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An Open Adelic Image Theorem for Abelian Schemes

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In this note, we prove an adelic open image theorem for abelian schemes and, more generally, families of 1-motives. This result, which can be interpreted as an adelic specialization theorem, shows for instance the abundancy of Galois-generic closed points on most connected Shimura varieties.

1 Introduction

Let k be a field of characteristic 0. Let X be a smooth, separated, and geometrically connected scheme over k with generic point η and set of closed points $|X|$. Write $\bar{X} := X \times_k \bar{k}$. Given $x \in X$, write $k(x)$ for the residue field of X at x. Fix an algebraically closed field Ω containing all the residue fields $k(x)$, $x \in X$ and let $\bar{\eta} : \mathrm{spec}(\Omega) \to X$ denote the corresponding geometric point. As X is geometrically connected, the sequence of profinite groups

$$(*)\ 1 \to \pi_1(\bar{X}; \bar{\eta}) \to \pi_1(X; \bar{\eta}) \xrightarrow{\pi} \pi_1(\mathrm{spec}(k); \bar{\eta}) \to 1$$

induced by functoriality of étale fundamental group from the sequence of morphisms $\bar{X} \to X \to \mathrm{spec}(k)$ is short exact. Furthermore, every $x \in X$, viewed as a morphism

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$\mathrm{spec}(k(x)) \to X$, induces a commutative diagram of profinite groups

$$(**) \quad \begin{array}{ccc} \pi_1(X;\bar\eta) & \xrightarrow{\pi} & \pi_1(\mathrm{spec}(k);\bar\eta) \\ & \nwarrow^{\sigma_x} & \uparrow \sigma^x \\ & & \pi_1(\mathrm{spec}(k(x));\bar\eta)\,. \end{array}$$

When $x \in |X|$, σ_x and σ^x are injective. When $x = \eta$, σ_η is surjective (recall that X is normal).

In the following, we will omit fiber functors in our notation for étale fundamental groups unless it helps understand the situation (see Section 1.2.1). Also, we will identify $\pi_1(\mathrm{spec}(k))$ with the absolute Galois group of k and simply denote it by Γ_k.

1.1 Specializing Galois extensions

Given a surjective morphism $\rho : \pi_1(X) \twoheadrightarrow \Gamma$ of profinite groups, write $\Gamma_x := \mathrm{Im}(\rho \circ \sigma_x)$ and define

$$X^{\mathrm{gen}} := \{x \in |X| \mid \Gamma_x \text{ is open in } \Gamma\}, \quad X^{ex} := |X| \setminus X^{\mathrm{gen}},$$

which we will call the (closed) generic locus and the (closed) exceptional locus, respectively.

Understanding the specialization of $\rho : \pi_1(X) \twoheadrightarrow \Gamma$ amounts to describing X^{gen}. A lemma of Serre [32, Section 10.6] asserts that, if Γ contains an open subgroup which is a finite product of ℓ-adic Lie groups, then the set $X^{ex} \cap X(k)$ is thin in $X(k)$. In particular, when k is Hilbertian (for instance finitely generated), this ensures that X^{gen} is non-empty (and even that there exists an integer $d \geq 1$ such that the set of all $x \in X^{\mathrm{gen}}$ with $[k(x):k] = d$ is infinite). Serre's lemma is the classical tool to transfer properties of ℓ-adic Galois representations from number fields to fields finitely generated over $\mathbb{Q}$ or conversely (see for instance [33, §1]).

Recently, Serre's lemma was strengthened by Tamagawa and the author as follows [7, Theorem 1.1]. Assume that k is finitely generated over $\mathbb{Q}$, that every open subgroup of $\bar\Gamma := \rho(\pi_1(\bar X))$ has finite abelianization, and that X is a curve. Then for every integer $d \geq 1$ the set of all $x \in X^{ex}$ with $[k(x):k] \leq d$ is finite. [7, Theorem 1.1] applies in particular to the ℓ-adic representations $\rho : \pi_1(X) \to \mathrm{GL}(\mathrm{H}^*(Y_{\bar\eta}, \mathbb{Q}_\ell))$ attached to a smooth proper morphism $Y \to X$ [6, §5.1] and gives information about the variation of arithmetico-geometrical invariants in the fibers of $Y \to X$ (uniform boundedness of the ℓ-primary torsion when $Y \to X$ is an abelian scheme [7, Corollary 4.3], degeneration of the motivated motivic Galois group and jumping of the Néron–Severi rank [4] etc.).

However, the above results are both ℓ-adic in nature and fail to ensure that X^{gen} is non-empty for more general profinite groups Γ, in particular adelic groups.

This is the problem we discuss in this paper in the particular case of adelic representations arising from the Tate module of an abelian scheme (and, more generally, a 1-motive).

1.2 **Main results**

1.2.1 *Notation*

For $x \in X$, let $k(\bar{x})$ denote the algebraic closure of $k(x)$ in Ω and consider the corresponding geometric point $\bar{x} : \text{spec}(k(\bar{x})) \to X$ over $x \in X$. Let $F_{\bar{x}}$ denote the fibre functor which sends an étale cover $X' \to X$ to the finite set $X'_{\bar{x}}(k(\bar{x}))$.

Let $A \to X$ be an abelian scheme. Set $g := \dim(A_\eta)$. For every integer $n \geq 1$, the kernel $A[n] \to X$ of the multiplication-by-n morphism $[n] : A \to A$ is an étale cover. Write

$$H_{n,\bar{x}} := F_{\bar{x}}(A[n])(\simeq (\mathbb{Z}/n)^{2g}).$$

The inclusion $k(\bar{x}) \hookrightarrow \Omega$ defines an étale path $\alpha_x : F_{\bar{\eta}} \tilde{\to} F_{\bar{x}}$ hence an isomorphism

$$\alpha_x(A[n]) : H_{n,\bar{\eta}} \tilde{\to} H_{n,\bar{x}},$$

which is compatible with the induced isomorphism of étale fundamental groups

$$\alpha_x \circ - \circ \alpha_x^{-1} : \pi_1(X; \bar{\eta}) \tilde{\to} \pi_1(X; \bar{x}).$$

Modulo these isomorphisms, the 'usual' Galois representation

$$\pi_1(\text{spec}(k(x)); \bar{x}) \to \pi_1(X; \bar{x}) \to \text{GL}(H_{n,\bar{x}})$$

identifies with the representation

$$\pi_1(\text{spec}(k(x)); \bar{\eta}) \stackrel{\sigma_x}{\to} \pi_1(X; \bar{\eta}) \to \text{GL}(H_{n,\bar{\eta}}).$$

So, in the following, we will identify $\pi_1(X; \bar{\eta})$ and $\pi_1(X; \bar{x})$, $\pi_1(\text{spec}(k(x)); \bar{\eta})$ and $\pi_1(\text{spec}(k(x)); \bar{x})$, $H_{n,\bar{\eta}}$ and $H_{n,\bar{x}}$ and simply denote them by $\pi_1(X)$, $\Gamma_{k(x)}$, H_n, respectively.

To sum it up, one attaches to $A \to X$ a projective system of continuous linear representations

$$\rho_n : \pi_1(X) \to \text{GL}(H_n)$$

with the property that for every point $x \in X$ the "local" representation

$$\rho_{n,x} := \rho_n \circ \sigma_x : \Gamma_{k(x)} \to \mathrm{GL}(H_n)$$

identifies with the "usual" Galois representation of $\rho_{n,x} : \Gamma_{k(x)} \to \mathrm{GL}(A_x[n])$.

Given a prime ℓ, write $H_{\ell^\infty} := \overset{\lim}{\longleftarrow} H_{\ell^n} (\simeq (\mathbb{Z}_\ell)^{2g})$ and

$$\rho_{\ell^\infty} : \pi_1(X) \to \mathrm{GL}(H_{\ell^\infty}), \quad \rho_{x,\ell^\infty} := \rho_{\ell^\infty} \circ \sigma_x : \Gamma_{k(x)} \to \mathrm{GL}(H_{\ell^\infty})$$

for the corresponding ℓ-adic representations.

We will use the following notation:

$$G_\ell := \mathrm{im}(\rho_\ell) \subset \mathrm{GL}(H_\ell); \quad G_{\ell^\infty} := \mathrm{im}(\rho_{\ell^\infty}) \subset \mathrm{GL}(H_{\ell^\infty})$$
$$\bar{G}_\ell := \rho_\ell(\pi_1(\bar{X})) \lhd G_\ell; \quad \bar{G}_{\ell^\infty} := \rho_{\ell^\infty}(\pi_1(\bar{X})) \lhd G_{\ell^\infty}$$
$$G_{x,\ell} := \mathrm{im}(\rho_{x,\ell}) \subset G_\ell; \quad G_{x,\ell^\infty} := \mathrm{im}(\rho_{x,\ell^\infty}) \subset G_{\ell^\infty}.$$

Note that G_{ℓ^∞}, $\bar{G}_{\ell^\infty}$ and G_{x,ℓ^∞}, being closed subgroups of $\mathrm{GL}(H_{\ell^\infty})$, are ℓ-adic Lie groups.

Set $V_{\ell^\infty} := H_{\ell^\infty} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and let $\mathcal{G}_{\ell^\infty}$, $\bar{\mathcal{G}}_{\ell^\infty}$, $\mathcal{G}_{x,\ell^\infty}$ denote the Zariski closure in $\mathrm{GL}_{V_{\ell^\infty}}$ of G_{ℓ^∞}, $\bar{G}_{\ell^\infty}$, G_{x,ℓ^∞} respectively.

Due to the Hodge–Tate property [2], G_{ℓ^∞} and G_{x,ℓ^∞} are open in $\mathcal{G}_{\ell^\infty}(\mathbb{Q}_\ell)$ and $\mathcal{G}_{x,\ell^\infty}(\mathbb{Q}_\ell)$, respectively, and as $\bar{\mathcal{G}}_{\ell^\infty}$ is semi-simple (see the beginning of Section 2.3), $\bar{G}_{\ell^\infty}$ is open in $\bar{\mathcal{G}}_{\ell^\infty}(\mathbb{Q}_\ell)$.

We will also consider the associated adelic (or product) representations

$$\rho = \Big(\prod_\ell \rho_{\ell^\infty}\Big) : \pi_1(X) \to \prod_\ell \mathrm{GL}(H_{\ell^\infty}), \quad \rho_x = \Big(\prod_\ell \rho_{x,\ell^\infty}\Big) : \Gamma_{k(x)} \to \prod_\ell \mathrm{GL}(H_{\ell^\infty})$$

and their images

$$G := \mathrm{im}(\rho), \quad \bar{G} := \rho(\pi_1(\bar{X})), \quad G_x := \mathrm{im}(\rho_x).$$

1.3 Statements

Let X^{gen} and X^{ex} denote the closed generic locus and closed exceptional locus attached to the adelic representation $\rho : \pi_1(X) \twoheadrightarrow G$ (see Section 1.1).

For every prime ℓ, let $X^{\mathrm{gen}}_{\ell^\infty} \subset X$ denote the set of all $x \in |X|$ such that the following three equivalent conditions hold:

(i) G_{x,ℓ^∞} is open in G_{ℓ^∞};

(ii) $\dim(G_{x,\ell^\infty}) = \dim(G_{\ell^\infty})$;

(iii) $\dim(\mathcal{G}_{x,\ell^\infty}) = \dim(\mathcal{G}_{\ell^\infty})$,

where dimension is in the sense of ℓ-adic Lie groups in (ii) and algebraic groups over $\mathbb{Q}_\ell$ in (iii). Also, write $X^{ex}_{\ell^\infty} := |X| \setminus X^{\mathrm{gen}}_{\ell^\infty}$

From now on, assume that k is a number field.

The following, observed by Hui [17], is a consequence of the Tate conjectures for abelian varieties [15] and the Borel-de Siebenthal Theorem.

Lemma 1.1. The set $X^{ex}_{\ell^\infty}$ is independent of ℓ (that is for every $x \in |X|$, G_{x,ℓ^∞} is open in G_{ℓ^∞} for one prime ℓ if and only if G_{x,ℓ^∞} is open in G_{ℓ^∞} for every prime ℓ). □

Thus, in the following, we will write $X^{\mathrm{gen}}_{-} := X^{\mathrm{gen}}_{\ell^\infty}$, $X^{ex}_{-} := X^{ex}_{\ell^\infty}$. Note that, by definition, $X^{\mathrm{gen}} \subset X^{\mathrm{gen}}_{-}$. Hence the best adelic sharpening of Lemma 1.1 one can expect is

Theorem 1.2. $X^{\mathrm{gen}} = X^{\mathrm{gen}}_{-}$ (that is for every $x \in |X|$, G_{x,ℓ^∞} is open in G_{ℓ^∞} for one prime ℓ if and only if G_x is open in G). □

Combined with the adelic unipotent part of the Mumford–Tate conjecture for 1-motives [21], Theorem 1.2 extends to 1-motives (see Theorem 3.2).

1.4 Extensions of Theorem 1.2

1.4.1 *To the adelic representation $\rho : \pi_1(X) \to \prod_\ell \mathrm{GL}(\mathrm{H}^*(Y_{\bar\eta}, \mathbb{Q}_\ell))$ attached to a smooth proper morphism $Y \to X$*

There are currently two heuristic obstructions to the extension of Theorem 1.2 to this general "motivic" setting, both related to the use of the Tate conjectures for abelian varieties. The first obstruction comes from the fact that, in general, the ℓ-independency of $X^{ex}_{\ell^\infty}$ is not known though it is predicted by the motivated (ℓ-adic) Tate conjectures which describe $X^{ex}_{\ell^\infty}$ as the degeneration locus of the motivated motivic Galois group [4]. Formally, one could overcome this obstruction by setting $X^{ex}_{-} := \cup_\ell X^{ex}_{\ell^\infty}$ (though, with this definition, X^{gen}_{-} might be empty). Then, our proof roughly decomposes into three steps. Fix $x \in X^{\mathrm{gen}}_{-}$.

(1) Reduction of Theorem 1.2 to an ℓ-adic statement: up to replacing X with a connected étale cover one may assume that $G = \prod_\ell G_{\ell^\infty}$, $\bar{G} = \prod_\ell \bar{G}_{\ell^\infty}$ and $G_x = \prod_\ell G_{x,\ell^\infty}$. In particular, $G_{x,\ell^\infty} = G_{\ell^\infty}$ for $\ell \gg 0$ if and only if $G_x \subset G$ is open.

