COUNTING REAL GALOIS COVERS OF THE PROJECTIVE LINE

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ABSTRACT. We consider the following problem about Galois covers of $\mathbb{P}^1$. Fixing their type of ramification that is, essentially, their monodromy group G and their branch locus, assumed to be defined over $\mathbb{R}$, the question is how many covers are defined over $\mathbb{R}$ and how many are not? J.-P. Serre showed the number of all Galois covers with given ramification type can be computed from the character table of G. We re-use Serre's method of calculation in the more refined situation of Galois covers defined over $\mathbb{R}$, for which there is a group-theoretic characterization due to P. Dèbes and M. Fried. We obtain explicit answers to our problem. As an application, we exhibit new families of covers not defined over their field of moduli and the monodromy group of which can be chosen arbitrarily large. We also give examples of Galois covers defined over the field $\mathbb{Q}^{tr}$ of totally real algebraic numbers with $\mathbb{Q}$-rational branch locus.

2000 *Mathematic Subject Classification.* Primary 12F12, 20C15; Secondary 14H30, 14H10.

INTRODUCTION

By Riemann's Existence Theorem there is a bijective correspondence between isomorphism classes of Galois covers $f : X \to \mathbb{P}^1_{\mathbb{C}}$ of the projective line with Galois group G and branch points $t_1, ..., t_r \in \mathbb{P}^1(\mathbb{C})$ and r-tuples $(g_1, ..., g_r) \in G$ of generators of G satisfying the relation $g_1 \cdots g_r = 1$. Fixing an r-tuple $\mathbf{C} = (C_1, ..., C_r)$ of conjugacy classes of G, we say f is of type $\mathbf{C}$ if the corresponding r-tuple $(g_1, ..., g_r) \in G$ has the extra property that there exists a permutation σ such that $g_i \in C_{\sigma(i)}$, for $i = 1, ..., r$.

An important and well-known formula proved by Serre in [11] chap.7 computes the number of r-tuples $(g_1, ..., g_r) \in G$ with $g_i \in C_i$ for $i = 1, ..., r$ and such that $g_1 \cdots g_r = 1$. In many cases, this formula can be used to compute the number of isomorphism classes of G-covers of $\mathbb{P}^1$ of type $\mathbf{C}$, with branch points $t_1, ..., t_r \in \mathbb{P}^1(\mathbb{C})$. This formula proved to be particularly powerful in the classical rigid situation and, for instance, led to the realization over $\mathbb{Q}$ of most of the sporadic groups (see [9] chap II for a systematic investigation of this method).

In this paper we consider the refined problem of counting the number of those G-covers of $\mathbb{P}^1$ with fixed branch locus and for which the field of the real numbers $\mathbb{R}$ is a field of definition. We also consider the related problem of how many G-covers have their field of moduli contained in $\mathbb{R}$. For these two questions P. Dèbes and M. Fried showed in [5] there is also a group theoretic characterization: the r-tuples $(g_1, ..., g_r) \in G$ should verify some additional conditions, involving the involutions of G (see §1).

Our results are the following. First, generalizing Serre's formula, and using Dèbes and Fried's results, we give a general formula for the number of r-tuples $(g_1, ..., g_r) \in G$ corresponding to G-covers $f : X \to \mathbb{P}^1$ with given branch locus and which are defined over $\mathbb{R}$. In the general situation, this formula is more complicated than the one given by Serre. In

order to simplify it and make it effective, we consider two special cases separately, where the branch locus consists either only of real points or only of pairs of complex conjugate points. We give then several applications. On the one hand, we deal with the existence of G-covers which are not defined over their field of moduli. Some criteria are already known that guarantee that the field of moduli is a field of definition, for instance when $Z(G)$ is a direct summand of G (see [2] prop.2.8). Most of these results rely on a cohomological approach (see for instance [3], [4] or [15]); ours is different and leads to criteria - one of them being an easy-to-check group-theoretic condition - for G-covers not to be defined over their field of moduli. Applying these criteria, we exhibit infinite families of groups for which one can always find such G-covers. On the other hand we explain how to use our computations to descend from $\mathbb{C}$ to the field $\mathbb{Q}^{tr}$ of all totally real algebraic numbers. It is known (cf [5] Theorem 5.7) that each finite group is the Galois group of a regular extension of $\mathbb{Q}^{tr}(X)$ but the proof does not show this can be done with a branch point divisor $\mathbf{t}$ defined over $\mathbb{Q}$. Our method - when it works - enables us to choose $\mathbf{t}$ this way. We conclude by considering the case of the Mathieu group M_{11}.

The paper is organized as follow. In §1 we introduce the main tools. In §2 we state the results and make some comments. §3 is devoted to the proofs and §4 to the examples and applications.

I wish to thank P. Dèbes for encouraging me to write this paper and making many helpful suggestions.

1. Preliminaries

Notations: For a finite group G, denote:
- the set of all inner automorphisms of G by Int(G).
- the set of all elements of order ≤ 2 in G by Inv(G).
- the set of all the irreducible complex characters of G by Irr(G) and the trivial character of G by χ_1.
- for all $g \in G$ the centralizer of g in G by $\mathrm{Cen}_G(g)$.

Recall a G-cover with group G is a pair (f, α) where $f : X \to \mathbb{P}^1$ is a Galois cover with group G and $\alpha : \mathrm{Aut}(f) \to G$ is a group isomorphism. One can attach to each G-cover of $\mathbb{P}^1_{\mathbb{C}}$ the three following invariants: the monodromy group G, the branch point set $\mathbf{t} = \{t_1, ..., t_r\} \subset \mathbb{P}^1(\mathbb{C})$ (that we sometimes view as a divisor $(t_1) + ... + (t_r)$ on $\mathbb{P}^1$) and for each $t \in \mathbf{t}$ the *associated inertia canonical conjugacy class* C_t . To summarize this, we will sometimes say the considered G-cover has *ramification type* $[G, \mathbf{C}, \mathbf{t}]$ (see [13] definition 2.12 p.37). Adopting the topological point of view, let us recall what these invariants correspond to: given $\mathbf{t} = \{t_1, ..., t_r\}$ introduce a *topological bouquet* $\underline{\gamma}$ of $\mathbb{P}^1_{\mathbb{C}}\backslash\mathbf{t}$, that is an r-tuple of homotopy classes of loops $\gamma_1, ..., \gamma_r$ based at some point $t_0 \notin \mathbf{t}$ such that
- $\gamma_1, ..., \gamma_r$ generate the topological fundamental group $\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\backslash\mathbf{t}, t_0)$ with the single relation $\gamma_1 ... \gamma_r = 1$.
- γ_i is a loop revolving once, counterclockwise, about t_i, $i = 1, ..., r$.

Now, considering a G-cover $f : X \to \mathbb{P}^1_{\mathbb{C}}$, the monodromy action defines a permutation representation $\pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C})\backslash\mathbf{t}, t_0) \to \text{Per}(f^{-1}(t_0))$. The image group G of this representation is the monodromy group (or, equivalently the Galois group) of f and the conjugacy class C_{t_i} of the image of γ_i in G is the inertia canonical class corresponding to t_i, $i = 1, ..., r$.

For any integer $r \geq 3$ let $\mathcal{U}^r \subset (\mathbb{P}^1_{\mathbb{C}})^r$ be the subset of $(\mathbb{P}^1_{\mathbb{C}})^r$ consisting of all r-tuples $\mathbf{t}' = (t_1, ..., t_r) \in (\mathbb{P}^1_{\mathbb{C}})^r$ such that $t_i \neq t_j$ for $1 \leq i \neq j \leq r$, let $\mathcal{U}_r = \mathcal{U}^r/S_r$ be the quotient space of $\mathcal{U}^r$ by the natural action of the symmetric group S_r and $\pi_r : \mathcal{U}_r \to \mathcal{U}^r/S_r$ the canonical projection. Given a finite group G let $\psi_{r,G} : \mathcal{H}_{r,G} \to \mathcal{U}_r$ be the coarse moduli space (fine assuming $Z(G) = \{1\}$) for the category of G-covers of $\mathbb{P}^1_{\mathbb{C}}$ with group G and r branch points, where $\psi_{r,G}$ is the application which to a given isomorphism class of G-covers associates its branch point set. For any r-tuple $\mathbf{C} = (C_1, ..., C_r)$ of non trivial conjugacy classes in G let $\mathcal{H}_{r,G}(\mathbf{C})$ be the corresponding *Hurwitz space* [7], that is the union of irreducible components of $\mathcal{H}_{r,G}$ parametrizing the isomorphism classes of G-covers with ramification type $[G, \mathbf{C}, \mathbf{t}]$. A point $\mathbf{h} = (h, (t_1, ..., t_r))$ of the fiber product $\mathcal{H}_{r,G}(\mathbf{C}) \times_{\mathcal{U}_r} \mathcal{U}^r$ then corresponds to a G-cover given with an ordering of its branch points, which allows us to define a monodromy application:

$$\begin{array}{rccc} \text{M:} & \mathcal{H}_{r,G}(\mathbf{C}) \times_{\mathcal{U}_r} \mathcal{U}^r & \to & \{C_1, ..., C_r\}^r \\ & (h, (t_1, ..., t_r)) & \to & (C_{t_1}, ..., C_{t_r}) \end{array}$$

This application, being continuous, is constant on each connected component of $\mathcal{H}_{r,G}(\mathbf{C}) \times_{\mathcal{U}_r} \mathcal{U}^r$. So, $M^{-1}(\mathbf{C})$ is a union of connected components of $\mathcal{H}_{r,G}(\mathbf{C}) \times_{\mathcal{U}_r} \mathcal{U}^r$; we will denote this variety by $\mathcal{H}'_{r,G}(\mathbf{C})$. We have a cartesian square:

$$\begin{array}{ccc} \mathcal{H}'_{r,G}(\mathbf{C}) & \xrightarrow{\Pi_r} & \mathcal{H}_{r,G}(\mathbf{C}) \\ {\scriptstyle \psi'_{r,G}}\downarrow & \square & \downarrow{\scriptstyle \psi_{r,G}} \\ \mathcal{U}^r & \xrightarrow{\pi_r} & \mathcal{U}_r \end{array}$$

We will freely use the general theory of Hurwitz spaces (see for instance [7] and [13]), and only recall here the description of the fibers of $\psi_{r,G}$ and $\psi'_{r,G}$ in terms of *Nielsen classes* $\text{Ni}(\mathbf{C}, G)$ and *straight Nielsen classes* $\text{Sni}(\mathbf{C}, G)$ respectively, where:

$$\text{Ni}(\mathbf{C}, G) = \left\{ (g_1, ..., g_r) \in G^r \;\middle|\; \begin{array}{l} (1)\ G =< g_1, ..., g_r > \\ (2)\ g_1 \cdots g_r = 1 \\ (3)\ g_i \in C_{\sigma(i)},\ i = 1, ..., r \text{ for some } \sigma \in S_r \end{array} \right\}$$

and $\text{Sni}(\mathbf{C}, G)$ is the set defined as $\text{Ni}(\mathbf{C}, G)$, but replacing (3) by

$$(3)'\ g_i \in C_i \text{ for } i = 1, ..., r.$$

We use the notations $\overline{\text{ni}}(\mathbf{C}, G)$ and $\overline{\text{sni}}(\mathbf{C}, G)$ for the corresponding quotient sets modulo the componentwise action of $\text{Int}(G)$.

Given $\mathbf{t} \in \mathcal{U}_r$, it is classical that $(\psi_{r,G})^{-1}(\mathbf{t})$ is in bijection with $\overline{\text{ni}}(\mathbf{C}, G)$. Furthermore, if we choose an ordering of the branch points $\mathbf{t}' = (t_1, ..., t_r)$ in $\mathbf{t}$, $\overline{\text{sni}}(\mathbf{C}, G)$ is in bijection with $(\psi'_{r,G})^{-1}(\mathbf{t}')$. The correspondence is given by the monodromy action. We will sometimes say abusively that *a G-cover with branch point set* $\mathbf{t} \in \mathcal{U}_r(\mathbb{C})$ *is in* $\overline{\text{ni}}(\mathbf{C}, G)$ when its isomorphism class has ramification type $[G, \mathbf{C}, \mathbf{t}]$ or that, *if an ordering* $\mathbf{t}' = (t_1, ..., t_r) \in \mathcal{U}^r(\mathbb{C})$ *is given, a*

G-cover is in $\overline{\text{sni}}(\mathbf{C}, G)$ when C_i is the inertia canonical class associated with t_i for $i = 1, ..., r$.

