Tannakian Cebotarev density theorem
(joint with A. Tamagawa)

Arithmetic of Algebraic varieties - BICMR, Beijing University China
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X_0 : smooth variety ($=$ separated, of finite type, geo. connected) over k_0

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$$\begin{array}{ccccc} Y_0 & \xrightarrow{f_0} & X_0 & \longrightarrow & k_0 \\ \uparrow & \square & \uparrow & \square & \uparrow \\ Y & \xrightarrow[f]{} & X & \longrightarrow & k \end{array}$$

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$\mathcal{U}$: set of non-principal ultrafilters on $\mathcal{L}$

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- For $\mathfrak{l} \in \mathcal{L} \cup \{p\}$, $\overline{\mathbb{Q}}_{\mathfrak{l}}$: algebraic closure of the completion of $\mathbb{Q}$ at $\mathfrak{l}$
- For $\mathfrak{u} \in \mathcal{U}$: $\underline{\mathbb{F}} := \prod_{\ell \in \mathcal{L}} \overline{\mathbb{F}}_\ell \twoheadrightarrow \overline{\mathbb{Q}}_{\mathfrak{u}}$ the corresponding ultraproduct

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$Q = \overline{\mathbb{Q}}_\ell$, $\ell \in \mathcal{L}$, $\overline{\mathbb{Q}}_\mathfrak{u}$, $\mathfrak{u} \in \mathcal{U}$, $\overline{\mathbb{Q}}_p$, $\mathcal{C}(X_0, Q)$ category of Q-coefficients over X_0 *i.e.*

- For $Q = \overline{\mathbb{Q}}_\ell$: Weil $\overline{\mathbb{Q}}_\ell$-sheaf on X_0
- For $Q = \overline{\mathbb{Q}}_\mathfrak{u}$: almost $\mathfrak{u}$-tame sheaf on X_0
- For $Q = \overline{\mathbb{Q}}_p$: overconvergent $\overline{\mathbb{Q}}_p$-F-isocrystals on X_0

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$$\begin{array}{ccc}
\mathcal{C}(X_0, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\simeq} & Rep^{cont}_{\overline{\mathbb{Q}}_\ell}(W(X_0)) \\
\uparrow & & \uparrow \\
\text{Étale } \overline{\mathbb{Q}}_\ell\text{-coeff.} & \xrightarrow{\simeq} & Rep^{cont}_{\overline{\mathbb{Q}}_\ell}(\pi_1(X_0))
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}(X_0, \overline{\mathbb{Q}}_\mathfrak{u}) & \hookrightarrow & Rep_{\overline{\mathbb{Q}}_\mathfrak{u}}(W(X_0)) \\
& \searrow & \uparrow \\
& & Rep_{\overline{\mathbb{Q}}_\mathfrak{u}}(\pi_1(X_0))
\end{array}$$

$(W(X_0) := \pi_1(X_0) \times_{\pi_1(k_0)} \varphi^{\mathbb{Z}})$

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$SS\mathcal{C}(X_0, Q) \hookrightarrow \mathcal{C}(X_0, Q)$ full subcategory of semisimple Q-coefficients

$\mathcal{C}(X, Q)$ geometric version

- For $Q = \overline{\mathbb{Q}}_\mathfrak{l}$: Weil=étale $\overline{\mathbb{Q}}_\ell$-sheaf on X
- For $Q = \overline{\mathbb{Q}}_\mathfrak{u}$: almost $\mathfrak{u}$-tame sheaf on X
- For $Q = \overline{\mathbb{Q}}_\mathfrak{l}$: overconvergent $\overline{\mathbb{Q}}_p$-isocrystals on X_0

+ canonical functor $\mathcal{C}(X_0, Q) \to \mathcal{C}(X, Q)$, $\mathcal{C}_0 \to \mathcal{C} := \mathcal{C}_0|_X$

Output of the ‘philosophy’ of motives

(Grothendieck, Serre, Tate, Deligne *etc.*)

