Lemma 3.6, which can be chosen as close to $R^{-1}(v_i)$ as desired; and
- 'remaining' components not of the first two types, but which are controlled by the Hausdorff convergence.

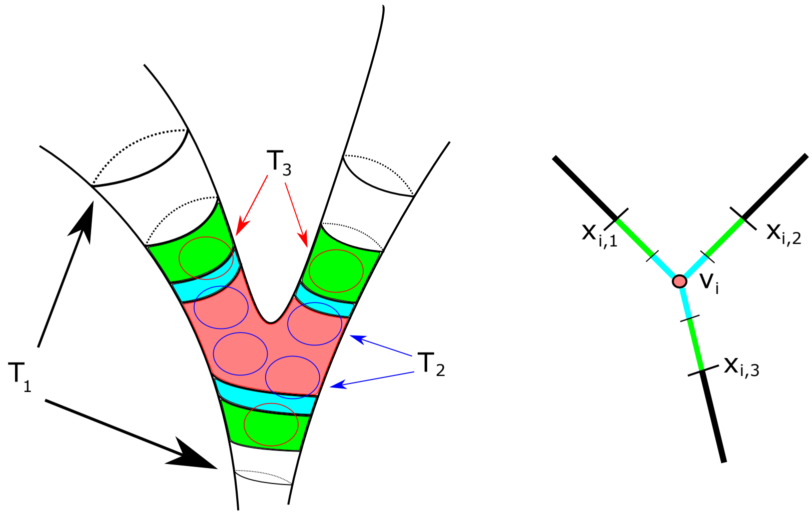

Figure 2: On the left: $R^{-1}(J_i)$ (the union of pink and light blue regions) contains $R^{-1}(v_i)$ (pink region) and type $T_{2,i}$ circles (blue). $R^{-1}(V_i)$ (the union of pink, light blue and light green regions) contains both type $T_{3,i}$ circles (red) and type $T_{2,i}$ circles. Type T_1 circles (black) are level sets outside the interior of $R^{-1}(V_i)$. On the right: we indicate a neighborhood of the vertex v_i in G, coloured to indicate the images of the respective regions on the left.

Proof. For $i = 1, \ldots, t$, let $\{U_{i,j}\}_{j=1}^{s_i}$ be the connected components of $G\backslash\{v_i\}$, where $s_i := \mathrm{val}(v_i)$ is the number of connected components of $G\backslash\{v_i\}$. Denote $\mu(U_{i,j})$ by $a_{i,j}$ so we have $m_i + \sum_{j=1}^{s_i} a_{i,j} = 1$. By our assumption on k, we have $a_{i,j} > \frac{3}{k+1}$ for all i, j. Let $r_{i,j} \in (0, \frac{1}{k+1}]$ be the unique number such that $a_{i,j} - r_{i,j}$ is an integer multiple of $\frac{1}{k+1}$.

For any v_i and any $j = 1, \ldots, s_i$, we define $x_{i,j} \in U_{i,j}$ to be the unique point such that $x_{i,j}$ is on an edge adjacent to v_i and the open interval between v_i and $x_{i,j}$ has μ-measure $r_{i,j}$. The existence of $x_{i,j}$ is guaranteed by the assumption $\frac{3}{k+1} < \mu(e)$, whilst its uniqueness comes from the fact that G is a tree (so there is a bijective correspondence between edges adjacent to v_i and connected components $U_{i,j}$). Moreover, again by the assumption $\frac{3}{k+1} < \mu(e)$, we have that $x_{i,j} \neq x_{i',j'}$ unless $i = i'$ and $j = j'$. Most importantly, by our choice of $r_{i,j}$, each connected component of $G\backslash\{x_{i,j}\}_{i,j}$ has μ-measure being an integer multiple of $\frac{1}{k+1}$. Denote the component of $G\backslash\{x_{i,j}\}_{i,j}$ containing v_i by V_i. By construction, we have $\mu(V_i) = m_i + \sum_{j=1}^{s_i} r_{i,j} \in (m_i, m_i + \frac{\mathrm{val}(v_i)}{k+1}]$ so $S_{k,i} < (k+1)\mu(V_i) \leqslant S_{k,i} + \mathrm{val}(v_i)$. Without loss of generality, we can assume that $J_i \subset V_i$ for all $i = 1, \ldots, t$ (see Figure 2).

By Lemma 3.6, we fix $S_{k,i}$ pairwise disjoint circles in $R^{-1}(J_i)$ bounding disjoint disks of area $\frac{1}{k+1}$ in $R^{-1}(J_i)$. Denote these circles by $L_{k,j}$ for

$$k - \sum_{u=1}^{i-1}(S_{k,u} + B_u) + 1 \leqslant j \leqslant k - \sum_{u=1}^{i-1}(S_{k,u} + B_u) + S_{k,i}.$$

The complement of the associated $S_{k,i}$ disjoint closed disks in $R^{-1}(V_i)$ is also a connected

[3] Recall that $S_{k,i} := \lfloor (k+1)m_i \rfloor$

open subset of $\mathbb{S}^2$. Therefore, we can put an additional set of $(k+1)\mu(V_i) - S_{k,i} - 1$ circles, each bounding a disc of area $\frac{1}{k+1}$ again, to obtain a further collection of circles $L_{k,j}$ indexed by k, j with

$$k - \sum_{u=1}^{i-1} (S_{k,u} + B_u) + S_{k,i} + 1 \leqslant j \leqslant k - \sum_{u=1}^{i-1} (S_{k,u} + B_u) + (k+1)\mu(V_i) - 1.$$

Combining the circles from the previous two steps, we obtain circles $L_{k,j}$ in $R^{-1}(V_i)$ indexed by k, j such that

$$k - \sum_{u=1}^{i-1} (S_{k,u} + B_u) + 1 \leqslant j \leqslant k - \sum_{u=1}^{i-1} (S_{k,u} + B_u) + (k+1)\mu(V_i) - 1$$

with the property that the collection of circles is pairwise disjoint, and every connected component in its complement has area $\frac{1}{k+1}$.

Any component of $G \backslash \{x_{i,j}\}_{i,j}$ not containing any v_i is an interval. We can subdivide that interval so that each sub-interval has μ-measure $\frac{1}{k+1}$. Let X be the union of $\{x_{i,j}\}_{i,j}$ and the additional points we added to subdivide the intervals. Then $R^{-1}(X)$ gives us a further $|X| = k - \sum_{i=1}^{t}((k+1)\mu(V_i) - 1)$ circles in $\mathbb{S}^2$. The union of

$$R^{-1}(X) \bigcup \{L_{k,j}\}$$

with

$$k - \sum_{u=1}^{i-1} (S_{k,u} + B_u) + 1 \leqslant j \leqslant k - \sum_{u=1}^{i-1} (S_{k,u} + B_u) + (k+1)\mu(V_i) - 1$$

for all $i = 1, \ldots, t$ is the desired k-component monotone link $\underline{L}_k$.

First note that $\underline{L}_k$ is indeed a k-component monotone link, because all the components of its complement have area $\frac{1}{k+1}$. To verify the bullets of the lemma, we now explain the labeling of the components of the link.

In general, we only have

$$(k+1)\mu(V_i) - 1 \leqslant S_{k,i} + \mathrm{val}(v_i) - 1 = S_{k,i} + B_i$$

but not an equality. Therefore, we need to label the $(S_{k,i} + B_i) - ((k+1)\mu(V_i) - 1)$-many circles in $R^{-1}(X)$ that are 'closest' to $R^{-1}(v_i)$ to be

$$\{L_{k,j}\} \text{ with } k - \sum_{u=1}^{i-1} (S_{k,u} + B_u) + (k+1)\mu(V_i) \leqslant j \leqslant k - \sum_{u=1}^{i} (S_{k,u} + B_u).$$

This can be done so that the third and final bullet of the lemma hold, because $\frac{1}{k+1}((S_{k,i} + B_i) - ((k+1)\mu(V_i) - 1)) \leqslant \frac{B_i}{k+1}$ is converging to zero as k goes to infinity. The first and the second bullet of the lemma hold by construction. $\square$

For each vertex $v_i \in G$, let $\chi_i := 2 - \mathrm{val}(v_i)$. We are now going to use the monotone links $\underline{L}_k$ constructed in Lemma 3.7 to prove the following.

Proposition 3.8. *Let $H : (\mathbb{S}^2, \omega) \to \mathbb{R}$ be an autonomous Hamiltonian with finitely many critical values. Then*

$$\lim_{k\to\infty}\left(k\mu_k(H) - (k+1)\int_{\mathbb{S}^2} H\right) = -\frac{1}{2}\sum_{i=1}^{t}\chi_i H_G(v_i) \tag{13}$$

Remark 3.9. When H is Morse, (13) reduces to

$$\lim_{k\to\infty}\left(k\mu_k(H) - (k+1)\int_{\mathbb{S}^2} H\right) = -\frac{1}{2}\sum_{j=1}^{s}(-1)^{\mathrm{ind}(p_j)}H(p_j)$$

where $\{p_j\}_{j=1}^s$ is the set of critical points of H. Compare to Equation (6).

Proof of Proposition 3.8. Let k be large enough and $J_{k,i}$ be an open neighborhood of v_i for $i = 1, \ldots, t$. We apply Lemma 3.7 to obtain a monotone link $\underline{L}_k$. Let $T_{k,1}$, $T_{k,2,i}$ and $T_{k,3,i}$ be as in Lemma 3.7. For $i = 1, \ldots, t$, let $V_{k,i}$ be the connected component of $\mathbb{S}^2 \backslash \bigcup_{j\in T_{k,1}} L_{k,j}$ containing $\bigcup_{j\in T_{k,2,i}\cup T_{k,3,i}} L_{k,j}$. Note that the area of $V_{k,i}$ is $\frac{S_{k,i}+B_i+1}{k+1} = \frac{S_{k,i}+\mathrm{val}(v_i)}{k+1}$. Let $V_k = \bigcup_{i=1}^t V_{k,i}$.

The Lagrangian control property (Remark 2.8) applied to the link $\underline{L}_k$ yields

$$k\mu_k(H) \in \sum_{j=1}^{k} H(L_{k,j}), \tag{14}$$

where $H(L_{k,j}) := \{H(y) | y \in L_{k,j}\}$.

We want to show that, by choosing $J_{k,i}$ to be sufficiently small, we obtain the following three identities:

$$\lim_{k\to\infty}\left(\left(\sum_{j\in T_{k,1}} H(L_{k,j})\right) - (k+1)\int_{\mathbb{S}^2\backslash V_k} H\right) = \frac{1}{2}\sum_{i=1}^{t}\mathrm{val}(v_i)H_G(v_i), \tag{15}$$

$$\lim_{k\to\infty}\left(\left(\sum_{j\in T_{k,2,i}} H(L_{k,j})\right) - (k+1)\int_{V_{k,i}} H\right) = -\mathrm{val}(v_i)H_G(v_i), \text{ for all } i, \tag{16}$$

$$\lim_{k\to\infty}\sum_{j\in T_{k,3,i}} H(L_{k,j}) = B_i H_G(v_i) = (\mathrm{val}(v_i) - 1)H_G(v_i), \text{ for all } i. \tag{17}$$

Note that these are limits of sets, since not all the $L_{k,j}$ are contained in level sets of H; however, the $J_{k,i}$ shrink with k and hence the diameters of the sets $H(L_{k,j})$ also tend to zero as k increases to infinity. Once these equalities are proved, by summing them up we will get

$$\lim_{k\to\infty}\left(\sum_{j=1}^{k} H(L_{k,j}) - (k+1)\int_{\mathbb{S}^2} H\right) = \frac{1}{2}\sum_{i=1}^{t}(\mathrm{val}(v_i) - 2)H_G(v_i). \tag{18}$$

The result then follows from (14) and the observation that the RHS of (18) is precisely $-\frac{1}{2}\sum_{i=1}^{t}\chi_i H_G(v_i)$.