Since we are interested in G-covers defined over $\mathbb{R}$, we will always suppose the branch point divisor is real, that is consists of $r = r_1 + 2r_2$ branch points with:

(bp) $\begin{cases} \text{- } r_1 \text{ real branch points } t_1, ..., t_{r_1}, \text{ which we assume to be ordered: } t_1 < ... < t_{r_1}. \\ \text{- } r_2 \text{ complex conjugated pairs } \{z_i, \overline{z}_i\} \subset \mathbb{P}^1(\mathbb{C})\backslash\mathbb{P}^1(\mathbb{R}). \text{ We will generally write} \\ z_i = t_{r_1+i},\ \overline{z}_i = t_{r+1-i},\ i = 1, ..., r_2. \text{ We may also, if needed, order them} \\ \text{according to their real and imaginary parts.} \end{cases}$

We now introduce the two following subsets of $\text{Sni}(\mathbf{C}, G)$, which play an important part in the sequel:

- the set $\text{Sni}^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)$, which is the subset of $\text{Sni}(\mathbf{C}, G)$ consisting of those $(g_1, ..., g_r)$ in $\text{Sni}(\mathbf{C}, G)$ verifying the additional condition:

  (4) there exists $g_0 \in G$ such that
  - $g_0(g_1...g_i)g_0^{-1} = (g_1...g_i)^{-1}$ for $i = 1, ..., r_1 - 1$
  - $g_0 g_{r_1+i} g_0^{-1} = g_{r+1-i}^{-1}$ and $g_0 g_{r+1-i} g_0^{-1} = g_{r_1+i}^{-1}$ for $i = 1, ..., r_2$

- the set $\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$, which is the subset of $\text{Sni}^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)$ consisting of those $(g_1, ..., g_r)$ in $\text{Sni}^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)$ for which

  (4)' in addition to (4) g_0 can be taken in $\text{Inv}(G)$

As above we write $\overline{\text{sni}}^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)$ and $\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$ for the corresponding quotient sets modulo the action of $\text{Int}(G)$. We have the following relation:

$$|\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)| = |\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|[G : Z(G)]$$

We will also need the "Σ-versions", $\Sigma^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)$ and $\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$ of $\text{Sni}^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)$ and $\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$ which are defined by conditions (2), (3)', (4) and (2), (3)', (4)' respectively (that is we drop the generating condition (1)). It readily follows from the definitions that

$$|\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)| = \frac{|\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|}{[G : Z(G)]} \leq \frac{|\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|}{[G : Z(G)]}$$

So, computing the cardinality of the "Σ-versions", which is easier, gives an upper bound for $|\text{Sni}^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)|$ and $|\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$. Moreover, in lots of situations $\text{Sni}^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2) = \Sigma^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)$ and $\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2) = \Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$ (see comment 2.2.3).

One of the main results of [5] is that, given $\mathbf{t}' \in \mathcal{U}^r$ ordered as in (bp), there exists an identification $(\Psi'_{r,G})^{-1}(\mathbf{t}') \simeq \overline{\text{sni}}(\mathbf{C}, G)$, as recalled above, such that $\overline{\text{sni}}^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)$ is exactly the set of those G-covers in $\overline{\text{sni}}(\mathbf{C}, G)$ with field of moduli contained in $\mathbb{R}$ and $\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$ the one of those G-covers in $\overline{\text{sni}}(\mathbf{C}, G)$ which are defined over $\mathbb{R}$.

A complete proof of this statement can be found in [5]. We only recall here the main ideas. Let $\mathbf{t} \in \mathcal{U}_r(\mathbb{R})$ be a real branch point divisor ordered as in (bp). The first step consists in

describing the action of complex conjugation c on the fundamental group $\pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C})\backslash\mathbf{t}, t_0)$ of $\mathbb{P}^1(\mathbb{C})\backslash\mathbf{t}$, which we denote by π^{top}. One can find $\Gamma_1, ..., \Gamma_r \in \pi^{\text{top}}$ which generate π^{top} with the single relation $\Gamma_1 \cdots \Gamma_r = 1$ and complex conjugation c acts on π^{top} by Hurwitz's formulas (see for instance [9]):

$$(*) \quad \begin{array}{ll} c.\Gamma_i = \Gamma_1 \cdots \Gamma_{i-1}\Gamma_i^{-1}(\Gamma_1 \cdots \Gamma_{i-1})^{-1} & \text{for } i = 1, ..., r_1 \\ c.\Gamma_{r_1+i} = \Gamma_{r+1-i}^{-1} & \text{for } i = 1, ..., r_2 \end{array}$$

We will denote by $\mathcal{C}$ the formal operator which maps each component Γ_i of an r-tuple $(\Gamma_1, ..., \Gamma_r)$ to the right hand side term of the formulas (*) (that is $c.\Gamma_i = \Gamma_i^{\mathcal{C}}$, $i = 1, ..., r$). Let $\Omega/\mathbb{C}(X)$ be the maximal algebraic extension of $\mathbb{C}(X)$ unramified outside $\mathbf{t}$; $\Omega/\mathbb{C}(X)$ is Galois with group $\text{Gal}(\Omega|\mathbb{C}(X)) =: \pi^{\text{alg}}$. And, by Riemann's Existence Theorem we get an isomorphism $\widehat{\pi^{\text{top}}} \xrightarrow{\sim} \pi^{\text{alg}}$, where $\widehat{\pi^{\text{top}}}$ is the profinite completion of π^{top} [11].

The second step is an if and only if condition for the "descent from $\mathbb{C}$ to $\mathbb{R}$": As the branch point divisor is real, $\Omega/\mathbb{R}(X)$ is Galois with group $\text{Gal}(\Omega|\mathbb{R}(X)) =: \pi_{\mathbb{R}}$. Furthermore, since $\mathbb{P}^1$ has real points, the short exact sequence (**) below splits and $\pi_{\mathbb{R}} \simeq \pi^{\text{alg}} \rtimes \mathbb{Z}/2\mathbb{Z}$. Now, if $K/\mathbb{C}(X)$ is the function field extension of an algebraic G-cover $f : X \to \mathbb{P}^1$ and $\psi : \pi^{alg} \to G$ is the corresponding epimorphism, f can be defined over $\mathbb{R}$ (so f is in $\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$) if and only if there exists a map $\tilde{\psi}$ such that the following diagram commutes:

$$(**) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \pi^{\text{alg}} & \longrightarrow & \pi_{\mathbb{R}} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \psi} & \swarrow{\scriptstyle \exists\tilde{\psi}} & & & & & \\ & & G & & & & & & \end{array}$$

For all $\psi \in \text{Hom}(\pi^{\text{alg}}, G)$, write $g_i = \psi(\Gamma_i)$ i=1,...,r. Then, ψ extends to $\tilde{\psi} \in \text{Hom}(\pi^{\text{alg}} \rtimes \mathbb{Z}/2\mathbb{Z}, G)$ if and only if there exists $g_0 \in \text{Inv}(G)$ for which $g_0 g_i g_0 = g_i^{\mathcal{C}}$, $i = 1, ..., r$ (see [5]; lemma 3.3). This provides the condition in the definition of $\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$.

Furthermore, if $f : X \to \mathbb{P}^1_{\mathbb{C}}$ corresponds to $(g_1, ..., g_r) \in \text{Sni}(\mathbf{C}, G)$, $f^c : X^c \to \mathbb{P}^1_{\mathbb{C}}$ corresponds to $(g_1^{\mathcal{C}}, ..., g_r^{\mathcal{C}}) \in \text{Sni}(\mathbf{C}^{\mathcal{C}}, G)$. So the set of all isomorphism classes of G-covers with field of moduli contained in $\mathbb{R}$ and branch points $\mathbf{t}'$ in $\overline{\text{sni}}(\mathbf{C}, G)$ corresponds to $\overline{\text{sni}}^{mod,\mathbb{R}}(\mathbf{C}; r_1, r_2)$. The extra condition $g_0^2 = 1$ that appears in $\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$ comes from Weil's cocycle condition [14].

Remark 1.1. If we fix $\mathbf{t} \in \mathcal{U}_r(\mathbb{R})$, the real points in the fiber $(\psi_{r,G})^{-1}(\mathbf{t})$ correspond to G-covers which have their field of moduli contained in $\mathbb{R}$. So, when working with moduli spaces, it is no longer posssible to distinguish between the G-covers defined over $\mathbb{R}$ and those which only have their field of moduli contained in $\mathbb{R}$. Some information is lost.

2. Statements and comments

2.1. **Statements.** Our main results are estimates of the cardinality of $\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$. What we actually compute is not $|\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$ but $|\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$, which is an upper bound for $|\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$. In the sequel, we will always assume $\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2) \neq \emptyset$.

We distinguish between the three following situations, depending on the branch points configuration:

- *General configuration* (R-C): $r_1, r_2 \geq 0$.

and the two special cases:
- *Real configuration* (R): $r_2 = 0$.
- *Complex pairs configuration* (C): $r_1 = 0$.
Though (R) and (C) are only special cases of (R-C), it is easier to compute $|\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$ in those two situations and the formulas obtained are much simpler than in the general case.

To make the formulas more legible, we will write:
- Z_i for the order of the centralizer of any element in the conjugacy class C_i.
- $\underline{\chi} \in \mathrm{Irr}(G)^r$ for any r-tuple $(\chi_1, ..., \chi_r) \in \mathrm{Irr}(G)^r$.
- $\underline{u} \in \mathrm{Inv}(G)^r$ for any r-tuple $(u_0, ..., u_{r-1}) \in \mathrm{Inv}(G)^r$.
We also fix $g_1, ..., g_r \in G$ with $g_i \in C_i$, $i = 1, ..., r$.

2.1.1. *Statement of theorem 2.1 (configuration (R)).* For all $\underline{\chi} \in \mathrm{Irr}(G)^r$ we set:

$$\mathbf{I}_{\underline{\chi}} = \sum_{\underline{u} \in \mathrm{Inv}(G)^r/G\cdot} \chi_1(u_0u_1)\chi_2(u_1u_2)\cdots\chi_r(u_{r-1}u_0)$$

where $\mathrm{Inv}(G)^r/G\cdot$ is the quotient set of the equivalence relation on $\mathrm{Inv}(G)^r$ which identifies two r-tuples $\underline{u}, \underline{u}' \in \mathrm{Inv}(G)^r$ if $(u_0, ..., u_{r-1}) = g.(u'_0, ..., u'_{r-1})$ for some $g \in G$. We also write:

$$\mathbf{n}^{\mathbb{R}}(\mathbf{C}; r, 0) = \frac{1}{Z_1 \cdots Z_r} \sum_{\underline{\chi} \in \mathrm{Irr}(G)^r} \chi_1(g_1)\cdots\chi_r(g_r)\ \mathbf{I}_{\underline{\chi}}$$

Theorem 2.1 (Real branch points)**.** *We have*

$$|\Sigma^{\mathbb{R}}(\mathbf{C}; r, 0)| = \mathbf{n}^{\mathbb{R}}(\mathbf{C}; r, 0)$$

Remark 2.2. This formula can be improved to give exactly $|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; r, 0)|$

$$|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; r, 0)| = \frac{|Z(G)|}{|G|Z_1 \cdots Z_r} \sum_{\underline{\chi} \in \mathrm{Irr}(G)^r} \chi_1(g_1)\cdots\chi_r(g_r)\ \mathbf{I}^*_{\underline{\chi}}$$

where $\mathbf{I}^*_{\underline{\chi}}$ is defined as $\mathbf{I}_{\underline{\chi}}$ with the only difference that the summation domain is the subset of $\mathrm{Inv}(G)^r/G\cdot$ of those r-tuples of representatives $\underline{u} \in \mathrm{Inv}(G)^r/G\cdot$ such that $G =< u_0u_1, ..., u_{r-2}u_{r-1} >$. This condition does not depend on the representive $\underline{u}$ since, if $g \cdot \underline{u} \in \mathrm{Inv}(G)^r$ for some $g \in G$ then $gu_igu_{i+1} = (gu_i)^{-1}gu_{i+1} = u_iu_{i+1},\ i = 0, ..., r-2$.

2.1.2. *Statement of theorem 2.3 (configuration (C)).* For any $\chi \in \mathrm{Irr}(G)$ and for any $g_0 \in G$ we denote the number of occurences of the trivial representation in the decomposition of $\chi|_{\mathrm{Cen}_G(g_0)}$ into a direct sum of irreducible linear representations by $\frac{\alpha_{\chi,g_0}}{|\mathrm{Cen}_G(g_0)|}$, that is (see [12]):

$$\alpha_{\chi,g_0} = \sum_{u \in \mathrm{Cen}_G(g_0)} \chi(u)$$

We also set:

$$\mathbf{A}_{\chi} = \sum_{g_0 \in \mathrm{Inv}(G)/Z(G)\cdot} \alpha_{\chi,g_0}$$

where $\mathrm{Inv}(G)/Z(G)\cdot$ is defined as $\mathrm{Inv}(G)^r/G\cdot$ above and:

$$\mathbf{n}^{\mathbb{R}}(\mathbf{C};0,s)=\frac{|C_1|\cdots|C_s|}{|G|}\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(g_1)\cdots\chi(g_s)}{\chi(1)^{s-1}}\,\mathbf{A}_\chi$$

Theorem 2.3 (Complex conjugate branch points)**.** *We have*

$$|\Sigma^{\mathbb{R}}(\mathbf{C};0,s)|\leq\mathbf{n}^{\mathbb{R}}(\mathbf{C};0,s)$$

with equality if $\Sigma^{\mathbb{R}}(\mathbf{C};0,s)=\mathrm{Sni}^{\mathbb{R}}(\mathbf{C};0,s)$

2.1.3. *Statement of theorem 2.4 (configuration (R-C)).* Our formula for the general case is more complicated. We set, for $r_1,r_2>0$ [1]:

$$\mathbf{n}_0^{\mathbb{R}}(\mathbf{C};r_1,r_2)=\sum_{\underline{\chi},\alpha,\beta,\underline{u}}\frac{\alpha(g_{r_1})\prod_{i=1}^{r_2}\beta(g_{r_1+i})}{\beta(1)^{r_2-1}}\prod_{i=1}^{r_1-1}(\chi_i(g_i)\chi_i(u_{i-1}u_i))\sum_{x\in G}\alpha(u_{r_1-1}x^{-1}u_0x)\beta(x)$$

where the first summation is taken over all $\underline{\chi}\in\mathrm{Irr}(G)^{r_1-1}$, all $\alpha,\beta\in\mathrm{Irr}(G)$, all $\underline{u}\in\mathrm{Inv}(G)^{r_1}/\sim$ and $\sim$ is an equivalence relation on $\mathrm{Inv}(G)^{r_1}$ which we will define in 3.3. We also write

$$\mathbf{n}^{\mathbb{R}}(\mathbf{C};r_1,r_2)=\frac{|C_{r_1+1}|\cdots|C_{r_1+r_2}|}{|G|Z_1\cdots Z_{r_1}}\,\mathbf{n}_0^{\mathbb{R}}(\mathbf{C};r_1,r_2)$$

Theorem 2.4 (Real and complex conjugate branch points)**.** *We have*

$$|\Sigma^{\mathbb{R}}(\mathbf{C};r_1,r_2)|\leq\mathbf{n}^{\mathbb{R}}(\mathbf{C};r_1,r_2)$$

2.2. **Comments.**

2.2.1. For a fixed $\mathbf{t}\in\mathcal{U}_r(\mathbb{R})$, the invariants of G and $\mathbf{C}$ on which the number of real G-covers in $\mathrm{Sni}(\mathbf{C},G)$ depends clearly appear in Theorems 2.1, 2.3 and 2.4. Compared with Serre's formula for the basic rigidity criterion, one can notice the important part played by the involutions of G.