(2) Reduction of the ℓ-adic statement $G_{x,\ell^\infty} = G_{\ell^\infty}$ for $\ell \gg 0$ to a modulo-ℓ statement: The restriction of the reduction-modulo-ℓ map $\bar{G}_{\ell^\infty} \to \bar{G}_\ell$ is a Frattini cover. In particular, if $\bar{G}_{\ell^\infty} \cap G_{x,\ell^\infty} \subset G_{\ell^\infty}$ maps surjectively onto $\bar{G}_\ell$ for $\ell \gg 0$ then $G_{x,\ell^\infty} = G_{\ell^\infty}$ for $\ell \gg 0$. The fact that $\bar{G}_{\ell^\infty} \cap G_{x,\ell^\infty} \subset G_{\ell^\infty}$ maps surjectively onto $\bar{G}_\ell$ for $\ell \gg 0$ is deduced from the following.

(3) Modulo-ℓ statement: One has $G_{x,\ell} = G_\ell$ for $\ell \gg 0$.

While the proofs in (1) and (2) work in the general motivic setting, the proof of the modulo-ℓ statement $G_{x,\ell} = G_\ell$, $\ell \gg 0$ relies heavily on the variants modulo ℓ for $\ell \gg 0$ of the Tate conjectures for abelian varieties. But it is not clear that such modulo-ℓ variants for $\ell \gg 0$ can be expected in the general motivic setting (for instance whether they can be deduced from the ℓ-adic Tate conjectures).

Let us also mention the group-theoretical result of [22] for arbitrary compatible systems of rational ℓ-adic representations $\rho_{\ell^\infty} : \mathcal{G} \to \mathrm{GL}_r(\mathbb{Q}_\ell)$ of an F-group $\mathcal{G}$. Let G_{ℓ^∞} denote the image of ρ_{ℓ^∞} and $\mathcal{G}_{\ell^\infty}$ its Zariski closure in $\mathrm{GL}_{r,\mathbb{Q}_\ell}$. The main result [22, Theorem 2.6] states that there exists a subset L of primes of Dirichlet density 1 such that $\mathcal{G}_{\ell^\infty}$ is unramified and that $G^{sc}_{\ell^\infty} := (p^{sc})^{-1} p^{ad}(G_{\ell^\infty}) \subset \mathcal{G}^{sc}_{\ell^\infty}(\mathbb{Q}_\ell)$ is hyperspecial for $\ell \in L$, where $p^{ad} : \mathcal{G}_{\ell^\infty} \to \mathcal{G}^{ad}_{\ell^\infty}$ denotes the quotient of $\mathcal{G}_{\ell^\infty}$ by its radical and $p^{sc} : \mathcal{G}^{sc}_{\ell^\infty} \to \mathcal{G}^{ad}_{\ell^\infty}$ the simply-connected cover of $\mathcal{G}^{ad}_{\ell^\infty}$. This result relies on delicate group-theoretical arguments (in particular a refined analysis of Frobenius tori, the Bruhat–Tits theory and the classification of finite simple groups) and is quite general. However, to replace L by the set of all primes $\ell \gg 0$, one roughly has to prove that the modulo-ℓ images are "as big as possible" for $\ell \gg 0$. In the special case of so-called type A motivic representations of the absolute Galois group of a number field investigated in [18–20], this can be done using the special features of semi-simple groups of type A (in particular, the fact that any proper subgroup has strictly smaller semi-simple rank) and an additional algebraico-geometric input (the tame inertia conjecture of Serre, proved by Caruso). Combining these results, those of [5] and the techniques of this paper, one can prove the analogue of Theorem 1.2 for the adelic representations $\rho : \pi_1(X) \to \prod_\ell \mathrm{GL}(\mathrm{H}^*(Y_{\bar{\eta}}, \mathbb{Q}_\ell))$ attached to a smooth proper morphism $Y \to X$ under the assumption that $\mathcal{G}_{\ell^\infty}$ is reductive for every prime ℓ and that there exists one prime ℓ such that all the simple factors of $\mathcal{G}_{\ell^\infty, \bar{\mathbb{Q}}_\ell}$ are of type A_n for $n = 6$ or ≥ 9. (Alternatively, one may consider the $\pi_1(X)$-semisimplifications.) Here is a sketch of the argument. Fix $x \in |X|$ such that G_{x,ℓ^∞} is open in G_{ℓ^∞} for some prime ℓ. One may assume that $\mathcal{G}_{x,\ell^\infty}$, $\mathcal{G}_{\ell^\infty}$, $\bar{\mathcal{G}}_{\ell^\infty}$ are connected (Theorem 2.1(3)). As ρ is motivic, it is enough to show that $G_{x,\ell^\infty} = G_{\ell^\infty}$ for $\ell \gg 0$ (Theorem 2.1(1)) and one may assume that $\bar{G}_{\ell^\infty}$ is generated by its ℓ-Sylow for $\ell \gg 0$ (Theorem 2.1(2)). As ρ is motivic and $\bar{\mathcal{G}}_{\ell^\infty}$ (which is independent of ℓ by comparison between Betti and ℓ-adic cohomology - see the beginning of Section 2.3) has only simple factors of type A, G_{x,ℓ^∞} is open in G_{ℓ^∞} (equivalently $\mathcal{G}_{x,\ell^\infty} = \mathcal{G}_{\ell^\infty}$) for every prime ℓ [5, Theorem 1.2(1)]. As, by assumption, $\mathcal{G}_{x,\ell^\infty}$ is reductive for every ℓ and has only absolutely simple factors of type A_n for $n=6$ or $n \geq 9$ for one prime ℓ, it follows from [20, Theorem 1] that $\mathcal{G}^{sc}_{\ell^\infty} (= \mathcal{G}^{sc}_{\ell^\infty,x})$ is unramified and that $G^{sc}_{x,\ell^\infty} \subset \mathcal{G}^{sc}_{\ell^\infty}(\mathbb{Q}_\ell)$ is hyperspecial for $\ell \gg 0$. But as $G^{sc}_{x,\ell^\infty} \subset G^{sc}_{\ell^\infty} \subset \mathcal{G}^{sc}_{\ell^\infty}(\mathbb{Q}_\ell)$ and $G^{sc}_{\ell^\infty}$ is compact, this forces $G^{sc}_{x,\ell^\infty} = G^{sc}_{\ell^\infty}$ for $\ell \gg 0$. This in turn implies that $G^{+}_{x,\ell^\infty} = G^{+}_{\ell^\infty}$ for $\ell \gg 0$, where, given an ℓ-adic Lie group Γ, we let $\Gamma^+ \subset \Gamma$ denote the subgroup generated by the ℓ-Sylow. But then $G^{+}_{x,\ell^\infty} \supset \bar{G}^{+}_{\ell^\infty} = \bar{G}_{\ell^\infty}$, which is enough to conclude (see the discussion after Theorem 2.1).

1.4.2 *To finitely generated fields of characteristic* 0

The statements of Theorems 1.2 and 3.2 extend by specialization from number fields to finitely generated fields of characteristic 0. Here is a sketch of the argument for Theorem 1.2. Assume that k is a finitely generated field of characteristic 0 and let $x \in |X|$ such that G_{x,ℓ^∞} is open in G_{ℓ^∞} for some prime ℓ. Without loss of generality, we may assume that $x \in X(k)$. Fix a model

$$(1) \quad \mathcal{A} \longrightarrow \mathcal{X} \underset{\mathfrak{x}}{\overset{\curvearrowleft}{\longrightarrow}} S \quad \text{of } (2) \quad A \longrightarrow X \underset{x}{\overset{\curvearrowleft}{\longrightarrow}} \mathrm{spec}(k),$$

that is S is a smooth, separated, geometrically connected scheme over a number field $k^\#$ and with generic point $\zeta : \mathrm{spec}(k) \to S$, $\mathcal{X} \to S$ is a smooth, separated morphism with geometrically connected fibers, $\mathcal{A} \to \mathcal{X}$ is an abelian scheme, $\mathfrak{x} : S \to \mathcal{X}$ is a section of $\mathcal{X} \to S$ and the pull-back of (1) by $\zeta : \mathrm{spec}(k) \to S$ is (2). One has the following commutative diagram of étale fundamental groups

$$\begin{array}{ccccc}
\pi_1(S) & \xrightarrow{\ \mathfrak{x}\ } & \pi_1(\mathcal{X}) & \longrightarrow & \prod_\ell \mathrm{GL}(H_{\ell^\infty})) \\
\zeta \uparrow & & \uparrow & \nearrow \rho & \\
\Gamma_k & \xrightarrow[\sigma_x]{} & \pi_1(X) & &
\end{array}$$

with $\Gamma_k \twoheadrightarrow \pi_1(S)$, $\pi_1(X) \twoheadrightarrow \pi_1(\mathcal{X})$ surjective (this uses the normality of S and $\mathcal{X}$). In particular, the image of $\pi_1(S) \to \pi_1(\mathcal{X}) \to \prod_\ell \mathrm{GL}(H_{\ell^\infty})$ is G_x and the image of $\pi_1(\mathcal{X}) \to \prod_\ell \mathrm{GL}(H_{\ell^\infty})$ is G. Fix $s \in |S|$ such that the image of $\Gamma_{k^\#(s)} \stackrel{s}{\to} \pi_1(S) \stackrel{\mathfrak{x}}{\to} \pi_1(\mathcal{X}) \to \mathrm{GL}(H_{\ell^\infty})$ is open in the image of $\pi_1(S) \stackrel{\mathfrak{x}}{\to} \pi_1(\mathcal{X}) \to \mathrm{GL}(H_{\ell^\infty})$ hence in the image of $\pi_1(\mathcal{X}) \to \mathrm{GL}(H_{\ell^\infty})$. (Such an s always exists, say, by [32, §10.6]. See Section 1.1.) By Theorem 1.2 applied to $\mathcal{A} \to \mathcal{X}$, one gets that the image of $\Gamma_{k^\#(s)} \stackrel{s}{\to} \pi_1(S) \stackrel{\mathfrak{x}}{\to} \pi_1(\mathcal{X}) \to \prod_\ell \mathrm{GL}(H_{\ell^\infty})$ (hence the image - which is G_x - of $\pi_1(S) \stackrel{\mathfrak{x}}{\to} \pi_1(\mathcal{X}) \to \prod_\ell \mathrm{GL}(H_{\ell^\infty})$) is open in the image - which is G - of $\pi_1(\mathcal{X}) \to \prod_\ell \mathrm{GL}(H_{\ell^\infty})$.

1.4.3 *To finitely generated field of positive characteristic $p > 0$*

It is plausible that Theorem 1.2 extends to positive characteristic $p > 0$. The proof in (1) works as it is for $p > 0$, the proof in (3) should be carefully adjusted and might work as well. But the proof in (2) uses that $\bar{\mathcal{G}}_{\ell^\infty}$ with its tautological representation is defined over $\mathbb{Q}$, independent of ℓ and carries a discrete finitely generated integral structure (coming from the comparison between Betti and ℓ-adic cohomology). The existence of this integral structure is not known in positive characteristic.

1.5 **Further comments, application**

1.5.1 *Arithmetico-geometric properties of the exceptional locus*

From the equality $X^{ex} = X^{ex}_{-} = X^{ex}_{\ell^\infty}$ and what was recalled in Section 1.1, for every number field K containing k the set $X^{ex} \cap X(K)$ is thin in $X(K)$ and for every integer $d \geq 1$ and curve $C \hookrightarrow X$, the set of all $x \in X^{ex} \cap C$ with $[k(x) : k] \leq d$ is finite. The latter property may suggest that X^{ex} is of bounded height but this is false as already shown by the example of the universal elliptic scheme over the modular curve $Y(n)$ ($n \geq 3$), where X^{ex} coincides with the C.M. locus.

1.5.2 *Connexion with the Mumford–Tate conjecture*

1. X^{ex} contains the set of all $x \in |X|$ where the Mumford–Tate group degenerates and, from the Mumford–Tate conjecture, it should coincide with it. In particular, one may expect that X^{ex} is "maigre" that is contained in a countable union of proper closed subschemes [12, 7.5; 1, Theorem 5.2, 1), 2)]; see also [24, Section 6]). (More precisely, the Mumford-Tate group $\mathcal{M}_{A_x}$ of A_x coincides with the motivated motivic Galois group $G(A_{\bar{x}})$ of $A_{\bar{x}}$ (defined for a large enough 'auxilliary category') [1, Section 6.3] and the latter coincides with $G(A_{\bar{\eta}})$ if

and only $G(A_{\bar{x}})(\mathbb{Q}_\ell)$ contains an open subgroup of $\bar{G}_{\ell^\infty}$ [1, Theorem 5.2, 1), 2); 4, Theorem 5.1]. The key ingredient in André's argument is the fixed part theorem of Deligne.)

2. Theorem 1.2 says nothing about the Mumford–Tate conjecture but if the Mumford–Tate conjecture holds it (or rather a variant of its proof) implies that the best possible adelic version of the Mumford–Tate conjecture holds (see Section 2.5).
3. Actually, underlying Theorem 1.2 is the natural idea that, when working with families of abelian varieties, one can 'go around' (and may be beyond) the Mumford–Tate conjecture by replacing the (derived subgroup of the) Mumford–Tate group $\mathcal{M}_{A_\eta}$ of the generic fiber with the connected component $\bar{\mathcal{G}}^\circ_{\ell^\infty}$ of $\bar{\mathcal{G}}_{\ell^\infty}$. More precisely, in general $\bar{\mathcal{G}}^\circ_{\ell^\infty}$ is strictly smaller than $D\mathcal{M}_{A_\eta,\mathbb{Q}_\ell}$. But, (i) as $D\mathcal{M}_{A_\eta,\mathbb{Q}_\ell}$, $\bar{\mathcal{G}}^\circ_{\ell^\infty}$ is a semi-simple algebraic group defined over $\mathbb{Q}$ and independent of ℓ (see Section 2.3), and (ii) the assumption that $x \in X^{\mathrm{gen}}$ *i.e.* $G_{x,\ell^\infty} \subset G_{\ell^\infty}$ is open (equivalently $\bar{G}_{x,\ell^\infty} \subset \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{Q}_\ell)$ is open) can be interpreted as a weak replacement for the MT-conjecture (see Section 2.5) which, often, is enough for arithmetic applications. The application of Theorem 1.2 we give below in Section 1.5.3 provides an example of this 'variational philosophy'.