Identity (15) will be proved 'edge by edge'. More precisely, let E be the set of edges of G. Each connected component of $\mathbb{S}^2\backslash\bigcup_{j\in T_{k,1}} L_{k,j}$ other than $V_{k,1},\ldots,V_{k,t}$ is topologically an annulus and is canonically labeled by an edge $e\in E$. Let $A^e_{k,1},\ldots,A^e_{k,h_{e,k}}$ be the connected components[4] that are labeled by e and denote their closures by $\overline{A}^e_{k,i}$ for $i=1,\ldots,h_{e,k}$. By possibly relabeling, we can assume that $\overline{A}^e_{k,i}\cap\overline{A}^e_{k,j}\neq\varnothing$ if and only if $j\in\{i-1,i,i+1\}$. Let $\partial\overline{A}^e_{k,i}=L^e_{k,i-1}\cup L^e_{k,i}$ for $i=1,\ldots,h_{e,k}$. Note that, every component of $\bigcup_{j\in T_{k,1}} L_{k,j}$ is of the form $L^e_{k,i}$ for precisely one $e\in E$ and $i\in\{0,\ldots,h_{e,k}\}$. By identifying $A^e_{k,i}$ with $([\frac{i-1}{k+1},\frac{i}{k+1}]\times\mathbb{R}/\mathbb{Z}, dz\wedge dy)$ using an S^1-equivariant area preserving diffeomorphism, in such a way that $L^e_{k,i}$ is identified with $\{i\}\times\mathbb{R}/\mathbb{Z}$, we have

$$\begin{aligned}
&\sum_{i=1}^{h_{e,k}}\left(H(L^e_{k,i})-(k+1)\int_{A^e_{k,i}}H\right)\\
&=(k+1)\sum_{i=1}^{h_{e,k}}\int_{A^e_{k,i}}H(L^e_{k,i})-H(z)dz\\
&=(k+1)\sum_{i=1}^{h_{e,k}}\int_{\frac{i-1}{k+1}}^{\frac{i}{k+1}}-H'(\frac{i}{k+1})(z-\frac{i}{k+1})+O\left(\frac{1}{(k+1)^2}\right)dz\\
&=\left(\frac{1}{2(k+1)}\sum_{i=1}^{h_{e,k}}H'(\frac{i}{k+1})\right)+h_{e,k}\,O\left(\frac{1}{(k+1)^2}\right)
\end{aligned}$$

where Taylor's theorem was used in passing from the second to the third line above. Note that $h_{e,k}<(k+1)$. By passing k to ∞ and applying the fundamental theorem of calculus, we get

$$\begin{aligned}
\lim_{k\to\infty}\sum_{i=1}^{h_{e,k}}\left(H(L^e_{k,i})-(k+1)\int_{A^e_{k,i}}H\right)&=\lim_{k\to\infty}\frac{1}{2}(H(L^e_{k,h_{e,k}})-H(L^e_{k,0}))\\
&=\frac{1}{2}(H_G(\partial^1 e)-H_G(\partial^0 e))
\end{aligned}$$

where $\partial^i e$ are the corresponding vertices adjacent to e. By adding back the term $\lim_k H(L^e_{k,0})$, we have

$$\lim_{k\to\infty}\left(\sum_{i=0}^{h_{e,k}}H(L^e_{k,i})-\sum_{i=1}^{h_{e,k}}(k+1)\int_{A^e_{k,i}}H\right)=\frac{1}{2}(H_G(\partial^1 e)+H_G(\partial^0 e))$$

This completes the calculation over a single edge e. By summing over all $e\in E$, we get (15).

[4]It may help the reader to note that superscripts e always index a choice of edge $e\in E$ in this argument.

Equation (17) follows from the fact that $|T_{k,3,i}| = B_i$ and the third bullet of Lemma 3.7. Therefore, it suffices to verify Equation (16).

Let $\epsilon_k > 0$ be such that $\lim_{k\to\infty} S_{k,i}\epsilon_k = 0$ for all i. Let $J_{k,i}$ be a sufficiently small neighborhood of v_i such that $H_G(J_{k,i}) \subset [H_G(v_i) - \epsilon_k, H_G(v_i) + \epsilon_k]$. It implies that

$$\sum_{j\in T_{k,2,i}} H(L_{k,j}) \subset [S_{k,i}H_G(v_i) - S_{k,i}\epsilon_k, S_{k,i}H_G(v_i) + S_{k,i}\epsilon_k]$$

On the other hand, recall that $\mu(v_i) = m_i \geqslant \frac{S_{k,i}}{k+1}$. Therefore, we can find an open disk $D_{k,i} \subset R^{-1}(J_{k,i})$ such that $\omega(D_{k,i}) = \frac{S_{k,i}}{k+1}$. It implies that

$$\lim_{k\to\infty}\left(\left(\sum_{j\in T_{k,2,i}} H(L_{k,j})\right) - (k+1)\int_{D_{k,i}} H\right) = \lim_{k\to\infty}[-2S_{k,i}\epsilon_k, 2S_{k,i}\epsilon_k] = 0 \text{ for all } i \quad (19)$$

Recall that $\omega(V_{k,i}\backslash D_{k,i}) = \omega(V_{k,i}) - \omega(D_{k,i}) = \frac{S_{k,i}+\mathrm{val}(v_i)}{k+1} - \frac{S_{k,i}}{k+1} = \frac{\mathrm{val}(v_i)}{k+1}$. Therefore, we also have

$$\lim_{k\to\infty}(k+1)\int_{V_{k,i}\backslash D_{k,i}} H = \lim_{k\to\infty}(k+1)\omega(V_{k,i}\backslash D_{k,i})H_G(v_i) = \mathrm{val}(v_i)H_G(v_i) \text{ for all } i \quad (20)$$

Equation (16) now follows from Equations (19) and (20). □

Proof of Theorem 3.3. We embed $\mathbb{D}^2$ into the northern hemisphere of $\mathbb{S}^2$. By Proposition 3.8 and (5), it suffices to show that $\sum_{i=1}^t \chi_i H_G(v_i)$ coincides with $\int_{\mathbb{R}} n_H(\xi)d\xi$. We can reinterpret $\int_{\mathbb{R}} n_H(\xi)d\xi$ using G as follows. Let v_1 be the vertex of G given by $R(\partial\mathbb{D}^2)$ (it is also the R-image of the entire southern hemisphere). Let e be an edge of G. Let the two vertices adjacent to e be ∂^+e and ∂^-e. Since G is a tree, there is no ambiguity to require that ∂^+e is further away from v_1 than ∂^-e (we allow that $\partial^-e = v_1$). If $H_G(\partial^+e) > H_G(\partial^-e)$, we define $n_e = 1$. If $H_G(\partial^+e) < H_G(\partial^-e)$, we define $n_e = -1$. It is clear from the definition of n_H that

$$\int_{\mathbb{R}} n_H(\xi)d\xi = \sum_e \int_{H_G(e)} n_e = \sum_e (H_G(\partial^+e) - H_G(\partial^-e))$$

where the sum is over all edges of G. For any vertex v of G other than v_1, there is a unique edge e_v of G such that $\partial^+e_v = v$ because G is a tree. Therefore, we have

$$\begin{aligned}\sum_e (H_G(\partial^+e) - H_G(\partial^-e)) &= -\mathrm{val}(v_1)H_G(v_1) + \sum_{i=2}^t (H_G(v_i) - (\mathrm{val}(v_i)-1)H_G(v_i)) \\ &= \sum_{i=1}^t \chi_i H_G(v_i)\end{aligned}$$

where the last equality uses that $H_G(v_1) = 0$ and $\chi_i = 2 - \mathrm{val}(v_i)$. It completes the proof. □

Remark 3.10. For higher genus surfaces, one can use a similar method to estimate $c_{\underline{L}}(H)$ for appropriate monotone links $\underline{L}$ most of whose components are level sets of H. However, the homogenized spectral invariant $\mu_{\underline{L}}$ will depend on the particular link $\underline{L}$, and not only on the number of components of $\underline{L}$. Therefore, for any fixed sequence of monotone links $\{\underline{L}_k\}_{k\in\mathbb{N}}$, this method should not give a robust estimate of the subleading asymptote of $\mu_{\underline{L}_k}$ for all autonomous Hamiltonians H simultaneously.

4 Non-simplicity for kernel of Calabi and for Hameo

In this section, we prove Theorem 1.2 whose statement we recall here.

Theorem (Theorem 1.2)**.** *The following groups are not simple:*

1. *The kernel of Calabi on* $\mathrm{Hameo}(\mathbb{D}^2, \omega)$.
2. *The group* $\mathrm{Hameo}(\mathbb{S}^2, \omega)$.

The goal of this section is to explain the proof.

Proper normal subgroups from subleading asymptotics

We begin by defining the normal subgroups which will turn out to be proper. To define our subgroups, we will use the subleading asymptotics of the quasimorphims arising from link spectral invariants which were introduced in Section 2.2.2.

First, consider the case of the disc. Denote by $\ker(\mathrm{Cal})$ the kernel of the Calabi homomorphism $\mathrm{Cal} : \mathrm{Hameo}(\mathbb{D}^2, \omega) \to \mathbb{R}$. Recall the quasimorphism $f_k : \mathrm{Homeo}(\mathbb{S}^2, \omega) \to \mathbb{R}$; its restriction to $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$ has defect bounded by $\frac{2}{k}$, see Lemma 1.12. Our normal subgroup will consist of those elements (of the kernel of Cal) for which the f_k have bounded subleading asymptotics. More precisely, define

$$N(\mathbb{D}^2) := \{\psi \in \ker(\mathrm{Cal}) : \text{the sequence } |k f_k(\psi)| \text{ is bounded}\}.$$

Proposition 4.1. $N(\mathbb{D}^2)$ *is a normal subgroup of* $\ker(\mathrm{Cal})$ *which contains all of its smooth elements.*

Proof. The argument here is very similar to that of the proof of Theorem 3.1 and so we will not provide all the details. $N(\mathbb{D}^2)$ is a subgroup because the f_k have defect $\frac{2}{k}$ and it is normal because the f_k, being homogeneous quasimorphisms, are invariant under conjugation.