2.2.2. From a practical point of view, the terms depending on involutions make formulas in configurations (R) and (R-C) complicated for direct computations. On the contrary, $\mathbf{n}^{\mathbb{R}}(\mathbf{C};0,s)$ is easy to compute once the character table of G and the centralizers of its involutions are known. When $\mathrm{Sni}^{\mathbb{R}}(\mathbf{C};0,s)$ is properly contained in $\Sigma^{\mathbb{R}}(\mathbf{C};0,s)$, $\mathbf{n}^{\mathbb{R}}(\mathbf{C};0,s)$ only gives an upper bound for $|\Sigma^{\mathbb{R}}(\mathbf{C};0,s)|$, but we explain in the next comment how this difficulty can be handled.

[1] For $r_2=0$ or $r_1=0$, the formulas are the ones given in 2.1 and 2.3

2.2.3. One can proceed as in the classical rigidity context, generalizing the method given by Serre in [11] to evaluate $|\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$ from $|\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$:

1. Evaluate $|\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$ by $\mathbf{n}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$, using the character table of G.
2. Compute $|\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2)| - |\text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$, by finding r-tuples $(g_1, ..., g_r) = \underline{g}$ in $\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$ which do not generate G (to do this, try to find r-tuples the entries of which are contained in a maximal subgroup of G). But we are to be careful: when an r-tuple $\underline{g} \in \Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2) - \text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$ has been found, the following should be done,
   - In situation (R): $\underline{g}$ has to be counted once as in the classical rigidity method.
   - In situation (C): an extra difficuly arises from the computation of $\mathbf{A}_\chi$. One has to compute $\text{Cen}_G(< g_1, ..., g_{2s} >)$ and notice that $\underline{g}$ corresponds to one single class of $\text{Inv}(G)/\text{Cen}_G(< g_1, ..., g_{2s} >)\cdot$. If this class can be written has the union of n classes of $\text{Inv}(G)/Z(G)\cdot$, $\underline{g}$ has to be counted n times.
   - Situation (R-C) has to be dealt with as situation (C).

The best situation is obviously when $\Sigma^{\mathbb{R}}(\mathbf{C}; r_1, r_2) = \text{Sni}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$. For instance this occurs when each non trivial conjugacy class of G appears at least once in $\mathbf{C}$ or, more generally when $\mathbf{C}$ is *g-complete* [6], that is for any $g_i \in C_i$, $i = 1, ..., r$, we have $G =< g_1, ..., g_r >$. Then theorem 2.3 directly provides $|\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$. Moreover, if $\Sigma(\mathbf{C}, G) = \text{Sni}(\mathbf{C}, G)$, one can also compute $|\overline{\text{sni}}(\mathbf{C}, G)|$ with Serre's formula [11] and consequently the proportion of G-covers defined over $\mathbb{R}$: $|\overline{\text{sni}}(\mathbf{C}, G)|/|\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$.

2.2.4. As in the rigidity context $|\overline{\text{sni}}(\mathbf{C}, G)|$ and $|\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$ provide some information about the field of moduli of the associated G-covers. For instance, the condition $|\overline{\text{sni}}(\mathbf{C}; r_1, r_2)| = |\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$ (under some technical assumptions) leads to G-covers defined over $\mathbb{Q}^{tr}$ (see 4.2.1 and 4.2.2 for some applications of this). Similarly, when $\overline{\text{sni}}(\mathbf{C}, G)$ contains a G-cover f defined over $\mathbb{Q}^{tr}$ and satisfying some other technical conditions, $|\overline{\text{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)|$ is an upper bound for the degree of a field extension $K/\mathbb{Q}$ over which f is defined (see [3] Th. 4.1).

2.2.5. As in theorems 2.1, 2.3 and 2.4, one can give formulas for G-covers with field of moduli contained in $\mathbb{R}$. They can be proved exactly as the ones for G-covers defined over $\mathbb{R}$, using in the proof, instead of condition (4), the equivalent one

(4)" there exists $g_0 \in G$ such that $g_0^2 \in Z(G)$ and
- $(g_0 g_1 ... g_i)^2 = g_0^2$ for $i = 1, ..., r_1 - 1$
- $g_0 g_{r_1+i} g_0^{-1} = g_{r+1-i}^{-1}$ for $i = 1, ..., r_2$

We write $Z(G)^{\frac{1}{2}} = \{g \in G | g^2 \in Z(G)\}$. We only state the results for configuration (R) and (C):

1. *Configuration (R)*: Set $E_{r,G} = \{\underline{u} \in G^r \mid \exists g_0 \in Z(G)^{\frac{1}{2}} \ ; \ u_i^2 = g_0^2 \text{ for } i = 0, ..., r-1\}/G\cdot$ and

$$\begin{cases} \mathbf{I}_{\underline{\chi}}^{mod} = \sum_{\underline{u}\in E_{r,G}} \chi_1(u_0u_1^{-1})\chi_2(u_1u_2^{-1})\cdots\chi_r(u_{r-1}u_0^{-1}) \quad \text{for any } \underline{\chi}\in \mathrm{Irr}(G)^r \\ \mathbf{n}^{mod,\mathbb{R}}(\mathbf{C};r,0) = \dfrac{1}{Z_1...Z_r}\sum_{\underline{\chi}\in\mathrm{Irr}(G)^r} \chi_1(g_1)...\chi_r(g_r)\ \mathbf{I}_{\underline{\chi}}^{mod} \end{cases}$$

Then we get: $|\Sigma^{mod,\mathbb{R}}(\mathbf{C};r,0)| = \mathbf{n}^{mod,\mathbb{R}}(\mathbf{C};r,0)$.

2. *Configuration (C)*: Set

$$\begin{cases} \mathbf{A}_{\chi}^{mod} = \sum_{g_0\in Z(G)^{\frac{1}{2}}/Z(G)\cdot} \alpha_{\chi,g_0} \quad \text{for any } \chi\in Irr(G) \\ \mathbf{n}^{mod,\mathbb{R}}(\mathbf{C};0,s) = \frac{|C_1|...|C_s|}{|G|}\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(g_1)...\chi(g_s)}{\chi(1)^{s-1}}\mathbf{A}_{\chi}^{mod} \end{cases}$$

Then we get: $|\Sigma^{mod,\mathbb{R}}(\mathbf{C};0,s)| \leq \mathbf{n}^{mod,\mathbb{R}}(\mathbf{C};0,s)$ with equality if $\Sigma^{mod,\mathbb{R}}(\mathbf{C};0,s) = \mathrm{Sni}^{mod,\mathbb{R}}(\mathbf{C};0,s)$.

3. Proofs

We give the proofs of theorems 2.1 and 2.3 in details; for theorem 2.4, we just explain the main changes, in particular we give the description of $\Sigma^{\mathbb{R}}(\mathbf{C};r_1,r_2)$ we use so as to explain the definition of $\sim$. For a detailed proof of theorem 2.4, see [1].

Following Serre's method, we are going to compute $|\Sigma^{\mathbb{R}}(\mathbf{C};r_1,r_2)|$ using the function

$$\epsilon = \frac{1}{|G|}\sum_{\chi\in\mathrm{Irr}(G)} \chi(1)\chi$$

which is 1 on 1_G and 0 elsewhere.

First, we prove the following technical lemma, which we will use in the sequel.

Lemma 3.1. *Given a finite group G, for any irreducible character $\chi\in\mathrm{Irr}(G)$ and for any $g_1,...,g_n,u,v\in G$ we have:*

$$\sum_{(\gamma_1,...,\gamma_n)\in G} \chi(ug_1^{\gamma_1}\cdots g_n^{\gamma_n}v) = \frac{|G|^n\prod_{i=1}^n\chi(g_i)}{\chi(1)^n}\chi(uv)$$

Proof. Let $R: G\to \mathrm{GL}(V)$ be a linear irreducible representation of G with character χ. Then

$$\sum_{\underline{\gamma}\in G} R(ug_1^{\gamma_1}\cdots g_n^{\gamma_n}v) = R(u)\left(\sum_{\gamma_1\in G} R(g_1^{\gamma_1})\cdots\sum_{\gamma_n\in G} R(g_n^{\gamma_n})\right)R(v)$$

But for any $g,h\in G$

$$\sum_{\gamma\in G} R(g^\gamma)R(h) = \sum_{\gamma\in G} R(g^\gamma h) = \sum_{\gamma\in G} R(hg^{h^{-1}\gamma}) = R(h)\sum_{\gamma\in G} R(g^{h^{-1}\gamma}) = R(h)\sum_{\gamma\in G} R(g^\gamma)$$

So, according to Schur's lemma (cf. for instance [12] proposition 4 chap.2):

$$\sum_{\gamma\in G} R(g^\gamma) = \lambda\mathrm{Id}_V \text{ with } \lambda = \frac{1}{\dim V}\mathrm{Tr}(\sum_{\gamma\in G} R(g^\gamma)) = \frac{|G|}{\chi(1)}\chi(g)$$

Consequently we get

$$\sum_{\gamma_1\in G} R(g_1^{\gamma_1})\cdots\sum_{\gamma_n\in G} R(g_n^{\gamma_n}) = \frac{|G|^n\prod_{i=1}^n\chi(g_i)}{\chi(1)^n}\mathrm{Id}_V$$

so,

$$\sum_{\gamma\in G} R(ug_1^{\gamma_1}\cdots g_n^{\gamma_n}v) = \frac{|G|^n\prod_{i=1}^n\chi(g_i)}{\chi(1)^n}R(uv)$$

And, taking traces yields the formula in lemma 3.1. □

3.1. **Real branch points.** We first note that conditions (2) and (4)' in the definition of $\Sigma^{\mathbb{R}}(\mathbf{C};r,0)$ are equivalent to

$$(*)\ \exists g_0\in G \text{ such that } (g_0g_1\cdots g_i)^2=1,\ i=1,...,r-1 \text{ and } g_1\cdots g_r=1$$

which in turn is equivalent to

$$(**)\ g_1=u_0u_1,...,g_{r-1}=u_{r-2}u_{r-1} \text{ and } g_r=u_{r-1}u_0 \text{ for some } (u_0,...,u_{r-1})\in \mathrm{Inv}(G)^r$$

(just take $u_i=g_0\cdots g_i,\ i=0,...,r-1$). In the the rest of the paragraph we will use the r-cycle $c=(0,...,r-1)\in S_r$ to shorten the formulas. For instance (**) can be re-written $g_{i+1}=u_iu_{c(i)},\ i=0,...,r-1$

Now, fix $g_1,...,g_r\in G$ with $g_i\in C_i,\ i=1,...,r$ and consider the set $E_{\underline{g}}$ of those r-tuples $\underline{\gamma}=(\gamma_1,...,\gamma_r)\in G^r$ such that $g_{i+1}^{\gamma_{i+1}}=u_iu_{c(i)},\ i=0,...,r-1$ for some $(u_0,...,u_{r-1})\in\mathrm{Inv}(G)^r$. The correspondence $\underline{\gamma}\to(g_1^{\gamma_1},...,g_r^{\gamma_r})$ provides a surjective map $E_{\underline{g}}\to\Sigma^{\mathbb{R}}(\mathbf{C};r,0)$. Note then that two distinct r-tuples $\underline{\gamma},\underline{\gamma'}\in G^r$ have the same image if and only if $\gamma_i^{-1}\gamma_i'\in\mathrm{Cen}_G(g_i)$, $i=1,...,r$. Consequently

$$|\Sigma^{\mathbb{R}}(\mathbf{C};r,0)| = \frac{|E_{\underline{g}}|}{Z_1\cdots Z_r}$$

which reduces the problem to computing $|E_{\underline{g}}|$.

We proceed this way: for each $(\gamma_1,...,\gamma_r)\in G^r$, we check for every r-tuple $(u_0,...u_{r-1})\in\mathrm{Inv}(G)^r$ whether $g_{i+1}^{\gamma_{i+1}}=u_iu_{c(i)}$, $i=0,...,r-1$, that is whether

$$\prod_{i=0}^{r-1}\epsilon(u_ig_{i+1}^{\gamma_{i+1}}u_{c(i)})=1$$

However, we should take into account that for a given $\underline{\gamma}\in G^r$, distinct r-tuples $\underline{u},\underline{u'}\in\mathrm{Inv}(G)^r$ can satisfy $g_{i+1}^{\gamma_{i+1}}=u_iu_{c(i)}$, $i=0,...,r-1$; this is equivalent to the condition $u_0u_0'=u_1u_1'=...=u_{r-1}u_{r-1}'$, which can also be written $G\cdot(u_0,...,u_{r-1})=G\cdot(u_0',...,u_{r-1}')$, where G acts on G^r by left translation. This defines the equivalence relation $G\cdot$ on $\mathrm{Inv}(G)^r$ which appears in the statement of Theorem 2.1.