1.5.3 *Application to Shimura varieties*

If $A \to X$ is the universal abelian scheme over a connected Shimura variety X of Hodge type (see [28, Construction 2.9]), Theorem 1.2 shows that X^{ex} contains all the non-Galois-generic (in the sense of [28, Section 6]) points. This, together with the sparsity of X^{ex} (see Comment (1)) produces "lots of" Galois-generic closed points on X and, if the standard Mumford–Tate conjecture holds, Theorem 1.2 shows that every Hodge-generic point on X is Galois-generic (hence reduces [28, Conjecture 6.8] to the standard Mumford–Tate conjecture; see [28, Remark 6.10]). In [28, Theorem 7.6], Pink deduces from equidistribution results of Clozel *et al.* [10] that, for the standard Shimura varieties associated to $GSp_{2g,\mathbb{Q}}$, every infinite subset of the generalized Hecke orbit of a Galois-generic point is Zariski dense (a special case of [28, Conjecture 1.6], itself implied by the Zilber-Pink conjecture—see [29, Theorem 3.3]). In this case and for g odd or $g=2,6$, the existence of "lots of" Galois-generic closed points is ensured by Serre's adelic open image theorem for g-dimensional abelian varieties with endomorphism ring $\mathbb{Z}$ and g odd or $g=2,6$ [34, Section 7, Corollary of Theorem 3 and Compl. 8.1]. But for g even, $g \neq 2,6$, the existence of closed Galois-generic points (or even of Galois-generic points other than the generic points) on such Shimura varieties does not seem to have been noticed before. [28, Theorem 7.6] can be generalized to most connected Shimura varieties (see [9]), hence Theorem 1.2 provides new evidences for [28, Conjecture 1.6].

Theorem 3.2 (the extension of Theorem 1.2 to 1-motives) can be used similarly to prove the existence of Galois generic points on mixed Shimura varieties.

1.6 Structure of the remaining parts of the paper

Section 2 is devoted to the proof of Theorem 1.2. For clarity, the strategy of the proof is described in the preliminary Section 2.1, which reviews Steps (1), (2), and (3) mentioned in Section 1.4.1. Sections 2.2–2.4 carry each of these three steps in details. The final Section 3 deals with the extension of Theorem 1.2 to 1-motives.

2 Proof of Theorem 1.2

2.1 Strategy of the proof

For a subgroup $G\subset\prod_\ell \mathrm{GL}(H_{\ell^\infty})$ and a rational prime λ, let G_{λ^∞} denote the image of G by the projection $\prod_\ell \mathrm{GL}(H_{\ell^\infty})\to \mathrm{GL}(H_{\lambda^\infty})$ onto the λth factor and let G_λ denote the image of G_{λ^∞} by the reduction-modulo-λ morphism $\mathrm{GL}(H_{\lambda^\infty})\to\mathrm{GL}(H_\lambda)$.

Our strategy to prove Theorem 1.2 decomposes into three main steps. We review them together with the main intermediary statements below. The detailed proofs are postponed to Sections 2.2–2.4.

Throughout this section, fix $x\in X^{\mathrm{gen}}_{-}$ that is, assume that G_{x,ℓ^∞} is open in G_{ℓ^∞} for one (or, equivalently, for every, see Lemma 1.1) prime ℓ.

Step 1 : *From adelic to ℓ-adic*. Recall that a closed subgroup $\Gamma\subset\prod_\ell \mathrm{GL}(H_{\ell^\infty})$ is said to be *ℓ-independent* if $\Gamma=\prod_\ell \Gamma_{\ell^\infty}$. One first proves

Theorem 2.1. There exists an open subgroup $G_\circ\subset G$ such that

(1) $G_\circ$ and $\bar G_\circ := G_\circ\cap\bar G\subset\prod_\ell \mathrm{GL}(H_{\ell^\infty})$ are ℓ-independent;

(2) $\bar G_{\circ,\ell}$ is generated by its order-ℓ elements and has trivial abelianization for $\ell\gg 0$;

(3) For every prime ℓ, the Zariski closures of $G_{\circ,\ell^\infty}$ and $\bar G_{\circ,\ell^\infty}$ in $\mathrm{GL}_{V_{\ell^\infty}}$ are connected;

and there exists an open subgroup $G_{x,\circ}\subset G_\circ\cap G_x$ such that (1′) $G_{x,\circ}\subset\prod_\ell \mathrm{GL}(H_{\ell^\infty})$ is ℓ-independent. □

Theorem 2.1 reduces the problem to an ℓ-adic situation. Indeed, as $G_\circ$, $\bar{G}_\circ$, and $G_{x,\circ}$ are open in G, $\bar{G}$, and G_x respectively, we may assume that $G = G_\circ$, $\bar{G} = \bar{G}_\circ$, and $G_x = G_{x,\circ}$. Then Theorem 1.2 amounts to showing that $G_{x,\ell^\infty} = G_{\ell^\infty}$ for $\ell \gg 0$. But as $G_x\bar{G} \subset G$ is open in G (this follows from the short exact sequence $(*)$ and the fact that the image of σ^x in diagram $(**)$ is open), the morphism

$$G_x \to G/\bar{G} \simeq \prod_\ell G_{\ell^\infty}/\bar{G}_{\ell^\infty}$$

has open image or, equivalently, the morphism $G_{x,\ell^\infty} \to G_{\ell^\infty}/\bar{G}_{\ell^\infty}$ is surjective for $\ell \gg 0$. Thus Theorem 1.2 actually amounts to showing that $\bar{G}_{\ell^\infty} \subset G_{x,\ell^\infty}$ for $\ell \gg 0$. The proof of these inclusions, in turn, decomposes into two steps.

Step 2 : *From ℓ-adic to modulo-ℓ.* Recall that a morphism of profinite groups $\pi : G \to H$ is called a *Frattini cover* if it is surjective and if G contains no strict subgroup mapping surjectively onto H (or, equivalently, $\ker(\pi)$ is contained in the Frattini subgroup $\Phi(G)$ of G).

Theorem 2.2. The restriction of the reduction-modulo-ℓ morphism

$$\bar{G}_{\ell^\infty} \to \bar{G}_\ell$$

is a Frattini cover for $\ell \gg 0$. □

With Theorem 2.2 in hand, it remains to show that the restriction of the reduction-modulo-ℓ morphism

$$\bar{G}_{x,\ell^\infty} := G_{x,\ell^\infty} \cap \bar{G}_{\ell^\infty} \to \bar{G}_\ell$$

remains surjective. This is deduced (Corollary 2.7) from the following

Step 3 : *Modulo-ℓ statement.*

Theorem 2.6. One has $G_{x,\ell} = G_\ell$ for $\ell \gg 0$.

2.2 Step 1: From adelic to ℓ-adic statement and preliminary reductions

2.2.1 *Proof of Theorem* 2.1

The assertions (1), (1′) are special cases of [8, Corollary 4.6] (see also [35] for (1′)) and the assertion (2) is a special case of [8, Corollary 3.3]. The assertions about the Zariski closure of $G_{\circ,\ell^\infty}$ in (3) follow from [33, Theorem p. 15]. The assertion about the Zariski closure of $\bar{G}_{\circ,\ell^\infty}$ in (3) follows from the comparison between Betti and ℓ-adic cohomology (see the beginning of Section 2.3 for more details).

To prove Theorem 1.2 one may freely replace X by a connected étale cover so, from now on, and till the end of Section 2, we assume that the conclusions (1), (1′), (2), (3) of Theorem 2.1 hold for G, $\bar{G}$, and G_x.

2.2.2 *A few more reductions*

Up to replacing again X with a connected étale cover, one may also assume that

$$(4)\ \mathrm{End}(A_\eta) = \mathrm{End}(A_{\bar{\eta}}).$$

The fact that $x \in X_{-}^{\mathrm{gen}}$ implies that for every prime ℓ the groups $\mathcal{G}_{x,\ell^\infty}$ and $\mathcal{G}_{\ell^\infty}$ have the same dimension. In particular, (3) for $\mathcal{G}_{\ell^\infty}$ forces

$$(3')\ \text{For every prime } \ell, \text{ the group } \mathcal{G}_{x,\ell^\infty} \text{ is connected.}$$

The fact that $x \in X_{-}^{\mathrm{gen}}$ also implies that the canonical monomorphism $\mathrm{End}(A_{\bar{\eta}}) \hookrightarrow \mathrm{End}(A_{\bar{x}})$ is an isomorphism [4, Par. after Corollary 5.4]. So, the commutative diagram

$$\begin{array}{ccc} \mathrm{End}(A_{\bar{\eta}}) & = & \mathrm{End}(A_{\bar{x}}) \\ \| & & \uparrow \\ \mathrm{End}(A_\eta) & \hookrightarrow & \mathrm{End}(A_x) \end{array}$$

shows that (4) forces

$$(4')\ \mathrm{End}(A_{\bar{x}}) = \mathrm{End}(A_x) = \mathrm{End}(A_\eta) = \mathrm{End}(A_{\bar{\eta}}).$$

2.3 Step 2: From ℓ-adic to modulo-ℓ

From now on, fix an algebraic closure $k \hookrightarrow \bar{k}$ and a complex embedding $\bar{k} \hookrightarrow \mathbb{C}$. Let $(-)^{an}$ denote the analytification functor and, for $x \in X(\mathbb{C})$, let $\bar{\Pi}$ denote the image of $\pi_1^{top}(X_{\mathbb{C}}^{an}; x)$ acting on the integral Betti cohomology $H := \mathrm{H}_B^1(A_x^{an}, \mathbb{Z})$. Hence $\bar{\Pi} \subset \mathrm{GL}(H) \simeq \mathrm{GL}_{2g}(\mathbb{Z})$ is a *finitely generated* subgroup. Let $\bar{\mathcal{G}}$ denote the Zariski closure of $\bar{\Pi}$ in GL_H. The generic fiber of $\bar{\mathcal{G}}$ is a semi-simple algebraic group [11, Corollary 4.2.9] which we may assume to be connected (after replacing X with a connected étale cover). Hence $\bar{\mathcal{G}}$, together with its tautological representation $\bar{\mathcal{G}} \hookrightarrow \mathrm{GL}_H$, restricts to a semi-simple group scheme over $\mathbb{Z}[\frac{1}{N}]$ for some integer N large enough. By comparison between Betti and ℓ-adic cohomology $\bar{\mathcal{G}}_{\ell^\infty} \hookrightarrow \mathrm{GL}_{V_{\ell^\infty}}$ can be identified with $\bar{\mathcal{G}}_{\mathbb{Q}_\ell} \hookrightarrow \mathrm{GL}_{H\otimes\mathbb{Q}_\ell}$. This identification will be implicit in the following; for instance, by a slight abuse of notation, we will write $\bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)$ instead of $\bar{\mathcal{G}}(\mathbb{F}_\ell)$.

Let

$$\pi_\ell : \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{Z}_\ell) \twoheadrightarrow \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)$$

denote the reduction-modulo-ℓ morphism restricted to $\bar{\mathcal{G}}_{\ell^\infty}(\mathbb{Z}_\ell)$. Then

Theorem 2.3 (Tamagawa). The cover

$$\pi_\ell|_{\bar{G}_{\ell^\infty}} : \bar{G}_{\ell^\infty} \twoheadrightarrow \bar{G}_\ell$$

is a Frattini cover (and $\bar{G}_{\ell^\infty} = \pi_\ell^{-1}(\bar{G}_\ell) \subset \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{Z}_\ell)$ for $\ell \gg 0$). □

Before turning to the proof of Theorem 2.3, recall that given a connected semi-simple algebraic group $\mathcal{G}$ over $\mathbb{Q}$ one has

Fact 2.4. The reduction-modulo-ℓ morphism $\pi_\ell : \mathcal{G}(\mathbb{Z}_\ell) \to \mathcal{G}(\mathbb{F}_\ell)$ is a Frattini cover for $\ell \gg 0$. □

Proof. See for instance [22, Proposition 2.6] or [23, Lemma 16.4.5], where it is proved more precisely that the Frattini subgroup of $\mathcal{G}(\mathbb{Z}_\ell)$ is the kernel of $\pi_\ell : \mathcal{G}(\mathbb{Z}_\ell) \to \mathcal{G}(\mathbb{F}_\ell)$ for $\ell \gg 0$. (Note that [23, Lemma 16.4.5] is formulated in the body of a proof where $\mathcal{G}$ is assumed to be simple, connected, and simply connected but the proof of [23, Lemma 16.4.5] works for arbitrary algebraic subgroups $\mathcal{G} \subset \mathrm{GL}_{r/\mathbb{Q}}$ whose reduction modulo ℓ (for $\ell \gg 0$) is exponentially generated in the sense of [25]. This is always satisfied by connected semi-simple algebraic subgroups $\mathcal{G} \subset \mathrm{GL}_{r,\mathbb{Q}}$.) ■

This applies in particular to $\bar{\mathcal{G}}$. To deduce Theorem 2.3 from Fact 2.4, the key ingredient is the following purely group-theoretical statement.

Lemma 2.5. Consider a commutative diagram of profinite groups with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & G & \stackrel{\alpha}{\longrightarrow} & Q & \longrightarrow & 1 \\
 & & \| & & \uparrow & \square & \uparrow & & \\
1 & \longrightarrow & N & \longrightarrow & \alpha^{-1}(R) & \longrightarrow & R & \longrightarrow & 1
\end{array}$$

Assume furthermore that

1. N is a finitely generated pro-ℓ group;

2. Ω is finite;
3. $\ell \nmid [\Omega : K_\Omega(R)]$, where $K_\Omega(R) := \cap_{q\in\Omega} qRq^{-1} \subset \Omega$ is the largest normal subgroup of Ω contained in R.

Then $\alpha : G \twoheadrightarrow \Omega$ is a Frattini cover if and only if $\alpha : \alpha^{-1}(R) \twoheadrightarrow R$ is a Frattini cover. $\square$

2.3.1 *Proof of Lemma* 2.5 $\Longrightarrow$ *Theorem* 2.3

Given a closed subgroup $\Gamma \subset \mathrm{GL}(H_{\ell^\infty})$, we write $\Gamma(1) \lhd \Gamma$ for the kernel of the reduction modulo ℓ morphism $\pi_\ell : \Gamma \subset \mathrm{GL}(H_{\ell^\infty}) \twoheadrightarrow \mathrm{GL}(H_\ell)$. Consider the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{Z}_\ell)(1) & \longrightarrow & \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{Z}_\ell) & \xrightarrow{\pi_\ell} & \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell) & \longrightarrow & 1 \\
 & & \| & & \uparrow & \square & \uparrow & & \\
1 & \longrightarrow & \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{Z}_\ell)(1) & \longrightarrow & \pi_\ell^{-1}(\bar{G}_\ell) & \xrightarrow{\pi_\ell} & \bar{G}_\ell & \longrightarrow & 1
\end{array}$$

Then, from Fact 2.4, the cover $\pi_\ell : \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{Z}_\ell) \twoheadrightarrow \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)$ is a Frattini cover. Since $\bar{\Pi}$ is finitely generated, one can apply [25, Theorem 5.1] to get

$$\bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)^+ \subset (\bar{\Pi}_\ell =)\bar{G}_\ell \subset \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)$$

for $\ell \gg 0$ (here $\bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)^+ \subset \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)$ denotes the (normal) subgroup of $\bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)$ generated by the order ℓ-elements). As $[\bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell) : \bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)^+] \leq 2^{2g-1}$ [25, Rem. 3.6], the condition $\ell \nmid [\bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell) : K_{\bar{\mathcal{G}}_{\ell^\infty}(\mathbb{F}_\ell)}(\bar{G}_\ell)]$ is satisfied as soon as $\ell > 2^{2g-1}$. Thus one can apply Lemma 2.5 to get that the cover $\pi_\ell : \pi_\ell^{-1}(\bar{G}_\ell) \twoheadrightarrow \bar{G}_\ell$ is a Frattini cover as well. But as $\bar{G}_{\ell^\infty} \subset \pi_\ell^{-1}(\bar{G}_\ell)$ maps surjectively onto $\bar{G}_\ell$, this implies that $\bar{G}_{\ell^\infty} = \pi_\ell^{-1}(\bar{G}_\ell)$, whence the conclusion.