The fact that $N(\mathbb{D}^2)$ contains all of the smooth elements in the kernel of Calabi is a consequence of Corollary 3.2; this is because for such ψ, we have $\mu_1(\psi) = \mathrm{Cal}(\psi) = 0$. □

In the case of the sphere, our normal subgroup is defined similarly, however we cannot use the quasimorphisms $f_k : \mathrm{Homeo}(\mathbb{S}^2, \omega) \to \mathbb{R}$ because although the restriction f_k :

$\mathrm{Homeo}_c(\mathbb{D}^2, \omega) \to \mathbb{R}$ has defect $\frac{2}{k}$, the f_k have defect $\frac{2}{k} + 2$; see Remark 2.10. We remedy this problem by working instead with the sequence of quasimorphisms

$$g_k := \mu_{2^k-1} - \mu_{2^{k-1}-1}, \tag{21}$$

for $k \geqslant 2$ on $\mathrm{Homeo}(\mathbb{S}^2, \omega)$ (see Remark 2.10).

Then, the defect of g_k is bounded by $\frac{2}{2^k-1} + \frac{2}{2^{k-1}-1}$. Define

$$N(\mathbb{S}^2) := \{\psi \in \mathrm{Hameo}(\mathbb{S}^2, \omega) : \text{the sequence } |(2^k - 1)g_k(\psi)| \text{ is bounded}\}.$$

Proposition 4.2. *$N(\mathbb{S}^2)$ is a normal subgroup of* $\mathrm{Hameo}(\mathbb{S}^2, \omega)$ *which contains all of its smooth elements.*

As in the case of $N(\mathbb{D}^2)$ the proof is similar to that of Theorem 3.1 and so we will omit it.

To prove properness of these normal subgroups, we will exhibit examples of hameomorphisms with unbounded subleading asymptotics.

4.1 A quickly twisting hameomorphism

The first part of the proof is to find a useful element that is in Hameo. As in our previous work, [8, 7, 6], the desired map will be a twist map. However, in our previous work, we studied "infinite twists" that were twisting so quickly that they were not in Hameo. Here, we find a map that is twisting slowly enough to define an element of Hameo, but quickly enough to have interesting, i.e. unbounded, subleading asymptotics. The construction of this map will be the topic of this section.

Let $T : \mathbb{S}^2 \to \mathbb{S}^2$ be defined as follows.

We view $\mathbb{S}^2$ as the standard unit sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$ and equip it with the symplectic form $\omega = \frac{1}{4\pi} d\theta \wedge dz$ where (θ, z) are the standard cylindrical coordinates on $\mathbb{R}^3$. Denote by p_- the point on $\mathbb{S}^2$ whose z–coordinate is -1. We pick a function $H : \mathbb{S}^2 \backslash \{p_-\} \to \mathbb{R}$ which is of the form

$$H(\theta, z) = h(z),$$

where $h : (-1, 1] \to \mathbb{R}$ is a smooth function which vanishes for $z \geqslant -\frac{1}{2}$ and, for $z \leqslant -\frac{3}{4}$, satisfies the identity

$$h(z) = \sqrt{\frac{2}{1+z}}. \tag{22}$$

The function H induces a well-defined flow ϕ_H^t on $\mathbb{S}^2$ which fixes the point p_- and its action on (θ, z), with $z > -1$, is given by the following equation

$$\phi_H^t(\theta, z) = (\theta + 4\pi h'(z)t, z).$$

We define

$$T := \phi_H^1.$$

Note that T is supported in the disc $\mathbb{D}^2 := \{(\theta, z) : -1 \leqslant z \leqslant 0\} \subset \mathbb{S}^2$ and so we can view it as an element of either of $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$ or $\mathrm{Homeo}(\mathbb{S}^2, \omega)$.

Proposition 4.3. $T \in \mathrm{Hameo}(\mathbb{D}^2, \omega)$. *Moreover,*

$$\mathrm{Cal}(T) = \frac{1}{2}\int_{-1}^{1} h(z)dz < \infty.$$

Note that the above proposition implies that $T \in \mathrm{Hameo}(\mathbb{S}^2, \omega)$ as well.

Remark 4.4. As mentioned earlier, the homeomorphism T twists slowly enough to be in Hameo, and so its Calabi invariant is well-defined, yet it twists fast enough not to be contained in $N(\mathbb{D}^2)$; the heuristic reasoning behind $T \notin N(\mathbb{D}^2)$ is that, since $H(p_-) = \infty$, T has "infinite Ruelle invariant."

In comparison, if we were to modify the function h in Equation (22) to

$$h(z) = \frac{2}{1+z}$$

we would obtain an "infinite twist" homeomorphism that spins too fast to be contained in Hameo; here the heuristic reasoning is that the condition $\int_{-1}^{1} h(z)dz = \infty$ forces the homeomorphism to have "infinite Calabi invariant." Indeed, this can be proven rigorously via the argument given in [8] (see also the proof of [6, Theorem 1.3]).

Proof of Proposition 4.3. By definition of Hameo, to prove that $T \in \mathrm{Hameo}(\mathbb{D}^2, \omega)$, we must find smooth Hamiltonians K_n supported in a compact subset of the interior of $\mathbb{D}^2 = \{(\theta, z) : -1 \leqslant z \leqslant 0\}$ such that

(A) $\phi_{K_n}^1 \xrightarrow{C^0} T$,

(B) $\phi_{K_n}^t$ is Cauchy for the C^0-distance, uniformly in $t \in [0, 1]$,

(C) the sequence K_n is Cauchy for Hofer's norm $\|\cdot\|$.

We start by picking Hamiltonians H_n as follows. Let $D_n := \{(\theta, z) : -1 \leqslant z \leqslant -1 + \frac{1}{2^{2n}}\} \subset \mathbb{D}^2$ be the disc of radius $\frac{1}{2^{2n}}$ (in z coordinate) centered at p_-; note that

$$\mathrm{Area}(D_n) = \frac{1}{2^{2n}}\mathrm{Area}(\mathbb{D}^2) = \frac{1}{2^{2n+1}}. \tag{23}$$

Now, pick the Hamiltonian H_n so that the following hold:

(i) H_n depends only on the z variable,

(ii) $H_n = H$ outside of D_n and $H_n \approx 2^n\sqrt{2}$ in the interior of D_n,

(iii) $\|H_{n+1} - H_n\| \leqslant 2^n\sqrt{2}$.

To see why H_n can be picked to satisfy the above, note that $H(-1+\frac{1}{2^{2n}}) = 2^n\sqrt{2}$ and so to obtain H_n it suffices to smoothly flatten H on the interior of D_n.

Note that $\phi^1_{H_n} \circ T^{-1} = \mathrm{Id}$ outside of D_n and hence $\phi^1_{H_n} \xrightarrow{C^0} T$. We will find Hamiltonians K_n such that $\phi^1_{K_n} = \phi^1_{H_n}$, the sequence K_n is Cauchy for Hofer's norm $\|\cdot\|$ and $\phi^t_{K_n}$ is Cauchy for the C^0-distance, uniformly in t. Note that once this is proven Theorem 2.4 yields

$$\mathrm{Cal}(T) = \lim_n \mathrm{Cal}(\phi^1_{K_n}) = \lim_n \mathrm{Cal}(\phi^1_{H_n}) = \lim_n \int_{\mathbb{D}^2} H_n\,\omega = \int_{\mathbb{D}^2} H\,\omega = \frac{1}{2}\int_{-1}^{1} h(z)dz.$$

We need the following lemma whose proof relies on ideas going back to Sikorav [37].

Lemma 4.5. *Let Δ be a Euclidean 2-disc equipped with an area form ω of total area A. Suppose $D \subset \Delta$ is diffeomorphic to $\mathbb{D}^2$ and that* $\mathrm{Area}(D) < \frac{A}{N}$ *for some integer $N > 0$. Let F be a smooth Hamiltonian supported in the interior of D. Then, we have*

$$d_H(\phi^1_F, \mathrm{Id}) \leqslant \frac{\|F\|}{N} + 2A.$$

where d_H denotes the Hofer distance on $\mathrm{Ham}_c(\Delta,\omega)$ *and* $\|F\| = \int_0^1(\max_\Delta F_t - \min_\Delta F_t)dt$ *is the Hofer norm of F.*

Proof. Lemma 4.3 of [7] proves this for $A = 1$. A straightforward adaptation of the proof therein yields the case where $A \neq 1$. □

Before proving this lemma, we will use it to construct the sequence of Hamiltonians K_n.

For each n, the Hamiltonian $H_{n+1} - H_n$ is supported in the disc D_n, by item (ii) above, and $\|H_{n+1} - H_n\| \leqslant 2^n\sqrt{2}$, by (iii). Let $\Delta_n \subset \mathbb{D}^2$ be the disc centered at p_- and of area $A_n = 2^{-n/2}$. By Equation (23), we have $\mathrm{Area}(D_n) < \frac{A_n}{N}$ for $N = 2^{\lfloor 3n/2 \rfloor}$. Hence, applying Lemma 4.5, we obtain Hamiltonians G_n supported in Δ_n which satisfy

- $\phi^1_{G_n} = \phi^1_{H_{n+1}-H_n} = \phi^{-1}_{H_n}\phi^1_{H_{n+1}}$,
- $\|G_n\| \leqslant \frac{\|H_{n+1}-H_n\|}{N} + 2A_n \leqslant \frac{2^n\sqrt{2}}{N} + 2A_n$.

In particular, for $N = 2^{\lfloor 3n/2 \rfloor}$, we get Hamiltonians G_n such that the series $\sum_{i=1}^{\infty}\|G_i\|$ is summable. Since G_n is supported in A_n, the C^0-distance $d_{C^0}(\phi^t_{G_n}, \mathrm{Id})$ is bounded by the diameter of A_n, which is $O(2^{-n/4})$. It follows that the series $\sum_{i=1}^{\infty} d_{C^0}(\phi^t_{G_i}, \mathrm{Id})$ is summable as well (uniformly in t).

Now let us define $K_1 := H_1$ and then recursively $K_{n+1} := K_n \sharp G_n$ for $n \geqslant 1$. Then,

$$\phi^1_{K_n} = \phi^1_{H_1}\phi^1_{G_1}\cdots\phi^1_{G_{n-1}} = \phi^1_{H_n}.$$

Moreover, since $\sum_{i=1}^{\infty} \|G_i\| = \sum_{i=1}^{\infty} \|K_{i+1} - K_i\|$ is summable, the sequence K_n is Cauchy with respect to the Hofer norm. Similarly, since $\sum_{i=1}^{\infty} d_{C^0}(\phi^t_{G_i}, \mathrm{Id})$ is summable, $\phi^t_{K_n}$ converges for the C^0 topology.

This completes the proof of Proposition 4.3 modulo the proof of the lemma which we provide below. □

Proof of Lemma 4.5. We will present the proof of the lemma under the simplifying assumption that the Hamiltonian F is time independent and leave the more general case, which is very similar, to the reader. Note that we have only applied Lemma 4.5 to time-independent Hamiltonians.

Fix $1 \leqslant k \leqslant N$ and pick pairwise disjoint discs $D_1, \ldots, D_k \subset \Delta$ such that each of these discs has the same area as D. There exist Hamiltonian diffeomorphisms $\psi_1, \ldots, \psi_k \in \mathrm{Ham}_c(\Delta, \omega)$ such that

- $\psi_i(D) = D_i$ for each $i = 1, \ldots, k$,
- $d_H(\psi_i, \mathrm{Id}) \leqslant \frac{A}{N}$.