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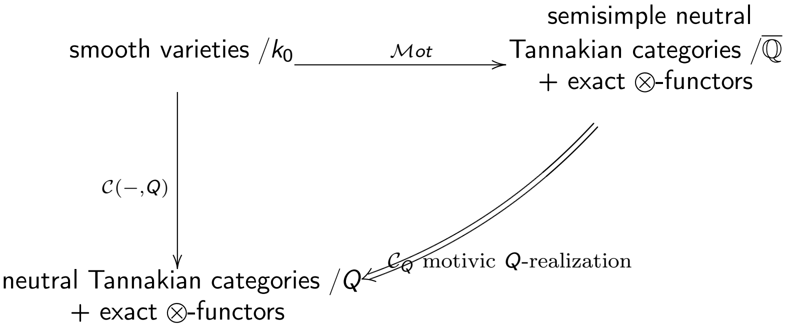

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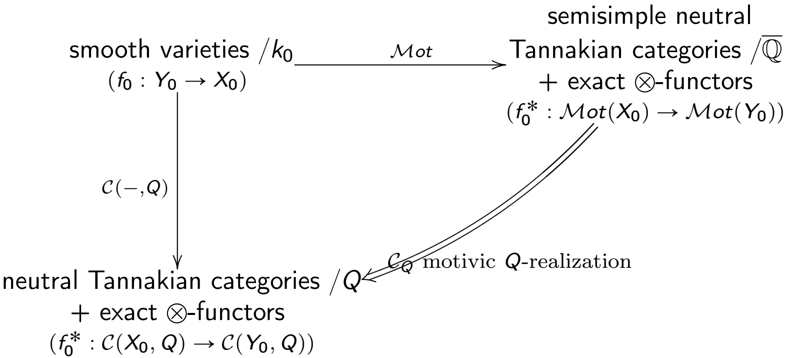

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$$\begin{array}{ccc} \mathcal{M}ot(X_0) \otimes_{\overline{\mathbb{Q}}} Q & \longrightarrow & \mathcal{C}(X_0, Q) \\ & \searrow^{\simeq} & \uparrow \\ & & SS\mathcal{C}(X_0, Q) \end{array}$$

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$$\mathcal{M}ot(X_0)\otimes_{\overline{\mathbb{Q}}} Q \xrightarrow{C_Q(X_0)} \mathcal{C}(X_0, Q)$$

$$\mathcal{M}ot(X_0)\otimes_{\overline{\mathbb{Q}}} Q \xrightarrow{\simeq} SS\mathcal{C}(X_0, \overline{\mathbb{Q}}_p), \quad \mathcal{M}ot(X_0)\otimes_{\overline{\mathbb{Q}}} Q \xrightarrow{\simeq} SS\mathcal{C}(X_0, \overline{\mathbb{Q}}_u), \quad \mathcal{M}ot(X_0)\otimes_{\overline{\mathbb{Q}}} Q \xrightarrow{\simeq} SS\mathcal{C}(X_0, \overline{\mathbb{Q}}_\ell), \quad \mathcal{M}ot(X_0)\otimes_{\overline{\mathbb{Q}}} Q \xrightarrow{\simeq} SS\mathcal{C}(X_0, Q)$$

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$$\mathcal{M}ot(X_0) \otimes_{\overline{\mathbb{Q}}} Q \xrightarrow{\mathcal{C}_Q(X_0)} \mathcal{C}(X_0, Q)$$

$$\mathcal{M}ot(X_0) \otimes_{\overline{\mathbb{Q}}} Q \xrightarrow{\simeq} SSC(X_0, \overline{\mathbb{Q}}_p), \quad SSC(X_0, \overline{\mathbb{Q}}_u), \quad SSC(X_0, \overline{\mathbb{Q}}_\ell), \quad SSC(X_0, Q) \hookrightarrow \mathcal{C}(X_0, Q)$$