Remark 2.6 (An abstract variant of Theorem 2.3)**.** The only input which is not purely group-theoretic in the proof of Theorem 2.3 is the comparison between Betti and ℓ-adic cohomology, which ensures that $\bar{\mathcal{G}}_{\ell^\infty}$ together with its tautological representation $\bar{\mathcal{G}}_{\ell^\infty} \hookrightarrow \mathrm{GL}_{V_{\ell^\infty}}$ is defined over $\mathbb{Q}$ and is the generic fiber of the Zariski closure of a finitely generated subgroup $\bar{\Pi} \subset \mathrm{GL}(H)$. The proof shows that, more generally, the following holds: *Let $N \geq 1$ be an integer and $\Pi \subset \mathrm{GL}_r(\mathbb{Z}[\frac{1}{N}])$ a finitely generated subgroup. Let $\mathcal{G} \hookrightarrow \mathrm{GL}_{r,\mathbb{Z}[\frac{1}{N}]}$ denote the Zariski closure of Π in $\mathrm{GL}_{r,\mathbb{Z}[\frac{1}{N}]}$ and, for every prime $\ell \nmid N$, let $\Pi_{\ell^\infty} \subset \mathrm{GL}_r(\mathbb{Z}_\ell)$ denote the topological closure of Π in $\mathrm{GL}_r(\mathbb{Z}_\ell)$. For $\ell \nmid N$, let $\pi_\ell : \mathcal{G}(\mathbb{Z}_\ell) \to \mathcal{G}(\mathbb{F}_\ell)$ denote the restriction of the reduction-modulo-ℓ morphism to $\mathcal{G}(\mathbb{Z}_\ell)$. Assume that $\mathcal{G}_\mathbb{Q}$ is connected and semi-simple. Then $\pi_\ell|_{\Pi_{\ell^\infty}} : \Pi_{\ell^\infty} \twoheadrightarrow \Pi_\ell$ is Frattini and, in particular, $\mathcal{G}(\mathbb{Z}_\ell)(1) \subset \Pi_{\ell^\infty}$ for $\ell \gg 0$.* $\square$

2.3.2 *Proof of Lemma* 2.5

For a subgroup $S \subset Q$, let (C_S) denote the property "$\alpha : \alpha^{-1}(S) \twoheadrightarrow S$ is a Frattini cover." Assume that Lemma 2.5 holds if R is normal in Q. Then from the equivalences

$$(C_Q) \Longleftrightarrow (C_{K_Q(R)}) \Longleftrightarrow (C_R)$$

Lemma 2.5 holds for any subgroup R of Q satisfying $\ell \not| [Q : K_Q(R)]$. So, we may assume that R is normal in Q.

Let $\Phi(N) \subset N$ denote the Frattini subgroup of N. As N is pro-ℓ, one has $\Phi(N) = N^\ell[N, N]$. Also, as $\Phi(N)$ is characteristic in N and N is normal in G, $\Phi(N)$ is also normal in G and one can consider the push forward short exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\alpha} & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & N/\Phi(N) & \longrightarrow & G/\Phi(N) & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

Claim 1. $(C_Q) \Longleftrightarrow G/\Phi(N) \twoheadrightarrow Q$ is a Frattini cover. □

Proof of Claim 1. The implication $\Longrightarrow$ is straightforward. As for the implication $\Longleftarrow$, let $S \subset G$ be a subgroup which maps surjectively onto Q. Then $(S\Phi(N))/\Phi(N)$ maps also surjectively onto Q as well hence $S\Phi(N) = G$. But then $N = (S\Phi(N)) \cap N = (S \cap N)\Phi(N)$, which implies $S \cap N = N \subset S$.

Applying Claim 1 to both $G \twoheadrightarrow Q$ and $\alpha^{-1}(R) \twoheadrightarrow R$, one may assume that N is a finite elementary abelian ℓ-group. Then N is endowed with the structure of $\mathbb{F}_\ell[Q]$-module and the extension

$$1 \to N \to G \to Q \to 1$$

corresponds to a cohomology class $\gamma^Q \in \mathrm{H}^2(Q, N)$. ■

Claim 2. (C_Q) is equivalent to any of the following two assertions:

- (C'_Q) For every simple quotient $N \twoheadrightarrow M$ as $\mathbb{F}_\ell[Q]$-module, $\gamma^Q \notin \ker(\mathrm{H}^2(Q, N) \to \mathrm{H}^2(Q, M))$;
- (C''_Q) For every non-zero quotient $N \twoheadrightarrow M$ as $\mathbb{F}_\ell[Q]$-module, $\gamma^Q \notin \ker(\mathrm{H}^2(Q, N) \to \mathrm{H}^2(Q, M))$; □

Proof of Claim 2. $(C_Q) \Longrightarrow (C''_Q)$: Assume that there exists a non-zero quotient $N \twoheadrightarrow M$ such that the push-forward exact sequence splits

$$1 \longrightarrow M \longrightarrow G/\ker(N \twoheadrightarrow M) \underset{\curvearrowleft\, s}{\longrightarrow} Q \longrightarrow 1.$$

By construction, the inverse image of $s(Q)$ in G maps surjectively onto Q. From (C_Q), this implies that the inverse image of $s(Q)$ in G is the whole G hence that $s(Q) = G/\ker(N \twoheadrightarrow M)$, which contradicts the fact that M is non-zero.

$(C'_Q) \Longrightarrow (C_Q)$: Assume that there exists a strict subgroup $S \subsetneq G$ which maps surjectively onto Q. Then $S \cap N \subsetneq N$ is a strict $\mathbb{F}_\ell[Q]$-submodule. In particular, $N/(S \cap N)$ is non-zero. But $S/(S \cap N)$ provides a splitting of the push-forward exact sequence

$$1 \longrightarrow N/(S \cap N) \longrightarrow G/(S \cap N) \longrightarrow Q \longrightarrow 1$$

hence the image $\gamma^Q_{N/(S\cap N)}$ of γ^Q in $\mathrm{H}^2(Q, N/(S \cap N))$ is zero and so, for every simple quotient, $N/(S \cap N) \twoheadrightarrow M$ one also has $\gamma^Q_{N/(S\cap N)} = 0$.

The proof of $(C''_Q) \Longrightarrow (C'_Q)$ is trivial. ■

We now prove Lemma 2.5 assuming that N is a finite elementary abelian ℓ-group. Write $H := \alpha^{-1}(R)$.

$(C_R) \Longrightarrow (C_Q)$: Let $N \twoheadrightarrow M$ be a non-zero quotient as $\mathbb{F}_\ell[Q]$-module. Let $\gamma^R \in \mathrm{H}^2(R, N)$ denote the cohomology class corresponding to the extension

$$1 \to N \to H \to R \to 1.$$

Then γ^R is the image of γ^Q *via* the restriction map $res : \mathrm{H}^2(Q, N) \to \mathrm{H}^2(R, N)$. From Claim 2 $(C_R) \Longrightarrow (C''_R)$, $\gamma^R \notin \ker(\mathrm{H}^2(R, N) \to \mathrm{H}^2(R, M))$. So, the conclusion follows from Claim 2 $(C''_Q) \Longrightarrow (C_Q)$ and the commutativity of the following natural diagram:

$$\begin{array}{ccc} \mathrm{H}^2(Q, N) & \longrightarrow & \mathrm{H}^2(Q, M) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{H}^2(R, N) & \longrightarrow & \mathrm{H}^2(R, M). \end{array}$$

$(C_Q) \Longrightarrow (C_R)$: Let $N \twoheadrightarrow M$ be a simple quotient as $\mathbb{F}_\ell[R]$-module. Set $K := \ker(N \twoheadrightarrow M)$. As R is normal in Q, for every $q \in Q$ the elementary abelian ℓ-group $qK \subset N$ is again an $\mathbb{F}_\ell[R]$-module. Let $M_q := N/qK$ denote the resulting $\mathbb{F}_\ell[R]$-module quotient. One has a canonical isomorphism of abelian groups

$$\begin{aligned} q\cdot :\ & N \tilde{\to} N \\ & n \to qn \end{aligned}$$

which sends K to qK hence induces an isomorphism of abelian groups

$$\begin{aligned} q\cdot :\ & M \tilde{\to} M_q \\ n \bmod K \to & (qn) \bmod qK. \end{aligned}$$

As $q\cdot : N \tilde{\to} N$ is compatible with the group automorphism

$$\begin{aligned} q - q^{-1} :\ & Q \tilde{\to} Q \\ & q' \to qq'q^{-1}, \end{aligned}$$

it induces a canonical isomorphism of cohomology groups

$$q\cdot : \mathrm{H}^2(Q, N) \tilde{\to} \mathrm{H}^2(Q, N)$$

which, actually, is the identity [31, VII, Section 5, Proposition 3]. Similarly, as the induced isomorphism $q\cdot : M \tilde{\to} M_q$ is compatible with the induced group automorphism $q - q^{-1} : R \tilde{\to} R$ (recall that R is normal in Q), it induces a canonical isomorphism of cohomology groups

$$q\cdot : \mathrm{H}^2(R, M) \tilde{\to} \mathrm{H}^2(R, M_q).$$

Restriction and functoriality yield a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(Q, N) & \xrightarrow{q\cdot = Id} & \mathrm{H}^2(Q, N) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{H}^2(R, N) & \xrightarrow{q\cdot} & \mathrm{H}^2(R, N) \\ \downarrow{\scriptstyle p_M} & & \downarrow{\scriptstyle p_{M_q}} \\ \mathrm{H}^2(R, M) & \xrightarrow{q\cdot = Id} & \mathrm{H}^2(R, M_q), \end{array}$$

where $p_M : N \twoheadrightarrow M$, $p_{M_q} : N \twoheadrightarrow M_q$ denote the projection maps. Let $\gamma^R \in \mathrm{H}^2(R, N)$ denote the cohomology class corresponding to the extension

$$1 \to N \to H \to R \to 1$$

and let γ^R_M and $\gamma^R_{M_q}$ denote the image of γ^R in $\mathrm{H}^2(R, M)$ and $\mathrm{H}^2(R, M_q)$, respectively. As $\gamma^R = res(\gamma^Q)$, the commutativity of the above diagram (and the fact that the upper horizontal arrow is the identity) shows that $q\gamma^R_M = \gamma^R_{M_q}$. So $\gamma^R_M \neq 0$ if and only if $\gamma^R_{M_q} \neq 0$. Now, consider the $\mathbb{F}_\ell[Q]$-submodule

$$\tilde{K} := \bigcap_{q \in Q} qK \subset N$$

and the resulting short exact sequence of $\mathbb{F}_\ell[Q]$-modules

$$0 \to \tilde{K} \to N \to \tilde{M} \to 0.$$

By construction, $\tilde{M}$ is a $\mathbb{F}_\ell[R]$-submodule of $\oplus_{q \in Q} M_q$. As every M_q is a simple $\mathbb{F}_\ell[R]$-module, there exists a subset $I \subset Q$ such that $\tilde{M} \simeq \oplus_{q \in I} M_q$ as an $\mathbb{F}_\ell[R]$-module. From the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(Q, N) & \longrightarrow & \mathrm{H}^2(Q, \tilde{M}) \\ \downarrow \scriptstyle{\mathrm{res}} & & \downarrow \scriptstyle{\mathrm{res}} \\ \mathrm{H}^2(R, N) & \longrightarrow & \mathrm{H}^2(R, \tilde{M}) \\ & & \| \\ & & \oplus_{q \in I} \mathrm{H}^2(R, M_q) \end{array}$$

one sees that $\gamma^R_M = 0$ if and only if $\gamma^R_{\tilde{M}} = (\gamma^R_{M_q})_{q \in I} = 0$, which, in turn, is equivalent to $\gamma^Q_{\tilde{M}} = 0$ because the restriction morphism $res : \mathrm{H}^2(Q, \tilde{M}) \hookrightarrow \mathrm{H}^2(R, \tilde{M})$ is injective (this is where we use the assumption that $\ell \nmid [Q : R]$). But this contradicts Claim 2 $(C'_Q) \Longrightarrow (C_Q)$. Whence $\gamma^R_M \neq 0$ and the conclusion follows from Claim 2 $(C''_R) \Longrightarrow (C_R)$.

2.4 Step 3: Proof of the modulo-ℓ statement

The aim of this section is to prove the following theorem.

Theorem 2.7. One has $G_{x,\ell} = G_\ell$ for $\ell \gg 0$. □

Combining Theorem 2.7 and Theorem 2.3, one can then conclude the proof of Theorem 1.2 observing the following corollary.

Corollary 2.8. The restriction of the reduction-modulo-ℓ morphism

$$\bar{G}_{x,\ell^\infty} := \bar{G}_{\ell^\infty} \cap G_{x,\ell^\infty} \to \bar{G}_\ell$$

is surjective for $\ell \gg 0$. □

Proof. Recall that, in the category of profinite groups, the snake lemma holds for a commutative diagram of profinite groups with exact rows

$$\begin{array}{ccccccc} & & G' & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 1 \\ & & {\scriptstyle\alpha'}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha''}\downarrow & & \\ 1 & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & & \end{array}$$

provided the cokernels of α' and α exist. More precisely, if $\mathrm{im}(\alpha')$ is normal in H' and $\mathrm{im}(\alpha)$ is normal in H, then there exists a morphism of profinite groups $\delta : \ker(\alpha'') \to \mathrm{coker}(\alpha')$ such that the following sequence of profinite groups is exact

$$\ker(\alpha') \to \ker(\alpha) \to \ker(\alpha'') \stackrel{\delta}{\to} \mathrm{coker}(\alpha') \to \mathrm{coker}(\alpha).$$

Also, note that if $\alpha : G \to H$ is surjective, then the normality of the image of G' in G automatically implies the normality of $\mathrm{im}(\alpha')$ in H' and the snake lemma applies. We apply these observations twice.