Consider the time-independent Hamiltonian

$$H := \frac{1}{k} \sum_{i=1}^{k} F \circ \psi_i^{-1}.$$

It is supported in the union of the discs D_i and $\|H\| \leqslant \frac{\|F\|}{k}$. Therefore,

$$d_H(\phi^1_F, \mathrm{Id}) \leqslant d_H(\phi^1_F, \phi^1_H) + d_H(\phi^1_H, \mathrm{Id}) \leqslant d_H(\phi^1_F, \phi^1_H) + \frac{\|F\|}{k}.$$

Hence, to prove the lemma, it is sufficient to show that $d_H(\phi^1_H, \phi^1_F) \leqslant \frac{2kA}{N}$. To do so, first observe that $\phi^1_H = \prod_{i=1}^{k} \psi_i \phi_F^{1/k} \psi_i^{-1}$ and $\phi^1_F = \prod_{i=1}^{k} \phi_F^{1/k}$. Hence,

$$\begin{aligned} d_H(\phi^1_H, \phi^1_F) &= d_H\left(\prod_{i=1}^{k} \psi_i \phi_F^{\frac{1}{k}} \psi_i^{-1}, \prod_{i=1}^{k} \phi_F^{\frac{1}{k}}\right) \\ &\leqslant \sum_{i=1}^{k} d_H(\psi_i \phi_F^{\frac{1}{k}} \psi_i^{-1}, \phi_F^{\frac{1}{k}}) \\ &\leqslant \sum_{i=1}^{k} d_H(\psi_i, \mathrm{Id}) + d_H(\psi_i^{-1}, \mathrm{Id}) \\ &= \sum_{i=1}^{k} 2\, d_H(\psi_i, \mathrm{Id}) \leqslant \frac{2kA}{N} \end{aligned}$$

Here, the inequalities on the second and the third line follow from the bi-invariance of Hofer's metric and the inequality on the final line follows from the fact that we picked ψ_i such that $d_H(\psi_i, \mathrm{Id}) \leqslant \frac{A}{N}$. □

4.2 The case of the disc

We now use the map T to prove the first item of Theorem 1.2.

We will prove that the normal subgroup $N(\mathbb{D}^2)$, from Proposition 4.1, is proper. This is a direct consequence of the following two lemmas.

Lemma 4.6. *Suppose that there exists an element $\psi \in \mathrm{Hameo}(\mathbb{D}^2, \omega)$ such that the sequence $|k \cdot (f_k(\psi) - \mathrm{Cal}(\psi))|$ is unbounded. Then, there exists some ψ' in the kernel of Calabi which does not belong to $N(\mathbb{D}^2)$.*

Proof. Let $\Theta \in \mathrm{Diff}_c(\mathbb{D}^2, \omega)$ be such that $\mathrm{Cal}(\Theta) = \mathrm{Cal}(\psi)$ and define

$$\psi' = \psi \circ \Theta^{-1}.$$

Then $\mathrm{Cal}(\psi') = 0$. On the other hand, by Lemma 1.12

$$|kf_k(\psi') - kf_k(\psi) - kf_k(\Theta^{-1})| \leqslant 2.$$

Now, we claim that the sequence $\{kf_k(\psi) + kf_k(\Theta^{-1})\}_k$ is unbounded which, in combination with the above inequality, implies that the sequence $|kf_k(\psi'))|$ is unbounded. This, in turn, implies that $\psi' \notin N(\mathbb{D}^2)$.

The fact that $\{kf_k(\psi) + kf_k(\Theta^{-1})\}_k$ is unbounded is an immediate consequence of Theorem 3.1: the sequence $\{k \cdot (f_k(\Theta^{-1}) - \mathrm{Cal}(\Theta^{-1}))\}_k$ is bounded, by the theorem, and the sequence $\{k \cdot (f_k(\psi) - \mathrm{Cal}(\psi))\}_k$ is unbounded, by assumption. Hence, the sum of these two sequences, which is exactly $\{kf_k(\psi) + kf_k(\Theta^{-1})\}_k$, is unbounded. This completes the proof. □

Lemma 4.7. *The sequence $|k(f_k(T) - \mathrm{Cal}(T))|$ is unbounded, where $T \in \mathrm{Hameo}(\mathbb{D}^2, \omega)$ is as in Proposition 4.3.*

Proof. Recall the (non-smooth) function H from (22) which we used in the definition of T. Let $H_n(z)$ be a sequence of smoothings of H, depending only on z, that agree with H except for $-1 \leqslant z \leqslant -1 + \frac{1}{2^{2n}}$. One could, for example, take H_n to be as in the proof of Proposition 4.3.

Note that $\phi^1_{H_n} \xrightarrow{C^0} T$ and so, by the C^0 continuity property of the f_k, we have

$$f_k(T) = \lim_{n \to \infty} f_k(\phi^1_{H_n}).$$

Now, since H and the H_n depend only on z, we can compute $f_k(\phi^1_{H_n})$ using the Lagrangian Control property; see Section 2.2.2. We have

$$f_k(\phi^1_{H_n}) = \frac{1}{k} \sum_{i=1}^{k} H_n(-1 + 2\frac{i}{k+1}).$$

Since $H_n = H$, except for $-1 \leqslant z \leqslant -1 + \frac{1}{2^{2n}}$, for n large enough we have $f_k(\phi^1_{H_n}) = \frac{1}{k} \sum_{i=1}^{k} H(-1 + 2\frac{i}{k+1})$ and hence

$$f_k(T) = \frac{1}{k}\sum_{i=1}^{k} H(-1+2\frac{i}{k+1}) = \frac{1}{k}\sum_{i=1}^{k} h(-1+2\frac{i}{k+1}).$$

Recall that $\mathrm{Cal}(T) = \frac{1}{2}\int_{-1}^{1} h(z)\,dz < \infty$. Hence, to prove that $kf_k(T) - k\mathrm{Cal}(T)$ is unbounded, it suffices to prove that the sequence whose kth term is given by

$$2(kf_k(T) - (k+1)\mathrm{Cal}(T)) = 2\sum_{i=1}^{k} h(-1+2\frac{i}{k+1}) - (k+1)\int_{-1}^{1} h(z)\,dz \tag{24}$$

is unbounded, which we will prove below.

Write $a_i = -1 + 2\frac{i}{k+1}$, for $i = 0, \dots, k+1$. Observe that (24) can be rewritten as

$$(k+1)\sum_{i=1}^{k}\left(\int_{a_{i-1}}^{a_i} h(a_i) - h(z)dz\right) - (k+1)\int_{a_k}^{a_{k+1}} h(z)dz.$$

The term $\int_{a_k}^{a_{k+1}} h(z)dz$ is zero since h is supported in $-1 \leqslant z \leqslant -\frac{1}{2}$. So we must prove unboundedness of the sum.

Since h is a convex function, we have $h(a_i) - h(z) \leqslant h'(a_i)(a_i - z)$. Thus

$$\begin{aligned}(k+1)\sum_{i=1}^{k}\left(\int_{a_{i-1}}^{a_i} h(a_i) - h(z)dz\right) &\leqslant (k+1)\sum_{i=1}^{k}\left(h'(a_i)\int_{a_{i-1}}^{a_i}(a_i - z)dz\right) \\ = (k+1)\sum_{i=1}^{k}\left(h'(a_i)\frac{2}{(k+1)^2}\right) &= \sum_{i=1}^{k}\left(h'(a_i)\frac{2}{k+1}\right).\end{aligned}$$

Now, since h' is non-decreasing, we have

$$\sum_{i=1}^{k}\frac{2}{k+1}h'(a_i) \leqslant \sum_{i=1}^{k}\int_{a_i}^{a_{i+1}} h'(z)dz = \int_{a_1}^{1} h'(z)dz = -h(a_1) \xrightarrow{k\to\infty} -\infty.$$

This shows that $(k+1)\sum_{i=1}^{k}\left(\int_{a_{i-1}}^{a_i} h(a_i) - h(z)dz\right)$ is unbounded and concludes the proof. □

4.3 The case of the sphere

Here, we prove the second item of Theorem 1.2.

Proof. Recall, from Proposition 4.2, the normal subgroup

$$N(\mathbb{S}^2) := \{\psi \in \mathrm{Hameo}(\mathbb{S}^2, \omega) : \text{the sequence } |(2^k - 1)g_k(\psi)| \text{ is bounded}\},$$

where g_k is the quasimorphism defined by (21).

We will show that $N(\mathbb{S}^2)$ is proper. For this, we define a variant T' of the map T from Proposition 4.3; that is, we let T' be the time-1 flow of $F(\theta, z) = f(z) = \sqrt{\frac{2}{1+z}}$, away from the south pole, and we set $T'(p_-) = p_-$. We claim that $T' \in \mathrm{Hameo}(\mathbb{S}^2, \omega)$; this follows directly from Proposition 4.3 via the observation that $T' \circ T^{-1}$ is smooth.

Define

$$S(k) = \sqrt{2^k} \sum_{i=1}^{2^k-1} \sqrt{\frac{1}{i}}.$$

Then, arguing as in the proof of Lemma 4.7 it can be shown that

$$(2^k - 1) g_k(T') = S(k) - \frac{2^k - 1}{2^{k-1} - 1} S(k-1) = S(k) - 2S(k-1) - \frac{1}{2^{k-1} - 1} S(k-1). \quad (25)$$

We will only provide an outline of the proof of the above formula as its derivation is similar to what was done in the proof of Lemma 4.7. Here is the outline: take an appropriate sequence of smoothings $F_n(z)$ of $F(z)$ which coincide with F away from a small neighborhood of p_-. Then, (25) follows from the following two items

1. $\phi^1_{F_n} \xrightarrow{C^0} T'$ and so, by the C^0 continuity of g_k, we have $g_k(T') = \lim_{n\to\infty} g_k(\phi^1_{F_n})$.
2. using the Lagrangian Control property, one obtains that

$$g_k(\phi^1_{F_n}) = \frac{1}{2^k - 1} S(k) - \frac{1}{2^{k-1} - 1} S(k-1).$$

It follows from (25) that to show that T' is not in $N(\mathbb{S}^2)$, we need to estimate the difference

$$S(k) - 2S(k-1) - \frac{1}{2^{k-1} - 1} S(k-1).$$

The crux of the issue is showing that $S(k) - 2S(k-1)$ is unbounded. To see this, write

$$\begin{aligned} S(k) - 2S(k-1) &= \sqrt{2^k} \sum_{i=1}^{2^k-1} \sqrt{\frac{1}{i}} - 2\sqrt{2^k} \sum_{i=1}^{2^{k-1}-1} \sqrt{\frac{1}{2i}} \\ &= \sqrt{2^k} \left(\sum_{i=0}^{2^{k-1}} \sqrt{\frac{1}{2i+1}} - \sum_{i=1}^{2^{k-1}} \sqrt{\frac{1}{2i}} \right) \\ &\geqslant \sqrt{2^k} \left(1 - \sqrt{\frac{1}{2}} \right), \end{aligned}$$

which is unbounded in k.