Putting these remarks together we get:

$$|E_{\underline{g}}| = \sum_{\substack{\underline{\gamma}\in G^r \\ \underline{u}\in \mathrm{Inv}(G)^r/G\cdot}} \prod_{i=0}^{r-1} \epsilon(u_i g_{i+1}^{\gamma_{i+1}} u_{c(i)}) = \sum_{\underline{u}\in \mathrm{Inv}(G)^r/G\cdot} \left(\sum_{\underline{\gamma}\in G} \prod_{i=0}^{r-1} \epsilon(u_i g_{i+1}^{\gamma_{i+1}} u_{c(i)}) \right)$$
$$= \sum_{\underline{u}\in \mathrm{Inv}(G)^r/G\cdot} \left(\prod_{i=0}^{r-1} \sum_{\gamma\in G} \epsilon(u_i g_{i+1}^{\gamma} u_{c(i)}) \right)$$

Using the formula $\epsilon = \frac{1}{|G|}\sum_{\chi\in \mathrm{Irr}(G)} \chi(1)\chi$ and lemma 3.1 we obtain, for $i = 0, ..., r-1$ and $\underline{u} \in \mathrm{Inv}(G)^r$:

$$\sum_{\gamma\in G} \epsilon(u_i g_{i+1}^{\gamma} u_{c(i)}) = \frac{1}{|G|} \sum_{\chi\in \mathrm{Irr}(G)} \chi(1) \sum_{\gamma\in G} \chi(u_i g_{i+1}^{\gamma} u_{c(i)})$$
$$= \sum_{\chi\in \mathrm{Irr}(G)} \chi(g_{i+1})\chi(u_i u_{c(i)})$$

Substituting this back in the previous formula leads to the announced result. Note the generating condition $G =< u_0u_1, ..., u_{r-2}u_{r-1} >$ can be taken into account to get $\mathrm{Sni}^{\mathbb{R}}(\mathbf{C}; r, 0)$: the only change is then that, in the sums above, the r-tuples $\underline{u}$ should run over the subset of $\mathrm{Inv}(G)^r/G\cdot$ of those r-tuples $\underline{u}$ of representatives satisfying this extra generating condition.This yields remark 2.2.

□

3.2. **Complex conjugate branch points:** This time note that conditions (2) and (4)' in the definition of $\Sigma^{\mathbb{R}}(\mathbf{C}; 0, s)$ are equivalent to

(∗) there exists $g_0 \in \mathrm{Inv}(G)$ such that $g_0\ g_i\ g_0\ g_{2s+1-i} = 1,\ i = 1, ..., s$ and $g_1 \cdots g_{2s} = 1$

which in turn is equivalent to

(∗∗) there exists $g_0 \in \mathrm{Inv}(G)$ such that $g_0\ g_i\ g_0\ g_{2s+1-i} = 1,\ i = 1, ..., s$ and $[g_1 \cdots g_s, g_0] = 1$ (where we write $[u, v]$ for the commutator $uvu^{-1}v^{-1}$ of $u, v \in G$).

As above, fix $g_1, ..., g_{2s} \in G$ with $g_i \in C_i$, $i = 1, ..., 2s$ and consider the set $E_{\underline{g}}$ of those $2s$-tuples $\underline{\gamma} = (\gamma_1, ..., \gamma_{2s}) \in G^r$ such that $g_0\ g_i^{\gamma_i}\ g_0\ g_{2s+1-i}^{\gamma_{2s+1-i}} = 1$ for $i = 1, ..., s$ and $[g_1^{\gamma_1} \cdots g_s^{\gamma_s}, g_0] = 1$ (or, equivalently $g_1^{\gamma_1} \cdots g_s^{\gamma_s} \in \mathrm{Cen}_G(g_0)$) for some $g_0 \in \mathrm{Inv}(G)$. Again, the correspondence $\underline{\gamma} \to (g_1^{\gamma_1}, ..., g_{2s}^{\gamma_{2s}})$ provides a surjective map $E_{\underline{g}} \to \Sigma^{\mathbb{R}}(\mathbf{C}; 0, s)$ and two distinct $2s$-tuples $\underline{\gamma}, \underline{\gamma'} \in G^r$ have the same image if and only if $\gamma_i^{-1}\gamma_i' \in \mathrm{Cen}_G(g_i)$ for $i = 1, ..., 2s$. Consequently

$$|\Sigma^{\mathbb{R}}(\mathbf{C}; 0, s)| = \frac{|E_{\underline{g}}|}{Z_1 \cdots Z_{2s}}$$

which reduces the problem to computing $|E_{\underline{g}}|$.

We proceed this way: for each $(\gamma_1, ..., \gamma_{2s}) \in G^{2s}$, we check for every $g_0 \in \mathrm{Inv}(G)$ whether

$g_0\ g_i^{\gamma_i}\ g_0\ g_{2s+1-i}^{\gamma_{2s+1-i}} = 1,\ i = 1, ..., 2s$ and $[g_1^{\gamma_1} \cdots g_s^{\gamma_s}, g_0] = 1$, that is whether

$$\epsilon([g_1^{\gamma_1}...g_s^{\gamma_s}, g_0]) \prod_{i=1}^{s} \epsilon(g_0 g_i^{\gamma_i} g_0 g_{2s+1-i}^{\gamma_{2s+1-i}}) = 1$$

As in 3.1 note that for a given $\underline{\gamma} \in G^r$, distinct involutions $g_0, g_0' \in \mathrm{Inv}(G)$ can satisfy condition (**). This is equivalent to the condition $g_0 g_0' \in \mathrm{Cen}_G(g_1^{\gamma_1}, ..., g_s^{\gamma_s})$ or $\mathrm{Cen}_G(g_1^{\gamma_1}, ..., g_s^{\gamma_s}).g_0 = \mathrm{Cen}_G(g_1^{\gamma_1}, ..., g_s^{\gamma_s}).g_0'$. And, as $Z(G) < \mathrm{Cen}_G(g_1^{\gamma_1}, ..., g_s^{\gamma_s})$, the preceeding equivalent conditions are implied by $Z(G) \cdot g_0 = Z(G) \cdot g_0'$ (see remark 3.2), where $\mathrm{Cen}_G(g_1^{\gamma_1}, ..., g_s^{\gamma_s})$ and $Z(G)$ act on G by left translation. Here again this gives the equivalence relation $Z(G)\cdot$ on $\mathrm{Inv}(G)$ which appears in the statement of theorem 2.3.

Putting these remarks together we get:

$$\begin{aligned}
|E_{\underline{g}}| &\leq \sum_{\substack{\underline{\gamma} \in G^{2s} \\ g_0 \in \mathrm{Inv}(G)/Z(G)\cdot}} \epsilon([g_1^{\gamma_1} \cdots g_s^{\gamma_s}, g_0]) \prod_{i=1}^{s} \epsilon(g_0\ g_i^{\gamma_i}\ g_0\ g_{2s+1-i}^{\gamma_{2s+1-i}}) \\
&\leq \sum_{\substack{(\gamma_1,...,\gamma_s) \in G \\ g_0 \in \mathrm{Inv}(G)/Z(G)\cdot}} \epsilon([g_1^{\gamma_1} \cdots g_s^{\gamma_s}, g_0]) \sum_{(\gamma_{s+1},...,\gamma_{2s}) \in G} \prod_{i=1}^{s} \epsilon(g_0\ g_i^{\gamma_i}\ g_0\ g_{2s+1-i}^{\gamma_{2s+1-i}}) \\
&\leq \sum_{\substack{(\gamma_1,...,\gamma_s) \in G \\ g_0 \in \mathrm{Inv}(G)/Z(G)\cdot}} \epsilon([g_1^{\gamma_1} \cdots g_s^{\gamma_s}, g_0]) \prod_{i=1}^{s} \sum_{\gamma \in G} \epsilon(g_0\ g_i^{\gamma_i}\ g_0\ g_{2s+1-i}^{\gamma})
\end{aligned}$$

As before lemma 3.1 combined with the formula defining ϵ gives:

$$\begin{aligned}
\prod_{i=1}^{s} \sum_{\gamma \in G} \epsilon(g_0 g_i^{\gamma_i} g_0 g_{2s+1-i}^{\gamma}) &= \prod_{i=1}^{s} \sum_{\chi \in \mathrm{Irr}(G)} \chi(g_0 g_i^{\gamma_i} g_0) \chi(g_{2s+1-i}) \\
&= \prod_{i=1}^{s} \sum_{\chi \in \mathrm{Irr}(G)} \chi(g_i) \chi(g_{2s+1-i})
\end{aligned}$$

Hence we have now:

$$|E_{\underline{g}}| \leq \left(\sum_{\substack{(\gamma_1,...,\gamma_s) \in G \\ g_0 \in \mathrm{Inv}(G)/Z(G)\cdot}} \epsilon([g_1^{\gamma_1} \cdots g_s^{\gamma_s}, g_0]) \right) \left(\prod_{i=1}^{s} \sum_{\chi \in \mathrm{Irr}(G)} \chi(g_i) \chi(g_{2s+1-i}) \right)$$

Noting that, for all $g_0 \in \mathrm{Inv}(G)$, $[u, v] = 1$ if and only if $u \in \mathrm{Cen}_G(v)$

$$\begin{aligned}
\sum_{(\gamma_1,...,\gamma_s) \in G} \epsilon([g_1^{\gamma_1} \cdots g_s^{\gamma_s}, g_0]) &= \sum_{u \in \mathrm{Cen}_G(g_0)} \sum_{(\gamma_1,...,\gamma_s) \in G} \epsilon(g_1^{\gamma_1}...g_s^{\gamma_s} u) \\
&= \sum_{u \in \mathrm{Cen}_G(g_0)} \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \sum_{(\gamma_1,...,\gamma_s) \in G} \chi(g_1^{\gamma_1} \cdots g_s^{\gamma_s} u)
\end{aligned}$$

So, using lemma 3.1 again,

$$\begin{aligned}\sum_{(\gamma_1,...,\gamma_s)\in G}\epsilon([g_1^{\gamma_1}...g_s^{\gamma_s},g_0]) &= |G|^{s-1}\sum_{u\in \mathrm{Cen}_G(g_0)}\sum_{\chi\in\mathrm{Irr}(G)}\frac{\prod_{i=1}^s\chi(g_i)}{\chi(1)^{s-1}}\chi(u)\\ &= |G|^{s-1}\sum_{\chi\in\mathrm{Irr}(G)}\frac{\prod_{i=1}^s\chi(g_i)}{\chi(1)^{s-1}}\sum_{u\in\mathrm{Cen}_G(g_0)}\chi(u)\end{aligned}$$

We recognize here $\alpha_{\chi,g_0}=\sum_{u\in\mathrm{Cen}_G(g_0)}\chi(u)$. Finally, we get:

$$|E_{\underline{g}}|\leq |G|^{s-1}\left(\prod_{i=1}^s\sum_{\chi\in\mathrm{Irr}(G)}\chi(g_i)\chi(g_{2s+1-i})\right)\left(\sum_{\chi\in\mathrm{Irr}(G)}\frac{\prod_{i=1}^s\chi(g_i)}{\chi(1)^{s-1}}\mathbf{A}_\chi\right)$$

To end the proof, just recall that we have assumed $|\Sigma^{\mathbb{R}}(\mathbf{C};0,s)|\neq\emptyset$, this implies in particular that $C_i=C_{2s+i-1}^{-1}$ for $i=1,...,s$, so $Z_i=Z_{2s+1-i}$ and $\chi(g_i)\chi(g_{2s+1-i})=|\chi(g_i)|^2$ whence

$$\sum_{\chi\in\mathrm{Irr}(G)}\chi(g_i)\chi(g_{2s+1-i})=\sum_{\chi\in\mathrm{Irr}(G)}|\chi(g_i)|^2=Z_i$$

for $i=1,...,s$, which leads to the announced result.

□

Remark 3.2. We only get an upper bound for $|\Sigma^{\mathbb{R}}(\mathbf{C};0,s)|$ because of the inclusions, which may be proper, $Z(G)<\mathrm{Cen}_G(g_1^{\gamma_1},...,g_r^{\gamma_r})$. But if $\Sigma^{\mathbb{R}}(\mathbf{C};0,s)=\mathrm{Sni}^{\mathbb{R}}(\mathbf{C};0,s)$, these inclusions become equalities and $|\Sigma^{\mathbb{R}}(\mathbf{C};0,s)|=|\mathrm{Sni}^{\mathbb{R}}(\mathbf{C};0,s)|$.