First, consider the commutative diagram of profinite groups with exact rows

$$\begin{array}{ccccccccc} & & \bar{G}_{\ell^\infty} & \longrightarrow & G_{\ell^\infty} & \longrightarrow & G_{\ell^\infty}/\bar{G}_{\ell^\infty} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \bar{G}_\ell & \longrightarrow & G_\ell & \longrightarrow & G_\ell/\bar{G}_\ell & & \end{array}$$

and where the vertical arrows are induced by reduction modulo-ℓ. The left and middle vertical arrows are surjective by definition. So, applying the snake lemma, one obtains that the kernel of $G_{\ell^\infty}/\bar{G}_{\ell^\infty} \to G_\ell/\bar{G}_\ell$ is a pro-ℓ group.

Next, consider the commutative diagram of profinite groups with exact rows

$$\begin{array}{ccccccccc} & & \bar{G}_{x,\ell^\infty} & \longrightarrow & G_{x,\ell^\infty} & \longrightarrow & G_{\ell^\infty}/\bar{G}_{\ell^\infty} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \bar{G}_\ell & \longrightarrow & G_\ell & \longrightarrow & G_\ell/\bar{G}_\ell & & \end{array}$$

and where the vertical arrows are induced by reduction modulo-ℓ. For $\ell \gg 0$ the middle vertical arrow is surjective by Theorem 2.7 and the right arrow in the top horizontal line is surjective by the preliminary reductions (see the discussion at the end of the paragraph Step 1 in Section 2.1). So, applying the snake lemma, one obtains that the kernel of $G_{\ell^\infty}/\bar{G}_{\ell^\infty} \to G_\ell/\bar{G}_\ell$, which is a pro-$\ell$ group, maps surjectively onto the cokernel of $\bar{G}_{x,\ell^\infty} \to \bar{G}_\ell$. But as $\bar{G}_\ell^{ab} = 0$ (recall condition (2) in Theorem 2.1), $\bar{G}_\ell$ has no solvable quotient, which forces the morphism $\bar{G}_{x,\ell^\infty} \to \bar{G}_\ell$ to be surjective. ■

Following ideas of Serre [34] and Nori [25], the strategy to prove Theorem 2.7 is to approximate G_ℓ and $G_{x,\ell}$ by the groups of $\mathbb{F}_\ell$-rational points of certain reductive subgroups $\mathcal{G}_\ell, \mathcal{G}_{x,\ell} \hookrightarrow \mathrm{GL}_{H_\ell}$ and prove that one has $\mathcal{G}_{x,\ell} = \mathcal{G}_\ell$ for $\ell \gg 0$.

2.4.1 *The reductive groups $\mathcal{G}_\ell$, $\mathcal{G}_{x,\ell}$*

To construct $\mathcal{G}_\ell$ and $\mathcal{G}_{x,\ell}$, we construct separately their (common) connected center $C_\ell \hookrightarrow \mathrm{GL}_{H_\ell}$ and their derived subgroups $\mathcal{S}_\ell$, $\mathcal{S}_{x,\ell} \hookrightarrow \mathrm{GL}_{H_\ell}$ in such a way that C_ℓ commutes with $\mathcal{S}_\ell$ and $\mathcal{S}_{x,\ell}$. The construction of C_ℓ relies on Serre's theory of abelian ℓ-adic representation [30]; the fact that C_ℓ commutes with $\mathcal{S}_\ell$ and $\mathcal{S}_{x,\ell}$ is a consequence of the modulo-ℓ Tate conjecture for endomorphisms of abelian varieties and the fact that $\mathcal{S}_\ell$ and $\mathcal{S}_{x,\ell}$ are reductive is a consequence of the modulo-ℓ Tate semi-simplicity conjecture. For the modulo-ℓ Tate conjectures, see for instance [15, IV, Section 4 (Comments after) (4.3) on p. 148].

2.4.1.1 *The tori C_ℓ.* We first define C_ℓ, which will happen to be the (common) connected center of $\mathcal{G}_\ell$ and $\mathcal{G}_{x,\ell}$, to be the 'reduction-modulo-ℓ' of the connected center C_{ℓ^∞} of $\mathcal{G}_{\ell^\infty} = \mathcal{G}_{x,\ell^\infty}$. The construction of C_{ℓ^∞}, due to Serre [34, p. 43] and which we recall now, shows that it is defined over $\mathbb{Q}$ hence that its reduction-modulo-ℓ is well-defined for $\ell \gg 0$.

From Condition (4′) in Section 2.2 $\mathrm{End}(A_\eta) \tilde{\to} \mathrm{End}(A_x) =: D$. Set $L := Z(D) \otimes_{\mathbb{Z}} \mathbb{Q}$. As L is a commutative semi-simple $\mathbb{Q}$-algebra, it decomposes as a product of number fields $(*)$ $L = \prod_{1 \le i \le s} L_i$. Set $T_{L_i} := \mathrm{Res}_{L_i|\mathbb{Q}}(\mathbb{G}_m)$, which is a $[L_i : \mathbb{Q}]$-dimensional torus over

$\mathbb{Q}$ and $T_L := \prod_{1\le i\le s} T_{L_i}$, which is a $[L:\mathbb{Q}]$-dimensional torus over $\mathbb{Q}$ (such that $T_L(\mathbb{Q}) = L^\times$). The decomposition $(*)$ corresponds to the decomposition up to isogeny of A_x into the product of its isotypical factors $A_x \sim \prod_{1\le i\le s} A_{x,i}$. Let $d_{x,i}$ denote the $L_i\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$-rank of $V_\ell(A_{x,i})$, set $\mathrm{GL}_L := \prod_{1\le i\le s}\mathrm{Res}_{L_i|\mathbb{Q}}(\mathrm{GL}_{d_{x,i},L_i})$ and write $\det_L := \mathrm{Res}_{L|\mathbb{Q}}(\det) : \mathrm{GL}_L \to T_L$ for the determinant morphism. The restriction of $\det_L$ to T_L is the isogeny

$$\begin{aligned}\delta:\ T_L &\to T_L\\ (t_1,\ldots,t_s) &\mapsto (t_1^{d_{x,1}},\ldots,t_s^{d_{x,s}}),\end{aligned}$$

For every prime ℓ, consider the abelian ℓ-adic representation

$$\lambda_{x,\ell^\infty} : \Gamma_{k(x)} \xrightarrow{\rho_{x,\ell^\infty}} \mathrm{GL}_L(\mathbb{Q}_\ell) \xrightarrow{\det_L(\mathbb{Q}_\ell)} T_L(\mathbb{Q}_\ell)$$

The $\lambda_{x,\ell^\infty} : \Gamma_{k(x)} \to T_L(\mathbb{Q}_\ell)$ (ℓ: prime) form a compatible family of rational semi-simple abelian ℓ-adic representations; hence, they are induced by some morphism of algebraic groups (with the notation of [30])

$$\lambda_x : S_{\mathfrak{m}_x} \to T_L.$$

Let $T_{\mathfrak{m}_x}$ denote the connected component of $S_{\mathfrak{m}_x}$ and let C_x denote the connected component of the fibre product

$$\begin{array}{ccc}\lambda_x(T_{\mathfrak{m}_x}) \times_{T_L,\delta} T_L & \longrightarrow & T_L\\ \downarrow & \square & \downarrow \delta\\ \lambda_x(T_{\mathfrak{m}_x}) & \hookrightarrow & T_L\end{array}$$

By construction, C_x is a torus over $\mathbb{Q}$ hence reduces modulo ℓ to a torus $C_{x,\ell}$ for $\ell \gg 0$. Also, by construction and the Tate conjecture for endomorphisms of abelian varieties [15], $C_{x,\mathbb{Q}_\ell} = C_{\ell^\infty}$ is the connected center of $\mathcal{G}_{x,\ell^\infty}$ so, from the equality $\mathcal{G}_{x,\ell^\infty} = \mathcal{G}_{\ell^\infty}$, one sees that C_x is independent of x. As a result, we will simply write $C := C_x$ and $C_\ell := C_{x,\ell}$.

2.4.1.2 *The semi-simple groups $\mathcal{S}_\ell$, $\mathcal{S}_{x,\ell}$.* Next, we turn to the definition of the derived subgroups $\mathcal{S}_\ell$, $\mathcal{S}_{x,\ell}$ of $\mathcal{G}_\ell$, $\mathcal{G}_{x,\ell}$.

Given a subgroup $S \subset \mathrm{GL}(H_\ell)$, write $S[\ell]$ for the order-ℓ elements in S and $S^+ \subset S$ for the (normal) subgroup generated by $S[\ell]$ (or equivalently, the subgroup generated by the ℓ-Sylow subgroups of S as soon as $\ell \ge 2g$). Following [25], define $\tilde{S} \hookrightarrow \mathrm{GL}_{H_\ell}$ to be the algebraic subgroup generated by the 1-parameter subgroups

$$e_g:\ \mathbb{A}^1_{\mathbb{F}_\ell}\to \mathrm{GL}_{H_\ell}$$
$$t\mapsto \exp(t\cdot\log(g))$$

associated to each $g\in S[\ell]$.

The following gathers the main results of [25].

Fact 2.9.

(1) $\tilde{S}$ is connected and generated by its unipotent elements and, if S acts semi-simply on H_ℓ, $\tilde{S}$ is semi-simple;
(2) For $\ell\gg 0$ (depending only on $2g$) one has $S^+=\tilde{S}(\mathbb{F}_\ell)^+$;
(3) The quotient $\tilde{S}(\mathbb{F}_\ell)/\tilde{S}(\mathbb{F}_\ell)^+$ is abelian of order $\leq 2^{2g-1}$. □

By the modulo-ℓ semi-simplicity Tate conjecture [15], G_ℓ (hence its normal subgroup $\bar{G}_\ell$) and $G_{x,\ell}$ act semi-simply on H_ℓ for $\ell\gg 0$ hence the groups

$$\bar{\mathcal{S}}_\ell:=\tilde{\bar{G}}_\ell,\quad \mathcal{S}_\ell:=\tilde{G}_\ell\quad\text{and}\quad \mathcal{S}_{x,\ell}:=\tilde{G}_{x,\ell}$$

are connected semi-simple algebraic groups for $\ell\gg 0$. Also, from Theorem 2.1(2), one has $\bar{G}_\ell=\bar{G}_\ell^+=\bar{\mathcal{S}}_\ell(\mathbb{F}_\ell)^+$ for $\ell\gg 0$.

2.4.1.3 *The reductive groups $\mathcal{G}_\ell$, $\mathcal{G}_{x,\ell}$.* From now on and till the end of this section, assume that ℓ is large enough so that C has good reduction modulo ℓ and that the groups $\bar{\mathcal{S}}_\ell$, $\mathcal{S}_\ell$ and $\mathcal{S}_{x,\ell}$ are semi-simple.

By construction, the connected semi-simple groups $\mathcal{S}_\ell$ and $\mathcal{S}_{x,\ell}$ commute with C_ℓ, hence it makes sense to define the reductive groups

$$\mathcal{G}_\ell:=\mathcal{S}_\ell C_\ell\quad\text{and}\quad \mathcal{G}_{x,\ell}:=\mathcal{S}_{x,\ell}C_\ell.$$

Furthermore, one has the following results, due to Serre (here, we implicitly use the fact that the k is a number field).

Fact 2.10 ([34, Theorem 1,2; 33, Section 3])**.**

(1) $\mathrm{rank}(\mathcal{G}_{x,\ell})=\mathrm{rank}(\mathcal{G}_{x,\ell^\infty})$ for $\ell\gg 0$ and, in particular, $\mathrm{rank}(\mathcal{G}_{x,\ell})$ is independent of ℓ for $\ell\gg 0$.
(2) There exists an integer (independent of x—though this is not explicitly stated in [34, Thm. 1], this can be seen from the proof) $d(g)\geq 1$ and a finite extension

k_x of $k(x)$ such that one has $\rho_{x,\ell}(\Gamma_{k_x}) \subset \mathcal{G}_{x,\ell}(\mathbb{F}_\ell)$ and $[\mathcal{G}_{x,\ell}(\mathbb{F}_\ell) : \rho_{x,\ell}(\Gamma_{k_x})] \leq d(g)$ for $\ell \gg 0$. □

2.4.2 *Approximation*

In this section, we compare G_ℓ with $\mathcal{G}_\ell(\mathbb{F}_\ell)$. For our purpose, the following lemma will be enough (but see Remark 2.13 for a more precise statement).

Lemma 2.11. There exist integers $\delta', \delta \geq 1$ such that for every prime ℓ one has

$$[\mathcal{G}_\ell(\mathbb{F}_\ell) : G_\ell \cap \mathcal{G}_\ell(\mathbb{F}_\ell)] \leq \delta, \quad [G_\ell : G_\ell \cap \mathcal{G}_\ell(\mathbb{F}_\ell)] \leq \delta'.$$ □

Proof. It is enough to prove that $[\mathcal{G}_\ell(\mathbb{F}_\ell) : G_\ell \cap \mathcal{G}_\ell(\mathbb{F}_\ell)]$ and $[G_\ell : G_\ell \cap \mathcal{G}_\ell(\mathbb{F}_\ell)]$ are bounded from above independently of ℓ for $\ell \gg 0$ (and then take $\delta := \max\{[\mathcal{G}_\ell(\mathbb{F}_\ell) : G_\ell \cap \mathcal{G}_\ell(\mathbb{F}_\ell)] \mid \ell\text{: prime}\}$, $\delta' := \max\{[G_\ell : G_\ell \cap \mathcal{G}_\ell(\mathbb{F}_\ell)] \mid \ell\text{: prime}\}$). To do so, we may freely replace X with a connected étale cover. In particular, we may replace X with X_{k_x} (where k_x is the finite extension of k introduced in Fact 2.10(2)) hence assume that $G_{x,\ell} \subset \mathcal{G}_{x,\ell}(\mathbb{F}_\ell)$ and $[\mathcal{G}_{x,\ell}(\mathbb{F}_\ell) : G_{x,\ell}] \leq d(g)$ for all but finitely many primes ℓ. Set $d_x := [k(x) : k]$. We assume that ℓ is large enough so that all the reduction carried out up to now hold and, also, that $\ell > d_x, 2g-1$. We proceed into several steps.

(1) *Claim* $1 : G_\ell^+ = \bar{G}_\ell G_{x,\ell}^+$ (in particular, $\bar{G}_\ell G_{x,\ell}^+$ is normal in G_ℓ).