To complete the proof, it therefore remains to show that the term $\frac{1}{2^{k-1}-1} S(k-1)$ is bounded in k. To do this, we write

$$\frac{1}{2^{k-1} - 1} S(k-1) = \frac{2^{k-1}}{2 \cdot 2^{k-1} - 1} \frac{S(k-1)}{2^{k-2}}.$$

The term $\frac{S(k-1)}{2^{k-2}}$ differs from the right Riemann sum, for the integrable function $\sqrt{\frac{2}{1+z}}$ on $-1 \leqslant z \leqslant 1$, by $\frac{1}{2^{k-2}}$, hence $\frac{1}{2^{k-1}-1}S(k-1)$ is bounded in k. We conclude from this the sequence $(2^k - 1)g_k(T')$ is unbounded and hence $T' \notin \mathrm{Hameo}(\mathbb{S}^2, \omega)$. This completes the proof of Theorem 1.2. □

5 Infinitely many extensions of Calabi

Having applied the two-term Weyl law to study the normal subgroup structure of $G = \mathrm{Homeo}_c(\mathbb{D}^2, \omega)$, we now invoke related asymptotic considerations to prove Theorem 1.9, which we recall for the reader states that the Calabi homomorphism admits infinitely many extensions to G. We also elaborate on the promise from the introduction that this perspective has value in identifying new normal subgroups whose quotients can be computed. We note for the benefit of the reader that while this section is thematically linked to the previous one, it does not cite results from there and so can be read independently.

5.1 The main theorem

We begin with the promised proof of Theorem 1.9, which collects considerations of the asymptotics of the f_k via a short argument.

Proof of Theorem 1.9. Define the group $R' := \mathbb{R}^{\mathbb{N}}/\sim$, where $s \sim t$ if and only if $s - t$ has limit 0. We can think of this as the "set of all limits of sequences". There is a natural map

$$S : G \to R', \qquad g \to (f_2(g), f_3(g), \ldots, f_n(g), \ldots). \tag{26}$$

(We have not included f_1 here, because as we have defined it, it is 0.) By Lemma 1.12, this is a group homomorphism. There is also a canonical homomorphism

$$\Delta : \mathbb{R} \to R', \quad x \mapsto (x, x, \ldots, x).$$

Now by the Weyl law (1), we have

$$S(h) = (\mathrm{Cal}, \mathrm{Cal}, \mathrm{Cal}, \ldots) \tag{27}$$

for every $h \in \mathrm{Diff}_c(\mathbb{D}^2, \omega)$.

We now find a section of the map Δ, as follows. The group R' is a vector space over $\mathbb{R}$. Take the vector $v_1 = (1, \ldots, 1, \ldots) \in R'$; by Zorn's Lemma, we can extend this to a basis β for R'. The section of Δ now comes from the splitting of R' with respect to this basis. More precisely, we define

$$s : R' \to \mathbb{R}, \quad s(v) = a_1, \qquad v = a_1 v_1 + \sum_{v_i \in \beta, v_i \neq v_1} a_i v_i.$$

It now follows from (27) that $s \circ S$ is the desired extension. Since there are infinitely many choices of extensions β, and the map S is surjective (see Proposition 5.3 below), it follows that there are infinitely many extensions. □

Remark 5.1. The above argument uses Zorn's lemma. As communicated to the authors by C. Rosendal, there are models of set theory where the axiom of choice is false and every homomorphism between Polish groups is continuous; in particular, the extensions from above do not exist in those models. Assuming dependent choice plus ZF, the existence of the above extension implies that there is a Vitali set. See [32, Thm. 5.] and [31].

Remark 5.2. In [6], the Calabi invariant was previously extended to a homomorphism Cal on $\mathrm{Hameo}(\mathbb{D}^2, \omega)$ by the rule

$$H \mapsto \int H\omega dt,$$

where H is any Hamiltonian for a given hameomorphism; such a Hamiltonian is not unique, but [6] showed that this extension does not depend on the choice of Hamiltonian. Any of the extensions to $\mathrm{Homeo}_c(\mathbb{D}^2, \omega)$ in Theorem 1.9 agree with this extension when restricted to Hameo; this follows from the fact that, similarly to (27),

$$S(h) = (\mathrm{Cal}, \mathrm{Cal}, \mathrm{Cal}, \ldots), \tag{28}$$

for any $h \in \mathrm{Hameo}(\mathbb{D}^2, \omega)$.

To see why the above equation is true, we note that, as in the proof of [6, Thm. 1.1], if $h \in \mathrm{Hameo}(\mathbb{D}^2, \omega)$ and $H : [0,1] \times \mathbb{D}^2 \to \mathbb{R}$ is a C^0 Hamiltonian for h, then, for any $\epsilon > 0$, we can find smooth Hamiltonians G_m such that

1. $\phi^1_{G_m}$ converges to h in the C^0 topology,

2. G_m uniformly converges to H.

Then,

$$|f_k(h) - f_k(G_m)| < \epsilon \quad \text{and} \quad \left| \int G_m - \int H \right| < \epsilon,$$

where in the first inequality above we have used the Hofer continuity property; see 2.7. Since $\epsilon > 0$ is arbitrary, (28) follows from the above inequalities and (1).

5.2 Normal subgroups with explicit quotients

It has been an open question since the proof of the Simplicity Conjecture mentioned in the introduction to identify the quotient of G by the normal subgroup of finite energy homeomorphisms FHomeo constructed there; see [8]. The circle of ideas around the proof of Theorem 1.9 allows us to resolve a variant of this question: we can find normal subgroups whose quotient can be calculated.

For example, define N to be the kernel of the map S from (26).

Proposition 5.3. *The map S from* (26) *is surjective. In particular, $G/N \simeq R'$.*

Proof. Given an element $s \in \mathbb{R}^{\mathbb{N}}$, we define a smooth autonomous Hamiltonian H on the complement of the north pole $p_+ \in \mathbb{S}^2$, and depending only on z, recursively as follows.

Call s_i the $(i-1)^{st}$ component of s. To motivate what follows, note that, given k, we can take our Lagrangian Link to correspond to the set $\{z = -1 + \frac{2i}{k+1} : 1 \leqslant i \leqslant k\}$. Fix also the data of a smooth function $E : [0,1] \to \mathbb{R}$ that is constant near 0 and 1 and satisfies $E(0) = 0$ and $E(1) = 1$.

We start by defining H to be equal to 0 on $\{-1 \leqslant z \leqslant 0\}$. Next, we define H to be equal to $2s_2$ on $\{z = 1/3\}$. We now extend H to a smooth function on $\{-1 \leqslant z \leqslant 1/3\}$ by defining it to be equal to $H(1/3)E(\frac{z}{1/3})$ on $\{0 \leqslant z \leqslant 1/3\}$. Note that any extension of H to a smooth function of z on the entire interval $[-1,1]$ will have f_2 of the extension equal to s_2, by the Lagrangian Control property (10).

Now assume, inductively, that we have extended H to a smooth function on $\{-1 \leqslant z \leqslant -1 + \frac{2k}{k+1}\}$, for some $k \geqslant 2$, that is constant near the endpoints of this interval and satisfies

$$f_i(H) = s_i, \ \ 2 \leqslant i \leqslant k$$

for any further extension of H to a smooth function on $[-1,1]$. We seek to extend H to a smooth function on $\{-1 \leqslant z \leqslant -1 + \frac{2k+2}{k+2}\}$ that is also constant near the endpoints of this interval and satisfies

$$f_i(H) = s_i, \ \ 2 \leqslant i \leqslant k+1$$

for any further extension of H to a smooth function on $[-1,1]$. Note, first of all, that $-1 + \frac{2k}{k+2} \leqslant -1 + \frac{2k}{k+1}$. In particular, the equation

$$H(-1 + \tfrac{2k+2}{k+2}) = (k+1)s_{k+1} - \sum_{i=1}^{k} H(-1 + \tfrac{2i}{k+2}),$$

makes sense, and we use it to define H on $\{z = -1 + \frac{2k+2}{k+2}\}$. We therefore have a function H defined on $\{-1 \leqslant z \leqslant -1 + \frac{2k}{k+1}\} \cup \{z = -1 + \frac{2k+2}{k+2}\}$, which is smooth on the first of these sets and constant near the endpoints of the first of these sets. Since $-1 + \frac{2k}{k+1} < -1 + \frac{2k+2}{k+2}$, there is no obstruction to further extending H smoothly to $\{-1 \leqslant z \leqslant -1 + \frac{2k+2}{k+2}\}$: more precisely, we define H to be

$$\left(H(-1 + \frac{2k+2}{k+2}) - H(-1 + \frac{2k}{k+1})\right) E\left(\frac{z - (-1 + \frac{2k}{k+1})}{\frac{2k+2}{k+2} - \frac{2k}{k+1}}\right) + H(-1 + \frac{2k}{k+1})$$

on $\{-1 + \frac{2k}{k+1} \leqslant z \leqslant -1 + \frac{2k+2}{k+2}\}$.

As above, we note that any further extension of H to a smooth function on $\{-1 \leqslant z \leqslant 1\}$ will have

$$f_{k+1}(H) = s_{k+1}, \tag{29}$$

by the Lagrangian Control property (10).

Given an element $s \in \mathbb{R}^{\mathbb{N}}$, we now define ψ to be the time-1 flow of the Hamiltonian H constructed above, away from p_+, and we set $\psi(p_+) = p_+$. We can view this as a compactly supported homeomorphism of the disc, which we also denote by ψ, and we claim that $S(\psi) = s$: indeed, for any fixed k, we can approximate ψ in C^0 by smooth flows corresponding to Hamiltonians that depend only on z, without changing the values of H on the components $\{z = -1 + \frac{2i}{k+1}\}$ of the Lagrangian Link, hence the claim follows from (29) together with the C^0 continuity of f_k (Theorem 2.9). $\square$

Remark 5.4. For a more familiar presentation of G/N via Proposition 5.3, we note that the group R' is isomorphic to $\mathbb{R}$. Indeed, both are uncountable vector spaces over $\mathbb{Q}$ of the same cardinality.

Remark 5.5. The map S allows us to define many other subgroups whose quotients can be identified. Indeed, we can take any subgroup $H \subset R'$, and then by Proposition 5.3, $N_H := S^{-1}(H)$ will be a normal subgroup with quotient H. One can think of the different N_H as "leading asymptotics subgroups": they correspond to different prescriptions of the leading asymptotics of the f_k. We may also produce groups by varying the target of S by taking different quotients of $\mathbb{R}^{\mathbb{N}}$. For example, if we quotient by the relation that $s \sim t$ if and only if $s - t$ remains bounded and take this to be the target of S, then the induced homomorphism out of G is still surjective, but one can show that its kernel contains FHomeo and Hameo, as introduced in the discussion at the end of the introduction.

6 The commutator group of G is simple

The goal of this section is to prove Theorem 1.11. We denote by G the kernel of the mass flow homomorphism $\mathrm{Homeo}_c(\Sigma, \omega) \to \mathbb{R}$, where Σ is a surface either compact or the interior of a compact surface with boundary. We denote by $[G, G]$ the commutator subgroup, i.e. the subgroup generated by commutators. We will denote by $[f, g] = f^{-1}g^{-1}fg$ the commutator of two elements f and g. Theorem 1.11 asserts that $[G, G]$ is simple.