3.3. **Real and complex conjugate branch points.** The method consists in rewriting conditions (2) and (4)' as in the two preceeding sections but replacing conditions $g_1\cdots g_{r_1}=1$ and $g_{r_1+1}\cdots g_{r_1+2r_2}=1$ by the weaker one $g_1\cdots g_{r_1}g_{r_1+1}\cdots g_{r_1+2r_2}=1$. So, in the general situation conditions (2) and (4)' are equivalent to

$$(*)\text{ there exists } g_0\in\mathrm{Inv}(G)\text{ such that }\begin{cases}g_1\cdots g_r=1\\ (g_0g_1\cdots g_i)^2=1,\ i=1,...,r_1-1\\ g_0\ g_{r_1+i}\ g_0\ g_{r+1-i}=1,\ i=1,...,r_2\end{cases}$$

which in turn is equivalent to

$$(**)\text{ there exists } (u_0,...,u_{r_1-1})\in\mathrm{Inv}(G)^{r_1}\text{ such that }\begin{cases}u_0\ u_{r_1-1}\ g_{r_1}\ [g_{r_1+1}\cdots g_{r_1+r_2},u_0]=1\\ g_{i+1}=u_iu_{i+1},\ i=0,...,r_1-2\\ u_0\ g_{r_1+i}\ u_0\ g_{r+1-i}=1,\ i=1,...,r_2\end{cases}$$

We still fix $g_1,...,g_r\in G$ with $g_i\in C_i$, $i=1,...,r$ and consider the set $E_{\underline{g},r_1,r_2}$ of those r-tuples $\underline{\gamma}=(\gamma_1,...,\gamma_{2s})\in G^r$ such that $(g_1^{\gamma_1},...,g_r^{\gamma_r})$ satisfies condition (**). As above

$$|\Sigma^{\mathbb{R}}(\mathbf{C};0,s)|=\frac{|E_{\underline{g},r_1,r_2}|}{Z_1\cdots Z_{2s}}$$

which once again reduces the problem to computing $|E_{\underline{g},r_1,r_2}|$. Then, for each $\underline{\gamma} = (\gamma_1, ..., \gamma_r) \in G^r$, to decide whether $\underline{\gamma} \in E_{\underline{g},r_1,r_2}$, we check for every $\underline{u} = (u_0, ..., u_{r_1-1}) \in \mathrm{Inv}(G)^{r_1}$ whether

$$\prod_{i=0}^{r_1-1} \epsilon(u_i g_{i+1}^{\gamma_{i+1}} u_{i+1}) \prod_{i=1}^{r_2} \epsilon(u_0 g_{r_1+i}^{\gamma_{r_1+i}} u_0 g_{r+1-i}^{\gamma_{r+1-i}}) \epsilon(u_0 u_{r_1-1} g_{r_1}^{\gamma_{r_1}} [g_{r_1+1}^{\gamma_{r_1+1}} \cdots g_{r_1+r_2}^{\gamma_{r_1+r_2}}, u_0]) = 1$$

Now, the introduction of $\sim$ derives from the usual remarks about counting exactly one time each element $\underline{\gamma} \in E_{\underline{g},r_1,r_2}$:
- for all $(\gamma_1, ..., \gamma_{r_1-1}) \in G^{r_1-1}$, $\underline{u}, \underline{u}' \in \mathrm{Inv}(G)^{r_1}$, the condition

$$u_i g_{i+1}^{\gamma_{i+1}} u_{i+1} = 1 = u_i' g_{i+1}^{\gamma_{i+1}} u_{i+1}', \ i = 0, ..., r_1 - 1$$

is equivalent to $u_0 u_0' = u_1 u_1' = ... = u_{r_1-1} u_{r_1-1}'$ which can also be written $G.(u_0, ...u_{r_1-1}) = G.(u_0', ...u_{r_1-1}')$, where G acts on G^{r_1} by left translation.
- for all $(\gamma_{r_1+1}, ..., \gamma_r) \in G^{2r_2}$, $u_0, u_0' \in \mathrm{Inv}(G)$ the condition

$$u_0 g_{r_1+i}^{\gamma_{r_1+i}} u_0 = (g_{r+1-i}^{\gamma_{r+1-i}})^{-1} = u_0' g_{r_1+i}^{\gamma_{r_1+i}} u_0', \ i = 1, ..., r_2$$

is equivalent to

$$u_0 u_0' \in \mathrm{Cen}_G(g_{r_1+1}^{\gamma_{r_1+1}}, ..., g_{r_1+r_2}^{\gamma_{r_1+r_2}})$$

that is $\mathrm{Cen}_G(g_{r_1+1}^{\gamma_{r_1+1}}, ..., g_{r_1+r_2}^{\gamma_{r_1+r_2}}) \cdot u_0 = \mathrm{Cen}_G(g_{r_1+1}^{\gamma_{r_1+1}}, ..., g_{r_1+r_2}^{\gamma_{r_1+r_2}}) \cdot u_0'$ which is implied by

$$N \cdot u_0 = N \cdot u_0'$$

where $N = \mathrm{Cen}_G(C_{r_1+1}, ..., C_{r_1+r_2})$ is the centralizer of the subgroup generated by the conjugacy classes of $g_{r_1+1}, ..., g_{r_1+r_2}$ and both $\mathrm{Cen}_G(g_{r_1+1}^{\gamma_{r_1+1}}, ..., g_{r_1+r_2}^{\gamma_{r_1+r_2}})$ and N act on G by left translation.

Hence, for all $\underline{\gamma} = (\gamma_1, ..., \gamma_{r_1-1}) \in G^{r_1-1}$ let $\sim_{\underline{\gamma}}$ the relation defined on $\mathrm{Inv}(G)^{r_1}$, by: for all $\underline{u}, \underline{u}' \in \mathrm{Inv}(G)^{r_1}$,

$$\begin{aligned} \underline{u} \sim_{\underline{\gamma}} \underline{u}' \iff & \mathrm{Cen}_G(g_{r_1+1}^{\gamma_{r_1+1}}, ..., g_{r_1+r_2}^{\gamma_{r_1+r_2}}) \cdot u_0 = \mathrm{Cen}_G(g_{r_1+1}^{\gamma_{r_1+1}}, ..., g_{r_1+r_2}^{\gamma_{r_1+r_2}}) \cdot u_0' \\ & \text{and } G \cdot (u_0, ..., u_{r_1-1}) = G \cdot (u_0', ..., u_{r_1-1}') \end{aligned}$$

and write $\sim$ for the relation one gets replacing $\mathrm{Cen}_G(g_{r_1+1}^{\gamma_{r_1+1}}, ..., g_{r_1+r_2}^{\gamma_{r_1+r_2}})$ by $\mathrm{Cen}_G(N)$ in the definition above. These relations are equivalence relations on $\mathrm{Inv}(G)^{r_1}$ and we obtain formula $\mathbf{n}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)$ in theorem 2.4 by summing on the equivalence classes $\mathrm{Inv}(G)^{r_1}/\sim$.

Remark 3.3. When, for all $\underline{\gamma} = (\gamma_1, ..., \gamma_{r_1-1}) \in G$, $N = \mathrm{Cen}_G(g_{r_1+1}^{\gamma_{r_1+1}}, ..., g_{r_1+r_2}^{\gamma_{r_1+r_2}})$, the inequality in Theorem 2.4 becomes an equality.

4. Applications

Except in 4.2.2, we will always assume we are in the complex pair configuration (C). We keep the notations from section 2, particularly concerning $\mathbf{A}_\chi$, $\mathbf{A}_\chi^{mod}$, α_{χ,g_0}, *etc.* In addition, say $\mathbf{C}$ is $\mathbb{C}g$*-complete symetric* if:

(1) $\Sigma(\mathbf{C}, G) = \mathrm{Sni}(\mathbf{C}, G)$ and
(2) $\mathbf{C} = (C_1, ..., C_s, C_s^{-1}, ..., C_1^{-1})$ (and so $\Sigma^{\mathbb{R}}(\mathbf{C}; 0, s) \neq \emptyset$)

If (1) is replaced by

$$(1)^{\mathbb{R}}\ \ \Sigma^{\mathbb{R}}(\mathbf{C};0,s)=\mathrm{Sni}^{\mathbb{R}}(\mathbf{C};0,s)$$

say $\mathbf{C}$ is $\mathbb{R}$*g-complete symetric*. In the following computations we will always make the hypothesis $\mathbf{C}$ is $\mathbb{C}$*g-complete symetric*. Clearly we have g-complete implies (1), which implies $(1)^{\mathbb{R}}$. Under condition (1) one can use directly Serre's formula to compute $|\mathrm{Sni}(\mathbf{C},G)|$, and under condition $(1)^{\mathbb{R}}$, formula $\mathbf{n}^{\mathbb{R}}(\mathbf{C};0,s)$ to compute $|\mathrm{Sni}^{\mathbb{R}}(\mathbf{C};0,s)|$.

In the examples, we describe the $2s$-tuples $\mathbf{C}$ satisfying (2) using the following notation $\mathbf{C}=[A_1^{(a_1)},...,A_n^{(a_n)}]$ to indicate that the $2s$-tuple $\mathbf{C}$ consists of
- s first entries where A_1 occurs a_1 times,..., A_n occurs a_n times (so $s=a_1+...+a_n$)
- s last entries which are the inverses of the s first ones, in reversed order.

When $\mathbf{C}$ is $\mathbb{C}$g-complete symetric, Serre's formula becomes:

$$|\overline{\mathrm{sni}}(\mathbf{C},G)|=|Z(G)|\left(\frac{|C_1|...|C_s|}{|G|}\right)^2\sum_{\chi\in\mathrm{Irr}(G)}\left(\frac{|\chi(g_1)|...\chi(g_s)|}{\chi(1)^{s-1}}\right)^2$$

hence:

$$\frac{|\overline{\mathrm{sni}}(\mathbf{C},G)|}{|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,s)|}=|C_1|...|C_s|\frac{\sum_{\chi\in\mathrm{Irr}G)}\left(\frac{|\chi(g_1)|...|\chi(g_s)|}{\chi(1)^{s-1}}\right)^2}{\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(g_1)...\chi(g_s)}{\chi(1)^{s-1}}A_\chi}$$

Remark 4.1. Note that $Z(G)=\cap_{\chi\in\mathrm{Irr}(G)}Z_\chi$ where $Z_\chi=\{g\in G|\ |\chi(g)|=\chi(1)\}$, $\chi\in\mathrm{Irr}(G)$, so, if $\mathbf{C}$ is g-complete symetric, $|\Sigma(\mathbf{C},G)|$ remains unchanged when adding central classes whereas $|\Sigma^{mod,\mathbb{R}}(\mathbf{C};0,s)|$ and $|\Sigma^{mod,\mathbb{R}}(\mathbf{C};0,s)|$ do not. So adding central classes in $\mathbf{C}$ can change the proportion $|\overline{\mathrm{sni}}(\mathbf{C},G)|/|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,s)|$ of G-covers defined over $\mathbb{R}$ (and with field of moduli contained in $\mathbb{R}$ as well).

4.1. **G-covers which are not defined over their field of moduli.** First we deal with the quaternion group $\mathbb{H}_8$ for which we exhibit G-covers not defined over their field of moduli. Then we generalize to obtain in particular a simple group-theoretic criterion for a finite group to be the Galois group of some G-cover not defined over its field of moduli. Lots of infinite families of groups verify this criterion.

4.1.1. *Quaternion group* $\mathbb{H}_8$. In the quaternion group $\mathbb{H}_8$ we have 4 non trivial conjugacy classes: $A=\{-1\}$, $A_i=\{\pm i\}$, $A_j=\{\pm j\}$, $A_k=\{\pm k\}$ Take

$$\mathbf{C}=[A^{(x)},A_i^{(a)},A_j^{(b)},A_k^{(c)}]\ (\text{so } s=x+a+b+c.)$$

To compute $\mathbf{n}^{\mathbb{R}}(\mathbf{C};0,s)$ note that $\mathrm{Inv}(\mathbb{H}_8)/Z(\mathbb{H}_8)\cdot=\{1\}$, consequently $\mathbf{A}_\chi=\alpha_{\chi,1}=8$ if $\chi=\chi_1$ and $\mathbf{A}_\chi=0$ otherwise, which leads to:

$$\begin{cases}\bullet\ \mathbf{n}^{\mathbb{R}}(\mathbf{C};0,s)=2^{a+b+c}\\ \bullet\ |\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,s)|=2^{a+b+c-2}\\ \bullet\ |\overline{\mathrm{sni}}(\mathbf{C},\mathbb{H}_8)|=2^{2(a+b+c)-3}\\ \bullet\ \dfrac{|\overline{\mathrm{sni}}(\mathbf{C},\mathbb{H}_8)|}{|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,s)|}=2^{a+b+c-1}\\ \bullet\ |\overline{\mathrm{sni}}(\mathbf{C},\mathbb{H}_8)|-|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,s)|=2^{a+b+c-2}(2^{a+b+c-1}-1)\end{cases}$$

If $a=b=1,\ x=c=0$ and so $r=2s=4$, we get $|\overline{\mathrm{sni}}(\mathbf{C},\mathbb{H}_8)|=2$ and $|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,2)|=1$ [2]. This gives rise to a new example of a G-cover with group $\mathbb{H}_8$ not defined over its field of moduli (recall that such an example was already given by K.Coombes and D.Harbater in [2] p.831 but with three rational branch points $(1,2,3)$ and canonical inertia invariant $\mathbf{C}=(\{\pm i\},\{\pm j\},\{\pm k\})$.) We give a precise argument in 4.1.2, but the general idea is that, given the branch points $(z_1,z_2,\overline{z_2},\overline{z_1})\in\mathcal{U}^r$ with z_1,z_2 not real, the fiber $(\psi'_{4,\mathbb{H}_8})^{-1}((z_1,z_2,\overline{z_2},\overline{z_1}))$ consists of two points $P'_1,\ P'_2$ corresponding to two G-covers $f_1,\ f_2$, one of which, say f_1, is defined over $\mathbb{R}$ and the other one, f_2, is not. If $P_1=\Pi_4(P'_1)$ and $P_2=\Pi_4(P'_2)$ are the corresponding points on $\mathcal{H}_{4,\mathbb{H}_8}(\mathbf{C})$ then, $P_1^c=P_1$ forces $P_2^c=P_2$ so P_2 is a real point i.e. f_2 has its field of moduli contained in $\mathbb{R}$.