Proof of Claim 1. Recall that $\bar{G}_\ell^+ = \bar{G}_\ell$. So one always has $G_\ell^+ \supset \bar{G}_\ell G_{x,\ell}^+$. For the converse inclusion, let $g \in G_\ell[\ell] \setminus \bar{G}_\ell[\ell]$. The image $\bar{g}$ of g in $Q_\ell := G_\ell/\bar{G}_\ell$ has order ℓ. On the other hand, the image $Q_{x,\ell}$ of $G_{x,\ell}$ in Q_ℓ has index $\leq d_x$. Since the orbits of $\bar{g}$ acting on $Q_\ell/Q_{x,\ell}$ have length 1 or ℓ, this forces $\bar{g} \in Q_{x,\ell}$ (recall that $\ell > d_x$) that is there exist $\gamma \in \bar{G}_\ell$ and $g_x \in G_{x,\ell}$ such that $g = g_x\gamma$. As $\bar{g} = \bar{g}_x$ is of order ℓ, ℓ divides the order of g_x and $g_x^\ell \in \bar{G}_\ell$. Write $|\langle g_x\rangle| = q_x\ell$ with $\ell \nmid q_x$ (recall that the assumption that $\ell \geq 2g$ implies that $\mathrm{GL}(H_\ell)$ contains no element of order ℓ^2). There exists $u, v \in \mathbb{Z}$ such that $u\ell + vq_x = 1$ hence

$$g = (g_x^{q_x})^v (g_x^\ell)^u \gamma$$

with $(g_x^{q_x})^v \in G_{x,\ell}[\ell]$ and $(g_x^\ell)^u\gamma \in \bar{G}_\ell$. ■

(2) *Claim* 2 : Let $S, T \hookrightarrow \mathrm{GL}_{r,\mathbb{F}_\ell}$ be algebraic subgroups such that S is semi-simple, S, T commute and the morphism

$$S \times T \to S \cdot T \hookrightarrow \mathrm{GL}_{r,\mathbb{F}_\ell}$$

is an isogeny. Then

$$[(ST)(\mathbb{F}_\ell) : S(\mathbb{F}_\ell)T(\mathbb{F}_\ell)] \leq 2^{r-1}.$$

Proof of Claim 2. Let $K \hookrightarrow S \times T$ denote the kernel of $S \times T \to S \cdot T$. From the exact sequence of flat sheaves

$$1 \to K \to S \times T \to S \cdot T \to 1$$

one obtains an exact sequence

$$1 \to K(\mathbb{F}_\ell) \to S(\mathbb{F}_\ell) \times T(\mathbb{F}_\ell) = (S \times T)(\mathbb{F}_\ell) \stackrel{(1)}{\to} (ST)(\mathbb{F}_\ell) \to \mathrm{H}^1_{fl}(\mathbb{F}_\ell, K).$$

As K is a finite, commutative group scheme over $\mathbb{F}_\ell$, it has a connected-étale decomposition

$$K = K^\circ \times K^{et}.$$

From the exact sequence of flat sheaves

$$1 \to K^{et} \to S \times T/K^\circ \to S \cdot T \to 1$$

one obtains an exact sequence

$$1 \to K^{et}(\mathbb{F}_\ell) \to (S \times T/K^\circ)(\mathbb{F}_\ell) \stackrel{(2)}{\to} (ST)(\mathbb{F}_\ell) \to \mathrm{H}^1_{fl}(\mathbb{F}_\ell, K^{et}).$$

But, as K^{et} is smooth over $\mathbb{F}_\ell$, one has

$$\mathrm{H}^1_{fl}(\mathbb{F}_\ell, K^{et}) = \mathrm{H}^1(\mathbb{F}_\ell, K^{et}(\bar{\mathbb{F}}_\ell)),$$

which, by the cohomology of cyclic groups (e.g. [31, Proposition 1 and Remark p. 197]), has order dividing $|K^{et}(\mathbb{F}_\ell)|$. But, by definition, K is a subgroup of the center of S, which is a semi-simple subgroup of $\mathrm{GL}_{r/\mathbb{F}_\ell}$ hence $|K^{et}(\mathbb{F}_\ell)| \leq 2^{r-1}$ (e.g. [25, p. 270]).

On the other hand, the morphism $S \times T \to S \times T/K^\circ$ is radicial hence induces an isomorphism onto $\mathbb{F}_\ell$-points. This shows that the morphisms (1) and (2) have the same image.

(3) As $\mathcal{S}_\ell$ is a connected semi-simple algebraic group and $\bar{\mathcal{S}}_\ell \hookrightarrow \mathcal{S}_\ell$ is a normal semi-simple algebraic subgroup, one has

$$\mathcal{S}_\ell = \bar{\mathcal{S}}_\ell \cdot \mathcal{S}'_\ell,$$

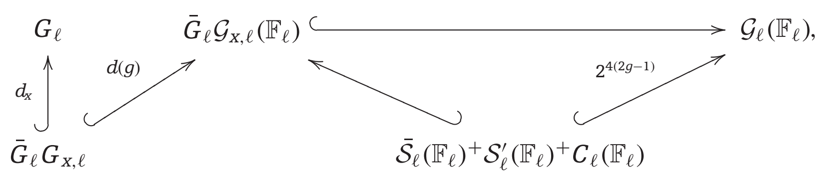
where $\mathcal{S}'_\ell \hookrightarrow \mathcal{S}_\ell$ is a normal semi-simple algebraic subgroup commuting with $\bar{\mathcal{S}}_\ell$. From Claim 1, one has

$$\bar{G}_\ell \mathcal{S}'_\ell(\mathbb{F}_\ell)^+ = \bar{\mathcal{S}}_\ell(\mathbb{F}_\ell)^+ \mathcal{S}'_\ell(\mathbb{F}_\ell)^+ \subset \mathcal{S}_\ell(\mathbb{F}_\ell)^+ = G_\ell^+ = \bar{G}_\ell G_{x,\ell}^+ = \bar{G}_\ell \mathcal{S}_{x,\ell}(\mathbb{F}_\ell)^+$$

(4) Consider the following diagram of inclusions:

$$\begin{aligned} G_\ell \supset \bar{G}_\ell G_{x,\ell} \subset \bar{G}_\ell \mathcal{G}_{x,\ell}(\mathbb{F}_\ell) \supset \bar{G}_\ell \mathcal{S}_{x,\ell}(\mathbb{F}_\ell) C_\ell(\mathbb{F}_\ell) \supset \bar{G}_\ell \mathcal{S}_{x,\ell}(\mathbb{F}_\ell)^+ C_\ell(\mathbb{F}_\ell) \\ \supset \bar{G}_\ell \mathcal{S}'_\ell(\mathbb{F}_\ell)^+ C_\ell(\mathbb{F}_\ell) = \bar{\mathcal{S}}_\ell(\mathbb{F}_\ell)^+ \mathcal{S}'_\ell(\mathbb{F}_\ell)^+ C_\ell(\mathbb{F}_\ell) \subset \bar{\mathcal{S}}_\ell(\mathbb{F}_\ell) \mathcal{S}'_\ell(\mathbb{F}_\ell) C_\ell(\mathbb{F}_\ell) \subset \mathcal{G}_\ell(\mathbb{F}_\ell). \end{aligned}$$

and just compute the successive index using Claim 2, $[\mathcal{G}_{x,\ell}(\mathbb{F}_\ell) : G_{x,\ell}] \leq d(g)$ (Fact 2.10) and the fact that for a semi-simple subgroup $\mathcal{S} \hookrightarrow \mathrm{GL}_{2g\mathbb{F}_\ell}$ one always has $[\mathcal{S}(\mathbb{F}_\ell) : \mathcal{S}(\mathbb{F}_\ell)^+] \leq 2^{2g-1}$. More precisely, one has



where the figures on the arrows denote the maximal possible index. In particular, one has

$$[\mathcal{G}_\ell(\mathbb{F}_\ell) : G_\ell \cap \mathcal{G}_\ell(\mathbb{F}_\ell)] \leq d(g) 2^{4(2g-1)}, \;\; [G_\ell : G_\ell \cap \mathcal{G}_\ell(\mathbb{F}_\ell)] \leq d_x.$$

This concludes the proof of Lemma 2.11. ■

2.4.3 *The equality* $\mathcal{G}_{x,\ell} = \mathcal{G}_\ell$

Lemma 2.12. One has $\mathcal{G}_{x,\ell} = \mathcal{G}_\ell$ for $\ell \gg 0$. □

Proof. If one replaces k by a finite field extension k' then $\mathcal{G}_{x,\ell}$, $\mathcal{G}_\ell$ remain unchanged for $\ell \gg 0$ (more precisely, $\mathcal{G}_{x,\ell^\infty}$ hence C_ℓ remains unchanged and $G_{x,\ell}^+$, G_ℓ^+ hence $\mathcal{S}_{x,\ell}$, $\mathcal{S}_\ell$ remain unchanged for $\ell > [k' : k]$) so we may assume that $k = k(x) = k_x$ and that $G_{x,\ell} \subset \mathcal{G}_{x,\ell}(\mathbb{F}_\ell)$ (Fact 2.10(2)).

As $\mathcal{G}_{x,\ell} \subset \mathcal{G}_\ell$ are connected reductive groups, one can apply the Borel-de Siebenthal Theorem ($\ell \neq 2, 3$) [16, 26]. For this, one has to show that

(i) $\mathrm{rank}(\mathcal{G}_{x,\ell}) = \mathrm{rank}(\mathcal{G}_\ell)$;

(ii) $Z(\mathcal{G}_{x,\ell}) = Z(\mathcal{G}_\ell)$,

where here, given a group G, we let $Z(G)$ denotes its center. Note that, if (i) holds, the inclusion $\mathcal{G}_{x,\ell} \subset \mathcal{G}_\ell$ implies that $Z(\mathcal{G}_\ell) \subset Z(\mathcal{G}_{x,\ell})$ (consider a maximal torus of $\mathcal{G}_\ell$ contained in $\mathcal{G}_{x,\ell}$). Then (ii) amounts to showing

$$\text{(ii}'\text{)} \ \mathrm{End}_{\mathcal{G}_{x,\ell}}(H_\ell) \subset \mathrm{End}_{\mathcal{G}_\ell}(H_\ell). \qquad \blacksquare$$

Proof of (i). As $x \notin X^{ex}$ one has $\mathrm{rank}(\mathcal{G}_{x,\ell^\infty}) = \mathrm{rank}(\mathcal{G}_{\ell^\infty}) =: \rho$. Also, recall that we chose $\ell \gg 0$ so that $\mathrm{rank}(\mathcal{G}_{x,\ell}) = \mathrm{rank}(\mathcal{G}_{x,\ell^\infty}) = \rho$. As a result, it is enough to show that $\mathrm{rank}(\mathcal{G}_\ell) = \rho$ for $\ell \gg 0$. As $\mathcal{G}_{x,\ell} \subset \mathcal{G}_\ell$, one already has $\rho \leq \mathrm{rank}(\mathcal{G}_\ell)$, so it is enough to prove that $\mathrm{rank}(\mathcal{G}_\ell) < \rho + 1$ for $\ell \gg 0$. But this is, somewhat, the easy inequality of [34, Section 6] and one can use the same argument. First, the image of $\mathcal{G}_{x,\ell^\infty} = \mathcal{G}_{\ell^\infty}$ by the characteristic polynomial morphism

$$ch : \mathrm{GL}_{H_{\ell^\infty}} \to \mathbb{A}^{2g-1}_{\mathbb{Q}_\ell} \times \mathbb{G}_{m,\mathbb{Q}_\ell}$$

is the base-change of a subvariety P defined over $\mathbb{Q}$, with dimension ρ and which is independent of ℓ. In particular, P extends to a subvariety over a non-empty open subscheme of $\mathrm{spec}(\mathbb{Z})$ hence has good reduction for $\ell \gg 0$. Write P_ℓ for the reduction of P modulo ℓ.

Assume that $\rho_\ell = \mathrm{rank}(\mathcal{G}_\ell) \geq \rho + 1$. Let $T_\ell \hookrightarrow \mathcal{G}_\ell$ be a maximal torus. Then

$$|T_\ell(\mathbb{F}_\ell)| \geq (\ell - 1)^\rho \geq \frac{\ell^{\rho+1}}{2^{\rho+1}}$$

[25, Lemma 3.5] and

$$[T_\ell(\mathbb{F}_\ell) : T_\ell(\mathbb{F}_\ell) \cap G_\ell] = [T_\ell(\mathbb{F}_\ell) : T_\ell(\mathbb{F}_\ell) \cap (\mathcal{G}_\ell(\mathbb{F}_\ell) \cap G_\ell)] \leq [\mathcal{G}_\ell(\mathbb{F}_\ell) : G_\ell \cap \mathcal{G}_\ell(\mathbb{F}_\ell)] \leq \delta.$$

Consequently

$$|T_\ell(\mathbb{F}_\ell) \cap G_\ell| \geq \frac{\ell^{\rho+1}}{2^{\rho+1}\delta}.$$

And as the characteristic polynomial morphism

$$ch_{\mathbb{F}_\ell} : T_\ell \to \mathbb{A}^{2g-1}_{\mathbb{F}_\ell} \times \mathbb{G}_{m\mathbb{F}_\ell}$$

is finite over its image, of degree at most $(2g)!$ one gets

$$|ch_{\mathbb{F}_\ell}(G_\ell)| \geq |ch_{\mathbb{F}_\ell}(T_\ell(\mathbb{F}_\ell) \cap G_\ell)| \geq \frac{|T_\ell(\mathbb{F}_\ell) \cap G_\ell|}{(2g)!} \geq \frac{\ell^{\rho+1}}{(2g)!\delta 2^{\rho+1}}.$$

But, by definition, $ch_{\mathbb{F}_\ell}(G_\ell) = ch(G_{\ell^\infty}) \bmod \ell \subset P_\ell(\mathbb{F}_\ell)$. Thus $|ch_{\mathbb{F}_\ell}(G_\ell)| \leq |P_\ell(\mathbb{F}_\ell)| = O(\ell^\rho)$, which yields the contradiction. (Note that $G_{\ell^\infty} \subset \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$ is compact hence conjugate in $\mathrm{GL}_{2g}(\mathbb{Q}_\ell)$ to a subgroup of $\mathrm{GL}_{2g}(\mathbb{Z}_\ell)$. As the characteristic polynomial morphism is invariant by conjugacy, this implies $ch(G_{\ell^\infty}) \subset P(\mathbb{Z}_\ell)$.) ■

Proof of (ii′). By the modulo-ℓ variant of the Tate conjecture for endomorphisms of abelian varieties [15] (and Condition (4′) in Section 2.2) one has, for $\ell \gg 0$

$$\mathrm{End}_{G_\ell}(H_\ell) = \mathrm{End}(A_\eta) \otimes_{\mathbb{Z}} \mathbb{F}_\ell = \mathrm{End}(A_x) \otimes_{\mathbb{Z}} \mathbb{F}_\ell = \mathrm{End}_{G_{x,\ell}}(H_\ell).$$