$M \in \mathcal{M}ot(X_0)$, $k(x_0) \xrightarrow{x_0} X_0$,

$$\begin{array}{ccc} Rep_Q(G(M)_Q) \simeq \langle M \rangle^{\otimes} \otimes_{\overline{\mathbb{Q}}} Q & \xrightarrow{\simeq} & \langle \mathcal{C}_Q(M) \rangle^{\otimes} \simeq Rep_Q(G(\mathcal{C}_Q(M)) \\ \downarrow x_0^* & & \downarrow x_0^* \\ Rep_Q(G(x_0^* M)_Q) \simeq \langle x_0^* M \rangle^{\otimes} \otimes_{\overline{\mathbb{Q}}} Q & \xrightarrow[\simeq]{} & \langle x_0^* \mathcal{C}_Q(M) \rangle^{\otimes} \simeq Rep_Q(G(x_0^* \mathcal{C}_Q(M)) \end{array}$$

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 \swarrow \simeq & \swarrow \simeq & \downarrow \simeq & \searrow \simeq & \uparrow \\
 SSC(X_0,\overline{\mathbb{Q}}_p) & SSC(X_0,\overline{\mathbb{Q}}_u) & SSC(X_0,\overline{\mathbb{Q}}_\ell) & & SSC(X_0,Q)
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$$
\begin{array}{ccc}
G(M)_Q & \xrightarrow{\simeq} & G(\mathcal{C}_Q(M)) \\
\uparrow x_0^* & & \uparrow x_0^* \\
G(x_0^* M)_Q & \xrightarrow[\simeq]{} & G(x_0^*\mathcal{C}_Q(M)) = \overline{(\varphi_{x_0}^{\mathbb{Z}})^{zar}}
\end{array}
$$

commutative
up to conjugacy

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$$\mathcal{M}ot(X_0)\otimes_{\overline{\mathbb{Q}}} Q \xrightarrow{C_Q(X_0)} \mathcal{C}(X_0, Q)$$

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$M\in\mathcal{M}ot(X_0)$, $k(x_0)\xrightarrow{x_0} X_0$,

$$\begin{array}{ccc} G(M)_{\overline{\mathbb{Q}}_\ell} & \xrightarrow{\simeq} & G(\mathcal{C}_{\overline{\mathbb{Q}}_\ell}(M)) \\ \uparrow x_0^* & & \uparrow x_0^* \\ G(x_0^* M)_{\overline{\mathbb{Q}}_\ell} & \xrightarrow{\simeq} & G(x_0^*\mathcal{C}_{\overline{\mathbb{Q}}_\ell}(M)) = \overline{(\varphi_{x_0}^{\mathbb{Z}})^{zar}} \end{array}$$

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$M \in \mathcal{M}ot(X_0)$, $k(x_0) \xrightarrow{x_0} X_0$, $\mathcal{C}_{\overline{\mathbb{Q}}_\ell}(M)$ étale $\leftrightarrow \pi_1(X_0) \curvearrowright M_{\overline{\mathbb{Q}}_\ell}$

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$G(\mathcal{C}_{\overline{\mathbb{Q}}_\ell}(M))$: Zar. closure of image of $\pi_1(X_0) \curvearrowright M_{\overline{\mathbb{Q}}_\ell}$

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$S\subset|X_0|$ of upper density >0 (resp. 1), ℓ-adic Cebotarev density theorem $\Rightarrow$ the Zar. closure of the conjugacy classes of the φ_{x_0}, $x_0\in S$ is dense in at least one connected component of (resp. is dense in) $G(\mathcal{C}_{\overline{\mathbb{Q}}_\ell}(M))$

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$G(\mathcal{C}_{\overline{\mathbb{Q}}_\ell}(M))$: Zar. closure of image of $\pi_1(X_0) \curvearrowright M_{\overline{\mathbb{Q}}_\ell}$

$S \subset |X_0|$ of upper density > 0 (resp. 1), $\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}$
$\Rightarrow$ the Zar. closure of the conjugacy classes of the φ_{x_0}, $x_0 \in S$ is dense in at least one connected component of (resp. is dense in) $G(M)$

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Main result

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$\mathcal{C}_0$ Q-coefficient (not necessarily semisimple) on X_0