We can also use formula $\mathbf{n}^{mod,\mathbb{R}}(\mathbf{C};0,s)$, which is more precise. For this, note that $Z(\mathbb{H}_8)^{\frac{1}{2}}/Z(\mathbb{H}_8)\cdot=\mathbb{H}_8/Z(\mathbb{H}_8)\cdot=\{1,i,j,k\}$ so $\mathbf{A}_\chi^{mod}=\alpha_{\chi,1}+\alpha_{\chi,i}+\alpha_{\chi,j}+\alpha_{\chi,1}=20$ if $\chi=\chi_1$ and $\mathbf{A}_\chi^{mod}=4$ otherwise, which leads to

$$\begin{cases}\bullet\ \mathbf{n}^{mod\,\mathbb{R}}(\mathbf{C};0,s)=2^{a+b+c-1}\times(5+(-1)^{b+c}+(-1)^{a+c}+(-1)^{a+b})\\ \bullet\ |\overline{\mathrm{sni}}^{mod,\mathbb{R}}(\mathbf{C};0,s)|=2^{a+b+c-3}\times(5+(-1)^{b+c}+(-1)^{a+c}+(-1)^{a+b})\end{cases}$$

Taking $a=b=1,\ x=c=0$ gives $|\overline{\mathrm{sni}}^{mod,\mathbb{R}}(\mathbf{C};0,2)|=2$, as expected. But we get more since $\Delta^{mod}(\mathbf{C};0,s):=|\overline{\mathrm{sni}}^{mod,\mathbb{R}}(\mathbf{C};0,s)|-|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,s)|=2^{a+b+c-3}(3+(-1)^{b+c}+(-1)^{a+c}+(-1)^{a+b})>0$, so there are exactly $\Delta^{mod}(\mathbf{C};0,s)$ G-covers in $\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,s)$ which are not defined over $\mathbb{R}$ but with their field of moduli contained in $\mathbb{R}$.

4.1.2. *General criteria.* With the usual notations write

$$\Delta^{mod}(\mathbf{C};r_1,r_2)=|\overline{\mathrm{sni}}^{mod,\mathbb{R}}(\mathbf{C};r_1,r_2)|-|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};r_1,r_2)|$$

When $\mathbf{C}$ is $\mathbb{R}$g-complete symetric and $(r_1,r_2)=(r,0)$ or $(0,s)$,

$$\Delta^{mod}(\mathbf{C};r_1,r_2)=\mathbf{n}^{mod\,\mathbb{R}}(\mathbf{C};r_1,r_2)-\mathbf{n}^{\mathbb{R}}(\mathbf{C};r_1,r_2)$$

Thus we obtain the following simple criterion:

Proposition 4.2. *Let G be a finite group. For any $\mathbb{R}$g-complete r-tuple $\mathbf{C}=(C_1,...,C_r)$ of non trivial conjugacy classes in G and for any r-tuple $\mathbf{t}'=(t_1,...,t_r)\in\mathcal{U}^r(\mathbb{R})$ with $t_1<...<t_r$, of all the isomorphism classes of G-covers in the straight Nielsen class $\overline{\mathrm{sni}}(\mathbf{C},G)$ with ordered branch point set $\mathbf{t}'$, exactly*

$$\Delta^{mod}(\mathbf{C};r,0)=\frac{|Z(G)|}{|G|Z_1\cdots Z_r}\sum_{\underline{\chi}\in\mathrm{Irr}(G)^r}\chi_1(g_1)...\chi_r(g_r)(\mathbf{I}_{\underline{\chi}}^{mod}-\mathbf{I}_{\underline{\chi}})$$

have field of moduli contained in $\mathbb{R}$ but are not defined over $\mathbb{R}$.
Similarly, for any $\mathbb{R}$g-complete symetric r-tuple $\mathbf{C}=(C_1,...,C_r)$ of non trivial conjugacy classes in G and for any r-tuple $\mathbf{t}'=(z_1,...,z_s,\overline{z_s},...,\overline{z_1})\in\mathcal{U}^r(\mathbb{C})$ with z_i not real, $i=1,...,s$,

[2]We can give here explicit representatives: $sni(\mathbf{C},\mathbb{H}_8)=\{(i,j,-j,-i),(i,j,j,i)\}$ and $\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,2)=\{(i,j,-j,-i)\}$.

of all the isomorphism classes of G-covers in the straight Nielsen class $\overline{\mathrm{sni}}(\mathbf{C},G)$ *with ordered branch point set* $\mathbf{t}'$*, exactly*

$$\Delta^{mod}(\mathbf{C};0,s)=\frac{|C_1|\cdots|C_s|}{[G:Z(G)]|G|}\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(g_1)\cdots\chi(g_s)}{\chi(1)^{s-1}}(\mathbf{A}_\chi^{mod}-\mathbf{A}_\chi)$$

have field of moduli contained in $\mathbb{R}$ *but are not defined over* $\mathbb{R}$*.*

Proposition 4.2 shows in particular that, once a $\mathbb{R}$g-complete symetric canonical inertia invariant $\mathbf{C}$ and a branch point configuration - (R) or (C) - are given, the number of G-covers in $\overline{\mathrm{sni}}(\mathbf{C},G)$ with field of moduli contained in $\mathbb{R}$ but not defined over $\mathbb{R}$ can be computed explicitly and is independent of the branch points.

Corollary 4.3. *Given a finite group G, there are G-covers with group G and branch point configuration (C) with field of moduli contained in* $\mathbb{R}$ *but not defined over* $\mathbb{R}$ *if and only if* $Z(G)$ *has an element which is a square in G but not in* $Z(G)$*.*

Proof. Let us compute $\mathbf{A}_\chi^{mod}-\mathbf{A}_\chi=\sum_{g_0}\alpha_{\chi,g_0}$ for any $\chi\in\mathrm{Irr}(G)$ and where g_0 ranges over a system of representatives of the set $Z(G)^{\frac{1}{2}}/Z(G)\cdot\backslash\mathrm{Inv}(G)/Z(G)\cdot$ For this, just note that for all $g_0\in Z(G)^{\frac{1}{2}}$ there exists $z\in Z(G)$ such that $(zg_0)^2=1$ (that is, $Z(G)g_0\in\mathrm{Inv}(G)/Z(G)\cdot$) if and only if g_0^2 is a square in $Z(G)$. Consequently, setting $E_G=\{g_0\in Z(G)^{\frac{1}{2}}|g_0^2\notin\{z^2\}_{z\in Z(G)}\}$, we get $\mathbf{A}_\chi^{mod}-\mathbf{A}_\chi=\sum_{g_0\in E_G/Z(G)\cdot}\alpha_{\chi,g_0}$. Also note that it follows from their definition that the α_{χ,g_0} are non negative integers and for $\chi=\chi_1$, they also are non zero ($\alpha_{\chi_1,g_0}=|\mathrm{Cen}_{g_0}(G)|$), so the $\mathbf{A}_\chi^{mod}-\mathbf{A}_\chi$ are non negative integers. Now, suppose there is a $\mathbb{R}$g-complete symetric $2s$-tuple $\mathbf{C}$ of non-trivial conjugacy classes of G such that $\Delta^{mod}(\mathbf{C};0,s)>0$. Then there is $\chi\in\mathrm{Irr}(G)$ such that $\mathbf{A}_\chi^{mod}-\mathbf{A}_\chi>0$, which obviously implies $E_G\neq\emptyset$.

Conversely, let $C_1,...,C_s$ be a listing of all the non trivial conjugacy classes in G and set $\mathbf{C}=(C_1,C_1^{-1},...,C_s,C_s^{-1},C_s,C_s^{-1},...,C_1,C_1^{-1})$. This $2s$-tuple is $\mathbb{C}$g-complete symetric. So one gets

$$\begin{aligned}\Delta^{mod}(\mathbf{C};0,s) &= \frac{(|C_1|\cdots|C_s|)^2}{[G:Z(G)]|G|}\sum_{\chi\in\mathrm{Irr}(G)}\frac{|\chi(g_1)|^2\cdots|\chi(g_s)|^2}{\chi(1)^{2s-1}}(\mathbf{A}_\chi^{mod}-\mathbf{A}_\chi)\\ &= \frac{(|C_1|\cdots|C_s|)^2}{[G:Z(G)]|G|}\left(\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ deg(\chi)=1}}(\mathbf{A}_\chi^{mod}-\mathbf{A}_\chi)+\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ deg(\chi)>1}}\frac{|\chi(g_1)|^2\cdots|\chi(g_s)|^2}{\chi(1)^{2s-1}}(\mathbf{A}_\chi^{mod}-\mathbf{A}_\chi)\right)\end{aligned}$$

and, since $E_G\neq\emptyset$, $\mathbf{A}_{\chi_1}^{mod}-\mathbf{A}_{\chi_1}=\sum_{g_0\in E_G/Z(G)\cdot}|\mathrm{Cen}_{g_0}(G)|>0$. □

Remark 4.4. (a) Corollary 4.3 can be proved directly, only using the definitions of $\overline{\mathrm{sni}}^{mod,\mathbb{R}}(\mathbf{C};0,s)$ and $\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,s)$ [1]. This alternate proof uses the same $4s$-tuple $\mathbf{C}$, which appears naturally in the proof above, to construct a G-cover in $\overline{\mathrm{sni}}^{mod,\mathbb{R}}(\mathbf{C};0,s)\backslash\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};0,s)$.

(b) Lots of groups satisfy the condition of corollary 4.3 - and so are groups of G-covers not defined over their field of moduli. For instance:

- $\mathrm{Gl}_n(p^m)$ with $n \geq 2$, $m \geq 1$, $p \geq 3$ prime,
- D_{2n} with $n \geq 4$ such that $4|n$,
- $\mathrm{O}_2(p^m, \mathrm{q}^h)$ with $m \geq 1$, $p \geq 3$ prime and q^h the hyperbolic form on $\mathbb{F}_{p^m}^2$,
- any group G such that $\mathrm{Inv}(G) \subset Z(G)$ and $2|[G : Z(G)]$ (for instance $\mathrm{Sl}_2(p^m)$ with $m \geq 1$, $p \geq 3$ irreducible, T_{4n} with $n \geq 2$)...

To my knowledge, the only example of families of G-covers not defined over their field of moduli and in which the group G can be taken arbitarily large was given by S.Wewers [15] with group $\mathrm{Sl}_2(p)$ where $p \not\equiv \pm 1[8]$ is an odd prime, canonical inertia invariant $(4A, pA, pB)$ and branch points (t_1, t_2, t_3) where $t_1 \in \mathbb{Q}$ and $\{t_2, t_3\}$ is $\mathbb{Q}$-rational.

Computing $\Delta^{mod}(\mathbf{C}; r_1, r_2)$ can be difficult. The following proposition gives a weaker but more practical criterion for the existence of G-covers not defined over their field of moduli. We give here the statement and proof for situation (C) but it can immediately be generalized to situations (R) and (R-C).

Proposition 4.5. *Suppose given a finite group G and a symetric 2s-tuple $\mathbf{C}$ of non trivial conjugacy classes in G. If $|\overline{\mathrm{sni}}(\mathbf{C}, G)| - |\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, s)|$ is odd then for any 2s-tuple of branch points $\mathbf{t}' = (z_1, ..., z_s, \overline{z_s}, ..., \overline{z_1}) \in \mathcal{U}^{2s}(\mathbb{C})$ with z_i not real for i=1,...,s, there is, in $\overline{\mathrm{sni}}(\mathbf{C}, G)$, at least one isomorphism class of G-covers with field of moduli contained in $\mathbb{R}$ but not defined over $\mathbb{R}$.*

Proof. Write $m = |\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, s)|$, $n = |\overline{\mathrm{sni}}(\mathbf{C}, G)|$, and let $P_1', ..., P_m'$ be the points in $(\psi'_{2s,G})^{-1}(\mathbf{t}')$ corresponding to the G-covers defined over $\mathbb{R}$ and $P_{m+1}', ..., P_n'$ the points corresponding to the G-covers which are not. Set $P_i = \Pi_{2s}(P_i')$, $i = 1, ..., n$, $E = \{P_1, ..., P_m\}$, $F = \{P_{m+1}, ..., P_n\}$. So, with $\mathbf{t} = \pi_{2s}(\mathbf{t}')$ we have $E \cup F \subset (\psi_{2s,G})^{-1}(\mathbf{t})$. Then observe that $E \cup F = \Pi_{2s}((\psi'_{2s,G})^{-1}(\mathbf{t}'))$ is left invariant by complex conjugation c. Indeed $(\psi'_{2s,G})^{-1}(\mathbf{t}')$ is the set of all G-covers f for which C_i is the inertia canonical class associated with z_i and C_i^{-1} is the inertia canonical class associated with $\overline{z_i} = z_{2s+1-i}$ for $i = 1, ..., s$ whereas $(\psi'_{2s,G})^{-1}(\mathbf{t}')^c$ is the set of all G-covers f^c, for which - by Fried's "Branch cycle argument" - C_i^{-1} is the inertia canonical class associated with $\overline{z}_i = z_{2s+1-i}$ and $(C_i^{-1})^{-1} = C_i$ is the inertia canonical class associated with $\overline{\overline{z}}_i = z_i$ for $i = 1, ..., s$. Now, since $P_1, ..., P_m$ are real points on $(\psi_{2s,G})^{-1}(\mathbf{t})$, we have $E^c = E$, which forces $F^c = F$. Hence, as $|F|$ is odd, F has at least one point P invariant under c. This point P is real, which means it corresponds to an isomorphism class of G-covers with field of moduli contained in $\mathbb{R}$ but, by definition of F, not defined over $\mathbb{R}$. □

4.1.3. *Dicyclic groups T_{4n} of order $4n$.* We give here an application of proposition 4.5. The quaternion group $\mathbb{H}_8$ is the first term of the family of dicyclic groups $(T_{4n})_{n\geq 2}$. The group T_{4n} can be defined by generators and relations:

$$T_{4n} = < a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} >$$

and contains $n + 2$ non trivial conjugacy classes:
- n classes $A_1, ..., A_n$ with $A_i = \{a^i, a^{-i}\}$, $i = 1, ..., n$ (note that $A_n = \{a^n\}$).
- $B_1 = \{a^{2j}b\}_{0\leq j\leq n-1}$ and $B_2 = \{a^{2j+1}b\}_{0\leq j\leq n-1}$.