On the other hand, by construction

$$\mathrm{End}_{\mathcal{G}_\ell}(H_\ell) \supset \mathrm{End}(A_\eta) \otimes_{\mathbb{Z}} \mathbb{F}_\ell = \mathrm{End}(A_x) \otimes_{\mathbb{Z}} \mathbb{F}_\ell \subset \mathrm{End}_{\mathcal{G}_{x,\ell}}(H_\ell).$$

But, since $G_{x,\ell} \subset \mathcal{G}_{x,\ell}(\mathbb{F}_\ell)$, one also has $\mathrm{End}_{\mathcal{G}_{x,\ell}}(H_\ell) \subset \mathrm{End}_{G_{x,\ell}}(H_\ell)$ hence $\mathrm{End}_{\mathcal{G}_{x,\ell}}(H_\ell) = \mathrm{End}_{G_{x,\ell}}(H_\ell)$ for $\ell \gg 0$. ■

2.4.4 *End of the proof of Theorem* 2.7

From the preliminary reductions (see the discussion at the end of Step 1 in Section 2.1), $G_\ell = G_{x,\ell}\bar{G}_\ell$ and $\bar{G}_\ell = \bar{G}_\ell^+$ for $\ell \gg 0$ so it is enough to show that $(\bar{G}_\ell^+ =)\bar{G}_\ell \subset G_{x,\ell}$ for $\ell \gg 0$. But this follows from Fact 2.9(2) and Lemma 2.12 since

$$\begin{array}{ccccc} & \mathcal{G}_\ell(\mathbb{F}_\ell) & = & \mathcal{G}_{x,\ell}(\mathbb{F}_\ell) & \\ & \cup & & \cup & \\ & \mathcal{G}_\ell(\mathbb{F}_\ell)^+ & = & \mathcal{G}_{x,\ell}(\mathbb{F}_\ell)^+ & \\ & \| & & \| & \\ \bar{G}_\ell = \bar{G}_\ell^+ \subset & G_\ell^+ & = & G_{x,\ell}^+ & \subset G_{x,\ell}. \end{array}$$

Remark 2.13 (A corollary of the proof)**.** From Theorem 2.7, up to replacing X with a connected étale cover, for every closed point $x \notin X^{ex}$ and for all but finitely many primes ℓ one has $G_{x,\ell} = G_\ell$. But then, from Fact 2.10 (2) and Lemma 2.12, fixing $x \notin X^{ex}$ and up to replacing X with X_{k_x}, one has $G_\ell \subset \mathcal{G}_\ell(\mathbb{F}_\ell)$ and $[\mathcal{G}_\ell(\mathbb{F}_\ell) : G_\ell]$ bounded from above independently of ℓ. In other words, we have improved Lemma 2.11 to get the exact analogue of Fact 2.10 (2) for finitely generated fields of characteristic 0: Up to replacing X with a connected étale cover, one has $G_\ell \subset \mathcal{G}_\ell(\mathbb{F}_\ell)$ for $\ell \gg 0$ and $[\mathcal{G}_\ell(\mathbb{F}_\ell) : G_\ell]$ is bounded from above independently of ℓ. (To reduce the case of representations of the absolute Galois group to the case of representations of the étale fundamental group, use that if A is an abelian variety over the function field $k(X)$ of a smooth, separated and geometrically connected scheme X over k then it extends to an abelian scheme over a non-empty open subscheme $U \subset X$ and the Galois representation of $\Gamma_{k(X)}$ onto $A[\ell]$ factors through $\Gamma_{k(X)} \twoheadrightarrow \pi_1(U)$.) □

2.5 Interlude on Mumford–Tate group versus étale fundamental group

The method used to prove Theorem 1.2 also shows that, *provided the Mumford–Tate conjecture holds*, the following refined adelic Mumford–Tate conjecture (Theorem 2.14) holds as well. Before stating it, let us recall the statement of the Mumford–Tate conjecture.

Let k be a number field, $k \hookrightarrow \mathbb{C}$ a complex embedding and A an abelian variety over k.

Let $H := \mathrm{H}^1(A^{an}_{\mathbb{C}}, \mathbb{Z})$ and $V := H \otimes \mathbb{Q}$ denote the integral and rational Betti cohomology. Similarly, let $H_{\ell^\infty} := \mathrm{H}^1(A_{\bar{k}}, \mathbb{Z}_\ell)$ and $V_{\ell^\infty} := H_{\ell^\infty} \otimes \mathbb{Q}_\ell$ denote the integral and rational ℓ-adic cohomology.

The Mumford–Tate group $\mathcal{M}_A \hookrightarrow \mathrm{GL}_V$ of A is the Galois group of the Tannakian category generated by V in the category of $\mathbb{Q}$-mixed Hodge structures. It is a connected reductive algebraic group over $\mathbb{Q}$. Let $\mathcal{M}_A^{ad}$ denote the adjoint group of $\mathcal{M}_A$; it is a connected semi-simple algebraic group over $\mathbb{Q}$. Write again $\mathcal{M}_A \hookrightarrow \mathrm{GL}_H$ for the Zariski-closure of $\mathcal{M}_A$ in GL_H. This is a connected reductive group scheme over $\mathbb{Z}[\frac{1}{N}]$ (for some integer $N \gg 0$) and, up to increasing N, $\mathcal{M}_A^{ad}$ and $\mathcal{M}_A \to \mathcal{M}_A^{ad}$ extend to a connected semi-simple group scheme and a morphism of group schemes over $\mathbb{Z}[\frac{1}{N}]$.

The Zariski closure $\mathcal{G}_{\ell^\infty} \hookrightarrow \mathrm{GL}_{V_{\ell^\infty}}$ of the image of the ℓ-adic representation $\Gamma_k \to \mathrm{GL}(V_{\ell^\infty})$ is the Galois group of the Tannakian category generated by V_{ℓ^∞} in the category of finite-dimensional continuous $\mathbb{Q}_\ell$-representations of Γ_k. It is a (not necessarily connected) reductive algebraic group over $\mathbb{Q}_\ell$.

The Mumford–Tate conjecture—a by-product of the Hodge and the Tate conjectures— predicts that modulo the comparison isomorphism $V \otimes \mathbb{Q}_\ell \simeq V_{\ell^\infty}$ between Betti and ℓ-adic cohomology, one has $\mathcal{G}^\circ_{\ell^\infty} = \mathcal{M}_{A,\mathbb{Q}_\ell}$. The inclusion $\mathcal{G}^\circ_{\ell^\infty} \subset \mathcal{M}_{A,\mathbb{Q}_\ell}$ is known [3, 14, 27]. Hence, possibly after replacing k with a finite extension, we may and will assume that $\mathcal{G}_{\ell^\infty} = \mathcal{G}^\circ_{\ell^\infty} \subset \mathcal{M}_{A,\mathbb{Q}_\ell}$ for every prime ℓ so that the adelic representation

$$\Gamma_k \to \prod_\ell \mathrm{GL}(H_{\ell^\infty})$$

factors through

$$\Gamma_k \to \mathcal{M}_A(\hat{\mathbb{Z}}) \left(= \prod_\ell \mathrm{GL}(H_{\ell^\infty}) \cap \mathcal{M}_A(\mathbb{Q}_\ell) \right).$$

Theorem 2.14. Assume the Mumford–Tate conjecture holds for A. Then the image of the adelic representation

$$\Gamma_k \to \mathcal{M}_A(\hat{\mathbb{Z}}) \to \mathcal{M}_A^{ad}(\hat{\mathbb{Z}})$$

contains $\prod_{\ell \gg 0} \mathcal{M}_A^{ad}(\mathbb{Z}_\ell)^+$, where, given a ℓ-adic Lie group M, we write $M^+ \subset M$ for the (normal open) subgroup generated by the ℓ-Sylow subgroups of M. □

Proof (sketch of). Let G_{ℓ^∞} and G_ℓ denote the images of the ℓ-adic and modulo ℓ-representations $\Gamma_k \to \mathcal{M}_A(\mathbb{Z}_\ell)$ and $\Gamma_k \to \mathcal{M}_A(\mathbb{F}_\ell)$, respectively. To prove Theorem 2.14, one may freely replace k by a finite field extension hence assume that the image G of the adelic representation $\Gamma_k \to \mathcal{M}_A(\widehat{\mathbb{Z}})$ satisfies $G = \prod_\ell G_{\ell^\infty}$. If the Mumford–Tate conjecture holds for A, $\mathcal{G}_{\ell^\infty}$ coincides with $\mathcal{M}_{A,\mathbb{Q}_\ell}$. Let $\mathcal{G}_\ell$ denote the reductive group constructed from G_ℓ as in Section 2.4.1. Eventually, let $\mathcal{D}_A$ denote the derived subgroup of $\mathcal{M}_A$. Using that $\mathcal{M}_A$ is an extension of $\mathcal{D}_A$ by a torus and that, for $\ell \gg 0$ (so that it makes sense), the reduction $\mathcal{D}_{A,\mathbb{F}_\ell}$ of $\mathcal{D}_A$ modulo ℓ is exponentially generated, one obtains that $\tilde{\mathcal{M}_A}(\mathbb{F}_\ell) = \mathcal{D}_{A,\mathbb{F}_\ell}$ hence $\mathcal{G}_\ell \subset \mathcal{M}_{A,\mathbb{F}_\ell}$. We would like to show that $\mathcal{G}_\ell = \mathcal{M}_{A,\mathbb{F}_\ell}$. For this, one can again apply the Borel-de Siebenthal Theorem as in the proof of Lemma 2.12. Namely, one has

$$\mathrm{rank}(\mathcal{G}_\ell) = \mathrm{rank}(\mathcal{G}_{\ell^\infty}) = \mathrm{rank}(\mathcal{M}_A) = \mathrm{rank}(\mathcal{M}_{A,\mathbb{F}_\ell}), \quad \ell \gg 0$$

hence $Z(\mathcal{M}_{A,\mathbb{F}_\ell}) \subset Z(\mathcal{G}_\ell)$. So it is enough to show that $\mathrm{End}_{\mathcal{G}_\ell}(H_\ell) \subset \mathrm{End}_{\mathcal{M}_{A,\mathbb{F}_\ell}}(H_\ell)$. But after replacing k with a finite extension, $G_\ell \subset \mathcal{G}_\ell(\mathbb{F}_\ell)$ hence

$$\mathrm{End}_{\mathcal{G}_\ell}(H_\ell) \subset \mathrm{End}_{G_\ell}(H_\ell) = \mathrm{End}(A) \otimes \mathbb{F}_\ell = \mathrm{End}_{\mathcal{M}_A}(H) \otimes \mathbb{F}_\ell = \mathrm{End}_{\mathcal{M}_{A,\mathbb{F}_\ell}}(H_\ell), \quad \ell \gg 0.$$

The equality $\mathcal{G}_\ell = \mathcal{M}_{A,\mathbb{F}_\ell}$ ensures that $G_\ell^+ = \mathcal{D}_A(\mathbb{F}_\ell)^+$ maps onto $\mathcal{M}_A^{ad}(\mathbb{F}_\ell)^+$. So, if $G_{\ell^\infty}^{ad}$ denotes the image of Γ_k in $\mathcal{M}_A^{ad}(\mathbb{Z}_\ell)$, then $G_{\ell^\infty}^{ad,+}$ maps onto $\mathcal{M}_A^{ad}(\mathbb{F}_\ell)^+$ hence, applying Lemma 2.5 with $\pi_\ell : \mathcal{M}_A^{ad}(\mathbb{Z}_\ell) \to \mathcal{M}_A^{ad}(\mathbb{F}_\ell)$ for $\alpha : G \to Q$ and $\mathcal{M}_A^{ad}(\mathbb{F}_\ell)^+ \hookrightarrow \mathcal{M}_A^{ad}(\mathbb{F}_\ell)$ for $R \hookrightarrow Q$, one obtains that

$$G_{\ell^\infty}^{ad} \supset G_{\ell^\infty}^{ad,+} = \pi_\ell^{-1}(\mathcal{M}_A^{ad}(\mathbb{F}_\ell)^+) = \mathcal{M}_A^{ad}(\mathbb{Z}_\ell)^+.$$ ■

Remark 2.15. Theorem 2.14 is the best general statement one can expect for the (adjoint) adelic Mumford–Tate conjecture since one can construct examples of an abelian variety A over a number field k (obtained by specializing universal abelian schemes over *ad* *hoc* Shimura varieties) such that the image of $\Gamma_k \to \mathcal{M}_A^{ad}(\mathbb{Z}_\ell)$ is contained in $\mathcal{M}_A^{ad}(\mathbb{Z}_\ell)^+$ and $\mathcal{M}_A^{ad}(\mathbb{Z}_\ell)^+ \subsetneq \mathcal{M}_A^{ad}(\mathbb{Z}_\ell)$ (hence, *a fortiori*, the image of $\Gamma_k \to \mathcal{M}_A^{ad}(\mathbb{Z}_\ell)$ is not equal to $\mathcal{M}_A^{ad}(\mathbb{Z}_\ell)$) for $\ell \gg 0$. □

3 Adelic Open Image Theorem for 1-Motives

In this last section, we extend Theorem 1.2 to the adelic representation attached to a 1-motive. We retain our convention about k and X.