As was mentioned in the introduction, it is known (See [8, Prop. 2.2][5]) that any normal subgroup of G contains $[G, G]$ and in particular the commutator group of $[G, G]$, which is normal in G, contains $[G, G]$, hence $[G, G]$ is perfect. Another consequence of this fact is that the simplicity of $[G, G]$ (Theorem 1.11) follows from the next lemma.

Lemma 6.1. *Any normal subgroup of $[G, G]$ is normal in G.*

Proof. Let H be a normal subgroup of $[G, G]$. To prove that H is normal in G, we need to prove that for all $h \in H$ and $g \in G$, the conjugate $g^{-1}hg$ belongs to H. We will prove it in several steps, gradually increasing the set of elements h and g for which we establish this property.

First step. We will show that $g^{-1}hg \in H$ for any $h \in H$, $g \in G$ satisfying the following two conditions:

[5]Proposition 2.2 in [8] is only stated on the disc, but holds on any compact surface by the same argument.

(I) the open set $U = \Sigma \backslash \mathrm{supp}(h)$ is non empty,

(II) there exists $f \in G$ such that $f(\mathrm{supp}(g)) \subset U$ and $f(\mathrm{supp}(g)) \cap \mathrm{supp}(g) = \varnothing$.

Note that (II) is satisfied as soon as g is supported in a disk of sufficiently small area.

Let $h \in H$ and $g \in G$ satisfy (I) and (II) and let $f \in G$ be as in (II). We claim that

$$g^{-1}hg = [f^{-1}, g]^{-1}h[f^{-1}, g]. \tag{30}$$

Indeed, denoting $S = \mathrm{supp}(g)$, then $[f^{-1}, g]$ is supported in $S \cup f(S)$. By (II) this is a disjoint union and $f(S) \subset U$. Moreover $[f^{-1}, g]$ coincides with g in the complement of $f(S)$.

For any $x \in \Sigma \backslash f(S)$, we therefore have

$$[f^{-1}, g]^{-1}h[f^{-1}, g](x) = [f^{-1}, g]^{-1}hg(x).$$

Since $f(S)$ is included in the complement of the support of h, the subsets $f(S)$ and its complement are both invariant by h. Thus $hg(x) \notin f(S)$, hence

$$[f^{-1}, g]^{-1}h[f^{-1}, g](x) = [f^{-1}, g]^{-1}hg(x) = g^{-1}hg(x).$$

For any $x \in f(S)$, the points $g(x)$ and $[f^{-1}, g](x)$ are also both in $f(S)$. As a consequence h acts trivially on them and we deduce the two identities

$$[f^{-1}, g]^{-1}h[f^{-1}, g](x) = x, \quad g^{-1}hg(x) = x.$$

We have shown that for all $x \in \Sigma$, the equality $[f^{-1}, g]^{-1}h[f^{-1}, g](x) = g^{-1}fg(x)$ holds. This establishes (30). Now since H is normal in $[G, G]$, we deduce that $g^{-1}hg$ belongs to H.

Second step. We will now show that $g^{-1}hg \in H$ for any $h \in H$ satisfying condition (I) and any $g \in G$. This case will essentially follow from the first step and fragmentation.

Let $h \in H$ satisfying (I) and $g \in G$. We let $(S_i)_{i \in I}$ be a finite open cover of Σ by disks of sufficiently small area in the sense that for each $i \in I$, there exists a map $f_i \in G$ satisfying $f_i(S_i) \subset U$ and $f_i(S_i) \cap S_i = \varnothing$.

By Fathi's fragmentation theorem [14, Thm. 6.6] the map g can be written as a product $g = g_1 \cdots g_N$ of elements in G such that each g_j is supported in one of the S_i's. By construction, the maps $h_j = g_j^{-1} \cdots g_1^{-1}hg_1 \cdots g_j$ and g_{j+1} satisfy conditions (I) and (II) for each j. Thus, we may use the first step by induction and deduce that $h_N = g^{-1}hg$ belongs to H.

Third step. We finally show that $g^{-1}hg \in H$ for any $h \in H$ and $g \in G$. This will rely on the second step and the following lemma.

Lemma 6.2. *Let $h \in H$ and let z be a fixed point of h (which exists by Arnold conjecture [24]). Then for every sufficiently small open neighborhood U of z, there exists $\ell \in H$ such that* $\mathrm{supp}(\ell) \neq \Sigma$ *and ℓ coincides with h on U.*

We postpone the proof of this lemma and use it to conclude the proof of the third step. Let $h \in H$ and $g \in G$. Let $\ell \in H$ be as provided by Lemma 6.2. Then, $h\ell^{-1}$ and ℓ belong to H and both satisfy condition (I). Our second step shows that $g^{-1}(h\ell^{-1})g \in H$ and $g^{-1}\ell g \in H$. As a consequence, their product $g^{-1}hg$ belongs to H. This concludes the proof of the third step and of Lemma 6.1. $\square$

Proof of Lemma 6.2. Let U be a small neighborhood of z. How small it is will be made precise below. Since z is fixed, it is known[6] that for every open neighborhood V of z, there exists an element $\alpha \in G$ which coincides with h in a neighborhood of z and is supported in V. We may assume that U is so small that $\alpha = h$ on U. We will use such an α to build our map ℓ.

Let x be a point such that $h(x) \neq x$. Note that we may assume without loss of generality that such a point exists. Taking a point y close to x but distinct from x, we obtain a configuration of four pairwise distinct points x, y, $h(x)$, $h(y)$. Let $f \in G$ be such that $f(x) = y$. Let A be an open neighborhood of x. If A is chosen small enough, then the four open sets A, $B = h(A)$, $C = f(A)$ and $D = h(C)$ are pairwise disjoint. In this situation, it is easy to check that for any $g \in G$ supported in A we have

$$\mathrm{supp}([f^{-1}, g]) \subset A \cup C \quad \text{and} \quad [f^{-1}, g] = g \text{ on } A.$$

Similarly, since $B \cup D = h(A \cup C)$, we have for any $g \in G$ supported in A

$$\mathrm{supp}([h^{-1}, [f^{-1}, g]]) \subset A \cup B \cup C \cup D \quad \text{and} \quad [h^{-1}, [f^{-1}, g]] = g \text{ on } A.$$

Since $h \in H$ and H is normal in $[G, G]$, the element $[h^{-1}, [f^{-1}, g]]$ belongs to H. Thus, we have shown that any element of G supported in A coincides on A with an element of H supported in $A \cup B \cup C \cup D \neq \Sigma$. We will apply this fact to an appropriate conjugate of the map α from the beginning of the proof.

Let $\beta \in [G, G]$ be a map that sends z to x (this can be found for instance among diffeomorphisms). Then, if the open sets U and V are chosen sufficiently small, the map $\beta\alpha\beta^{-1}$ is supported in A. By the above observation, there exists an element $\gamma \in H$ which coincides with $\beta\alpha\beta^{-1}$ on A and whose support is not the whole of Σ. Then, $\ell = \beta\gamma\beta^{-1}$ suits our needs. Indeed, ℓ coincides with α on V hence with h on U. Moreover, since H is normal in $[G, G]$, $\ell \in H$ and its support is not the whole of Σ. $\square$

[6] This is a standard folklore statement that can be proved by a combination of the Schoenflies and the Oxtoby-Ulam theorem.

7 Heuristic argument for recovering Ruelle

7.1 Hutchings' heuristic

If (X, ω) is a 4 dimensional Liouville domain, there is a sequence of spectral invariants $c_k(X)$ defined using Hutchings' theory of 'embedded contact homology'. These invariants satisfy

$$\lim_k c_k(X)^2/k = 4\mathrm{Vol}(X)$$

which is an analogue of the fact that – and motivated the result that – link spectral invariants asymptotically recover the Calabi invariant. (Under a tentative dictionary between ECH and quantitative Heegaard Floer homology, this becomes more than analogy.). Going further, Hutchings introduces

$$e_k(X) := c_k(X) - 2\sqrt{k\mathrm{Vol}(X)}$$

and conjectures [20] that for a 'nice and generic' star-shaped domain $X \subset \mathbb{R}^4$, one has a subleading asymptotic

$$\lim_k e_k(X) = -1/2\mathrm{Ru}(X) \tag{31}$$

where the Ruelle invariant in fact depends only on the contact boundary $Y = \partial X$ of X, and is given by $\mathrm{Ru}(X) := \int_{\partial X} \rho\, \lambda \wedge d\lambda$, with λ the restriction of the standard primitive $\sum_{i=1}^2 (x_i dy_i - y_i dx_i)$ to Y and $\rho : Y \to \mathbb{R}$ a rotation function which measures an asymptotic winding along the Reeb flow. Hutchings gives a heuristic derivation of (31) which has, broadly speaking, two parts: first, a recasting of the rotation function in terms which explicitly involves the action of ECH generators (which are certain periodic orbits of the Reeb flow); and second a description of the action of Reeb orbits in terms of integrals over the whole of Y, invoking a strong equidistribution result for those orbits. The discussion below can be seen rather directly as a translation of Hutchings' heuristic to our setting.

7.2 Actions of capped chords

In the next section we derive an expression for $k \cdot \mu_k(H)$ for a not necessarily autonomous Hamiltonian. The derivation appeals to different expressions for the action of a chord carrying the spectral invariant, computed either on the disc or on its symmetric product. We begin by reviewing the 'cappings' used to define the action.

Let $(\mathbb{D}^2, \omega = d\lambda)$ be a disk with area $\frac{1}{2}$ and let $H \in C_c^\infty([0,1] \times \mathbb{D}^2)$ be a compactly supported Hamiltonian function. We symplectically identify $(\mathbb{D}^2, \omega)$ with the northern hemisphere of $\mathbb{S}^2$, and $H \in C_c^\infty([0,1] \times \mathbb{D}^2) \subset C^\infty([0,1] \times \mathbb{S}^2)$. Let $\underline{L}$ be a Lagrangian link in $\mathbb{S}^2$ with k components bounding pairwise disjoint disks of area $\frac{1}{k+1}$. We also assume that $\underline{L}$ is disjoint from the south pole of $\mathbb{S}^2$, and use stereographic projection to identify the complement of the south pole with $\mathbb{C}$.

The Lagrangian Floer cochain complex of $\mathrm{Sym}(\underline{L})$ with respect to the Hamiltonian $\mathrm{Sym}(H)$ is denoted by $CF(\mathrm{Sym}(\underline{L}), \mathrm{Sym}(H))$; it is generated by the intersection points between $\mathrm{Sym}(\underline{L})$ and $\mathrm{Sym}(\phi_H^1(\underline{L}))$ together with cappings [6, Section 6.1].

Remark 7.1. More precisely, we need to perturb H inside $C_c^\infty([0,1]\times\mathbb{C})$ to make $\mathrm{Sym}(\phi_H^1(\underline{L}))$ intersect transversally with $\mathrm{Sym}(\underline{L})$ to define $CF(\mathrm{Sym}(\underline{L}),\mathrm{Sym}(H))$. To ease notation, we will continue to denote the Hamiltonian by H and assume that it is non-degenerate.