Take

$$\mathbf{C} = [A_n^{(\alpha_n)}, A_1^{(\alpha_1)}, ..., A_{n-1}^{(\alpha_{n-1})}, B_1^{(\beta_1)}, B_2^{(\beta_2)}]$$

and also write $\alpha = \alpha_1 + \cdots + \alpha_n$. So $s = \alpha + \beta_1 + \beta_2$. We have $\mathrm{Inv}(T_{4n})/Z(T_{4n})\cdot = \{1\}$, consequently $\mathbf{A}_\chi = \alpha_{\chi,1} = 4n$ if $\chi = \chi_1$ and $\mathbf{A}_\chi = 0$ otherwise. Using the character tables of these groups, which can be found in [8] p.385, and taking into account that, for $\mathbf{C}$ to be g-complete we need $\beta_1 + \beta_2 \geq 1$, we obtain

$$\begin{cases} \bullet\ \mathbf{n}^{mod\,\mathbb{R}}(\mathbf{C}; 0, s) = 2^\alpha n^{\beta_1+\beta_2} \\ \bullet\ \overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, s)| = 2^{\alpha-1} n^{\beta_1+\beta_2-1} \\ \bullet\ \overline{\mathrm{sni}}(\mathbf{C}, T_{4n})| = 2^{2\alpha-1} n^{2(\beta_1+\beta_2)-2} \\ \bullet\ \dfrac{|\overline{\mathrm{sni}}(\mathbf{C}, T_{4n})|}{|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, s)|} = 2^\alpha n^{\beta_1+\beta_2-1} \\ \bullet\ |\overline{\mathrm{sni}}(\mathbf{C}, T_{4n})| - |\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, s)| = 2^{\alpha-1} n^{\beta_1+\beta_2-1}(2^\alpha n^{\beta_1+\beta_2-1} - 1) \end{cases}$$

For $\alpha_1 = 1$, $\beta_1 \geq 1$, $\beta_2 \geq 0$, $\alpha_1 = ... = \alpha_n = 0$, $\mathbf{C}$ is g-complete symetric and $|\overline{\mathrm{sni}}(\mathbf{C}, T_{4n})| - |\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}, T_{4n})| = n^{\beta_1+\beta_2-1}(2n^{\beta_1+\beta_2-1} - 1)$ is odd when n is (and, when $\beta_1 + \beta_2 = 1$, it is always odd). So, for each $n \geq 2$, for each $\mathbf{t} = \{z_1, \overline{z}_1, ..., z_s, \overline{z}_s\} \in \mathcal{U}_r(\mathbb{R})$ with z_i not real, $i = 1, ...s$, there is at least one isomorphism class of G-cover f_n with ramification type $[T_{4n}, \mathbf{C}, \mathbf{t}]$ which is not defined over $\mathbb{R}$ but has its field of moduli contained in $\mathbb{R}$.

4.2. **Descent from $\mathbb{C}$ to $\mathbb{Q}^{tr}$.** We give here a combinatorial method to determine if a finite group G admits G-covers defined over $\mathbb{Q}^{tr}$ with a prescribed ramification type $[G, \mathbf{C}, \mathbf{t}]$. For this, we look for r-tuples of non trivial conjugacy classes $\mathbf{C}$ in G such that $|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; r_1, r_2)| = |\overline{\mathrm{sni}}(\mathbf{C}, G)|$. In that case and if $\mathbf{t} \in \mathcal{U}_r(\mathbb{R})$, the image on $\mathcal{H}_{r,G}(\mathbf{C})$ of the fiber $(\psi'_{r,G})^{-1}(\mathbf{t}') \cap \mathcal{H}'_{r,G}(\mathbf{C})$ above an ordering $\mathbf{t}'$ of $\mathbf{t}$ as in (bp), consists of real points; we denote it by

$$E^0_{r,G,\mathbf{t}}(\mathbf{C}) := \Pi_r((\psi'_{r,G})^{-1}(\mathbf{t}') \cap \mathcal{H}'_{r,G}(\mathbf{C})) \subset (\psi_{r,G})^{-1}(\mathbf{t})$$

Let us also write

$$E_{r,G,\mathbf{t}}(\mathbf{C}) := \bigcup_{m \geq 1 \mid (|G|,m)=1} E^0_{r,G,\mathbf{t}}(\mathbf{C}^m)$$

Then if for any $m \geq 1$ such that $(|G|, m) = 1$ we have $|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}^m; r_1, r_2)| = |\overline{\mathrm{sni}}(\mathbf{C}^m, G)|$ $E_{r,G,\mathbf{t}}(\mathbf{C})$ consists of real points. But, if we also assume $\mathbf{t} \in \mathcal{U}_r(\mathbb{Q})$, Fried's "Branch cycle argument" asserts that $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ stabilizes $E_{r,G,\mathbf{t}}(\mathbf{C})$. So, any point in this set is $\mathbb{Q}^{tr}$-rational.

We will only deal here with the odd dihedral groups D_{2n} (that is with n odd) in configuration (R) and (C). We show for instance that for any odd integer $n \geq 3$ and for any $r \geq 3$, D_{2n} is the Galois group of a regular extension of $\mathbb{Q}^{tr}(X)$ with exactly r rational branch points (compare for instance with Conjecture 5.2 in [5]). In configuration (C), we can only assert this occurs with 4 branch points.

4.2.1. *Configuration (C).* Recall D_{2n} is given by the generators and relations

$$D_{2n} = < u, v \mid u^n = v^2 = 1, vuv = v^{-1} >$$

and has $\frac{n-1}{2} + 1$ non trivial conjugacy classes:
- $\frac{n-1}{2}$ classes $A_1, ..., A_{n-1/2}$ with $A_i = \{u^i, u^{-i}\}$, $i = 1, ..., \frac{n-1}{2}$,

- $B = \{vu^i\}_{0\leq i\leq n-1}$.
Now, take

$$\mathbf{C} = [A_1^{(a_1)}, ..., A_{n-1/2}^{(a_{n-1/2})}, B^{(b)}] \text{ (so } s = a_1 + ... + a_{n-1/2} + b)$$

and also write $a = a_1 + ... + a_{n-1/2}$. Here $\mathrm{Inv}(D_{2n})/Z(D_{2n})\cdot = \{1, \{vu^i\}_{0\leq i\leq n-1}\}$ so

$$\begin{cases} \alpha_{\chi,1} = 2n & \text{if } \chi = \chi_1 \\ \alpha_{\chi,1} = 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \begin{cases} \alpha_{\chi,v} = 2 & \text{if } \chi = \chi_1 \\ \alpha_{\chi,v} = 0 & \text{if } \chi = \chi_2 \\ \alpha_{\chi,v} = 2 & \text{otherwise} \end{cases}$$

(where χ_2 is the irreducible character of D_{2n} defined by $\chi_2(u^k) = 1$, $k = 1, ..., n$ and $\chi_2(v) = -1$). So we get $\mathbf{A}_{\chi_1} = 4n$, $\mathbf{A}_{\chi_2} = 0$ and $\mathbf{A}_\chi = 2n$ if $\chi \neq \chi_1, \chi_2$, which, noticing that for $\mathbf{C}$ to be g-complete symetric we need $b \geq 1$, leads to:

$$\begin{cases} \bullet\ \mathbf{n}^{\mathbb{R}}(\mathbf{C}; 0, s) = 2^{a+1}n^b \\ \bullet\ |\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, s)| = 2^a n^{b-1} \\ \bullet\ |\overline{\mathrm{sni}}(\mathbf{C}, D_{2n})| = 2^{2a-1}n^{2b-2} \\ \bullet\ \dfrac{|\overline{\mathrm{sni}}(\mathbf{C}, D_{2n})|}{|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, s)|} = 2^{a-1}n^{b-1} \end{cases}$$

For instance, if $a_1 = b = 1$ and $a_2 = ... = a_{n-1/2} = 0$ we get $|\overline{\mathrm{sni}}(\mathbf{C}^m, D_{2n})| = 2 = |\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}^m; 0, s)|$ for all $m \geq 1$ such that $(2n, m) = 1$. As a result, if we choose a $2s$-tuple of branch points $\mathbf{t} = (z_1, ..., z_s, \overline{z_s}, ..., \overline{z_1}) \in \mathcal{U}^r(\mathbb{C})$ the associated divisor of which is rational the discussion above shows that any point in $E_{4,D_{2n},\mathbf{t}}(\mathbf{C})$ is a $\mathbb{Q}^{tr}$-point and, since $Z(D_{2n}) =< 1 >$, we conclude all the G-covers in $\overline{\mathrm{sni}}(\mathbf{C}, D_{2n})$ are defined over $\mathbb{Q}^{tr}$. Notice that $\mathbf{C} = (C_1, C_v, C_v, C_1)$ is not rational, so all the G-covers in $\overline{\mathrm{sni}}(\mathbf{C}, D_{2n})$ are defined over $\mathbb{Q}^{tr}$ but none of them is over $\mathbb{Q}$.

Remark 4.6. Generalizing the situation above, one gets the following descent criterion: For any finite group G, for any r-tuple $\mathbf{C} = (C_1, ..., C_r)$ of non trivial conjugacy classes in G, for any $\mathbf{t}^0 \in \mathcal{U}^r$the associated branch point divisor of which is rational, the two following conditions:

(1) $Z(G) = 1$ and
(2) for any $n \geq 1$ such that $(|G|, n) = 1$, $|\overline{\mathrm{sni}}(\mathbf{C}^n, G)| = |\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}^n; r_1, r_2)|$

imply all the G-covers in $\overline{\mathrm{sni}}(\mathbf{C}, G)$ are defined over $\mathbb{Q}^{tr}$.
Condition (2) can even be replaced by the stronger one - easier to check when $\mathbf{C}$ is not g-complete:

(2)' for any $n \geq 1$ such that $(|G|, n) = 1$, $|\Sigma(\mathbf{C}^n, G)| = |\Sigma^{\mathbb{R}}(\mathbf{C}^n; r_1, r_2)|$ and $\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}^n; r_1, r_2) \neq \emptyset$

4.2.2. *Configuration (R).* The example we give here corresponds to situation (R). For any $r \geq 3$ we exhibit G-covers with group D_{2n} defined over $\mathbb{Q}^{tr}$ (but not over $\mathbb{Q}$) and with r rational branch points. It also illustrates the difficulties one can encounter when trying to compute $\mathbf{n}^{\mathbb{R}}(\mathbf{C}; r, 0)$ directly. We will use the commutative diagram:

$$\begin{array}{ccc} \mathrm{Inv}(G)^r & \xrightarrow{\theta} & G^{r-1} \\ \downarrow{\scriptstyle \pi} & \nearrow_{\overline{\theta}} & \\ \mathrm{Inv}(G)^r/G\cdot & & \end{array}$$

where π is the canonical surjection and θ is the map given by the correspondence $(u_0,...,u_{r-1}) \to (u_0u_1, u_1u_2, ..., u_{r-2}u_{r-1})$. Rewrite $\mathbf{n}^{\mathbb{R}}(\mathbf{C};r,0)$ as follows (where c denotes as in §3.1 the r-cycle $(0,1,...,r-1)$):

$$\begin{aligned}\mathbf{n}^{\mathbb{R}}(\mathbf{C};r,0) &= \frac{1}{Z_1...Z_r} \sum_{\substack{\chi_1,...,\chi_r\in \mathrm{Irr}(G)\\ (u_0,...,u_{r-1})\in \mathrm{Inv}(G)^r/G\cdot}} \prod_{1\le i\le r} (\chi_i(g_i)\chi_i(u_{i-1}u_{c(i-1)}))\\ &= \frac{1}{Z_1...Z_r} \sum_{(u_0,...,u_{r-1})\in \mathrm{Inv}(G)^r/G\cdot} \prod_{1\le i\le r}\left(\sum_{\chi\in\mathrm{Irr}(G)} \chi(g_i)\chi(u_{i-1}u_{c(i-1)})\right)\end{aligned}$$

and also recall the general form of Serre's formula:

$$|\Sigma(\mathbf{C},G)| = \frac{|C_1|...|C_r|}{|G|}\sum_{\chi\in\mathrm{Irr}(G)} \frac{\prod_{1\le i\le r}\chi(g_i)}{\chi(1)^{r-2}}$$

When $G = D_{2n}$ we have $\overline{\chi} = \chi$ for any irreducible character $\chi \in \mathrm{Irr}(D_{2n})$, so we get $\sum_{\chi\in\mathrm{Irr}(G)} \chi(g_i)\chi(u_{i-1}u_{c(i-1)}) = \sum_{\chi\in\mathrm{Irr}(G)} \overline{\chi(g_i)}\chi(u_{i-1}u_{c(i-1)})$ is equal to Z_i if g_i and $u_{i-1}u_{c(i-1)}$ are conjugate and is equal to 0 otherwise, for $i = 1,...,r$. Consequently, the only tuples $\underline{u} = (u_0,...,u_{r-1}) \in \mathrm{Inv}(G)/G\cdot$ we will need in our computation are the $(\overline{\theta}^{-1}(g_1^{\gamma_1},...,g_{r-1}^{\gamma_{r-1}}))_{\gamma_1,...,\gamma_{r-1}\in G}$ when they exist. So,

$$\mathbf{n}^{\mathbb{R}}(\mathbf{C};r,0) = \frac{1}{Z_r}\sum_{\underline{u}\in\overline{\theta}^{-1}(C_1\times...\times C_{r-1})}\sum_{\chi\in\mathrm{Irr}(G)} \overline{\chi(g_r)}\chi(u_{r-1}u_0)$$