$x_0 \in |X_0|$, $\Phi^{\mathcal{C}_0}_{x_0} \subset G_0(\mathcal{C}_0)$: conj. class of φ_{x_0} (well-defined)

$S \subset |X_0|$, $\Phi^{\mathcal{C}_0}_S := \bigcup_{x_0 \in S} \Phi^{\mathcal{C}_0}_{x_0}$

Thm. (C.-Tamagawa, 2019)

$S \subset |X_0|$ of upper density > 0 (resp. 1) $\Rightarrow \overline{\Phi_S}^{zar}$ contains at least one connected component of $G(\mathcal{C}_0)$ (resp. $\overline{\Phi_S}^{zar} = G(\mathcal{C}_0)$)

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- Main ingredient : **theory of companions** ($Q \to \overline{\mathbb{Q}}_\ell$) to reduce to the case of étale $\overline{\mathbb{Q}}_\ell$-coefficients

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- Main ingredient : **theory of companions** ($Q \to \overline{\mathbb{Q}}_\ell$) to reduce to the case of étale $\overline{\mathbb{Q}}_\ell$-coefficients
- For $Q = \overline{\mathbb{Q}}_p$, $\mathcal{C}_0$ semisimple, X_0 curve : Hartl-Pàl (arXiv :1811.07084)

$\mathcal{C}_0 = \oplus_{i \in I} \mathcal{I}_{i,0}^{(\alpha_i)}$ with $\mathcal{I}_{i,0}$ irreducible with finite det., $\alpha_i \in \overline{\mathbb{Q}}_p^\times$

$$\mathcal{C}_0 \in \langle \underbrace{(\mathcal{C}_0^{der} := \oplus_{i \in I} \mathcal{I}_{i,0})}_{\text{semisimple mono., pure of weight } 0} \oplus \underbrace{(\mathcal{C}_0^{cst} := \oplus_{i \in I} \overline{\mathbb{Q}}_p^{(\alpha_i)})}_{\text{geo. constant}} \rangle$$

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$\mathcal{C}_0 = \oplus_{i \in I} \mathcal{I}_{i,0}^{(\alpha_i)}$ with $\mathcal{I}_{i,0}$ irreducible with finite det., $\alpha_i \in \overline{\mathbb{Q}}_p^\times$

$$\mathcal{C}_0 \in \langle \underbrace{(\mathcal{C}_0^{der} := \oplus_{i \in I} \mathcal{I}_{i,0})}_{\text{semisimple mono., pure of weight } 0} \oplus \underbrace{(\mathcal{C}_0^{cst} := \oplus_{i \in I} \overline{\mathbb{Q}}_p^{(\alpha_i)})}_{\text{geo. constant}} \rangle$$

May assume $\mathcal{C}_0 = \mathcal{C}_0^{der} \oplus \mathcal{C}_0^{cst}$, $G(\mathcal{C}_0) \sim G(\mathcal{C}_0^{der}) \times G(\mathcal{C}_0^{cst})$

Main result

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Rem : $\mathcal{C}'_0$ is then automatically irreducible with finite determinant

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- For higer dimensional X_0, reduce to the case of curves by geometric methods (no motives...)

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Rem. Unipotent radical of Cartan of G_0 has dimension $\leqslant 1$!

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Apply the classical Cebotarev density theorem to the *continuous* $\pi_1(X_0) \twoheadrightarrow \pi_0(G_0)$ + non resp. part of Thm. (Exo)

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- For $p \neq \ell \gg 0$ one can choose $\iota_\ell : \overline{\mathbb{Q}}_\ell \tilde{\rightarrow} \mathbb{C}$ such that $\iota_\ell^{-1} \iota(\alpha_i) \in \overline{\mathbb{Z}}_\ell^\times$ (Exo)

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$\Rightarrow$ rank(G_0) =rank$(G_{0,\ell})$

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The companion correspondance $\mathcal{O}b(\langle \mathcal{C}_0 \rangle^{\otimes})/ \simeq \mapsto \mathcal{O}b(\langle \mathcal{C}_{0,\ell} \rangle^{\otimes})/ \simeq$ induces a canonical semiring isomorphism

$$\mathbb{C}^+[G_0] \tilde{\to} \mathbb{C}^+[G_{0,\ell}],$$

characterized by the fact that it preserves local L-functions, and which maps irreducible representations to irreducible representations.