Equivalently, a generator of $CF(\mathrm{Sym}(\underline{L}),\mathrm{Sym}(H))$ is a Hamiltonian chord $x(t) = (x_1(t),\dots,x_k(t))_{t\in[0,1]}$ in the unordered configuration space $\mathrm{Conf}^k(\mathbb{C})$ from $\mathrm{Sym}(\underline{L})$ to itself together with a capping [6, Section 6.2]. Since $\pi_1(\mathrm{Sym}^k(\mathbb{S}^2),\mathrm{Sym}(\underline{L}))$ is trivial, every Hamiltonian chord x represents the trivial relative homotopy class. Let $\mathbf{x}:[0,1]\to\mathrm{Sym}(\underline{L})$ be a constant path, viewed as a reference base chord from $\mathrm{Sym}(\underline{L})$ to itself. A cap of x with respect to $\mathbf{x}$ is a continuous map $\hat{x}:[0,1]\times[0,1]\to\mathrm{Sym}^k(\mathbb{S}^2)$ such that $\hat{x}(0,t)=x(t)$, $\hat{x}(s,0),\hat{x}(s,1)\in\mathrm{Sym}(\underline{L})$ and $\hat{x}(1,t)=\mathbf{x}$.

Recall that two caps are equivalent if their symplectic areas are the same [6, Section 6.1][7].

Let $\hat{x}$ be a cap of x with image in $\mathrm{Sym}^k(\mathbb{C})$. Any other cap of x is equivalent to $\hat{x}$ up to adding a disc class in $\pi_2(\mathrm{Sym}^k(\mathbb{S}^2),\mathrm{Sym}(\underline{L}))$. Since $\mathrm{Sym}(\underline{L})$ is monotone [6, Lemma 4.19], adding two different primitive positive area disc classes from $\pi_2(\mathrm{Sym}^k(\mathbb{S}^2),\mathrm{Sym}(\underline{L}))$ to the fixed $\hat{x}$ gives the same equivalence class of caps. Therefore, by choosing such a primitive positive disc class to be in $\pi_2(\mathrm{Sym}^k(\mathbb{C}),\mathrm{Sym}(\underline{L}))$, we see that every equivalence class of cap is represented by one lying in $\mathrm{Sym}^k(\mathbb{C})$. We will restrict to using caps in $\mathrm{Sym}^k(\mathbb{C})$ from now on.

For any cap $\hat{x}$ in $\mathrm{Sym}^k(\mathbb{C})$, we define

$$\hat{x} \;\mapsto\; p_{\hat{x}} := (\hat{x}(s,0))_s \# (\hat{x}(1,t))_t \# (\hat{x}(1-s,1))_s.$$

Note that $p_{\hat{x}}(t)\in\mathrm{Sym}(\underline{L})$ for all $t\in[0,1]$. For any $t\in[0,1]$ and $i=1,\dots,k$, we let $p_{\hat{x},i}(t)$ to be the point on the i^{th} component of $\underline{L}$ such that $p_{\hat{x}}(t)=(p_{\hat{x},1}(t),\dots,p_{\hat{x},k}(t))$. By changing $p_{\hat{x}}$ by homotopy, if necessary, we can assume that $p_{\hat{x},i}$ is a smooth path for all i.

The action of $\hat{x}$ is defined to be

$$\begin{aligned}A(\hat{x}) &:= \int_0^1 \mathrm{Sym}(H)(x(t))dt - \int \hat{x}^*\,\mathrm{Sym}(\omega)\\ &= \int_0^1 \mathrm{Sym}(H)(x(t))dt - \int p_{\hat{x}}^*\,\mathrm{Sym}(\lambda) + \int x^*\,\mathrm{Sym}(\lambda)\\ &= \sum_{i=1}^k \left(\int_0^1 H(x_i(t))dt + \int x_i^*\lambda - \int p_{\hat{x},i}^*\lambda\right).\end{aligned}$$

By adding elements of $\pi_2(\mathrm{Sym}^k(\mathbb{C}),\mathrm{Sym}(\underline{L}))$ to $\hat{x}$, we can change the relative homotopy class of $p_{\hat{x},i}$ by a multiple of the circle which is the i^{th} component of $\underline{L}$ and it would change the integration $\int p_{\hat{x},i}^*\lambda$ by the area of the disc enclosed by the i^{th} component of $\underline{L}$. Let $\hat{x}_0$ be a cap of x such that $0\leqslant\sum_{i=1}^k\int p_{\hat{x}_0,i}^*\lambda<1$.

[7] In [6, Section 6.1], the definition of the equivalence of caps is slightly more complicated because we have the parameter η. Our case is simpler because $\eta=0$.

Lemma 7.2 ([25] (10.3.3))**.** *Let $F : \mathbb{D}^2 \to \mathbb{R}$ be the compactly supported function such that $dF = (\phi_H^1)^*\lambda - \lambda$. Then*

$$\int_0^1 H(x_i(t))dt + \int x_i^*\lambda = F(x_i(0)).$$

Proof. The result follows immediately from the quoted reference, up to checking sign conventions. The conventions for the Hamiltonian vector field are the same in this paper and in [25], namely $\iota_{X_H}\omega = dH$. However, the primitive λ used in their work differs from ours by sign, i.e. if their choice is λ_{MS} then we have $\lambda = -\lambda_{MS}$. Therefore, we have $dF = -((\phi_H^1)^*\lambda - \lambda) = -dF_{MS}$, where $dF_{MS} := (\phi_H^1)^*\lambda_{MS} - \lambda_{MS}$. Converting their equation (10.3.3), we get

$$F(x_i(0)) = -F_{MS}(x_i(0)) = -\int x_i^*\lambda_{MS} + \int_0^1 H(x_i(t))dt = \int x_i^*\lambda + \int_0^1 H(x_i(t))dt$$

□

Lemma 7.3 ([17])**.** *For the same F as above, we have*

$$\int_{\mathbb{D}^2} F\omega = 2\int_{\mathbb{D}^2} H\omega$$

Proof. The above formula is proven in [17] but with the opposite sign. The reason for this difference is that [17] and this paper have opposite conventions for the sign of the Hamiltonian vector field. □

Applying Lemma 7.2, we have

$$A(\hat{x}) = \sum_{i=1}^{k}\left(F(x_i(0)) - \int p_{\hat{x}_0,i}^*\lambda\right) + \frac{k_{\hat{x}}}{k+1}$$

where $k_{\hat{x}}$ is the image of $p_{\hat{x}}\# - p_{\hat{x}_0} \in \pi_1(\mathrm{Sym}(\underline{L}))$ to $\mathbb{Z}$ under the obvious map which sends the distinguished positive generators to 1.

We next need to relate $k_{\hat{x}}$ with $\mathrm{Cal}(H)$ and $\mathrm{Ru}(H)$. We have

$$\deg(\hat{x}) = \deg(\hat{x}_0) + 2k_{\hat{x}}.$$

To compute $\deg(\hat{x}_0)$, we use the canonical trivialization of $T\mathbb{C}^k = T\,\mathrm{Sym}^k(\mathbb{C})$ to pull-back a trivialization on $\hat{x}_0^*T\mathbb{C}^k$.

The degree $\deg(\hat{x}_0)$ is the Maslov index of the path of Lagrangians coming from concatenating $(T_{p_{\hat{x}_0}(1-t)}\,\mathrm{Sym}(\underline{L}))_{t\in[0,1]}$ and $(D\phi_H^t(T_{x(0)}\,\mathrm{Sym}^k(\underline{L})))_{t\in[0,1]}$.

Let Ω_{std} be the holomorphic volume form on $\mathrm{Conf}^k(\mathbb{C})$ induced from the standard holomorphic volume form on $\mathbb{C}^k$. Let

$$e^{2\pi i\theta_{std}(t)} = \left(\frac{\Omega_{std}(D\phi_H^t(T_{x(0)}\,\mathrm{Sym}^k(\underline{L})))}{|\Omega_{std}(D\phi_H^t(T_{x(0)}\,\mathrm{Sym}^k(\underline{L})))|}\right)^2$$

and

$$e^{2\pi i\theta_p(t)} = \left(\frac{\Omega_{std}(T_{p_{\hat{x}_0}(1-t)} \operatorname{Sym}(\underline{L}))}{|\Omega_{std}(T_{p_{\hat{x}_0}(1-t)} \operatorname{Sym}(\underline{L}))|} \right)^2.$$

Then (see [36, Section 11(j)], [35], [30]),

$$\deg(\hat{x}_0) = \lceil \theta_{std}(1) - \theta_{std}(0) + \theta_p(1) - \theta_p(0) \rceil.$$

By choosing $\underline{L}$ appropriately, we can assume that $p_{\hat{x}_0,i}$ is an embedding that is less than a full circle for all $i = 1, \dots, k$. That ensures that

$$-2k \leqslant \theta_p(1) - \theta_p(0) \leqslant 2k.$$

On the other hand, the standard holomorphic volume form Ω_0 on $\mathbb{C}$ also induces a holomorphic volume form Ω_{sing} on $\operatorname{Conf}^k(\mathbb{C})$. A direct computation shows that the forms Ω_{std} and Ω_{sing} are related by the Vandermonde determinant in the following sense:

Lemma 7.4. *Writing $\Omega_{sing} = dz_1 \wedge \cdots \wedge dz_k$ and $\Omega_{std} = dw_1 \wedge \cdots \wedge dw_k$, where w_i is the i^{th} elementary symmetric polynomial of the variables $z_1, \dots, z_k$, then*

$$\prod_{i<j}(z_i - z_j)\Omega_{sing} = \Omega_{std}$$

It follows that

$$\theta_{std}(1) - \theta_{std}(0) = \sum_{i=1}^{k} \left(\theta_0^i(1) - \theta_0^i(0)\right) + 2\sum_{i<j} (R_{i,j}(1) - R_{i,j}(0))$$

where $e^{2\pi i\theta_0^i(t)} = \left(\frac{\Omega_0(D\phi_H^t T_{x_i(0)} L_i)}{|\Omega_0(D\phi_H^t T_{x_i(0)} L_i)|}\right)^2$, and $e^{2\pi i R_{i,j}(t)} = \frac{x_i(t)-x_j(t)}{|x_i(t)-x_j(t)|}$.

Putting things together, we have

$$\begin{aligned} A(\hat{x}) =& \sum_{i=1}^{k} \left(F(x_i(0)) - \int p^*_{\hat{x}_0,i}\lambda \right) + \frac{\deg(\hat{x})}{2(k+1)} \\ &- \frac{1}{2(k+1)} \left(\lceil \sum_{i=1}^{k} \left(\theta_0^i(1) - \theta_0^i(0)\right) + 2\sum_{i<j} (R_{i,j}(1) - R_{i,j}(0)) + \theta_p(1) - \theta_p(0) \rceil \right) \end{aligned} \tag{32}$$

7.3 An expression for non-autonomous Hamiltonians

Now we can use (32) to estimate $\mu_k(H)$.