With the notations of 4.2.1 let us try and apply these remarks to the specific r-tuple

$$\mathbf{C} = (B, A_{i_1},...,A_{i_t},B) \text{ (so } r = t+2)$$

where we choose $1 \le i_1,...,i_t \le \frac{n-1}{2}$ so that $\mathbf{C}$ is g-complete. A representative of $\overline{\theta}^{-1}(vu^k, u^{\epsilon_1 i_1},...,u^{\epsilon_t i_t})$ is $(1, vu^k, vu^{k+\epsilon_1 i_1},..., vu^{k+\epsilon_1 i_1+...+\epsilon_t i_t})$, $k = 0,...,n-1$, $\epsilon_1,...,\epsilon_t \in \{\pm 1\}$. Moreover, since $u_{r-1}u_0 = vu^{k+\epsilon_1 i_1+...+\epsilon_t i_t} \in B$, we obtain :

$$\mathbf{n}^{\mathbb{R}}(\mathbf{C};r,0) = \frac{1}{2^2 n^t}\sum_{\epsilon_1,...,\epsilon_t\in\{\pm1\}}\sum_{k=0}^{n-1} 2\times n^t\times 2 = 2^t n$$

Hence on the one hand for all $m \ge 1$ such that $(2n,m) = 1$ we have $|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}^m;r,0)| = 2^{t-1}$ and on the other hand, by Serre's formula: $|\overline{\mathrm{sni}}(\mathbf{C},D_{2n})| = 2^{t-1}$. So if we fix a r-tuple of rational branch points $\mathbf{t} = (t_1,...,t_r) \in \mathcal{U}^r(\mathbb{Q})$, using the same argument as above we get that all the G-covers in $\overline{\mathrm{sni}}(\mathbf{C},D_{2n})$ are defined over $\mathbb{Q}^{tr}$. Moreover choosing for instance $i_1 = ... = i_t = 1$, we can assert those G-covers are not defined over $\mathbb{Q}$.

Remark 4.7. The computation we made above can be generalized to any tuple

$$\mathbf{C} = (B, A_{i_{1,1}},...,A_{i_{1,u_1}},B,B,A_{i_{2,1}},...,A_{i_{2,u_2}},B,B,...,B,A_{i_{t,1}},...,A_{i_{t,u_t}},B)$$

with $r = 2t+u_1+...+u_t$, we obtain $|sni^{\mathbb{R}}(\mathbf{C};r,0)| = 2^{u_1+...+u_t-1}n^{t-1}$ and $|\overline{\mathrm{sni}}(\mathbf{C},D_{2n})| = 2^{u_1+...+u_t-1}n^{2t-2}$, so $\frac{|\overline{\mathrm{sni}}(\mathbf{C},D_{2n})|}{|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C};r,0)|} = n^{t-1}$ which only depends on t.

4.2.3. *Application to regular realizations of D_{2a^∞} (with $a \geq 3$ odd) over $\mathbb{Q}^{tr}(X)$.* The results obtained in 4.2.1 and 4.2.2 do not depend on $n \geq 3$ odd, which yields regular realizations of the profinite groups $D_{2a^\infty} := \mathrm{limproj}_{\mathrm{n}\to+\infty} \mathrm{D}_{2\mathrm{a}^\mathrm{n}} \simeq \mathbb{Z}_\mathrm{a} \rtimes \mathbb{Z}/2\mathbb{Z}$, $a \geq 3$ odd, over $\mathbb{Q}^{tr}(X)$. Indeed, for any $a \geq 3$ odd and for any $n \geq 1$ write
- $A_{1,a,n}, ..., A_{a^n-1/2,a,n}$ with $A_i = \{u^i, u^{-i}\}$, $i = 1, ..., \frac{a^n-1}{2}$,
- $B_{a,n} = \{vu^i\}_{0\leq i\leq a^n-1}$.
for the $\frac{a^n+1}{2}$ non trivial conjugacy classes of D_{2a^n}. Also set $\mathbf{C}_{a,n} = (A_{1,a,n}, B_{a,n}, B_{a,n}, A_{1,a,n})$. This gives rise to a tower of Hurwitz spaces

$$\cdots \to \mathcal{H}'_{4,D_{2a^{n+1}}}(\mathbf{C}_{a,n+1}) \to \mathcal{H}'_{4,D_{2a^n}}(\mathbf{C}_{a,n}) \to \cdots \to \mathcal{H}'_{4,D_{2a}}(\mathbf{C}_{a,1})$$

Fix $\mathbf{t}' = (z_1, \overline{z}_1, z_2, \overline{z}_2) \in \mathcal{U}^4(\mathbb{C})$ with $z_i \in \mathbb{P}^1(\mathbb{C})\backslash\mathbb{P}^1(\mathbb{R})$ and $\{z_i, \overline{z}_i\} \in \mathcal{U}_2(\mathbb{Q})$, $i = 1, 2$, and consider the projective system of finite sets of $\mathbb{Q}^{tr}$-points (see 4.2.1)

$$\cdots \to \Pi_4(\mathcal{H}'_{4,D_{2a^{n+1}}}(\mathbf{C}_{a,n+1})_{\mathbf{t}'}) \to \Pi_4(\mathcal{H}'_{4,D_{2a^n}}(\mathbf{C}_{a,n})_{\mathbf{t}'}) \to \cdots \to \Pi_4(\mathcal{H}'_{4,D_{2a^1}}(\mathbf{C}_{a,1})_{\mathbf{t}'})$$

then $\mathrm{limproj}_{\mathrm{n}\to+\infty}\Pi_4(\mathcal{H}'_{4,\mathrm{D}_{2\mathrm{a}^\mathrm{n}}}(\mathbf{C}_{\mathrm{a,n}})_{\mathbf{t}'}) \neq \emptyset$ and any $\mathbf{p} \in \mathrm{limproj}_{\mathrm{n}\to+\infty}\Pi_4(\mathcal{H}'_{4,\mathrm{D}_{2\mathrm{a}^\mathrm{n}}}(\mathbf{C}_{\mathrm{a,n}})_{\mathbf{t}'})$ corresponds to a regular Galois realization of D_{2a^∞} over $\mathbb{Q}^{tr}(X)$ with branch points $\mathbf{t}'$ and inertia canonical invariant $(A_{1,a,\infty}, B_{a,\infty}, B_{a,\infty}, A_{1,a,\infty})$ (where $B_{a,\infty} = \{vu^i\}_{i\geq 0}$ and $A_{i,a,\infty} = \{u^i, u^{-i}\}$, $i \geq 1$).

Likewise, using the results of 4.2.2, one gets regular Galois realization of D_{2a^∞} over $\mathbb{Q}^{tr}(X)$ with rational branch points $\mathbf{t}' = (t_1, ..., t_{t+2}) \in \mathcal{U}^{t+2}(\mathbb{Q})$ and inertia canonical invariant $(B_{a,\infty}, A_{i_1,a,\infty}, ..., A_{i_t,a,\infty}, B_{a,\infty})$ where $i_1, ..., i_t \geq 1$ such that, for instance, $(i_j, a) = 1$, $j = 1, ..., t$.

4.3. **The Mathieu group M_{11}.** Our formulas are manageable even for more complicated groups, particularly in the branch point configuration (C). In our last example, the group is the Mathieu group M_{11}.

According to the Atlas $|M_{11}| = 11.5.3^2.2^4$ and M_{11} has 10 conjugacy classes: $1A$, $2A$, $3A$, $4A$, $5A$, $6A$, $8A$, $B*$, $11A$, $B**$. The difficulty here is to compute $\mathrm{Cen}_{M_{11}}(2A)$. We apply theorem 2.3 to the specific 4-tuple (8A,B*,11A,B**) to do this. We will use that $|\mathrm{Cen}_{M_{11}}(2A)| = 3.2^4$ and that any 2-Sylow S_2 of $\mathrm{Cen}_{M_{11}}(2A)$ is semidihedral with order 16 i.e. $S_2 = < x, a \mid x^2 = 1 = a^8, xax = a^3 > = SD_{16}$ (cf. [10] Ex. 7.4.4 p.205) to prove the following lemma, which is needed to carry out computations of $\mathbf{n}^{\mathbb{R}}(\mathbf{C}; 0, s)$.

Lemma 4.8. $\mathrm{Cen}_{M_{11}}(2A)$ *contains: 1 element with order 1, 13 elements with order 2, 8 elements with order 3, 6 elements with order 4, 8 elements with order 6, 12 elements with order 8 (6 in each conjugacy class).*

First, note that SD_{16} contains:
- 4 elements with order 8: a, a^3, a^5, a^7
- 6 elements with order 4: a^2, a^6, xa, xa^3, xa^5, xa^7
- 5 elements with order 2: a^4, xa^2, xa^4, xa^6, x
- 1 element with order 1: 1

Moreover, $Z(SD_{16}) =< a^4 >$ and SD_{16} has 3 kinds of subgroups of index 2: $\mathbb{Z}/8\mathbb{Z} =< a >$, $D_8 =< a^2, x >$, $\mathbb{H}_8 =< a^2, xa >$.

We are now able to describe $\mathrm{Cen}_{M_{11}}(2A)$ more precisely. According to the Atlas, there is

an unsplit short exact sequence: $1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Cen}_{M_{11}}(2A) \xrightarrow{\theta} S_4 \to 1$. So, as the center of S_4 is trivial, we get the inclusions $< 2A > \subset Z(\mathrm{Cen}_{M_{11}}(2A)) \subset \mathbb{Z}/2\mathbb{Z}$, which obviously are equalities. Consequently, for all $\sigma \in S_4$:
- If σ has order $2k+1, k = 0, 1$ then $\theta^{-1}(\sigma)$ contains an element with order $2k+1$ and an element with order $4k+2$.
- If σ has order 2 then $\theta^{-1}(\sigma)$ contains either two elements with order 2 or two elements with order 4 or two elements with order 6. Let us denote by n the number of elements with order 6 we obtain this way ($0 \leq n \leq 6$).
- If σ has order 4 then $\theta^{-1}(\sigma)$ contains either two elements with order 4 or two elements with order 8.
In particular, we have exactly 8 elements with order 3 and $8+n$ elements with order 6 in $\mathrm{Cen}_{M_{11}}(2A)$. All the other ones have order 1,2,4 or 8, so are contained in the 2-Sylow subgroups of $\mathrm{Cen}_{M_{11}}(2A)$. Let us write n_p for the number of p-Sylows in $\mathrm{Cen}_{M_{11}}(2A)$. From the above we deduce $n_3 = 4$. Furthermore, since $n_2|3$ and $n_2 \equiv 1$ [mod2] we have $n_2 = 1, 3$. But if $n_2 = 1$, $|\mathrm{Cen}_{M_{11}}(2A)| = 32 + n$: a contradiction, hence $n_2 = 3$. Still according to the Atlas $\mathrm{Cen}_{M_{11}}(2A)$ contains a normal subgroup with order 8, V, and as the 2-Sylows of $\mathrm{Cen}_{M_{11}}(2A)$ are conjugate, for all $S, T \in \mathcal{S}_2(\mathrm{Cen}_{M_{11}}(2A))$ we get $S \cap T = V$. Consequently, computing the order of $\mathrm{Cen}_{M_{11}}(2A)$ we get now $|\mathrm{Cen}_{M_{11}}(2A)| = 48+n$, which leads to $n = 0$.
There are 4 possibilities for V:
1/ $V = \mathbb{Z}/8\mathbb{Z}$ and we have in $\mathrm{Cen}_{M_{11}}(2A)$: 1 element with order 1, 13 elements with order 2, 8 elements with order 3, 14 elements with order 4, 8 elements with order 6, 4 elements with order 8 (2 in each conjugacy class).
2/ $V = D_8$ and we have in $\mathrm{Cen}_{M_{11}}(2A)$: 1 element with order 1, 5 elements with order 2, 8 elements with order 3, 14 elements with order 4, 8 elements with order 6, 12 elements with order 8 (6 in each conjugacy class).
3/ $V = \mathbb{H}_8$ and we have in $\mathrm{Cen}_{M_{11}}(2A)$: 1 element with order 1, 13 elements with order 2, 8 elements with order 3, 6 elements with order 4, 8 elements with order 6, 12 elements with order 8 (6 in each conjugacy class).

Here are the computations corresponding to the three configurations above:

	A_{χ_1}	A_{χ_2}	A_{χ_3}	A_{χ_4}	A_{χ_5}	A_{χ_6}	A_{χ_7}	A_{χ_8}	A_{χ_9}	$A_{\chi_{10}}$
$V = \mathbb{Z}/8\mathbb{Z}$	15840	10560	0	0	7920	0	0	15840	2640	5280
$V = D_8$	15840	7920	2640	2640	2640	0	0	10560	5280	7920
$V = \mathbb{H}_8$	15840	7920	0	0	7920	0	0	15840	0	7920

Finally, since the maximal subgroups of M_{11} have order 720, 660, 144, 120, 48 and none of these orders can be divided by both 8 and 11, we conclude that (8A, B*, 11A, B**) is g-complete symetric. Now, the first two configurations give $|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, 2)| = \frac{538}{3}$ and $|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, 2)| = \frac{536}{3}$ respectively whereas the third one gives $|\overline{\mathrm{sni}}^{\mathbb{R}}(\mathbf{C}; 0, 2)| = 180$. So $V = \mathbb{H}_8$, which gives a description of the centralizer of the involution class in M_{11}. For this 4-uple Serre's formula gives $|\overline{\mathrm{sni}}(\mathbf{C}, M_{11})| = 8752$.

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