- $G_0 = G_0^\circ \Leftrightarrow G_{0,\ell} = G_{0,\ell}^\circ$
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Step 4,5 : Apply the companion conjecture (connected case)

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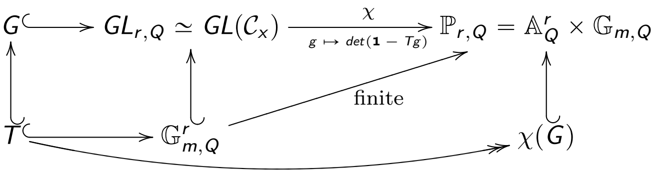

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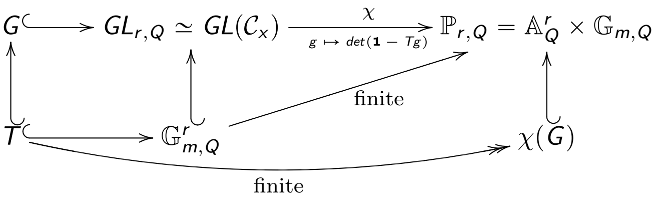

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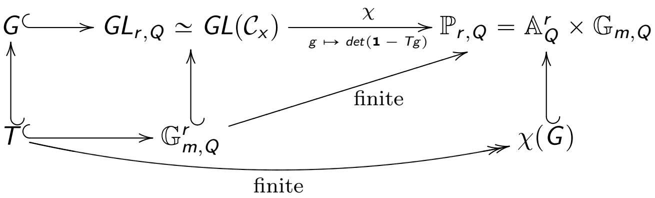

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$$\chi(G_{0,\ell}) \hookleftarrow \overline{Z_S}^{zar} \hookrightarrow \chi(G_0)$$

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Cebotarev for $\mathcal{C}_{0,\ell}$

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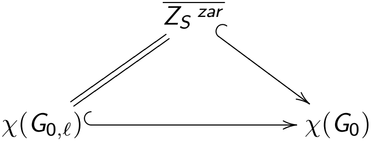

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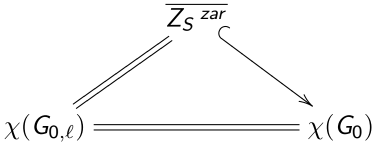

$\dim(\chi(G_0)) = \mathrm{rank}(G_0) = \mathrm{rank}(G_{0,\ell}) = \dim(\chi(G_{0,\ell}))$

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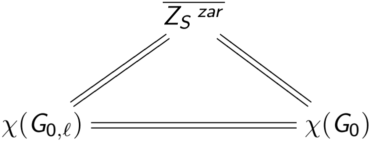

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$$\overline{Z_S}^{zar}$$

$$\chi(G_{0,\ell}) = \overline{Z_S}^{zar} = \chi(G_0), \qquad \chi(G_{0,\ell}) = \chi(G_0)$$

dim$(\chi(G_0))$ =rank(G_0) =rank$(G_{0,\ell})$ =dim$(\chi(G_{0,\ell}))$ $\Rightarrow \overline{\Phi_S^{\mathcal{C}_0}}^{zar} = G_0$

A criterion for Zariski-density

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$g \in G$ semisimple lifting $\overline{g} \in \pi_0(G)$, $C_g \subset Z_{G^\circ}(g)^\circ$ Cartan,

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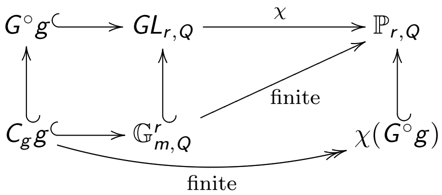