Fix a class $c \in QH(\operatorname{Sym}(\underline{L}))$. For each $N \in \mathbb{N}$, let $\hat{x}_N$ be a capped generator of $CF(\operatorname{Sym}(\underline{L}), \operatorname{Sym}(H^{\sharp N}))$ such that $A(\hat{x}_N) = kc_k(H^{\sharp N}; c)$. So we have

$$k\mu_k(H) = \lim_N \frac{kc_k(H^{\sharp N}; c)}{N} = \lim_N \frac{A(\hat{x}_N)}{N}.$$

For each $\hat{x}_N$, we can write it as $\hat{x}_{N,0}\#A_n$ for a relative homotopy class $A_n \in \pi_2(\mathbb{C}^k, \mathrm{Sym}(\underline{L}))$ such that $0 \leqslant \sum_{i=1}^k \int p^*_{\hat{x}_{N,0,i}}\lambda < 1$.

We denote the F, θ_p, θ_0^i and $R_{i,j}$ for $\hat{x}_{N,0}$ by F_N, $\theta_{N,p}$, $\theta^i_{N,0}$ and $R_{N,i,j}$ respectively so we have $-2k \leqslant \theta_{N,p}(1) - \theta_{N,p}(0) \leqslant 2k$ for all N.

As a result, we have the following expression for $k\mu_k(H)$:

$$\lim_N \left(\sum_{i=1}^k \frac{F_N((x_N)_i(0))}{N} - \frac{1}{2N(k+1)} \left(\sum_{i=1}^k \left(\theta^i_{N,0}(1) - \theta^i_{N,0}(0)\right) + 2\sum_{i<j} \left(R_{N,i,j}(1) - R_{N,i,j}(0)\right) \right) \right).$$

7.4 Heuristics

The discussion up to this point has been rigorous. Now we impose some optimistic assumptions on the behaviour of Riemann sums and show that they can be used to obtain the subleading asymptote of $\mu_k(H)$.

The first assumption concerns the Riemann sum of F over the equidistributed points $\{(x_N)_i(0)\}_{i=1}^k$ (as k goes to infinity). We assume for simplicity that there is no subleading error.

Assumption 7.5. We assume that

$$\lim_k \left(\left(\lim_N \sum_{i=1}^k \frac{F_N((x_N)_i(0))}{N} \right) - 2k\mathrm{Cal}(H) \right) = \lim_k \lim_N \sum_{i=1}^k \left(\frac{F_N((x_N)_i(0))}{N} - \int_{\mathbb{D}^2} F \right) = 0.$$

Remark 7.6. The term $\frac{F_N((x_N)_i(0))}{N}$ equals $\frac{1}{N}\sum_{j=0}^{N-1} F(\phi_H^j((x_N)_i(0)))$. The previous Assumption is therefore close to the conclusion of the Birkhoff ergodic theorem [33]. We recall that this theorem concerns the behaviour of length N time averages with a common start point (so taken from a common orbit). In the uniquely ergodic case, one can choose the start points arbitrarily, since any weak limit of measures along orbit segments converges to an invariant measure. Away from the uniquely ergodic case, if one can perturb the start-points of the length N orbit sums in open sets whose total measure diverges, one can recover the same result using the Borel-Cantelli lemma. Our situation is neither uniquely ergodic, nor do we have such generous control over the initial points $(x_N)_i(0)$ of our orbit segments (standard transversality arguments would let them vary in open sets of exponentially decreasing size).

The second assumption has a similar ergodic flavour, and concerns the Riemann sum of the linking number between pairs of points, and the interpretation of Calabi as the average asymptotic linking number [15, 17]. We again give a formulation with no subleading error.

Assumption 7.7. We assume that

$$\lim_k \left(\left(\lim_N \frac{1}{N(k+1)} \sum_{i<j} R_{N,i,j}(1) - R_{N,i,j}(0) \right) - \frac{k^2}{k+1}\mathrm{Cal}(H) \right) = 0.$$

The last assumption is mild. It is the interpretation of Ruelle as the average asymptotic rotation number. Since we are not going to prove the previous assumptions, we also leave out the details of this statement, but expect its verification would be straightforward.

Assumption 7.8. We assume that

$$\lim_k \lim_N \frac{1}{2N(k+1)} \left(\sum_{i=1}^{k} \left(\theta^i_{N,0}(1) - \theta^i_{N,0}(0) \right) \right) = \frac{1}{2}\mathrm{Ru}(H).$$

Under Assumptions 7.5, 7.7 and 7.8, we get

$$\lim_k \left(k\mu_k(H) - (k+1)\mathrm{Cal}(H) \right) = \lim_k \left(k\mu_k(H) - (2k - \frac{k^2}{k+1})\mathrm{Cal}(H) \right) = -\frac{1}{2}\mathrm{Ru}(H).$$

which coincides with the calculations for autonomous Hamiltonians.

References

[1] A. Banyaga. Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique. *Comment. Math. Helv.*, 53(2):174–227, 1978.

[2] L. Buhovsky. On two remarkable groups of area-preserving homeomorphisms. *arXiv:2204.08020*, 2022.

[3] E. Calabi. On the group of automorphisms of a symplectic manifold. *Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969)*, pages 1–26, 1970.

[4] G. Chen. Closed-open morphisms on Periodic Floer Homology. *arXiv:2111.11891*, 2021.

[5] C.-H. Cho and M. Poddar. Holomorphic orbi-discs and Lagrangian Floer cohomology of symplectic toric orbifolds. *J. Differential Geom.*, 98(1):21–116, 2014.

[6] D. Cristofaro-Gardiner, V. Humilière, C. Y. Mak, S. Seyfaddini, and I. Smith. Quantitative Heegaard Floer cohomology and the Calabi invariant. *arXiv:2105.11026*, 2021.

[7] D. Cristofaro-Gardiner, V. Humilière, and S. Seyfaddini. PFH spectral invariants on the two-sphere and the large scale geometry of Hofer's metric. *arXiv:2102.04404, to appear in JEMS*, 2021.

[8] D. Cristofaro-Gardiner, V. Humilière, and S. Seyfaddini. Proof of the simplicity conjecture. *arXiv:2001.01792*, 2021.

[9] D. Cristofaro-Gardiner, M. Hutchings, and V. G. B. Ramos. The asymptotics of ECH capacities. *Invent. Math.*, 199(1):187–214, 2015.

[10] D. Cristofaro-Gardiner, R. Prasad, and B. Zhang. Periodic Floer Homology and the smooth closing lemma for area-preserving surface diffeomorphisms. *arXiv:2110.02925*, 2021.

[11] D. Cristofaro-Gardiner and N. Savale. Sub-leading asymptotics of ECH capacities. *Selecta Mathematica*, 26(65), 2020.

[12] O. Edtmair and M. Hutchings. PFH spectral invariants and C^∞ closing lemmas. *arXiv:2110.02463*, 2021.

[13] M. Entov and L. Polterovich. Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.*, (30):1635–1676, 2003.

[14] A. Fathi. Structure of the group of homeomorphisms preserving a good measure on a compact manifold. *Ann. Sci. École Norm. Sup. (4)*, 13(1):45–93, 1980.

[15] A. Fathi. Sur l'homomorphisme de Calabi $\mathrm{Diff}_c^\infty(\mathbb{R}^2, m) \to \mathbb{R}$. *Appears in: Transformations et homéomorphismes préservant la mesure. Systèmes dynamiques minimaux., Thèse Orsay*, 1980.

[16] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian Floer theory on compact toric manifolds II: bulk deformations. *Selecta Math. (N.S.)*, 17(3):609–711, 2011.

[17] J.-M. Gambaudo and E. Ghys. Enlacements asymptotiques. *Topology*, 36(6):1355–1379, 1997.

[18] E. Ghys. Knots and dynamics. In *International Congress of Mathematicians. Vol. I*, pages 247–277. Eur. Math. Soc., Zürich, 2007.

[19] M. Hutchings. Quantitative embedded contact homology. *J. Differential Geom.*, 88(2):231–266, 2011.

[20] M. Hutchings. ECH capacities and the Ruelle invariant. *arXiv:1910.08260*, 2019.

[21] F. Le Roux. Simplicity of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \mathrm{Area})$ and fragmentation of symplectic diffeomorphisms. *J. Symplectic Geom.*, 8(1):73–93, 2010.

[22] C. Y. Mak and I. Smith. Non-displaceable Lagrangian links in four-manifolds. *Geom. Funct. Anal.*, 2021.

[23] K. Mann. The structure of homeomorphism and diffeomorphism groups. *Notices Amer. Math. Soc.*, 68(4):482–492, 2021.

[24] S. Matsumoto. Arnold conjecture for surface homeomorphisms. In *Proceedings of the French-Japanese Conference "Hyperspace Topologies and Applications" (La Bussière, 1997)*, volume 104, pages 191–214, 2000.

[25] D. McDuff and D. Salamon. *Introduction to symplectic topology.* Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, third edition, 2017.

[26] S. Müller. The group of Hamiltonian homeomorphisms in the L^∞-norm. *J. Korean Math. Soc.*, 45(6):1769–1784, 2008.

[27] Y.-G. Oh and S. Müller. The group of Hamiltonian homeomorphisms and C^0–symplectic topology. *J. Symplectic Geom.*, 5(2):167–219, 2007.

[28] L. Polterovich. *The geometry of the group of symplectic diffeomorphisms.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001.

[29] L. Polterovich and E. Shelukhin. Lagrangian configurations and Hamiltonian maps. *arXiv:2102.06118*, 2021.

[30] J. Robbin and D. Salamon. The Maslov index for paths. *Topology*, 32(4):827–844, 1993.

[31] C. Rosendal. Automatic continuity of group homomorphisms. *The Bulletin of Symbolic Logic*, 15(2), 2009.

[32] C. Rosendal. Continuity of universally measurable homomorphisms. *Forum of Mathematics, Pi*, 7(e5), 2019.

[33] D. J. Rudolph. *Fundamentals of measurable dynamics.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990. Ergodic theory on Lebesgue spaces.

[34] D. Ruelle. Rotation numbers for diffeomorphisms and flows. *Ann. Inst. H. Poincaré Phys. Théor.*, 42(1):109–115, 1985.

[35] P. Seidel. Graded Lagrangian submanifolds. *Bull. Soc. Math. France*, 128(1):103–149, 2000.

[36] P. Seidel. *Fukaya categories and Picard-Lefschetz theory.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[37] J.-C. Sikorav. Systèmes hamiltoniens et topologie symplectique. Dipartimento di Matematica dell' Universita di Pisa, ETS, EDITRICE PISA, 1990.

Dan Cristofaro-Gardiner
Mathematics Department, University of Maryland, College Park, USA.
e-mail: dcristof@umd.edu

Vincent Humilière
Sorbonne Université and Université de Paris, CNRS, IMJ-PRG, F-75006 Paris, France
& Institut Universitaire de France.
e-mail: vincent.humiliere@imj-prg.fr

Cheuk Yu Mak
School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, Edinburgh, EH9 3FD, U.K.
e-mail: cheukyu.mak@ed.ac.uk

Sobhan Seyfaddini
Sorbonne Université and Université de Paris, CNRS, IMJ-PRG, F-75006 Paris, France.
e-mail: sobhan.seyfaddini@imj-prg.fr

Ivan Smith
Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, CB3 0WB, U.K.
e-mail: is200@cam.ac.uk