Let $M := [Y \xrightarrow{u} G]$ be a (torsion-free) 1-motive over X, that is a complex of group schemes over X concentrated in degree -1, 0, where G is a semi-abelian group scheme, extension of an abelian scheme A (which, to simplify, we will assume to be *non-zero*) by a torus T and Y is étale locally isormorphic to $\mathbb{Z}^{\oplus r}$ (see [13, Section 10]). To M is attached a ℓ-adic Tate module $T_\ell(M)$, which is a ℓ-adic sheaf on X fitting in an exact sequence of ℓ-adic sheaves

$$0 \to T_\ell(G) \to T_\ell(M) \to Y \otimes \mathbb{Z}_\ell \to 0,$$

where $T_\ell(G) := \varprojlim G[\ell^n]$. Write $H^-_{\ell^\infty} := T_\ell(-)_{\bar\eta}$ and $V^-_{\ell^\infty} := H^-_{\ell^\infty} \otimes \mathbb{Q}_\ell$ for $- = A, G, T, Y$ etc. Applying the fibre functor $F_{\bar\eta}$, one obtains an extensions of $\pi_1(X)$-modules

$$0 \to H^G_{\ell^\infty} \to H^M_{\ell^\infty} \to Y \otimes \mathbb{Z}_\ell \to 0.$$

Let

$$\rho^M_{\ell^\infty} : \pi_1(X) \to \mathrm{GL}(H^M_{\ell^\infty})$$

denote the corresponding ℓ-adic representation. As in Section 1.2.1, the local representation

$$\rho^M_{x,\ell^\infty} := \rho^M_{\ell^\infty} \circ \sigma_x : \Gamma_{k(x)} \to \mathrm{GL}(H^M_{\ell^\infty})$$

identifies with the Galois representation attached to M_x. We use the same notation as in Section 1.2.1, indicating by a labeling $(-)^M$ the 1-motive to which the object we consider is attached (for instance, we will write $G^M_{\ell^\infty} := \mathrm{im}(\rho^M_{\ell^\infty})$, $\bar G^M_{\ell^\infty} := \rho^M_{\ell^\infty}(\pi_1(X_{\bar k}))$, $G^M_{x,\ell^\infty} := \mathrm{im}(\rho^M_{x,\ell^\infty})$ *etc.*). Eventually, write

$$U^M_{\ell^\infty} := \ker(G^M_{\ell^\infty} \twoheadrightarrow G^A_{\ell^\infty}), \quad U^M_{x,\ell^\infty} := \ker(G^M_{x,\ell^\infty} \twoheadrightarrow G^A_{x,\ell^\infty}).$$

Lemma 3.1. Given a closed point $x \in X$ the following are equivalent:

(i) G^M_{x,ℓ^∞} is open in $G^M_{\ell^\infty}$ for one prime ℓ;

(ii) G^M_{x,ℓ^∞} is open in $G^M_{\ell^\infty}$ for every prime ℓ. □

Proof. By definition G^M_{x,ℓ^∞} is open in $G^M_{\ell^\infty}$ if and only if G^A_{x,ℓ^∞} is open in $G^A_{\ell^\infty}$ and U^M_{x,ℓ^∞} is open in $U^M_{\ell^\infty}$. So, let ℓ be a prime such that G^M_{x,ℓ^∞} is open in $G^M_{\ell^\infty}$. Then, on the one hand, from Lemma 1.1, G^A_{x,λ^∞} is open in $G^A_{\lambda^\infty}$ for every prime λ. And, on the other hand, from [21, Theorem 1], the dimensions of $U^M_{\ell^\infty}$ and U^M_{x,ℓ^∞} are independent of ℓ. In particular,

$$\dim(U^M_{\lambda^\infty}) = \dim(U^M_{\ell^\infty}) = \dim(U^M_{x,\ell^\infty}) = \dim(U^M_{x,\lambda^\infty}),$$

which shows that U^M_{x,λ^∞} is open in $U^M_{\lambda^\infty}$ for every prime λ. ■

So, define X^{gen}_M to be the set of all $x \in |X|$ such that G^M_{x,ℓ^∞} is open in $G^M_{\ell^\infty}$ for one (or, equivalently, every) prime ℓ. (If $G = A$ is an abelian scheme and A_η contains no non-trivial k-isotrivial abelian subvariety, one can show that every open subgroup of $\bar{G}^M_{\ell^\infty}$ has finite abelianization. This ensures, again, that when X is a curve, for every integer $d \geq 1$, the set of all $x \in X^{ex}_M$ such that $[k(x):k] \leq d$ is finite [7, Theorem 1.1].) Set $X^{ex}_M := |X| \setminus X^{\mathrm{gen}}_M$

Theorem 3.2 (Adelic open image theorem for 1-motives). For every $x \in X^{\mathrm{gen}}_M$ the group G^M_x is open in G^M. □

Proof. As in the proof of Theorem 1.2, one may freely replace X with a connected étale cover. By definition $X^{ex}_A \subset X^{ex}_M$. In particular, from Theorem 1.2, up to replacing X with a connected étale cover, for every $x \in X^{\mathrm{gen}}_M$ the group G^A_x is open in G^A. Also, from [8, Corollary 4.5] and [35, Theorem 1], up to replacing again X with a connected étale cover, one may assume that

$$G^M = \prod_\ell G^M_{\ell^\infty}, \quad G^M_x = \prod_\ell G^M_{x,\ell^\infty},$$

and

$$G^A = \prod_\ell G^A_{\ell^\infty}, \quad G^A_x = \prod_\ell G^A_{x,\ell^\infty}.$$

As a result,

$$\ker(G^M \to G^A) = \prod_\ell U^M_{\ell^\infty}, \quad \ker(G^M_x \to G^A_x) = \prod_\ell U^M_{\ell^\infty,x}$$

and it is enough to show that for every $x \in X^{\mathrm{gen}}_M$ one has $U^M_{x,\ell^\infty} = U^M_{\ell^\infty}$ for $\ell \gg 0$. This follows from [21, Theorem 6.2] after observing that the construction of the semi-abelian scheme $U(M)$ in [21, Section 3.2] can be carried out in families. More precisely, for every prime ℓ

let $W_{-1}\,\mathrm{End}_{\mathbb{Q}_\ell}(V^M_{\ell^\infty})$ denote the Lie sub-algebra induced by the ascending filtration

$$W_i V^M_{\ell^\infty} = V^M_{\ell^\infty}, \quad i \geq 0, \quad W_{-1} V^M_{\ell^\infty} = V^G_{\ell^\infty}, \quad W_{-2} V^M_{\ell^\infty} = V^T_{\ell^\infty}, \quad W_i V^M_{\ell^\infty} = 0,\ i \leq -3.$$

that is,

$$W_i\mathrm{End}_{\mathbb{Q}_\ell}(V^M_{\ell^\infty}) := \{g \in \mathrm{End}_{\mathbb{Q}_\ell}(V^M_{\ell^\infty}) \mid g W_i V^M_{\ell^\infty} \subset W_{i-1} V^M_{\ell^\infty},\ i = 0, 1, 2\}.$$

Then, one can construct a semi-abelian scheme $U(M)$ over X, a section $u_M : X \to U(M)$ and, for every prime ℓ, a canonical isomorphism of $\pi_1(X)$-modules

$$\alpha^M_\ell : V^{U(M)}_{\ell^\infty} \tilde{\to} W_{-1}\mathrm{End}_{\mathbb{Q}_\ell}(V^M_{\ell^\infty})$$

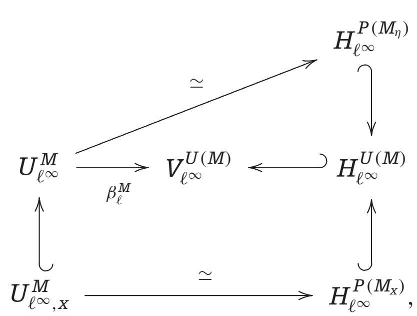
with the following property. (For instance, when $G = A$ is an abelian scheme and $Y \simeq \mathbb{Z}^{\oplus r}_X$ is constant, $U(M) = \underline{\mathrm{Hom}}(Y, A) \simeq A^{\oplus r}$ and $u_M : X \to A^{\oplus r}$ is the section corresponding to u: $Y \to A \in \underline{\mathrm{Hom}}(Y, A)(X)$, see [21, §3 and §6] for the general case, which is more technical.) Set

$$\beta^M_\ell : U^M_{\ell^\infty} \overset{log}{\hookrightarrow} W_{-1}\mathrm{End}_{\mathbb{Q}_\ell}(V^M_{\ell^\infty}) \overset{(\alpha^M_\ell)^{-1}}{\to} V^{U(M)}_{\ell^\infty}.$$

For every $t \in X$, let $P(M_t)$ denote the connected component of the Zariski closure of $\mathbb{Z}u_M(t)$ in $U(M)_t$. Then $\beta^M_\ell(U^M_{\ell^\infty,t})$ is contained and open in $V^{P(M_t)}_{\ell^\infty}$. Furthermore, for $\ell \gg 0$ (depending on t) one has

$$\beta^M_\ell(U^M_{\ell^\infty,t}) = H^{P(M_t)}_{\ell^\infty}.$$

Thus, for $x \in |X|$ (and fixing an étale path $\alpha : F_{\bar{\eta}} \tilde{\to} F_{\bar{x}}$), one obtains a commutative diagram



where the inclusions

$$H^{P(M_x)}_{\ell^\infty} \hookrightarrow H^{U(M)}_{\ell^\infty}(\simeq H^{U(M)_x}_{\ell^\infty}), \quad H^{P(M_\eta)}_{\ell^\infty} \hookrightarrow H^{U(M)}_{\ell^\infty}(\simeq H^{U(M)_\eta}_{\ell^\infty})$$

are those induced from the inclusions of semi-abelian schemes $P(M_x) \hookrightarrow U(M)_x$ and $P(M_\eta) \hookrightarrow U(M)_\eta$. In particular, the cokernel of $H_{\ell^\infty}^{P(M_x)} \hookrightarrow H_{\ell^\infty}^{P(M_\eta)}$ is torsion-free. But if $x \in X_M^{\mathrm{gen}}$, $H_{\ell^\infty}^{P(M_x)}$ is open in $H_{\ell^\infty}^{P(M_\eta)}$, which forces $H_{\ell^\infty}^{P(M_x)} = H_{\ell^\infty}^{P(M_\eta)}$. ■

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References

[1] André, Y. "Pour une théorie inconditionnelle des motifs." *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 83, no. 1 (1996): 5–49.

[2] Bogomolov, F. "Sur l'algébricité des représentations ℓ-adiques." *Comptes Rendus de l'Académie des Sciences Paris* 290 (1980): 701–103.

[3] Borovoi, M. "The action of the Galois group on the rational cohomology classes of type (p, p) of Abelian varieties." *Mathematics of the USSR-Sbornik* 94 (1974): 649–752.

[4] Cadoret, A. "Motivated Cycles Under Specialization." In *Groupes de Galois Géométriques et differentiels*, edited by P. Boalch and J.-M. Couveignes, 25–55. Séminaires et Congrès 27. S.M.F., 2012.

[5] Cadoret, A. "ℓ-independency of the exceptional cohomological loci." (2014): preprint.

[6] Cadoret, A. and A. Tamagawa. "A uniform open image theorem for ℓ-adic representations I." *Duke Mathematical Journal* 161 (2012): 2605–34.

[7] Cadoret, A. and A. Tamagawa. "A uniform open image theorem for ℓ-adic representations II." *Duke Mathematical Journal* 162 (2013): 2301–44.

[8] Cadoret, A. and A. Tamagawa. "On subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$ and representations of the étale fundamental group." (2013): preprint.

[9] Cadoret, A. and A. Kret. "Galois generic points on Shimura varieties." In preparation.

[10] Clozel, L., H. Oh, and E. Ullmo. "Hecke operators and equidistribution of Hecke points." *Inventiones Mathematicae* 144 (2001): 327–51.

[11] Deligne, P. "Théorie de Hodge, II." *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 40 (1971): 5–57.

[12] Deligne, P. "La conjecture de Weil pour les surfaces $K3$." *Inventiones Mathematicae* 15 (1972): 206–26.

[13] Deligne, P. "Théorie de Hodge III." *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 44 (1974): 5–78.

[14] Deligne, P. "Cycles on Abelian Varieties (Notes by J.S. Milne)." In *Motives, Hodge Cycles and Shimura Varieties*, p. 7100. Lecture Notes in Mathematics 900, Springer, Berlin, Heidelberg, New York 1982.

[15] Faltings, G. and G. Wüstholz (eds.), *Rational Points*. Aspects of Mathematics E6. Friedr. Vieweg & Sohn, Braunschweig, 1984.

[16] Gille, P. "The Borel-de Siebenthal Theorem." (2010): preprint, available on www.math.ens.fr/~gille/prenotes/bds.pdf.

[17] Hui, C.-Y. "Specialization of monodromy and ℓ-independence." *Comptes Rendus de l'Acadmie des Sciences* 350 (2012): 5–7.

[18] Hui, C.-Y. "Monodromy of Galois representations and equal-rank subalgebra equivalence." *Mathematical Research Letters* 20 (2013): 1–24.

[19] Hui, C.-Y. "ℓ-independence for compatible systems of (mod ℓ) Galois representations." *Compositio Mathematica*, to appear, arXiv:1305.2001.

[20] Hui, C.-Y. and M. Larsen. "Type A Images of Galois Representations and Maximality." (2014): preprint arXiv:1305.1989.

[21] Jossen, P. "On the Mumford-Tate conjecture for 1-motives." *Inventiones Mathematicae* 195 (2014): 393–439.

[22] Larsen, M. "Maximality of Galois actions for compatible systems." *Duke Mathematical Journal* 80 (1995): 601–30.

[23] Lubotzky, A. and D.Segal. *Subgroup growth*. Birkhäuser, Basel, 2003.

[24] Milne, J. S. "Shimura varieties and Moduli." (2011): preprint, available on http://www.jmilne.org/math/xnotes/svh.pdf.

[25] Nori, M. V. "On subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$." *Inventiones Mathematicae* 88 (1987): 257–75.

[26] Pépin Le Halleur, S. "Subgroups of maximal ranks of reductive groups." (2012): preprint, available on www.math.ens.fr/~gille/actes/pepin.pdf.

[27] Piatetskii-Shapiro, I. I. "Interrelation between the Tate and the Hodge conjectures for Abelian varieties." *Mathematics of the USSR-Sbornik* 94 (1971): 615–25.

[28] Pink, R. "A Combination of the Conjectures of Mordell-Lang and André-Oort." In *Geometric Methods in Algebra and Number Theory*, 251–82. edited by Bogomolov, F. and Y. Tschinkel, Progress in Mathematics 253. Birkhäuser, Basel, 2005.

[29] Pink, R. "A Common generalization of the conjectures of André-Oort, Manin-Mumford and Mordell-Lang." Unpublished Preprint 2005.

[30] Serre, J.-P. *Abelian ℓ-adic Representations and Elliptic Curves*. Research Notes in Mathematics. W.A. Benjamin, 1968.

[31] Serre, J.-P. *Corps Locaux*. Paris: Hermann, 1968.

[32] Serre, J.-P. *Lectures on the Mordell-Weil Theorem*. Aspects of Mathematics E15. Friedr. Vieweg & Sohn, Braunschweig, 1989.

[33] Serre, J.-P. *Lettres à Ken Ribet du 1/1/1981 et du 29/1/1981*, Oeuvres. Collected papers. IV, 1985–1998, Springer, Berlin, Heidelberg, New York, 2000.

[34] Serre, J. P. *Lettre à Marie-France Vigneras du 10/2/1986*, Oeuvres. Collected papers. IV, 1985–1998, Springer, Berlin, Heidelberg, New York, 2000.

[35] Serre, J.-P. "Un critère d'indépendance pour une famille de représentations ℓ-adiques." *Commentarii Mathematici Helvetici* 88 (2013): 543–76.