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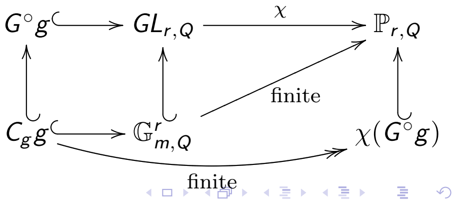

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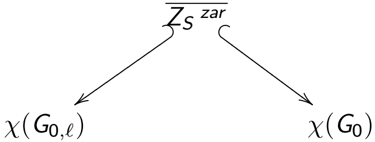

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- $\ker(\pi_1(X_0) \to G_{0,\ell} \to \pi_0(G_{0,\ell})) \leftrightarrow p : \tilde{X}_0 \to X_0$ Galois cover
$$\begin{array}{ccccc} G_0 & \twoheadrightarrow & \pi_0(G_0) \simeq \mathrm{Aut}(p) \simeq \pi_0(G_{0,\ell}) & \twoheadleftarrow & \pi_0(G_{0,\ell}) \\ g & \to & \overline{g} & \leftarrow & g_\ell \end{array}$$
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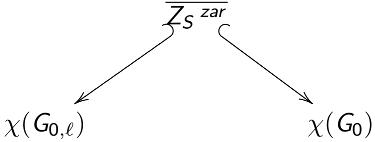

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  (Classical Cebotarev) may assume $\Phi_S^{\mathcal{C}_0} \subset G_0^\circ g$, $\Phi_S^{\mathcal{C}_{0,\ell}} \subset G_{0,\ell}^\circ g_\ell$
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$$\chi(G_{0,\ell}^\circ g_\ell) \hookleftarrow \overline{Z_S}^{zar} \hookrightarrow \chi(G_0^\circ g)$$

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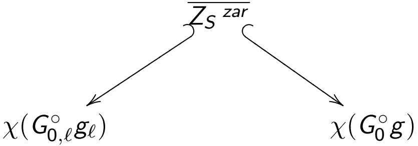

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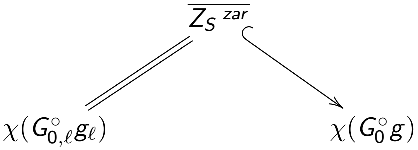

Cebotarev for $\mathcal{C}_{0,\ell}$

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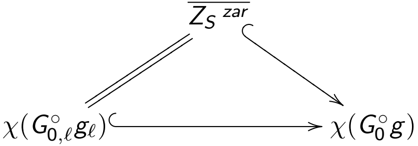

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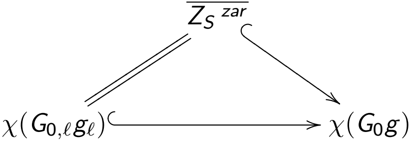

$$\mathrm{rank}(Z_{G_0^\circ}(g)^\circ) = \dim(\chi(G_0^\circ g)) \geqslant \dim(\chi(G_{0,\ell}^\circ g_\ell)) = \mathrm{rank}(Z_{G_{0,\ell}^\circ}(g_\ell)^\circ)$$

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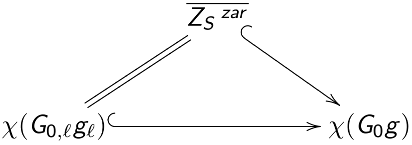

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$$\begin{array}{ccc} \mathcal{O}b(\langle p^*\mathcal{C}_0\rangle^{\otimes})/\simeq & \longleftrightarrow & \mathcal{O}b(\langle p^*\mathcal{C}_{0,\ell}\rangle^{\otimes})/\simeq \\ \uparrow g^* & & \uparrow g_\ell^* \\ \mathcal{O}b(\langle p^*\mathcal{C}_0\rangle^{\otimes})/\simeq & \longleftrightarrow & \mathcal{O}b(\langle p^*\mathcal{C}_{0,\ell}\rangle^{\otimes})/\simeq \end{array} \qquad \longleftrightarrow$$

$$\begin{array}{ccc} \mathbb{C}^+[G_0^\circ] & \longleftrightarrow & \mathbb{C}^+[G_{0,\ell}^\circ] \\ \uparrow g-g^{-1} & & \uparrow g_\ell - g_\ell^{-1} \\ \mathbb{C}^+[G_0^\circ] & \longleftrightarrow & \mathbb{C}^+[G_{0,\ell}^\circ] \end{array} \quad \overset{\text{[K-L-V,14]}}{\longleftrightarrow} \quad \begin{array}{ccc} G_0^\circ & \longleftrightarrow & G_{0,\ell}^\circ \\ \downarrow g-g^{-1} & & \downarrow g_\ell - g_\ell^{-1} \\ G_0^\circ & \longleftrightarrow & G_{0,\ell}^\circ \end{array}$$

Step 6,7 : General case

$G_0 := G(\mathcal{C}_0)$, $G := G(\mathcal{C}_0|_X)$
(Classical) Cebotarev $\Rightarrow$ may assume $\Phi_S \subset G_0^\circ g$ with $g = \varphi_{x_0}^{ss}$ for some $x_0 \in |X_0|$

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 & & \downarrow & & \| & & \downarrow & & \\
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Weight filtration + Weil II $\Rightarrow$
$G_0 = R_u(G_0) \times G_0^{red}$;
$\Leftrightarrow$ $G_0 = \mathbb{G}_{a,Q}^{\epsilon} \times \times G_0^{red}$ with $\epsilon = 0, 1$;
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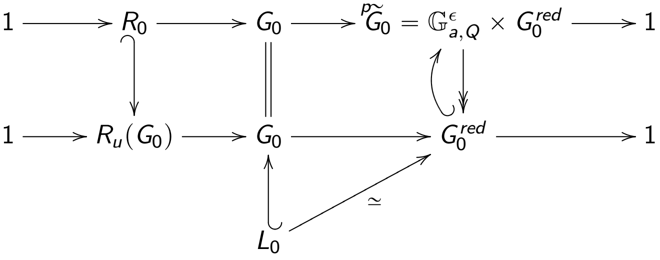

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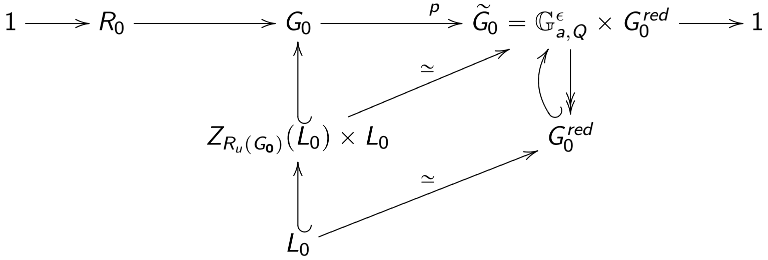

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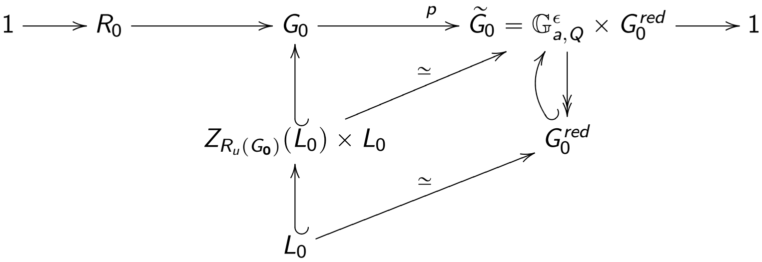

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  Key point : $\widetilde{\mathcal{C}}_0 \sim \widetilde{\mathcal{C}}_{0,\ell} = \mathcal{U}_{0,\ell} \oplus \mathcal{C}_{0,\ell}^{ss}$ with $\mathcal{C}_0^{ss} \sim \mathcal{C}_{0,\ell}^{ss}$ and

$$\mathcal{U}_{0,\ell} \leftrightarrow \pi_1(X_0) \twoheadrightarrow \pi_1(k_0) \stackrel{\varphi\mapsto U}{\to} GL_2(\overline{\mathbb{Q}}_\ell), \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

謝謝